Module - 4

Point Sources and Arrays

Power Theorem: If the poynting vector is known at all the points on a sphere of radius 'r' from a point source in a lossless medium, the total power radiated by the source is the integral over the surface of the sphere of the radial component S_r of average poynting vector.

$$P = \iint \vec{S}_r \cdot \vec{ds}$$

$$= \iint S_r \vec{a}_r \cdot r^2 \sin\theta \, d\theta \, d\phi \, \vec{a}_r$$

$$= r^2 S_r \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= r^2 S_r \left[-\cos\theta\right]_0^{\pi} \cdot 2\pi = 4\pi r^2 S_r$$

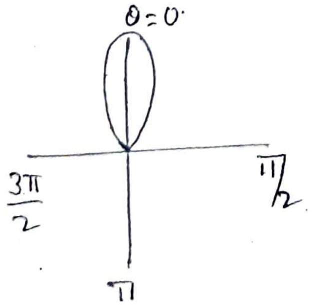
$$\therefore S_r = \frac{P}{4\pi r^2} \quad \therefore S_r r^2 = \frac{P}{4\pi} = U = \text{radiation intensity}$$

$$P = \iint U \, d\Omega$$

Consider an isotropic point source with radiation intensity of U_0 W/sr. The power radiated by the isotropic point source is given by,

$$P = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} U_0 \sin\theta \, d\theta \, d\phi = U_0 \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi = 4\pi U_0 \quad -(1)$$

1) $U = U_m \cos\theta$, unidirectional



$$P = \iint U \, d\Omega$$

$$= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} U_m \cos\theta \sin\theta \, d\theta \, d\phi$$

$$= U_m \int_{\theta=0}^{\pi/2} \sin\theta\cos\theta\, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= \frac{U_m}{2} \int_0^{\pi/2} 2\sin\theta\cos\theta\, d\theta \cdot 2\pi$$

$$= \pi U_m \int_0^{\pi/2} \sin 2\theta\, d\theta$$

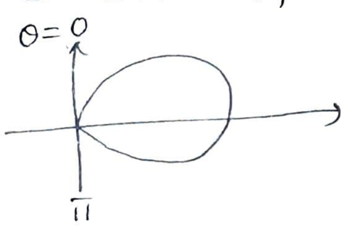
$$= \pi U_m \left[-\frac{\cos 2\theta}{2} \right]_0^{\pi/2} = \pi U_m \quad \text{--- (2)}$$

from (1) and (2)

$$\pi U_m = 4\pi U_0$$

$$\boxed{D = \frac{U_m}{U_0} = 4}$$

2) $U = U_m \sin\theta$, unidirectional



$$P = \iint U\, d\Omega = \int_{\theta=0}^{\pi} \int_{\phi=\pi/2}^{3\pi/2} U_m \sin\theta \sin\theta\, d\theta\, d\phi$$

$$= U_m \int_0^{\pi} \sin^2\theta\, d\theta \int_{\pi/2}^{3\pi/2} d\phi$$

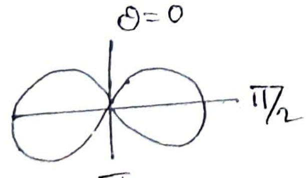
$$= \pi^2 U_m/2 \quad \text{--- (3)}$$

from (1) and (3)

$$4\pi U_0 = \pi^2 U_m/2$$

$$\therefore \boxed{D = \frac{U_m}{U_0} = \frac{8}{\pi}}$$

3) $U = U_m \sin\theta$, bidirectional



$$P = \iint U\, d\Omega = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} U_m \sin^2\theta\, d\theta\, d\phi$$

$$= U_m \int_{\theta=0}^{\pi} \sin^2\theta\, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$P = U_m \int_{\theta=0}^{\pi} \frac{1-\cos 2\theta}{2}\, d\theta \cdot 2\pi$$

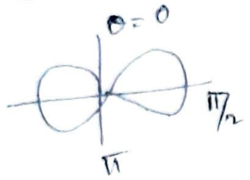
$$= 2\pi U_m \cdot \pi/2 = \pi^2 U_m \quad \text{— (4)}$$

Comping (1) + (4) we get

$$4\pi U_0 = \pi^2 U_m$$

$$\therefore \boxed{D = \frac{U_m}{U_0} = \frac{4}{\pi}}$$

4) $U = U_m \sin^2\theta$, bidirectional



$$P = \iint U\, d\Omega = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} U_m \sin^3\theta\, d\theta\, d\phi$$

$$= \int_{\theta=0}^{\pi} U_m \sin^3\theta\, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= \frac{U_m}{4} \int_{\theta=0}^{\pi} (3\sin\theta - \sin 3\theta)\, d\theta \cdot 2\pi$$

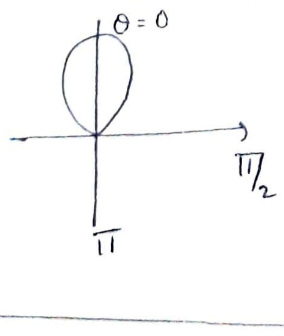
$$= \frac{8\pi U_m}{3} \quad \text{— (5)}$$

comparing (1) & (5) we get

$$4\pi U_0 = \frac{8\pi U_m}{3}$$

$$\therefore \boxed{D = \frac{U_m}{U_0} = 1.5}$$

5) $U = U_m \cos^2\theta$, unidirectional



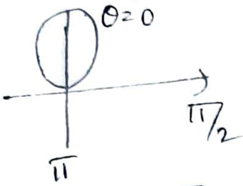
$$P = \iint U\, d\Omega = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} U_m \cos^2\theta \sin\theta\, d\theta\, d\phi$$

$$= 2\pi U_m / 3 \quad \text{— (6)}$$

$$P = 4\pi U_0 = \frac{2\pi U_m}{3}$$

$$\therefore \boxed{D = \frac{U_m}{U_0} = 6}$$

6) Show that the directivity of a source with unidirectional power pattern given by $U = U_m \cos^n\theta$ is given by $D = 2(n+1)$



$$P = \iint U_m \cos^n\theta \, d\Omega = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} U_m \cos^n\theta \sin\theta \, d\theta \, d\phi$$

$$P = \int_{\theta=0}^{\pi/2} U_m \cos^n\theta \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi = \frac{2\pi U_m}{n+1} \quad -(7)$$

from (1) and (7) we get

$$4\pi U_0 = \frac{2\pi U_m}{n+1}$$

$$\therefore \boxed{D = \frac{U_m}{U_0} = 2(n+1)}$$

7) Calculate the exact and approxiate directivity of the given patterns.

i) $U = U_m \sin\theta \sin^2\phi, \quad 0 \le \theta, \phi \le \pi$

Exact method:

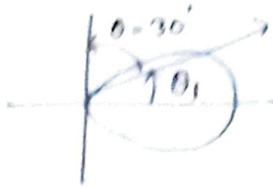
$$P = \iint U \, d\Omega = \int_{\theta=0}^{\pi} \int_{\phi=0}^{\pi} U_m \sin\theta \sin^2\phi \sin\theta \, d\theta \, d\phi$$

$$= \int_{\theta=0}^{\pi} U_m \sin^2\theta \, d\theta \int_{\phi=0}^{\pi} \sin^2\phi \, d\phi$$

$$= U_m \left(\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right) = \pi^2 U_m / 4$$

$$P = \frac{\pi^2 U_m}{4} = 4\pi U_0 \qquad \therefore D = \frac{U_m}{U_0} = \frac{16}{\pi}$$

Approximate method:



To get θ_{HP} put $\phi = 90°$

$U = U_m \sin\theta \sin^2 90°$

$U = U_m \sin\theta$

At half power points, $U = \frac{U_m}{2}$

$\therefore \frac{U_m}{2} = U_m \sin\theta \quad \therefore \theta = 30°, \quad \therefore \theta_1 = 90° - \theta = 60°$

$\theta_{HP} = 2\theta_1 = 120°$

To determine ϕ_{HP} put $\theta = 90°$

$U = U_m \sin 90°, \sin^2\phi$

At half power points, $U = \frac{U_m}{2}$

$\frac{U_m}{2} = U_m \sin^2\phi \quad \therefore \sin\phi = \frac{1}{\sqrt{2}} \quad \therefore \phi = 45°; \quad \phi_1 = 90° - \phi = 45°$

$\phi_{HP} = 2\phi_1 = 90°$

$D = \frac{41,253}{\theta_{HP}\phi_{HP}} = \frac{41253}{(120)(90)} = 3.8$

(ii) $U = U_m \sin\theta \sin^3\phi, \quad 0 \le \theta, \phi \le \pi$

Exact method:

$P = \iint U d\Omega = \iint U_m \sin^2\theta \sin^3\phi \, d\theta d\phi = \frac{2\pi U_m}{3}$

$\therefore P = 4\pi U_0 = \frac{2\pi U_m}{3} \quad \therefore D = \frac{U_m}{U_0} = 6$

Approximate method:

for θ_{HP} put $\phi = 90°$

$U = U_m \sin\theta \quad \therefore \frac{U_m}{2} = U_m \sin\theta$

$\therefore \theta = 30°, \quad \theta_1 = 90 - \theta = 60°, \quad \theta_{HP} = 2\theta_1 = 120°$

for ϕ_{HP} put $\theta = 90°$

$U = U_m \sin^3\phi, \therefore \frac{U_m}{2} = U_m \sin^3\phi$

$\phi = 52.52^\circ$, $\phi_1 = 90 - \phi = 37.46^\circ$, $\phi_{HP} = 2\phi_1 = 75^\circ$

$$D = \frac{41253}{\theta_{HP}\phi_{HP}} = 4.58$$

iii) $U = U_m \sin^2\theta \sin^3\phi$, $0 \le \theta \le \pi$

Exact method :

$$P = \iint U \, d\Omega = \int_0^{\pi}\int_0^{\pi} U_m \sin^2\theta \sin^3\phi \sin\theta \, d\theta \, d\phi$$

$$= U_m \int_{\phi=0}^{\pi} \sin^3\phi \, d\phi \int_{\theta=0}^{\pi} \sin^3\theta \, d\theta$$

$$= U_m \int_{\theta=0}^{\pi} \frac{3\sin\theta - \sin 3\theta}{4} d\theta \int_{\phi=0}^{\pi} \frac{3\sin\phi - \sin 3\phi}{4} d\phi$$

$$= U_m \left[\frac{3}{4} (-\cos\theta)_0^{\pi} - \frac{1}{4} \left(- \frac{\cos 3\theta}{3} \right)_0^{\pi} \right]$$

$$= U_m \cdot \frac{4}{3} \cdot \frac{4}{3} = \frac{16 U_m}{9}$$

$$4\pi U_0 = \frac{16 U_m}{9}$$

$$\therefore D = \frac{U_m}{U_0} = 7.07$$

Approximate method:

For θ_{HP} Put $\phi = 90^\circ$

$\therefore U = U_m \sin^2\theta$ $\therefore \frac{U_m}{2} = U_m \sin^2\theta$

$\therefore \sin\theta = \frac{1}{\sqrt{2}}$ $\therefore \theta = 45^\circ$ $\theta_1 = 90 - \theta = 45^\circ$,

$\theta_{HP} = 2\theta_1 = 90^\circ$

For ϕ_{HP} put $\theta = 90°$;

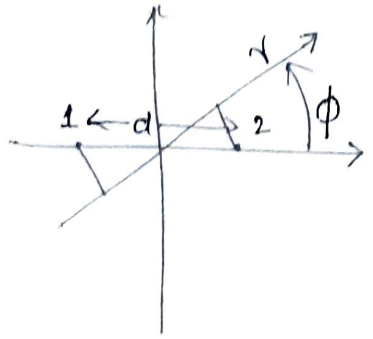
$U = U_m \sin^3\phi$, $\therefore \frac{U_m}{2} = U_m \sin^3\phi$

$\therefore \sin\phi = \frac{1}{\sqrt[3]{2}}$ $\therefore \phi = 52.52°$, $\phi_1 = 90 - \phi = 37.46°$

$\phi_{HP} = 2\phi_1 = 75°$

$\therefore D = \frac{41253}{\theta_{HP}\,\phi_{HP}} = \frac{41253}{(90)(75)} = 6.117$

Array of point sources: consider an array of two isotropic point sources which are separated by a distance of 'd' from each other. The field at a point at a distance of r from center is required.



Case 1: Two isotropic point sources of same amplitude and phase

$d_r = \frac{2\pi}{\lambda} d$

$E = E_0\, e^{\frac{j d_r \cos\phi}{2}} + E_0\, e^{-\frac{j d_r \cos\phi}{2}}$

let $\psi = d_r \cos\phi$

$\therefore E = E_0\, e^{j\psi/2} + E_0\, e^{-j\psi/2}$

$= 2E_0 \cos\psi/2$

Considering $2E_0 = 1$, $E = \cos\psi/2$

let $d = \lambda/2$, $d_r = \frac{2\pi}{\lambda} d = \pi$ $\therefore \psi = \pi\cos\phi$

$\therefore E = \cos\left(\frac{\pi\cos\phi}{2}\right)$

i) maximum points:

$\cos\left(\frac{\pi}{2}\cos\phi\right) = \pm 1$ $\therefore \frac{\pi}{2}\cos\phi = K\pi$, $K = 0, 1, 2, \ldots$

consider $K = 0$

$\frac{\pi}{2}\cos\phi = 0 \Rightarrow \phi = \pi/2, 3\pi/2$

ii) Null points:

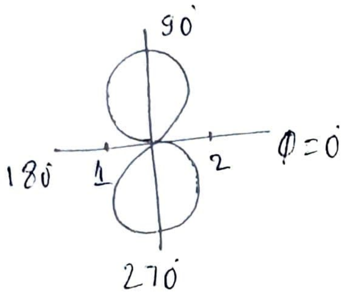
$$E = \cos\left[\frac{\pi}{2}\cos\phi\right] = 0 \qquad \therefore \frac{\pi}{2}\cos\phi = \pm(2K+1)\frac{\pi}{2}$$

put $K=0 \Rightarrow \phi = 0, \pi$

iii) Half power points:

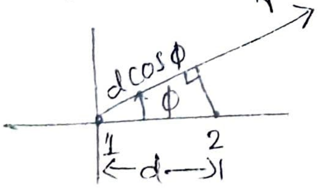
$$E = \cos\left[\frac{\pi}{2}\cos\phi\right] = \pm\frac{1}{\sqrt{2}}, \quad \frac{\pi}{2}\cos\phi = \pm(2K+1)\frac{\pi}{4}, \; K=0,1,2\ldots$$

put $K=0 \Rightarrow \phi = 60^\circ, 120^\circ, 240^\circ, 300^\circ$



- pattern is a figure of 8 with maximum at $\phi = 90^\circ$ and 270° and nulls at $\phi = 0^\circ$ & $\phi = 180^\circ$
- maximum field occurs in a plane which is perpendicular to the placement of antenna array elements.
- This array is known as broadside array.

If the sources are placed on one side of origin,



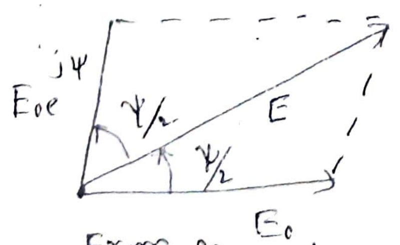
$$E = E_0 + E_0 e^{j\psi} = 2E_0 e^{j\psi/2}\cos(\psi/2)$$

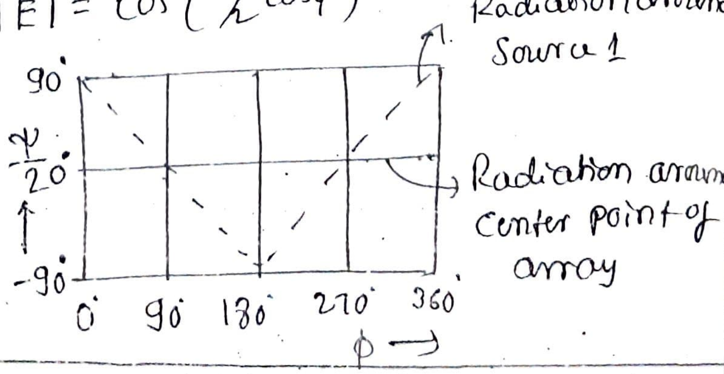
$d_r = \frac{2\pi}{\lambda}d$ at $d = \lambda/2$ and $d_r = \pi$

$$\therefore E = e^{j\psi/2}\cos\left(\frac{\pi}{2}\cos\phi\right)$$

$$\therefore |E| = \cos\left(\frac{\pi}{2}\cos\phi\right)$$

phase pattern





$\psi/2 = \frac{d_r \cos\phi}{2} = \frac{\pi}{2}\cos\phi$

ϕ	$\psi/2$
0	$\pi/2$
90°	0
180°	$-\pi/2$
270°	0
360°	$\pi/2$

Case 2: Array of two isotropic point sources of same amplitude and opposite phase.

$$E = E_0 e^{j\psi/2} e^{j0} + E_0 e^{-j\psi/2} e^{j\pi}$$

$$= E_0 e^{j\psi/2} - E_0 e^{-j\psi/2}$$

$$= 2jE_0 \sin(\psi/2)$$

$2jE_0 = 1 \quad \therefore E = \sin(\psi/2)$

$d = \lambda/2, \quad \therefore \psi = d_r \cos\phi = \pi\cos\phi \quad \therefore E = \sin(\frac{\pi}{2}\cos\phi)$

i) maximum points:

$\sin\left(\frac{\pi}{2}\cos\phi\right) = \pm 1, \quad \frac{\pi}{2}\cos\phi = \pm(2K+1)\frac{\pi}{2}, \quad K = 0, 1, 2 \ldots$

Take $K = 0 \quad \therefore \phi = 0, \pi$

ii) Null points:

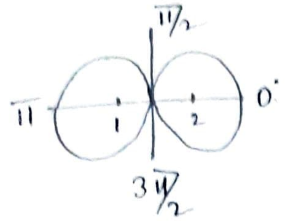
$E = \sin\left(\frac{\pi}{2}\cos\phi\right) = 0 \quad \therefore \frac{\pi}{2}\cos\phi = \pm K\pi, \quad K = 0, 1, 2, \ldots$

Take $K = 0 \quad \therefore \phi = \pi/2, 3\pi/2$

iii) Half power points:

$E = \sin\left(\frac{\pi}{2}\cos\phi\right) = \pm\frac{1}{\sqrt{2}}, \quad \frac{\pi}{2}\cos\phi = \pm(2K+1)\frac{\pi}{4}$, take $K = 0$

$\phi = 60°, 120°, 240°, 300°$



- The field pattern is maximum in the direction of sources.
- The array is called as end fire array.

Case 3: Two isotropic point sources of same amplitude and in phase quadrature.

Consider two isotropic point sources which are separated by a distance d between them and are located equidistant from each other.

$$E = E_0 e^{j(\psi/2 + \pi/4)} + E_0 e^{-j(\psi/2 + \pi/4)}$$

$$= 2E_0 \left[\frac{e^{j(\psi/2 + \pi/4)} + e^{-j(\psi/2 + \pi/4)}}{2} \right]$$

$$= 2E_0 \cos(\psi/2 + \pi/4)$$

Normalize the field $2E_0 = 1$

$$E = \cos\left(\psi/2 + \frac{\pi}{4}\right)$$

If $d = \lambda/2$, $d_r = \frac{2\pi}{\lambda} d = \pi$

$$\therefore E = \cos\left(\pi/2 \cos\phi + \frac{\pi}{4}\right)$$

Maximum points:

$$E = \cos\left(\frac{\pi}{2}\cos\phi + \frac{\pi}{4}\right) = \pm 1$$

$$\frac{\pi}{2}\cos\phi + \frac{\pi}{4} = \pm K\pi, \text{ Put } K = 0$$

$$\frac{1}{2}\cos\phi + \frac{1}{4} = 0 \quad \therefore \phi = 120°, 240°$$

Null points:

$$E = \cos\left(\frac{\pi}{2}\cos\phi + \frac{\pi}{4}\right) = 0$$

$\therefore \frac{\pi}{2}\cos\phi + \frac{\pi}{4} = \pm(2k+1)\frac{\pi}{2}, \quad k = 0,1,2,3\ldots$

Take $k=0$

$\therefore \cos\phi + \frac{1}{2} = \pm 1$

$\therefore \phi = 60^\circ, 300^\circ$

Half power points:

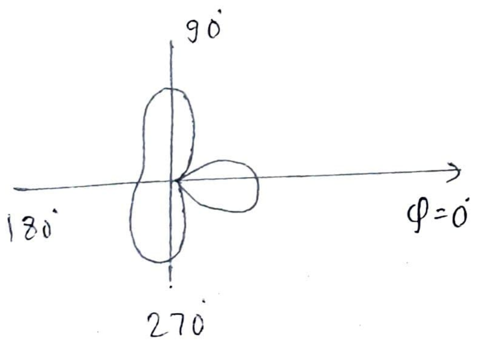
$$E = \cos\left(\frac{\pi}{2}\cos\phi + \frac{\pi}{4}\right) = \pm\frac{1}{\sqrt{2}}$$

$$\frac{\pi}{2}\cos\phi + \frac{\pi}{4} = \pm(2k+1)\frac{\pi}{4}, \quad k = 0,1,2,3,\ldots$$

Take $k=0$

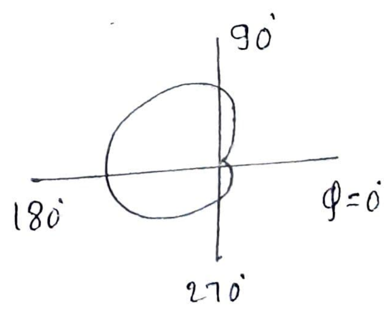
$\frac{\pi}{2}\cos\phi + \frac{\pi}{4} = \pm\frac{\pi}{4}$, $\cos\phi + \frac{1}{2} \pm \pm 1$

$\therefore \phi = 90^\circ, 180^\circ, 270^\circ$



Consider $d = \lambda/4$, $\psi = d_r\cos\phi$

$\therefore E = \cos\left(\frac{\pi}{4}\cos\phi + \frac{\pi}{4}\right)$



maximum points: $\phi = 180^\circ$

Null points: $\phi = 0^\circ, 360^\circ$

Half power points: $\phi = 90^\circ, 270^\circ$

Case 4: General case of two isotropic point sources of equal amplitude and any phase difference.

$$E = E_0 e^{j\psi/2} + E_0 e^{-j\psi/2}$$

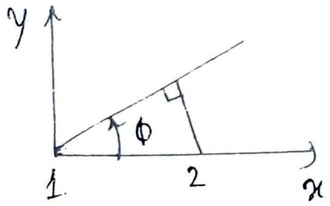
$$\psi = d_r \cos\phi + \delta$$

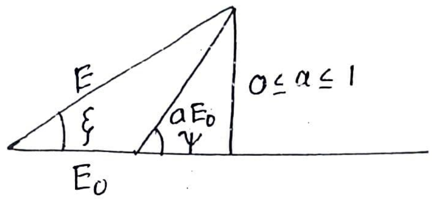
$\delta \Rightarrow$ phase difference between adjacent sources

$$\therefore E = 2E_0 \cos(\psi/2)$$

For normalized field, $E = \cos(\psi/2)$

Case 5: most general case of two isotropic sources of unequal amplitude and any phase difference.





$$E = \left[(E_0 + aE_0\cos\psi)^2 + (aE_0\sin\psi)^2\right]^{1/2} \Big/ \tan^{-1}\left(\frac{aE_0\sin\psi}{E_0 + aE_0\cos\psi}\right)$$

$$= E_0\left(1 + a^2 + 2a\cos\psi\right)^{1/2} \Big/ \tan^{-1}\left(\frac{a\sin\psi}{1 + a\cos\psi}\right)$$

Pattern multiplication

Consider an array of two isotropic point sources with any phase difference ψ between them. The field due to these sources is given by $E = E_0 e^{j\psi/2} + E_0 e^{-j\psi/2}$

$$\therefore E = 2E_0 \cos\psi/2$$

If the isotropic point sources are replaced by non-isotropic point sources, then the field pattern changes. It is assumed that the two non-isotropic point sources are oriented in the same direction, and also have the same field pattern.

$\therefore \frac{\pi}{2}\cos\phi + \frac{\pi}{4} = \pm(2K+1)\frac{\pi}{2}, \quad K = 0,1,2,3 \ldots$

Take $K=0$

$\therefore \cos\phi + \frac{1}{2} = \pm 1$

$\therefore \phi = 60^\circ, 300^\circ$

Half power points:

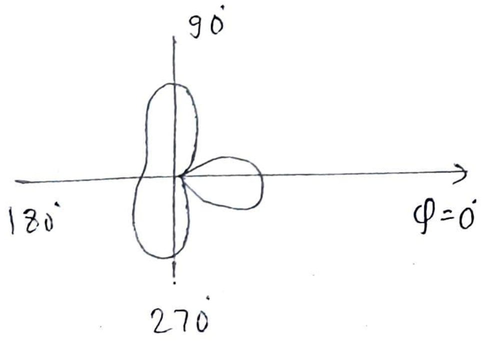
$$E = \cos\left(\frac{\pi}{2}\cos\phi + \frac{\pi}{4}\right) = \pm\frac{1}{\sqrt{2}}$$

$$\frac{\pi}{2}\cos\phi + \frac{\pi}{4} = \pm(2K+1)\frac{\pi}{4}, \quad K=0,1,2,3,\ldots$$

Take $K=0$

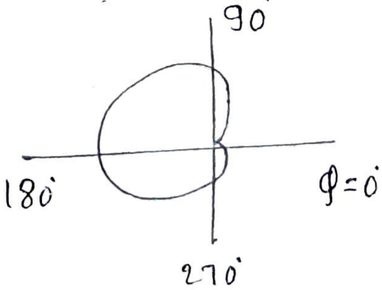
$$\frac{\pi}{2}\cos\phi + \frac{\pi}{4} = \pm\frac{\pi}{4}, \quad \cos\phi + \frac{1}{2} = \pm 1$$

$\therefore \phi = 90^\circ, 180^\circ, 270^\circ$



Consider $d = \lambda/4$, $\psi = d_r\cos\phi$

$$\therefore E = \cos\left(\frac{\pi}{4}\cos\phi + \frac{\pi}{4}\right)$$



Maximum points: $\phi = 180^\circ$

Null points: $\phi = 0^\circ, 360^\circ$

Half power points: $\phi = 90^\circ, 270^\circ$

consider the placement of sources as shown below:

let these sources be described by $E_0 = E_0' \sin\phi$

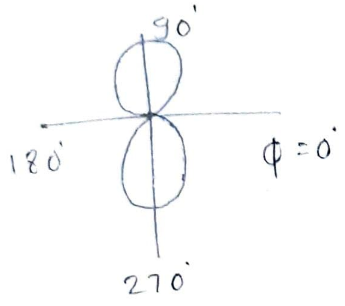
Then the total field is given by:

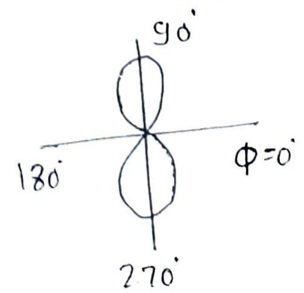
$$E = 2E_0' \sin\phi \cos\psi/2$$

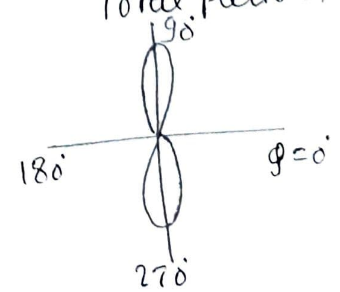
principle of pattern multiplication: The field pattern of an array of non isotropic but similar point sources is the product of the pattern of individual source and the pattern of the array of isotropic point sources having a same locations, relative amplitudes and phase as the non-isotropic point sources.

Ex:

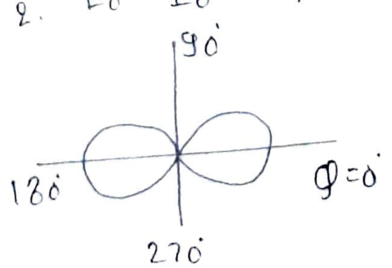
1 $E_0 = E_0' \sin\phi$ array pattern Total pattern

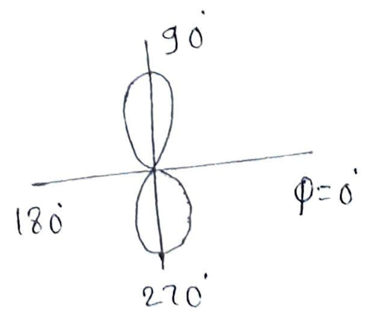


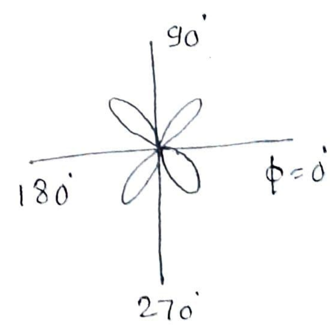




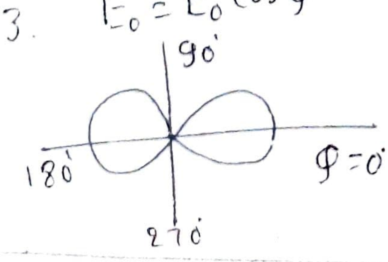
2. $E_0 = E_0' \cos\phi$

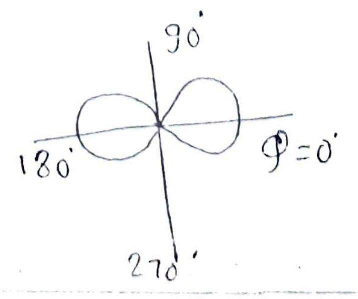


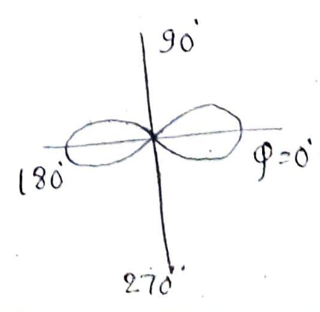




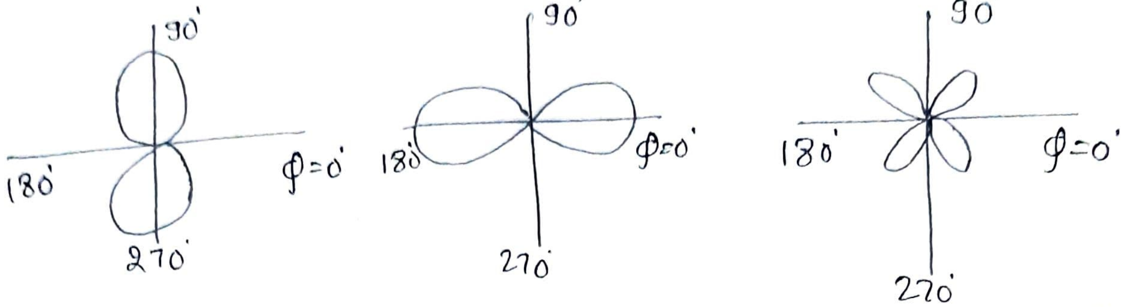
3. $E_0 = E_0' \cos\phi$



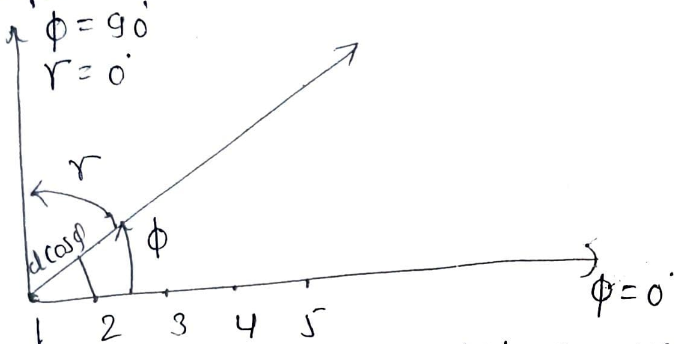




4. $E_0 = E_0' \sin\phi$



Linear array of N isotropic point sources of equal amplitude and spacing.



consider N isotropic point sources which are arranged in the form of an array. The distance between successive sources is 'd' and it is assumed that source 2 leads source1, source 3 leads source 2 and so on by equal amount.

Field at any point due to these source is given by:

$$E = 1 + e^{j\psi} + e^{j2\psi} + \dots + e^{j(n-1)\psi}$$

$$= \frac{e^{jn\psi} - 1}{e^{j\psi} - 1} = \frac{e^{jn\psi/2}\left[e^{jn\psi/2} - e^{-jn\psi/2}\right]}{e^{j\psi/2}\left[e^{j\psi/2} - e^{-j\psi/2}\right]} = \frac{e^{j(n-1)\frac{\psi}{2}} \sin\left(\frac{n\psi}{2}\right)}{\sin \psi/2}$$

$$E_{max} = \lim_{\psi \to 0} \left[\frac{\sin(n\psi/2)}{\sin \psi/2}\right] = n$$

$$\therefore E = \frac{1}{n} e^{j(n-1)\frac{\psi}{2}} \frac{\sin\left(\frac{n\psi}{2}\right)}{\sin(\psi/2)}$$

$$|E| = \frac{1}{n} \frac{\sin(n\psi/2)}{\sin(\psi/2)}$$ This is known as array factor.

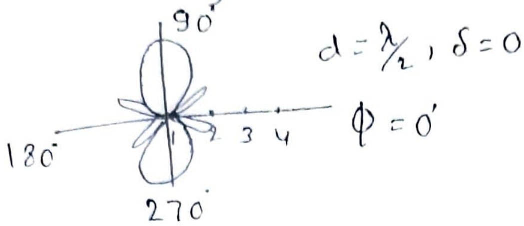
Consider a broad side array in which $\psi = d_r\cos\phi + \delta$

$\delta = 0$ for broad side array. $\therefore \psi = 0 = d_r\cos\phi + 0$

$\therefore d_r\cos\phi = 0 \quad \therefore \phi = (2K+1)\pi/2, \; K = 0, 1, 2 \ldots$

$\therefore \phi = \pi/2, \; 3\pi/2, \ldots$

The array has maximum at $\pi/2$ and $3\pi/2$

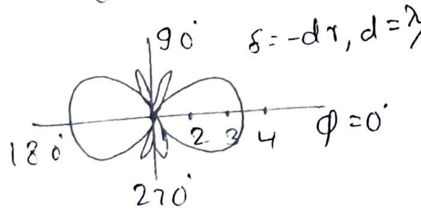


Consider an endfire array

$\psi = d_r\cos\phi + \delta, \quad \psi = 0 \Rightarrow d_r\cos\phi + \delta = 0$

Endfire array has its maximum at $\phi = 0, \pi \ldots$

$\therefore d_r\cos 0 + \delta = 0 \quad \therefore \boxed{\delta = -d_r}$



End fire array with increased directivity

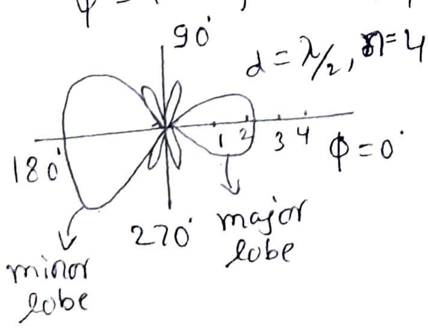
Hansen and woodyard proposed that the directivity of endfire array can be increased by taking $\delta = -d_r - \frac{\pi}{n}$

where n is the number of point sources.

$\therefore \psi = d_r\cos\phi - d_r - \pi/n$

when $n = 4$, $d = \lambda/2$, $d_r = \frac{2\pi}{\lambda} \cdot d = \pi$

$\psi = \pi\cos\phi - \pi - \pi/4 = \pi\cos\phi - 5\pi/4$



Array with maximum field in the arbitrary direction

consider $\psi = d_r \cos\phi + \delta$

$0 = d_r \cos\phi + \delta$

when $d = \lambda/2$, $d_r = \frac{2\pi}{\lambda} \cdot \lambda/2 = \pi$

$\therefore 0 = \pi \cos\phi + \delta$

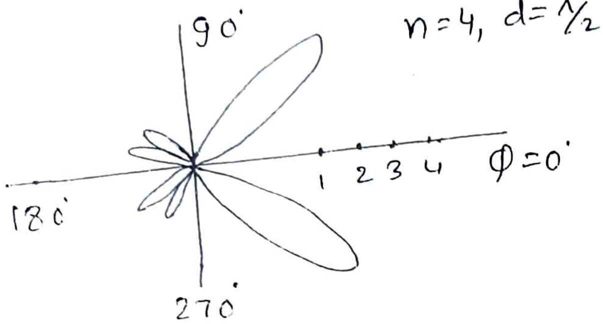
Assuming that maximum field is directed at $\phi = 60^\circ$

$\delta = -\pi \cos\phi = -\pi/2$

$\therefore \psi = \pi \cos\phi - \pi/2$

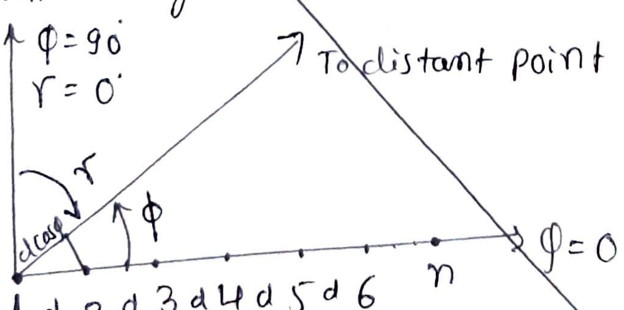
Considering 4 isotropic point sources,

$n = 4$, $d = \lambda/2$, $\delta = -\pi/2$



Linear arrays of n - isotropic point sources of equal amplitude and spacing:

consider a case of n isotropic point sources of equal amplitude and spacing arranged as a linear array, where n is any positive integer.

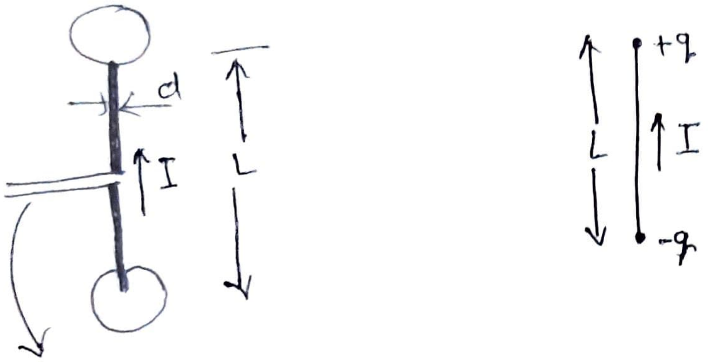


The total field E at a large distance in the direction ϕ is given by

$$E = 1 + e^{j\psi} + e^{j2\psi} + e^{j3\psi} + \cdots + e^{j(n-1)\psi} \quad -(1)$$

Electric Dipoles

The short electric dipole: Any linear antenna may be considered as consisting of a large number of very short conductors connected in series. A short linear conductor is called a short dipole. A short dipole is always of finite length.



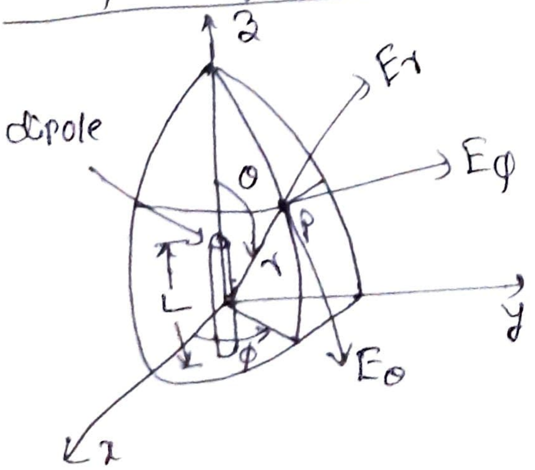
a) short dipole antenna (b) equivalent of short dipole.

Consider a short dipole. The length L is very short compared to the wave length ($L << \lambda$). plates at the ends of the dipole provide capacitive loading. The short length and the presence of these plates result in a uniform current I along the entire length L of the dipole. The dipole is energized by a balanced transmission line. The diameter d of the dipole is small compared to its length ($d<<L$)

The current and charge are related by,

$$I = \frac{dq}{dt} \quad -(1)$$

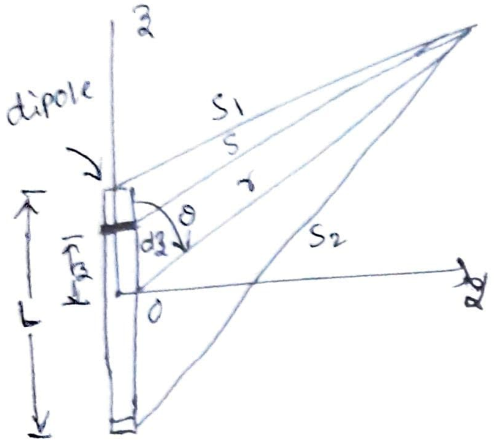
The fields of a short dipole: Let the dipole of length L be placed coincident with z-axis and with its center at the origin. It is assumed that the medium surrounding the dipole is air or vaccum. If a current is flowing in the short dipole, the effect of the current is not felt ~~sim~~ instantaneously at the point P, but only after an interval equal to the time required for the disturbance to propogate over the distance r.



The retarded current [I] is given by,

$$[I] = I_0 e^{j\omega[t - r/c]} \quad -①$$

The equation states that the disturbance at a time t and at a distance r from a current element is caused by a current [I] that occurred at an earlier time $t - r/c$. The time difference r/c is the interval required for the disturbance to travel the distance r, where c is the velocity of light.



The retarded vector potential of the electric current has only one component, A_z. Its value is:

$$A_z = \frac{\mu_0}{4\pi} \int_{-L/2}^{L/2} \frac{[I]}{s} dz \quad -②$$

where [I] is the retarded current given by,

$$[I] = I_0 e^{j\omega[t - s/c]} \quad -③$$

where I_0 - peak value in time of current

z - distance to a point on the conductor.

If the distance from the dipole is large compared to its length ($r >> L$) and ($\lambda >> L$), s may be approximated to r, neglecting the phase differences of the field contributions from different parts of the wire, we get

$$A_z = \frac{\mu_0 L I_0 e^{j\omega[t - r/c]}}{4\pi r} \quad -④$$

The retarded scalar potential V of a charge distribution is

$$V = \frac{1}{4\pi\epsilon_0}\int_V \frac{[\rho]}{S}\, d\tau \quad - (5)$$

where $[\rho]$ is the retarded charge density given by

$$[\rho] = \rho_0 e^{j\omega[t - S/c]} \quad - (6)$$

and $d\tau$ - infinitesimal volume element

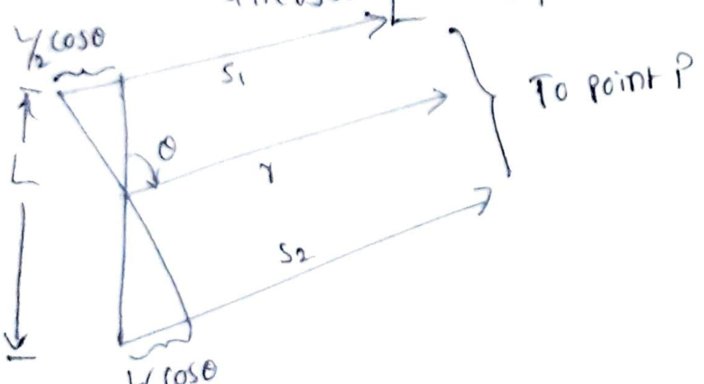
Since the region of charge in the case of dipole being considered is confined to the points at the ends, eqn (5) reduces to

$$V = \frac{1}{4\pi\epsilon_0}\left\{\frac{[q]}{S_1} - \frac{[q]}{S_2}\right\} \quad - (7)$$

$$[q] = \int [I]\, dt = I_0 \int e^{j\omega(t - S/c)}\, dt = \frac{[I]}{j\omega} \quad - (8)$$

putting (8) in (7)

$$V = \frac{I_0}{4\pi\epsilon_0 j\omega}\left[\frac{e^{j\omega[t - \frac{S_1}{c}]}}{S_1} - \frac{e^{j\omega[t - S_2/c]}}{S_2}\right] \quad - (9)$$



when $r >> L$, the lines connecting the ends of the dipole and the point P may be considered as parallel, so that

$$S_1 = r - \frac{L}{2}\cos\theta \quad - (10) \qquad \text{and} \qquad S_2 = r + \frac{L}{2}\cos\theta \quad - (11)$$

putting (10) and (11) in (9) we get

$$E_r = \frac{I_0 L \cos\theta\, e^{j\omega(t - r/c)}}{2\pi\epsilon_0}\left(\frac{1}{cr^2} + \frac{1}{j\omega r^3}\right) \quad - (12)$$

and

$$E_\theta = \frac{I_0 l \sin\theta \, e^{j\omega[t - r/c]}}{4\pi\epsilon_0}\left(\frac{j\omega}{c^2 r} + \frac{1}{c r^2} + \frac{1}{j\omega r^3}\right) \quad - (13)$$

$$\nabla \times A = \frac{\vec{a}_r}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta) A_\theta - \frac{\partial A_\theta}{\partial\phi}\right] + \frac{\vec{a}_\theta}{r\sin\theta}\left[\frac{\partial A_r}{\partial\phi} - \frac{\partial(r\sin\theta)A_\phi}{\partial r}\right]$$

$$+ \frac{\vec{a}_\phi}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right] \quad - (14)$$

since $A_\phi = 0$, A_r and A_θ are independent of ϕ, magnetic field is given by,

$$|H| = H_\phi = \frac{I_0 l \sin\theta \, e^{j\omega[t - r/c]}}{4\pi}\left(\frac{j\omega}{cr} + \frac{1}{r^2}\right) \quad - (15) \quad \text{general case}$$

$$- (16)$$

$$H_r = H_\theta = 0$$

when r is very large, terms $\frac{1}{r^2}$ and $\frac{1}{r^3}$ may be neglected

$\therefore$ In the far field, E_r is negligible, only E_θ and H_ϕ are given by,

$$E_\theta = \frac{j\omega I_0 L \sin\theta \, e^{j\omega[t - r/c]}}{4\pi\epsilon_0 c^2 r} = \frac{j I_0 \beta L}{4\pi\epsilon_0 c r}\sin\theta \, e^{j\omega[t - \frac{r}{c}]} \quad - (17)$$

$$H_\phi = \frac{j\omega I_0 L \sin\theta \, e^{j\omega[t - \frac{r}{c}]}}{4\pi r c} = \frac{j I_0 \beta L}{4\pi r}\sin\theta \, e^{j\omega[t - \frac{r}{c}]} \quad - (18)$$

Far field case

$$\frac{E_\theta}{H_\phi} = \frac{1}{\epsilon_0 c} = \sqrt{\frac{\mu_0}{\epsilon_0}} = 376.7\,\Omega \quad - (19)$$

At low frequencies:

consider $[I] = I_0 e^{j\omega(t - r/c)} = j\omega[q]$ — (20)

eqn (12) and (13) are written as,

$$E_r = \frac{[q] L \cos\theta}{2\pi\epsilon_0}\left(\frac{j\omega}{c r^2} + \frac{1}{r^3}\right) \quad - (21)$$

and $$E_\theta = \frac{[q] L \sin\theta}{4\pi\epsilon_0}\left(-\frac{\omega^2}{c^2 r} + \frac{j\omega}{c r^2} + \frac{1}{r^3}\right) \quad - (22)$$

The retarded scalar potential V of a charge distribution is:

$$V = \frac{1}{4\pi\epsilon_0}\int_V \frac{[\rho]}{S}\, d\tau \quad - (5)$$

where $[\rho]$ is the retarded charge density given by,

$$[\rho] = \rho_0 e^{j\omega[t - S/c]} \quad - (6)$$

and $d\tau$ - infinitesimal volume element

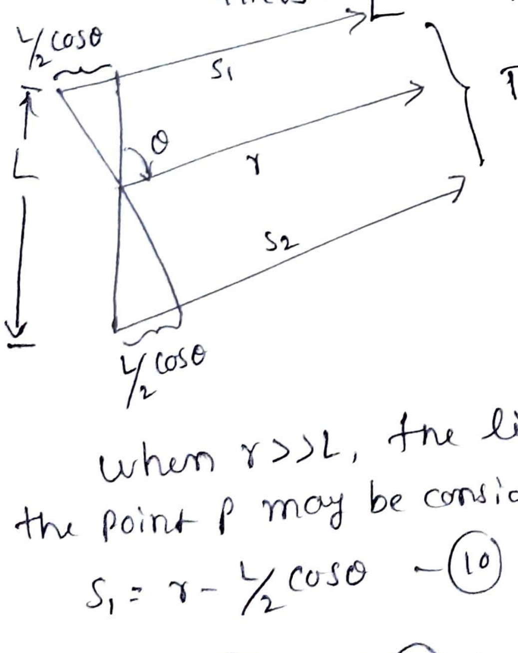
since the region of charge in the case of dipole being considered is confined to the points at the ends, eqn (5) reduces to

$$V = \frac{1}{4\pi\epsilon_0}\left\{\frac{[q]}{S_1} - \frac{[q]}{S_2}\right\} \quad - (7)$$

$$[q] = \int [I]\, dt = I_0 \int e^{j\omega(t - S/c)}\, dt = \frac{[I]}{j\omega} \quad - (8)$$

putting (8) in (7)

$$V = \frac{I_0}{4\pi\epsilon_0 j\omega}\left[\frac{e^{j\omega[t - \frac{S_1}{c}]}}{S_1} - \frac{e^{j\omega[t - S_2/c]}}{S_2}\right] \quad - (9)$$



when $r \gg L$, the lines connecting the ends of the dipole and the point P may be considered as parallel, so that

$$S_1 = r - \frac{L}{2}\cos\theta \quad - (10) \quad \text{and} \quad S_2 = r + \frac{L}{2}\cos\theta \quad - (11)$$

putting (10) and (11) in (9) we get

$$E_r = \frac{I_0 L \cos\theta\, e^{j\omega(t - r/c)}}{2\pi\epsilon_0}\left(\frac{1}{c r^2} + \frac{1}{j\omega r^3}\right) \quad - (12)$$

and

$$E_\theta = \frac{I_o l \sin\theta \, e^{jw[t - r/c]}}{4\pi\epsilon_o} \left(\frac{jw}{c^2 r} + \frac{1}{c r^2} + \frac{1}{jw r^3} \right) \quad - (13)$$

$$\nabla \times A = \frac{\vec{a_r}}{r\sin\theta} \left[\frac{\partial}{\partial\theta}(\sin\theta) A_\theta - \frac{\partial A_\theta}{\partial\phi} \right] + \frac{\vec{a_\theta}}{r\sin\theta} \left[\frac{\partial A_r}{\partial\phi} - \frac{\partial (r\sin\theta) A_\phi}{\partial r} \right]$$

$$+ \frac{\vec{a_\phi}}{r} \left[\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta} \right] \quad - (14)$$

since $A_\phi = 0$, A_r and A_θ are independent of ϕ, magnetic field is given by,

$$|H| = H_\phi = \frac{I_o L \sin\theta \, e^{jw[t - \frac{r}{c}]}}{4\pi} \left(\frac{jw}{c r} + \frac{1}{r^2} \right) \quad - (15) \quad \text{general case}$$

$$- (16)$$

$$H_r = H_\theta = 0$$

when r is very large, terms $\frac{1}{r^2}$ and $\frac{1}{r^3}$ may be neglected

$\therefore$ In the far field, E_r is negligible, ony E_θ and H_ϕ are given by,

$$E_\theta = \frac{jw I_o L \sin\theta \, e^{jw[t - r/c]}}{4\pi\epsilon_o c^2 r} = j \frac{I_o \beta L}{4\pi\epsilon_o c r} \sin\theta \, e^{jw[t - \frac{r}{c}]} \quad - (17)$$

$$H_\phi = \frac{jw I_o L \sin\theta \, e^{jw[t - \frac{r}{c}]}}{4\pi r c} = \frac{j I_o \beta L}{4\pi r} \sin\theta \, e^{jw[t - \frac{r}{c}]} \quad - (18)$$

Far field case

$$\frac{E_\theta}{H_\phi} = \frac{1}{\epsilon_o c} = \sqrt{\frac{\mu_o}{\epsilon_o}} = 376.7\Omega \quad - (19)$$

<u>At low frequencies:</u>

consider $[I] = I_o e^{jw(t - r/c)} = jw[q] \quad - (20)$

eqn (12) and (13) are written as,

$$E_r = \frac{[q] L \cos\theta}{2\pi\epsilon_o} \left(\frac{jw}{c r^2} + \frac{1}{r^3} \right) \quad - (21)$$

and $$E_\theta = \frac{[q] L \sin\theta}{4\pi\epsilon_o} \left(-\frac{w^2}{c^2 r} + \frac{jw}{c r^2} + \frac{1}{r^3} \right) \quad - (22)$$

The magnetic field is given by,

$$H_\phi = \frac{[I] L \sin\theta}{4\pi}\left(\frac{j\omega}{cr} + \frac{1}{r^2}\right) \quad - (23)$$

at low frequencies $\omega \to 0$, $[q] = q_0 e^{j\omega[t - r/c]} = q_0$ — (24)

and $[I] = I_0$ — (25)

∴ quasi-stationary case the field equations are;

$$E_r = \frac{q_0 L \cos\theta}{2\pi\epsilon_0 r^3} \quad - (26)$$

$$E_\theta = \frac{q_0 L \sin\theta}{4\pi\epsilon_0 r^3} \quad - (27) \qquad r >> L$$

$$H_\phi = \frac{I_0 L \sin\theta}{4\pi r^2} \quad - (28)$$

consider $|A| = \frac{1}{2r_\lambda}$, $|B| = \frac{1}{4\pi r_\lambda^2}$ and $|C| = \frac{1}{8\pi^2 r_\lambda^3}$

For the special case, where $\theta = 90$ and $r_\lambda >> 1/2\pi$

$$|H_\phi| = \frac{I_0 L \lambda}{2r} \text{ A/m} \quad - (29)$$

while $r_\lambda << 1/2\pi$

$$|H_\phi| = \frac{I_0 L}{4\pi r^2} \quad - (30)$$

The magnetic field at any distance r from an infinite linear conductor with direct current is given by,

$$H_\phi = \frac{I_0}{2\pi r} \quad - (31)$$

Rearranging the general field components for short electric dipole,

$$E_r = \frac{[I] L \lambda\, 2\cos\theta}{\lambda}\left[\frac{1}{2\pi r_\lambda^2} - j\frac{1}{4\pi^2 r_\lambda^3}\right] \quad - (32)$$

$$E_\theta = \frac{[I] L \lambda\, 2\sin\theta}{\lambda}\left[j\frac{1}{2r_\lambda} + \frac{1}{4\pi r_\lambda^2} - \frac{j}{8\pi^2 r_\lambda^3}\right] \quad - (33)$$

$$H_\phi = \frac{[I] L_\lambda \sin\theta}{\lambda}\left[\frac{j}{2r_\lambda} + \frac{1}{4\pi r_\lambda^2}\right] \quad -(34)$$

at radian distance $r_\lambda = \frac{1}{2\pi}$ the fields of (32), (33) and (34) reduce to

$$E_r = \frac{2\sqrt{2}\pi [I] L_\lambda 3\cos\theta}{\lambda} \angle -45^\circ \quad -(35)$$

$$E_\theta = \frac{\pi [I] L_\lambda 3\sin\theta}{\lambda} \quad -(36)$$

$$H_\phi = \frac{\sqrt{2}\pi [I] L_\lambda \sin\theta}{\lambda} \angle 45^\circ \quad -(37)$$

The magnitude of the average power flux or poynting vector in the θ direction is given by,

$$S_\theta = \frac{1}{2} Re\, E_r H_\phi^* = \frac{1}{2} E_r H_\phi\, Re\, 1 \angle -90^\circ = \frac{1}{2} E_r H_\phi \cos(-90^\circ) = 0 \quad -(38)$$

indicating no power is transmitted. However, the product $E_r H_\phi$ represents imaginary or reactive energy that oscillates back and forth from electric to magnetic energy twice per cycle.

The magnitude of power flux in the r direction is given by,

$$S_r = \frac{1}{2} E_\theta H_\phi \cos(-45^\circ) = \frac{1}{2\sqrt{2}} E_\theta H_\phi \quad -(39)$$

indicating energy flow in the r-direction.

much closer to dipole $r_\lambda << \frac{1}{2\pi}$, (32), (33) and (34) reduce approximately to

$$E_r = -j\frac{[I] L_\lambda 3\cos\theta}{4\pi^2 \lambda r_\lambda^3} \quad -(40), \qquad E_\theta = \frac{-j [I] L_\lambda 3\sin\theta}{8\pi^2 \lambda r_\lambda^3} \quad -(41)$$

$$H_\phi = \frac{[I] L_\lambda \sin\theta}{4\pi \lambda r_\lambda^2} \quad -(42)$$

Here $S_r = S_\theta = 0$, However the products $E_r H_\phi$ and $E_\theta H_\phi$ represent imaginary energy oscillating back and forth but not going anywhere. $\therefore$ close to the dipole there is a region of almost complete energy storage.

Remote from the dipole $\left[r_\lambda >> \frac{1}{2\pi}\right]$, (32), (33) and (34) reduce to

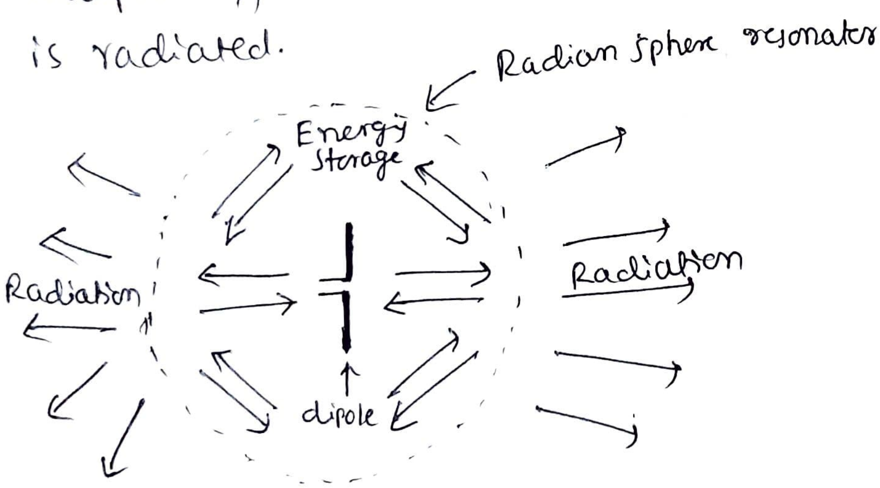
$$E_r = 0 \quad -(43)$$

$$E_\theta = \frac{j[I] L_\lambda Z \sin\theta}{2\lambda r_\lambda} \quad -(44)$$

$$H_\phi = \frac{j[I] L_\lambda \sin\theta}{2\lambda r_\lambda} \quad -(45)$$

Since $E_r = 0$, there is no energy flow in the θ direction ($S_\theta = 0$)

However, since $E_\theta H_\phi$ are in time phase, their product represents real power flow in the outward radial direction. This power is radiated.



Radiation resistance of short electric dipole

The average poynting vector is given by

$$S = \frac{1}{2} Re(\vec{E} \times \vec{H}^*) \quad -(1)$$

$$S_r = \frac{1}{2} Re\, E_\theta H_\phi^* \quad -(2)$$

where E_θ and H_ϕ are complex

$$E_\theta = H_\phi Z = H_\phi \sqrt{\frac{\mu}{\epsilon}} \quad -(3)$$

Thus $S_r = \frac{1}{2} Re\, Z H_\phi H_\phi^* = \frac{1}{2} |H_\phi|^2 Re\, Z = \frac{1}{2} |H_\phi|^2 \sqrt{\frac{\mu}{\epsilon}}$ — (4)

The total power P radiated is then:

$$P = \iint S_r\, ds = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}} \int_0^{2\pi}\int_0^{\pi} |H_\phi|^2 r^2 \sin\theta\, d\theta\, d\phi \quad -(5)$$

$$|H_\phi| = \frac{\omega I_0 L \sin\theta}{4\pi c r} \quad -(6)$$

Putting (6) in (5) we get

$$P = \frac{1}{32}\sqrt{\frac{\mu}{\epsilon}} \frac{\beta^2 I_0^2 L^2}{\pi^2} \int_0^{2\pi}\int_0^{\pi} \sin^3\theta\, d\theta\, d\phi \quad -(7)$$

$$P = \frac{1}{32}\sqrt{\frac{\mu}{\epsilon}} \frac{\beta^2 I_0^2 L^2}{\pi^2} \cdot \frac{8\pi}{3} = \sqrt{\frac{\mu}{\epsilon}} \frac{\beta^2 I_0^2 L^2}{12\pi} \quad -(8)$$

The power radiated must be equal to power delivered to the dipole.

$$\sqrt{\frac{\mu}{\epsilon}} \frac{\beta^2 I_0^2 L^2}{12\pi} = \left(\frac{I_0}{\sqrt{2}}\right)^2 R_r \quad -(9)$$

$$\therefore R_r = \sqrt{\frac{\mu}{\epsilon}} \frac{\beta^2 L^2}{6\pi} \quad -(10) \qquad \text{since } \sqrt{\frac{\mu}{\epsilon}} = 377,$$

$$R_r = 80\pi^2 \left(\frac{L}{\lambda}\right)^2 = 80\pi^2 L_\lambda^2 = 790 L_\lambda^2 \quad -(11)$$

For a general case, where current is not uniform on the dipole, the radiated power is:

$$P = \sqrt{\frac{\mu}{\epsilon}} \frac{\beta^2 I_{av}^2 L^2}{12\pi} \quad -(12)$$, where I_{av} = average current on dipole

The power delivered to the dipole is,

$$P = \frac{1}{2} I_0^2 R_r \quad -(13)$$ where I_0 = amplitude of terminal current of center-fed dipole.

from (12) and (13)

$$R_r = 790\left(\frac{I_{av}}{I_0}\right)^2 L_\lambda^2 \quad -(14)$$

For a short dipole without end loading, we have $I_{av} = \frac{1}{2} I_0$,

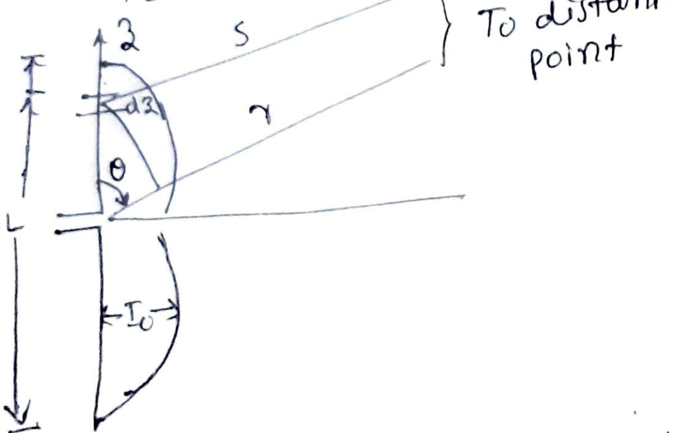
$$\therefore R_r = 197 L_\lambda^2 \quad -(15)$$

The thin linear antenna:

consider a symmetrical, thin, linear, center-fed antenna of length L. The retarded value of the current at any point z on the antenna referred to a point at a distance s is:

$$[I] = I_0 \sin\left[\frac{2\pi}{\lambda}\left(\frac{L}{2} \pm z\right)\right] e^{j\omega(t - r/c)} \quad -(1)$$

here the function $\sin\left[\frac{2\pi}{\lambda}\left(\frac{L}{2} \pm z\right)\right]$ is the form factor for the current on the antenna. The expression $L/2 + z$ is used when $z < 0$ and $L/2 - z$ is used when $z > 0$.



considering this antenna to be madeup of a series of infinitesimal dipoles of length dz, the field of the entire antenna may then be obtained by integrating the fields from all of dipoles making up the antenna with the result,

$$H_\varphi = \frac{j[I_0]}{2\pi r}\left[\frac{\cos[(\beta L \cos\theta)/2] - \cos(\beta L/2)}{\sin\theta}\right] \quad -(2)$$

$$E_\theta = \frac{j60[I_0]}{r}\left[\frac{\cos[(\beta L\cos\theta)/2] - \cos(\frac{\beta L}{2})}{\sin\theta}\right] \quad -(3)$$

where $[I_0] = I_0 e^{j\omega(t - r/c)}$ and $E_\theta = 120\pi H_\phi$

Radiation resistance of $\lambda/2$ antenna:

To find the radiation resistance, the Poynting vecter is integrated over a large sphere yielding the power radiated and this power is then equated to $\left(\frac{I_0}{\sqrt{2}}\right)^2 R_0$, where R_0 is the radiation resistance at a current maximum point and I_0 is the peak value in time of the current at this point.

$$P = \iint S_r\, ds = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}\int_0^{2\pi}\int_0^{\pi} |H_\phi|^2 r^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{15 I_0^2}{\pi}\int_0^{2\pi}\int_0^{\pi}\frac{\left[\cos[(\frac{\beta L}{2})\cos\theta] - \cos(\frac{\beta L}{2})\right]^2}{\sin\theta}\, d\theta\, d\phi \quad -(1)$$

$$= 30 I_0^2\int_0^{\pi}\frac{\left\{\cos[\frac{\beta L}{2}\cos\theta] - \cos(\frac{\beta L}{2})\right\}^2}{\sin\theta}\, d\theta \quad -(2)$$

Equating the radiated power to $P = I_0^2 R_0/2$ we get $-(3)$

$$R_0 = 60\int_0^{\pi}\frac{\left\{\cos[(\frac{\beta L}{2})\cos\theta] - \cos(\frac{\beta L}{2})\right\}^2}{\sin\theta}\, d\theta \quad -(4)$$

where radiation resistance R_0 is referred to the current maximum.

The radiation resistance of $\lambda/2$ antenna is:

$R_r = 30\, C_{in}(2\pi) = 73\,\Omega$

By including inductive reactance also the terminal impedance is: $Z = 73 + j\,42.5\,\Omega$