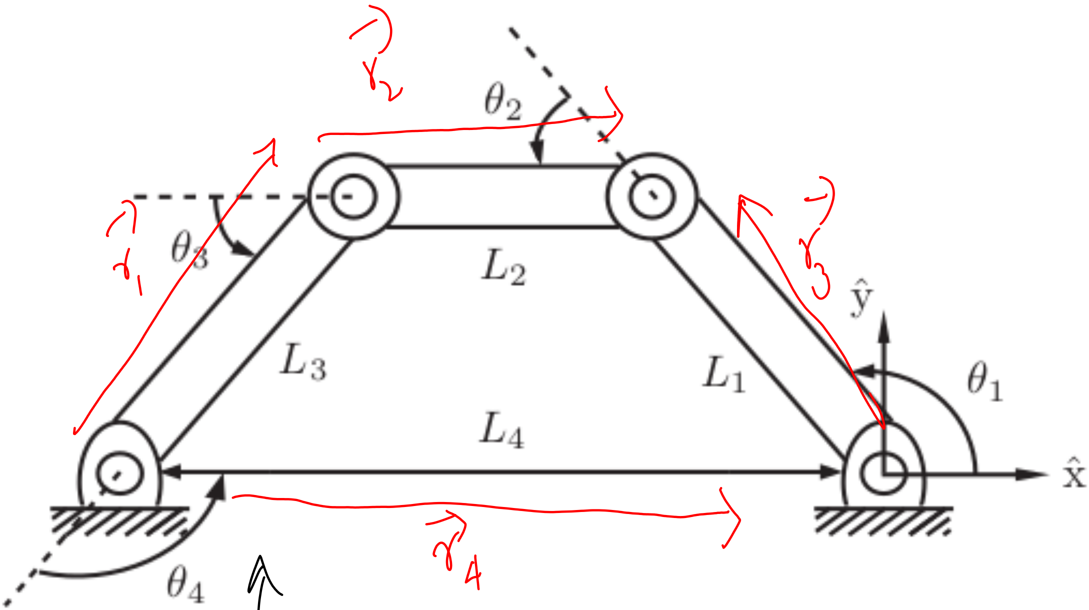
⇒ Configuration & Velocity Constrain



four bar linkage

The Constrains of the 4 bar system

$$L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2) + \dots + L_4 \cos(\theta_1 + \dots \theta_4) = 0$$
$$L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2) + \dots + L_4 \sin(\theta_1 + \dots \theta_4) = 0$$
$$\theta_1 + \theta_2 + \theta_3 + \theta_4 - 2\pi = 0$$

Loop Closure equation

For general robots C-space implicitly represented as $\theta = [\theta_1, \theta_2, \theta_3 \dots]^T$

$$g(\theta) = \begin{bmatrix} g(\theta_1, \dots, \theta_n) \\ \vdots \\ g(\theta_1 \dots \theta_n) \end{bmatrix} = 0$$

k = no. of independant eq

https://tutorial.math.lamar.edu/Classes/DE/LA_Matrix.aspx

Holonomic Constrain

Constraints that reduce the dimension of C-space.

C-space can be n-k dimension surface

When the robot is in motion

$$\frac{d}{dt} g(\theta(t)) = 0$$

Thus

$$\begin{bmatrix} \frac{\partial g_1}{\partial \theta_1}(\theta)\dot{\theta}_1 + \cdots + \frac{\partial g_1}{\partial \theta_n}(\theta)\dot{\theta}_n \\ \vdots \\ \frac{\partial g_k}{\partial \theta_1}(\theta)\dot{\theta}_1 + \cdots + \frac{\partial g_k}{\partial \theta_n}(\theta)\dot{\theta}_n \end{bmatrix} = 0.$$

By takin $\dot{\theta}$'s out

$$\begin{bmatrix} \frac{\partial g_1}{\partial \theta_1}(\theta) & \cdots & \frac{\partial g_1}{\partial \theta_n}(\theta) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_k}{\partial \theta_1}(\theta) & \cdots & \frac{\partial g_k}{\partial \theta_n}(\theta) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} = 0.$$

$\Rightarrow \quad \frac{\partial g}{\partial \theta}(\theta)\dot{\theta} = 0 \quad \Rightarrow$ Partial diffrentiation

$A(\theta)\dot{\theta} = 0 \quad \Rightarrow$ Pfaffian Constrain

Also $A(\theta) = \frac{\partial g(\theta)}{\partial \theta} \quad \Rightarrow \quad \int A(\theta) = g(\theta)$

Integrable Constrain

Non-holonomic Constrain

-> Pfaffian Constrain that are not integrable

-> reduces feasible velocities but not reachable C-space

eg -> Coin problem

=> Task Space & Workspace

Task Space - Space in which robot is expressed for pen and paper - $\mathbb{R}^2$

Workspace - Configuration that endeffector can reach

Rigid Body Motions

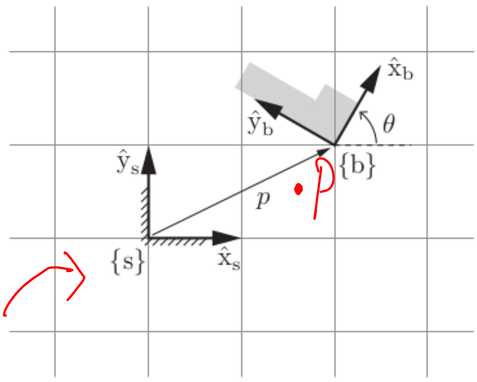
Rigid body velocity - $\mathbb{R}^6$ -> Spatial Velocity (Twist)

Rigid body motions in plane

w.r.t $\{s\}$

$$P = P_x \hat{x}_s + P_y \hat{y}_s$$

To reduce ambiguity
(indicate reference frame)



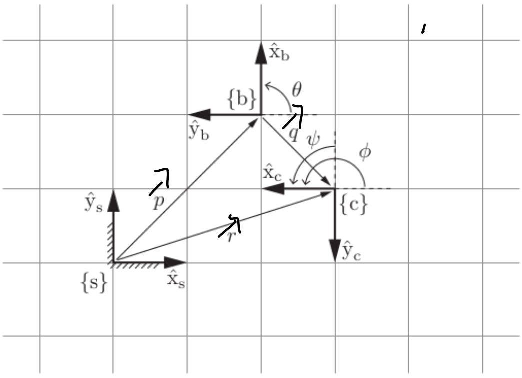
$\{b\}$ w.r.t $\{s\}$

$$\rightarrow \quad \hat{x}_b = \cos\theta\, \hat{x}_s + \sin\theta\, \hat{y}_s$$

$$\hat{y}_b = -\sin\theta\, \hat{y}_s + \cos\theta \cdot \hat{y}_s$$

point $p \in \mathbb{R}^2$

$$p = \begin{bmatrix} P_x \\ P_y \end{bmatrix}$$



Vector - $\vec{p}, \vec{q}, \vec{r}$

Points - p, q, r

$$p = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

$$x_b = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$y_b = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$P = \begin{bmatrix} \hat{x}_b & \hat{y}_b \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

rotation matrix

$(P, p) \rightarrow$ represent orientation & position

By knowing R_{bc} & R_{sb} - we can find R_{cs}

Using rotation matrix vector pair.

1) represent conf of rigid body
2) to change reference frame
3) to displace a vector or fram

$\rightarrow$ rotation followed by translation Screw motion

this can be represented by (β, s_x, s_y)

Exponential Coordinates

$\rightarrow$ placement obtained by simultaneous angular & linear velocity

$\rightarrow (\omega, v_x, v_y)$

Chasles - Mozzi theorem

every rigid body displacement is achieved by finite rotation and translation on screw axis

⇒ Rotation of Angular Velocities

Conditions of rotation matrix

r_{ij} : $i \downarrow$ $j \rightarrow$

Unit

$$r_{11}^2 + r_{21}^2 + r_{31}^2 = 1$$
$$r_{12}^2 + r_{22}^2 + r_{32}^2 = 1$$
$$r_{13}^2 + r_{23}^2 + r_{33}^2 = 1$$

Orthogonal

$$r_{11}r_{12} + r_{21}r_{22} + r_{31}r_{32} = 0$$
$$r_{12}r_{13} + r_{22}r_{23} + r_{32}r_{33} = 0$$
$$r_{13}r_{13} + r_{21}r_{23} + r_{31}r_{33} = 0$$

⇒ 6 constrains of rotational matrix

Definition 3-1

Special orthogonal group SO(3) ⇒ $R^T R = I$, $|R| = 1$

Definition 3-2

Special orthogonal group SO(2)

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Matrix equation

$$a = px + qy$$
$$b = pu + qv$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x & y \\ u & v \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

These are called groups because they satisfy required property of a mathematical group

→ A, B in a group AB is also in group

→ $AB(C) = A(BC)$

→ $AI = IA = A$

→ $A^{-1}A = AA^{-1} = I$

$$R^{-1} = R^T$$

$R_1 \times R_2 \rightarrow$ Rotation matrix

$(R_1 R_2) R_3 = R_1 (R_2 R_3)$ but $R_1 R_2 \neq R_2 R_1$

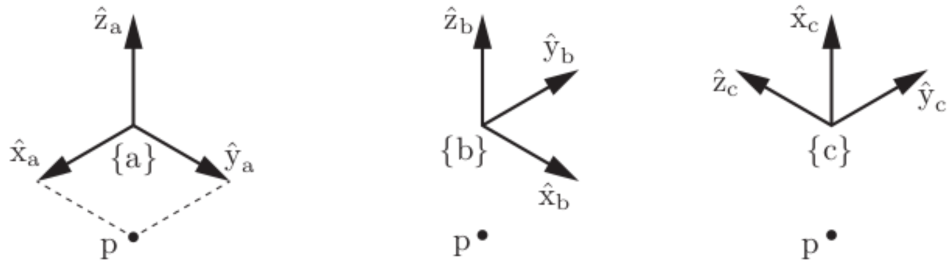
Uses of Rotation Matrix

(a) to represent an orientation;

(b) to change the reference frame in which a vector or a frame is represented;

(c) to rotate a vector or a frame.

Representing an orientation



$$R_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_b = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_c = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_a = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad P_b = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad P_c = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

$$\{a\} \xrightarrow{z\,-90^\circ} \{b\} \xrightarrow{y\,-90^\circ} \{c\}$$

$$R_{ac} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \quad R_{ca} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

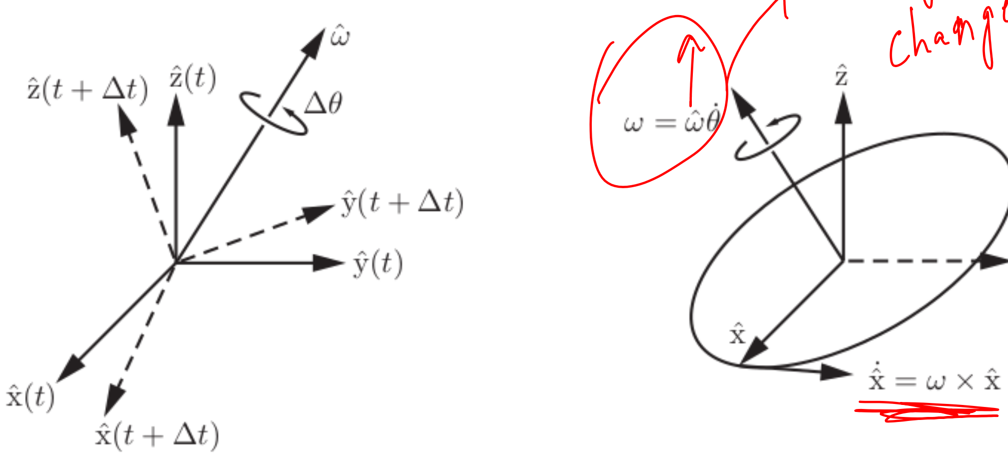
$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Angular Velocity

unit vector → rate of change of theta



$\hat{x} \Rightarrow$ derivate

$\Rightarrow \dot{\hat{x}} = w \times \hat{x}$

from time $t \to t+\Delta t \Rightarrow$ rotation $= \Delta\theta$ about unit axis w

$\dot{r}_i = w_s \times r_i \qquad (r_1(t) = \hat{x},\ r_2(t) = \hat{y},\ r_3(t) = \hat{z})$

w_s - Velocity

$\dot{R} = [w_s \times r_1 \quad w_s \times r_2 \quad w_s \times r_3] = w_s \times R$

rewrite as

$\dot{R} = [w_s] R \longleftarrow$ Skew symmetric

Definition for vector $x = [x_1\ x_2\ x_3] \in \mathbb{R}^3$

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

so(3) little so3

https://www.euclideanspace.com/maths/algebra/matrix/functions/skew/index.htm

$$\dot{R}_{sb} = [\omega_s] R_{sb}$$

$$\Rightarrow [\omega] = \dot{R} R^{-1}$$

Proposition

$$R[\omega]R^T = [R\omega]$$

$$\omega_s = R_{sb}\, \omega_b$$

$$\therefore \omega_b = R_{sb}^{-1} \omega_s = R^T \omega_s$$

Now writing skew-symmetric

$$[\omega_b] = [R^T \omega_s]$$

$$= R^T [\omega_s] R$$

$$= R^T (\dot{R} R^T) R$$

$$= R^T (\dot{R} \cancel{R^{-1}}) \cancel{R} \quad \text{(skew matrix)}$$

$$= R^T \dot{R} = R^{-1} \dot{R}$$

$$\boxed{\omega_b = R_{sb}^{-1} \omega_s \qquad [\omega_b] = R^{-1}\dot{R}}$$

$$\dot{R} R^{-1} = [\omega_s] \qquad \text{P-3.9}$$

$$R^{-1} \dot{R} = [\omega_b]$$