Module :- 3

Image Processing
VII Sem ECE

20

Image Restoration

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Image Degradation and Restoration Model

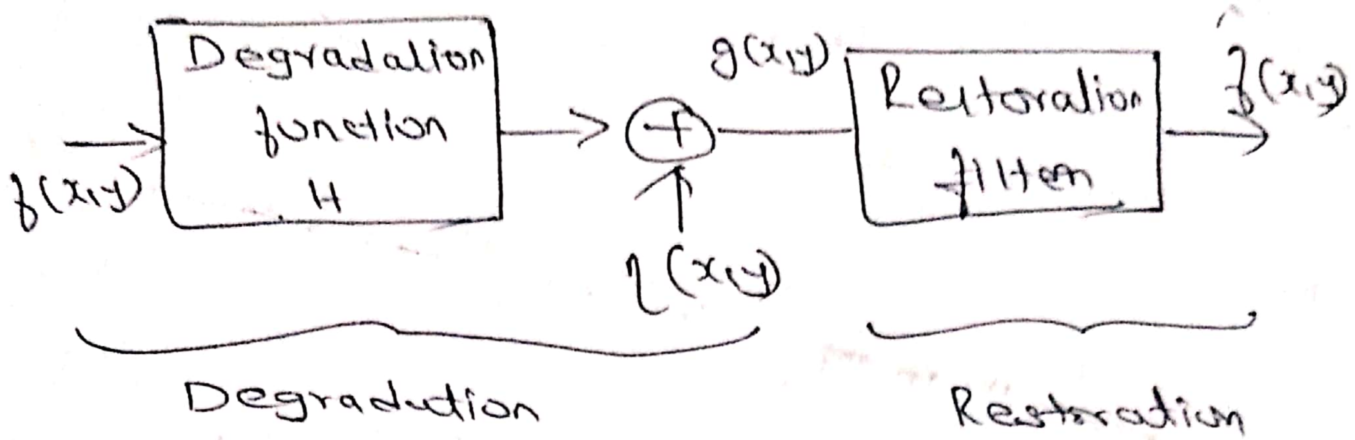


Image restoration is a process of recovering an image that has been degraded by some knowledge of degradation function H and additive noise term.

$f(x,y)$ = Input or original image

H = Degradation function

$\eta(x,y)$ = Additive noise term

$g(x,y)$ = Degraded image

$\hat{f}(x,y)$ = Recovered or restored image

In spatial domain

$$G(x,y) = h(x,y) \circledast f(x,y) + \eta(x,y)$$

$\circledast$ = Convolution

In Frequency domain

$$G(U,V) = H(U,V)\,F(U,V) + N(U,V)$$

Image Restoration

Image Restoration is a pre-processing method that supress a known degradations.

Noise Models

spatial and frequenty properties of noise

* Noise is assumed to be white noise i.e fourier spectrum of noise is constant
* Noise is assumed to be independent in spatial domain [i.e Noise is uncorrelated with image] i.e there is no correlation between pixel value of image and value of noise components.

Some important noise probablity density function (PDF)

* Noise which is independent of spatial locations are Gaussian, Rayleight, Gamma, exponential, Uniform noise
* Noise which is spatially dependent on spatial Location is periodic noise

Following are the most commonly occuring noise models

1) Gaussian Noise

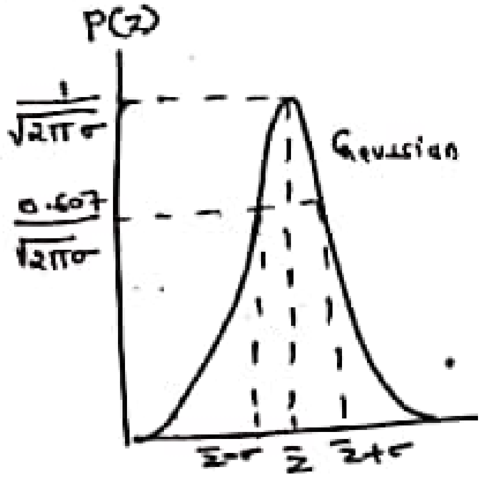
The PDF of a Gaussian random variable z is given by

$$P(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\bar{z})^2}{2\sigma^2}}$$

Where z = Intensity (gray) level

μ = Mean (average) value of z

σ = standard deviation.



* plot of $p(z)$ with respect to z is shown in fig. 70% of its values are in the range $[(\bar{z}-\sigma), (\bar{z}+\sigma)]$ and 95% of the values are in the range $[(\bar{z}-2\sigma), (\bar{z}+2\sigma)]$

2) Rayleigh Noise

The PDF of Rayleigh noise is given by

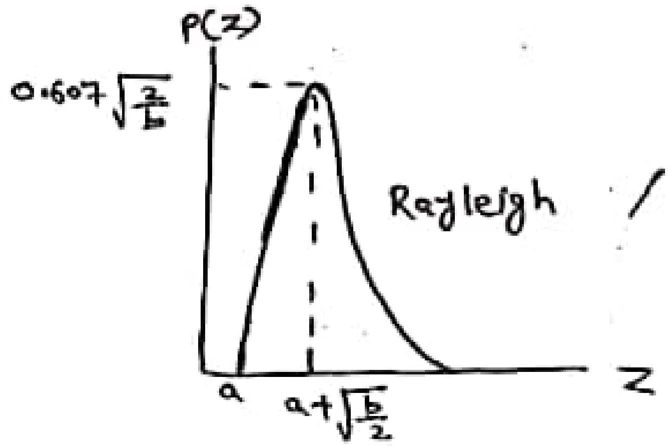
$$p(z) = \begin{cases} \frac{2}{b}(z-a)e^{-\frac{(z-a)^2}{b}} & \text{for } z \geq a \\ 0 & \text{for } z < a \end{cases}$$

Mean is given as

$$\bar{z} = a + \sqrt{\frac{\pi b}{4}}$$

Varience is given by

$$\sigma^2 = \frac{b(4-\pi)}{4}$$



* Plot of Rayleigh noise is shown in above fig. curve dosen't start from origin and it is not symmetrical w.r.t center of curve. Thus, Rayleigh density is useful for approximating skewed (non uniform) histograms.

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3) Erlang (Gamma) Noise

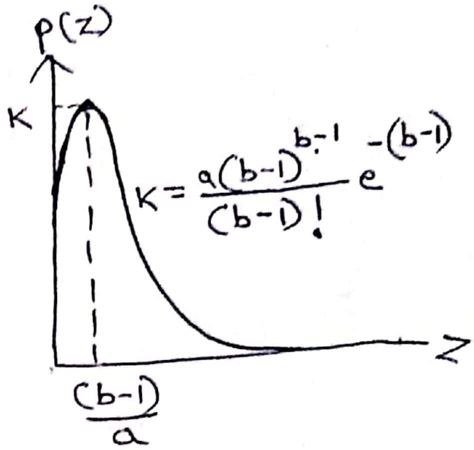
The PDF of Erlang Noise is given by

$$P(z) = \begin{cases} \dfrac{a^b z^{b-1}}{(b-1)!} e^{-az} & \text{for } z \geq 0 \\ 0 & \text{for } z < 0 \end{cases}$$

Here $a > 0$, b is +ve integer

Mean is given by $\mu = \dfrac{b}{a}$

Variance is given by $\sigma^2 = \dfrac{b}{a^2}$



* Fig shows plot of Erlang noise

4) Exponential Noise

PDF of exponential noise is given by

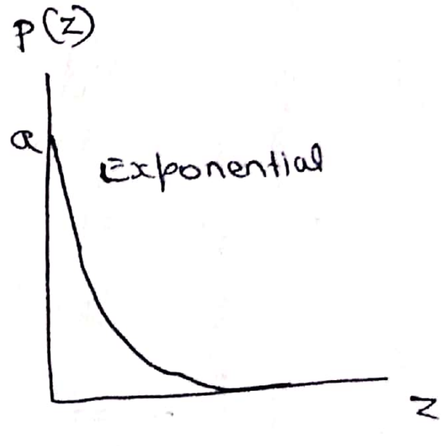
$$P(z) = \begin{cases} a e^{-az} & \text{for } z \geq 0 \\ 0 & \text{for } z < 0 \end{cases}$$

where $a > 0$,

Mean and variance is given by

$$\bar{z} = \frac{1}{a}$$

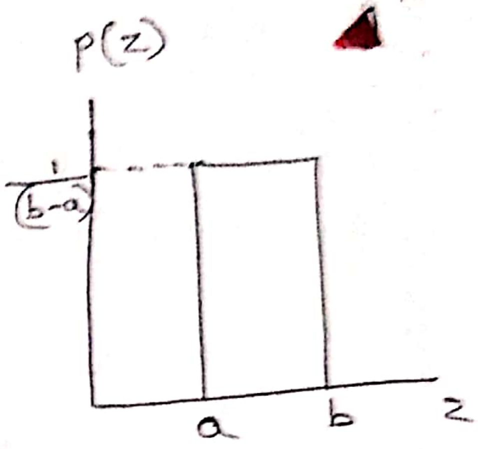
$$\& \quad \sigma^2 = \frac{1}{a^2}$$



* Fig shows the plot of exponential Noise

* PDF of exponential noise = PDF of erlang Noise when $b = 1$

⑤ Uniform noise



Uniform noise is given by

$$p(z) = \begin{cases} \frac{1}{b-a} & \text{if } a \le z \le b \\ 0 & \text{otherwise} \end{cases}$$

The mean and variance of uniform noise is given by

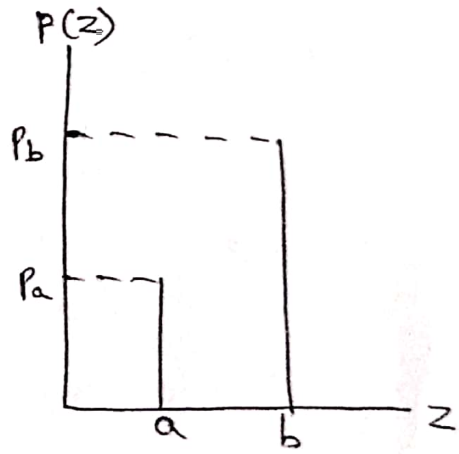
$$\mu = \frac{a+b}{2}, \quad \sigma^2 = \frac{(b-a)^2}{2}$$

⑥ Impulse (salt and pepper) Noise

The PDF of impulse noise is given by

$$p(z) = \begin{cases} P_a & \text{for } z=a \\ P_b & \text{for } z=b \\ 0 & \text{otherwise} \end{cases}$$

* If $b>a$, intensity (gray level) 'b' will appear as a light dot on the image and 'a' appears as a dark dot. This is bipolar noise.



* If $P_a=0$ or $P_b=0$, then it is unipolar noise
* If $P_a=0$ and only P_b exists, this is called pepper noise as only black dots are visible as noise
* If $P_b=0$ and only P_a exists, this is called salt noise as only white dots are visible on image as noise

Periodic Noise

* Periodic Noise is spatially dependent noise.
* During Image acquisition, electrical or electrochemical interference may cause such type of periodic noise.
* A strong periodic noise can be seen in the frequency domain as equi spaced dots at a particular radius around the center of the spectrum.

Estimation of Noise parameters

The parameters of periodic noise are estimated in frequency domain. If the sensor specifications are known, the noise PDF can be known from it. Consider the small strip (sub image) denoted by S

Let $P_s(z_i)$, $i = 0, 1, 2 ---- L-1$ denotes the probability estimates of intensities of pixels in S.

L = number of pixels in S. Then mean and Variance is given by

$$\bar{Z}(\mu) = \sum_{i=0}^{L-1} z_i P_s(z_i)$$

$$\sigma^2 = \sum_{i=0}^{L-1} (z_i - \mu)^2 P_s(z_i)$$

z_i are gray level of pixels in sub image S

Restoration in the presence of Noise only — spatial Filtering

[Image denoising in spatial domain]

* When the only degradation present in an image is noise then

$$g(x,y) = f(x,y) + \eta(x,y) \rightarrow \text{in spatial domain}$$

$$G(u,v) = F(u,v) + N(u,v) \rightarrow \text{in freq domain}$$

where $f(x,y)$ = image and $\eta(x,y)$ = noise

* Noise Removal methods

- Mean filters
  1) Arithmetic-Mean filter
  2) Geometric —ll—
  3) Harmonic —ll—
  4) Contra harmonic —ll—
- Order statistics filter
  1) Median filter
  2) Max & min filter
  3) Mid point filter
  4) Alpha trimmed filter
- Adaptive filters
  1) Adaptive local noise reduction filter
  2) Adaptive median filter.

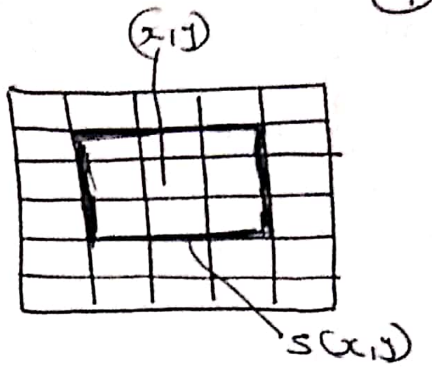
Mean Filters

1) Arithmetic Mean filters

Let S_{xy} represent the set of coordinates in the sub image of size $m \times n$, centered at point (x,y). The arithmetic Mean filter computes the average value of corrupted image $g(x,y)$ in the area defined by S_{xy}

i.e

$$\hat{f}(x,y) = \frac{1}{mn} \sum_{(s,t) \in S_{xy}} g(s,t)$$



Ex consider sub image s(x,y)

30	10	20
10	250	25
20	25	30

→

x	x	x
x	47	x
x	x	x

$= \frac{1}{9}[30+10+20+10+250+25+20+25+30] = 46.7$
$= \underline{\underline{47}}$

* In mean filter noise is removed as a result of blurring

2) Geometric Mean Filter

An image restored using a geometric mean filter is given by

$$\hat{f}(x,y) = \left(\prod_{(s,t) \in S_{xy}} g(s,t)\right)^{1/mn}$$

* Geometric mean filter removes noise and preserves more detail

* Here s(x,y) is the original image and the filter mask is $\underline{m}$ by $\underline{n}$ pixels

Ex

5	16	22
6	3	18
12	3	15

→

x	x	x
x	9	x
x	x	x

$= (5 \times 16 \times 22 \times 6 \times 3 \times 18 \times 12 \times 3 \times 15) \times \frac{1}{3 \times 3}$

$= 8.77 \Rightarrow \underline{\underline{9}}$

3) Harmonic mean filter

Harmonic mean filtered image is given by

$$\hat{f}(x,y) = \frac{mn}{\sum_{(s,t)\in S_{xy}} \frac{1}{g(s,t)}}$$

Harmonic mean filter works well for salt noise and guassian noise, but fails for pepper noise

Ex:

30	10	20
10	250	25
20	25	30

$\longrightarrow$

x	x	x
x	20	x
x	x	x

$$= \frac{9}{\frac{1}{30}+\frac{1}{10}+\frac{1}{20}+\frac{1}{10}+\frac{1}{250}+\frac{1}{25}+\frac{1}{20}+\frac{1}{25}+\frac{1}{30}}$$

$$= 19.97 \simeq 20$$

4) Contra harmonic filter

Restored image from contra harmonic filter is

$$\hat{f}(x,y) = \frac{\sum_{(s,t)\in S_{xy}} g(s,t)^{Q+1}}{\sum_{(s,t)\in S_{xy}} g(s,t)^{Q}}$$

* Here Q is the order of filter
* this filter reduces salt & pepper noise

for $Q>0$ it elliminates pepper noise

for $Q<0$, it elliminates salt noise

For $q=0$, $\hat{f}(x,y) = \dfrac{\sum_{(s,t)\in S_{xy}} g(s,t)^1}{\sum_{(s,t)\in S_{xy}} 1} = \dfrac{\sum_{(s,t)\in S_{xy}} g(s,t)}{mn}$

= Mean filter

Thus for $q=0$, contra harmonic filter becomes arithmetic mean filter

For $q=-1$, $\hat{f}(x,y) = \dfrac{\sum_{(s,t)\in S_{xy}} g(s,t)^0}{\sum_{(s,t)\in S_{xy}} \frac{1}{g(s,t)}} = \dfrac{mn}{\sum_{s,t\in S_{xy}} \frac{1}{g(s,t)}}$

= Harmonic mean filter

Thus for $q=-1$, it becomes harmonic mean filter

Order statistics filter

* Order statistics filter are non linear spatial filters
* Its response is based on ordering the pixels containing in sub image area. The response of the filter at any point is determined by the ranking result.

1) **Median filter**

Median filter replaces the pixel value by the median of the pixel value in the neighbourhood of the center pixel (x,y)

$$\hat{f}(x,y) = \underset{(s,t) \in S_{xy}}{\text{Median}} \{ g(s,t) \}$$

Ex:-

20	10	20
10	250	25
20	25	30

$\Longrightarrow$

10
10
20
20
25
25
30
30
250

$\Longrightarrow$ Median = 25

x	x	x
y	25	x
x	x	x

* It removes noise with less blurring than Linear smoothing filters
* This filter is effective for bipolar and unipolar impulse noise.

(2) Max and min filter

The max filter represents the 100^{th} percentile of the ranked set of numbers

$$\hat{f}(x,y) = \underset{(s,t) \in S_{xy}}{\max} \{ g(s,t) \}$$

* It is used for finding the brightest point in an image, pepper noise in image has very low values. It is reduced by max filter

Ex

30	10	20
10	250	25
20	25	30

$\xrightarrow[\text{filter}]{\text{Max}}$

x	x	x
x	250	x
x	x	x

The min filter represents 0th percentile of the ranked set of numbers

$$\hat{f}(x,y) = \min_{(s,t)\in S_{xy}} \{g(s,t)\}$$

* This filter is useful for finding the darkest point in image and also it reduces salt noise

Ex:

30	10	20
10	250	25
20	25	30

$\xrightarrow{\text{min filter}}$

X	X	X
X	10	X
X	X	X

③ Mid point filter

This filter computes the mid point of maximum and minimum values of intensities:

$$\hat{f}(x,y) = \frac{1}{2}\left[\max_{(s,t)\in S_{xy}} \{g(s,t)\} + \min_{(s,t)\in S_{xy}} \{g(s,t)\}\right]$$

* This filter is a combination of order statistics and averaging. It works well for guassian and uniform noise

Ex

30	10	20
10	250	25
20	25	30

$\xrightarrow{\text{Mid point filter}}$

X	X	X
X	130	X
X	X	X

$$= \frac{1}{2}[250 + 10]$$

$$= \underline{130}$$

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4) Alpha trimmed filters

Let there be $m \times n$ pixels in the neighbourhood S_{xy}. Remove $\frac{d}{2}$ lowest and $\frac{d}{2}$ highest gray level valued pixels. Number of remaining pixels are $(mn-d)$, which are represented by $g_r(s,t)$.

Restored image by Alpha trimmed filters is given by

$$\hat{f}(x,y) = \frac{1}{(mn-d)} \sum_{(s,t)\in S_{xy}} g_r(s,t)$$

* For $d=0$, alpha trimmed filter = Arithmetic filter
* For $d = \frac{mn-1}{2}$ Alpha Trimmed filter = Median filter
* For $d=2$, we remove $\frac{d}{2}=1$ min value (10 in below example) and $\frac{d}{2}=1$ max value (250)

30	10	20
10	250	25
20	25	30

$\xrightarrow[\text{for } d=2]{\text{Alpha Trimmed filter}}$

×	×	×
×	23	×
×	×	×

$$= \frac{1}{9-2}\left[30+20+10+25+20+25+30\right]$$

$$= 22.8 = \underline{\underline{23}}$$

* This filter removes the combination of salt and pepper and Gaussian noise

Adaptive filters

* Mean Filters and adaptive filters are not capable of distinguishing noise from pixel values. Adaptive filters replace all pixels values with mean/median which causes distortion
* In adaptive filter its behaviour adapts to change in the characteristics of ~~filter~~ image area being filtered.

① Adaptive local noise reduction filter

* This filter operate on a region S_{xy}. Average intensity of this region is given by mean. Contrast in the region is measured by variance
* Filter is operated in a local region S_{xy} centered at (x, y) where

$g(x,y)$ = noisy image of input image $f(x,y)$

σ_n^2 = Variance of noise that corrupts $f(x,y)$

m_L = Local mean of pixels in region S_{xy}

σ_L^2 = Local variance of pixels in region S_{xy}

$\hat{f}(x,y)$ = Filtered image

The behaviour of filter is as follows

1) In case of no noise, filter should not do any action. This reduces distortions,

i.e if $\sigma_n^2 = 0 \Rightarrow \hat{f}(x,y) = g(x,y)$

2) Edges should not be treated as noise. They should be preserved. edges are identified in S_{xy}. If $\sigma_L^2 > \sigma_n^2$ (Local variance is higher as compared to overall noise variance), no filter action should take place

$$\text{if } \sigma_L^2 > \sigma_n^2 \Rightarrow \hat{f}(x,y) \simeq g(x,y)$$

3) If the pixel is actually noisy, filter action should happen. When the local area has the same property as the overall image (i.e 2 variance are equal), noise reduction is done by averaging

$$\text{if } \sigma_L^2 = \sigma_n^2 \Rightarrow \hat{f}(x,y) = m_L$$

Thus adaptive filter is given by

$$\boxed{\hat{f}(x,y) = g(s,t) - \frac{\sigma_n^2}{\sigma_L^2}\left[g(s,t) - m_L\right]} \quad \text{—(1)}$$

Here σ_n^2 needs to be known in advance. σ_L^2 and m_L is estimated for selected area

Case (i)

In case of no noise $(\sigma_n^2 = 0)$

$\therefore$ eqn (1) becomes

$$\boxed{\hat{f}(x,y) = g(s,t)}$$

Case 2 :- In case of edges

$$\sigma_n^2 < \sigma_L^2$$

then $\dfrac{\sigma_n^2}{\sigma_L^2} \cong 0$

$\therefore$ eqn (1) becomes

$$\boxed{\hat{f}(x,y) = g(s,t)}$$

Case 3 :- In case of presence of noise

if $\sigma_n^2 = \sigma_L^2$ then $\dfrac{\sigma_n^2}{\sigma_L^2} \cong 1$

$\therefore$ eqn (1) becomes

$$\hat{f}(x,y) = g(s,t) - [g(s,t) - m_L]$$

$$= g(s,t) - g(s,t) + m_L$$

$$\boxed{\hat{f}(x,y) = m_L}$$

* Adaptive filter removes noise and introduces less blurring compared to mean filter

* If noise variance is not estimated correctly, filter gives undesirable result i.e

If estimated variance value is too low as compared to actual variance noise correction will be smaller than it should be.

* If the estimate is too high, the noise correction is large and output image may loose the dynamic range

② Adaptive Median filter

Normal median filter handles impulse noise with small probability. Adaptive median filter can handle ~~large~~ large impulse noise

The main objective of adaptive median filter is

(i) To remove salt and pepper (impulse) noise
(ii) To smoothen noise other than impulse noise
(iii) To reduce distortion of thinning & thickening of edges

Variables used in the algorithm are

S_{xy} = Rectangular window

Z_{min} = Minimum gray level value in S_{xy}

Z_{max} = Maximum gray level value in S_{xy}

Z_{med} = Median —— " ——

Z_{xy} = Gray level at (x, y)

S_{max} = Maximum allowed size S_{xy}

Adaptive median filter works in 2 stages denoted as A and B stage.

Stage A :-

If $Z_{min} < Z_{med} < Z_{max}$, go to stage B
else increase the window S_{xy} size
if window size $\leq S_{max}$ repeat level A
else output = Z_{med}

Stage B

If $z_{min} < z_{xy} < z_{max} \Rightarrow$ output $= z_{xy}$
(do not filter)

else output $= z_{med}$ (Replace pixel value with z_{med})

Explanation

The purpose of Stage A is to determine z_{xy} is an impulse or not
i.e if $z_{med} \neq$ impulse (z_{min} or z_{max}), then go to stage B)

The purpose of stage B is to determine, if the point in the center of window z_{xy} is itself an impulse.
i.e if $z_{xy} \neq$ impulse (z_{min} or z_{max}) then there is no need to filter and output value is same as z_{xy}.

If $z_{xy} =$ impulse ($z_{xy} = z_{min}$ or $z_{xy} = z_{max}$)
then output = median value

Advantage

(i) only noisy pixel is filtered
(ii) If filtering is done, we make sure that median values is not noise.

Periodic noise reduction by frequency domain filtering

* Periodic noise is spatially dependent noise and it occurs due to electrical or electromagnetic interference. It gives rise to regular noise pattern in an image. Frequency domain techniques are very effective in removing periodic noise.

Frequency domain Filters are [for periodic noise reduction]

(i) Band reject filter

Removing periodic noise from an image involves removing a particular range of frequencies from the origin

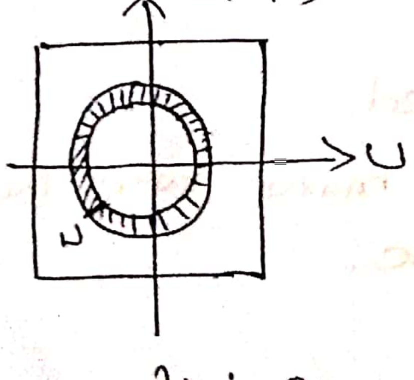
Transfer function of ideal band Reject filter is

$$H(u,v) = \begin{cases} 1 & D(u,v) < D_0 - W/2 \\ 0 & D_0 - \frac{W}{2} \le D(u,v) \le D_0 + \frac{W}{2} \\ 1 & D(u,v) > D_0 + \frac{W}{2} \end{cases}$$

Where $D(u,v)$ is the distance from the origin

W is the width (Bandwidth)

D_0 is the radial center



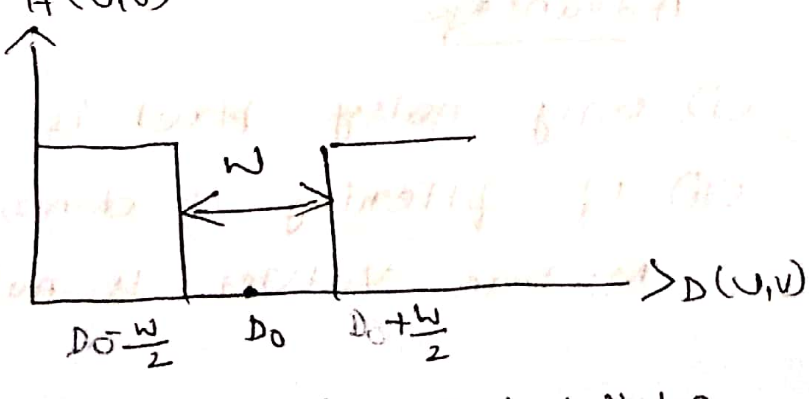


fig: Freq response of ideal band reject filter

Transfer function of butterworth band reject filter of order n is given by

$$H(u,v) = \frac{1}{1+\left[\frac{D(u,v)W}{D^2(u,v)-D_0^2}\right]^{2n}}$$

Gaussian band reject filter is given by

$$H(u,v) = 1 - e^{-\frac{1}{2}\left[\frac{D^2(u,v)-D_0^2}{D(u,v)W}\right]^2}$$

② <u>Band pass filter</u>

A Band pass filter performs opposite operation of bandreject filter. ∴ The Transfer function of bandpass filter is

$$H_{bp}(u,v) = 1 - H_{BR}(u,v)$$

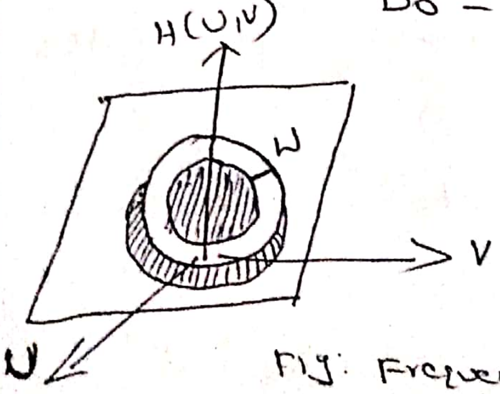
Ideal band pass filter is given by

$$H(u,v) = \begin{cases} 0 & D(u,v) \le D_0 - \frac{W}{2} \\ 1 & D_0 - \frac{W}{2} < D(u,v) < D_0 + \frac{W}{2} \\ 0 & D(u,v) \ge D_0 + \frac{W}{2} \end{cases}$$

where $D(u,v)$ = Distance from origin

W = Bandwidth

D_0 = radial center or cut off frequency



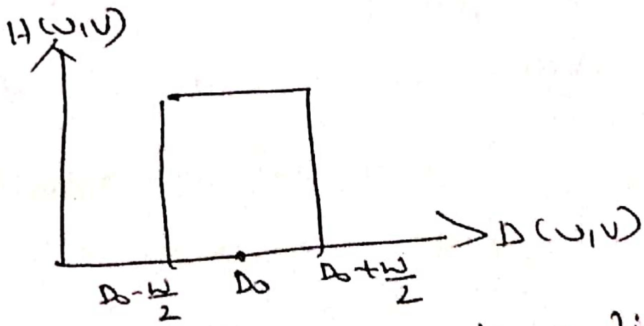


Fig: Frequency response of band pass filter.

Butterworth band pass filter of order n is given by

$$H(u,v) = 1 - H(u,v)_{\text{butterworth band reject}}$$

$$= 1 - \frac{1}{1 + \left[\frac{D(u,v)W}{D^2(u,v) - D_0^2}\right]^{2n}}$$

$$= \frac{\left[\frac{D(u,v)W}{D^2(u,v) - D_0^2}\right]^{2n}}{1 + \left[\frac{D(u,v)W}{D^2(u,v) - D_0^2}\right]^{2n}}$$

Similarly Gaussian band pass filter is given by

$$H(u,v) = 1 - H(u,v)_{\text{gaussian band } \cancel{\text{pass}} \text{ reject}}$$

$$= 1 - \left[1 - e^{-\frac{1}{2}\left[\frac{D^2(u,v) - D_0^2}{D(u,v)W}\right]^2}\right]$$

$$= e^{-1/2\left[\frac{D^2(u,v) - D_0^2}{D(u,v)W}\right]^2}$$

③ **Notch filter**

Notch filter pass or reject frequency in a predefined neighborhood about a center frequency.

The Transfer function of ideal notch reject filter with centers (u_0, v_0) and $(-u_0, -v_0)$ is

$$H(u,v) = \begin{cases} 0 & D_1(u,v) \leq D_0 \text{ or } D_2(u,v) \leq D_0 \\ 1 & \text{otherwise} \end{cases}$$

where

$D_1(u,v)$ and $D_2(u,v)$'s the distance from (u_0, v_0) and $(-u_0, -v_0)$

where $D_0 =$ ~~cutoff~~ frequency radius

$$D_1 = \left[\left(u - \frac{M}{2} - u_0\right)^2 + \left(v - \frac{N}{2} - v_0\right)^2\right]$$

$$D_2 = \left[\left(u - \frac{M}{2} + u_0\right)^2 + \left(v - \frac{N}{2} + v_0\right)^2\right]$$

The center of frequency rectangle has been shifted to point $\left(\frac{M}{2}, \frac{N}{2}\right)$

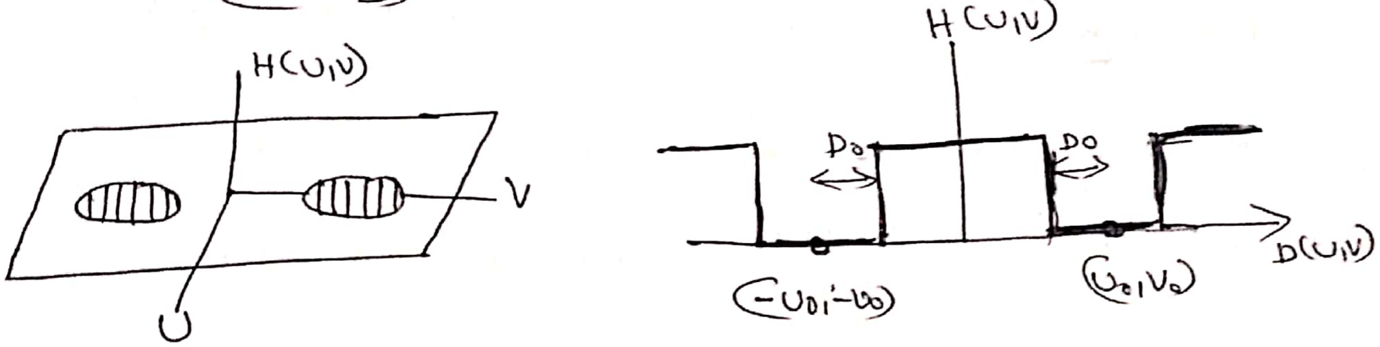


fig:- Frequency response of ideal notch reject filter

Transfer function of butterworth notch reject filter is

$$H(u,v) = \frac{1}{1 + \left[\frac{D_0^2}{D_1(u,v)D_2(u,v)}\right]^2}$$

TF of Gaussian notch reject filter is given by

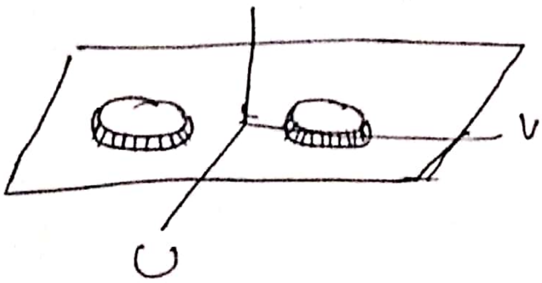
$$H(u,v) = 1 - e^{-\frac{1}{2}\left[\frac{D_1(u,v)\,D_2(u,v)}{D_0^2}\right]}$$

Notch ~~pass~~ filter

The Transfer function of notch ~~reject~~ pass filter can be obtained from notch reject filter

$$H_{np}(u,v) = 1 - H_{nR}(u,v)$$

i.e $H(u,v) = \begin{cases} 1 & D_1(u,v) \le D_0 \text{ or } D_2(u,v) \le D_0 \\ 0 & \text{otherwise} \end{cases}$



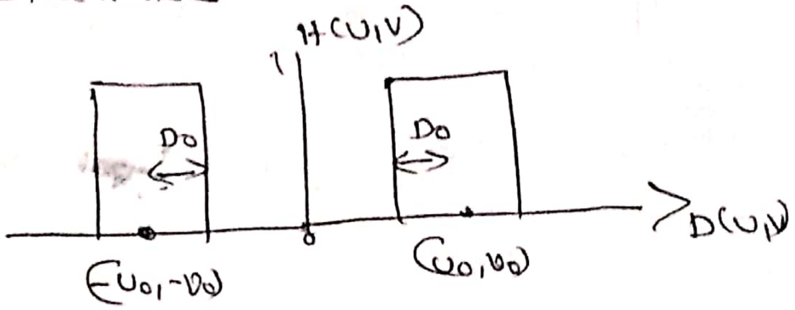


fig: Frequency response of ideal notch filter

Butterworth notch pass filter is given by

$H(u,v) = 1 - H(u,v)_{BW \text{ notch reject filter}}$

$$= 1 - \frac{1}{1 + \left[\frac{D_0^2}{D_1(u,v)\, D_2(u,v)}\right]^n}$$

$$= \frac{\left[\frac{D_0^2}{D_1(u,v)\, D_2(u,v)}\right]^n}{1 + \left[\frac{D_0^2}{D_1(u,v)\, D_2(u,v)}\right]^n}$$

Gaussian notch pass filter is given by

$H(u,v) = 1 - H(u,v)_{BW \text{ notch reject filter}}$

$$= 1 - \left[1 - e^{-1/2\left(\frac{D_0(u,v)\, D(u,v)}{D_0^2}\right)}\right]$$

$$= e^{-1/2\left[\frac{D_0(u,v)\, D_1(u,v)}{D_0^2}\right]}$$

Note:

* If $u_0 \neq v_0 = 0$, notch reject filter becomes high pass filter
* If $u_0 = v_0 = 0$, notch pass filter becomes low pass filter

Optimum Notch filter

The first step in optimum notch filter is to extract the principal frequency component of the interference pattern (noise) from noisy image. This can be done by placing a notch pass filter $H(u,v)$ at the location of each noise spike.

Fourier transform of interference noise pattern is given by

$$N(u,v) = H(u,v) \cdot G(u,v) \quad \text{——— (1)}$$

Where $G(u,v)$ = Fourier transform of corrupted image

$H(u,v)$ = Optimum Notch filter transfer function.

In spatial domain eqn (1) is written as

$$\eta(x,y) = F^{-1}\left[H(u,v) \cdot G(u,v)\right] \quad \text{——— (2)}$$

$$\eta(x,y) = F^{-1}[H(u,v)] \cdot g(x,y) \quad \text{——— (3)}$$

Corrupted image $g(x,y)$ is assumed to be formed by addition of uncorrupted image $f(x,y)$ & noise $\eta(x,y)$

$$g(x,y) = f(x,y) + \eta(x,y)$$

If the noise term $\eta(x,y)$ were known completely, it can be subtracted from $f(x,y)$, but if the noise is an approximation, then a weighted portion of $\eta(x,y)$ is subtracted from $f(x,y)$

$$\hat{f}(x,y) = g(x,y) - w(x,y)\,\eta(x,y) \quad \text{——— (4)}$$

$\hat{f}(x,y)$ is the estimate of $f(x,y)$ and $w(x,y)$ is to be determined.

Consider a Neighbourhood of size $(2a+1)\times(2b+1)$ about the point (x,y). The Local variance of $\hat{f}(x,y)$ at coordinates (x,y) can be estimated from the samples. [i.e select $w(x,y)$ so that variance of $\hat{f}(x,y)$ is minimized for size $(2a+1)\times(2b+1)$]

$$\boxed{\sigma^2(x,y) = \frac{1}{(2a+1)(2b+1)} \sum_{s=-a}^{a} \sum_{t=-b}^{b} \left[\hat{f}(x+s,y+t) - \overline{\hat{f}}(x,y)\right]^2} \rightarrow (5)$$

Where $\overline{\hat{f}}(x,y)$ = average value of $\hat{f}$ in neighbourhood of size $(2a+1)\times(2b+1)$

$$\text{where } \overline{\hat{f}} = \frac{1}{(2a+1)(2b+1)} \sum_{s=-a}^{a} \sum_{t=-b}^{b} \hat{f}(x+s,y+t) \rightarrow (6)$$

substituting eqn (4) in eqn (5)

$$\boxed{\sigma^2(x,y) = \frac{1}{(2a+1)(2b+1)} \sum_{s=-a}^{a} \sum_{t=-b}^{b} \left[g(x+s,y+t) - w(x,y)\,\eta(x+s,y+t)\right] - \left[\overline{g}(x,y) - \overline{w(x,y)\cdot\eta(x,y)}\right]^2} \rightarrow (7)$$

Assuming that $w(x,y)$ remains constant over the Neighbourhood gives the approximation $w(x+s,y+t) = w(x,y) \rightarrow (8)$

for $-a \le s \le a$ and $-b \le t \le b$. This assumption results in

$$\overline{w(x,y),\eta(x,y)} = w(x,y)\,\overline{\eta}(x,y) \longrightarrow (9)$$

∴ Equation 7 reduces to

$$\boxed{\sigma^2(x,y) = \frac{1}{(2a+1)(2b+1)} \sum_{s=-a}^{a} \sum_{t=-b}^{b} \left[g(x+s,y+t) - w(x,y)\,\eta(x+s,y+t)\right] - \left[\overline{g}(x,y) - w(x,y)\cdot\overline{\eta}(x,y)\right]^2}$$

To minimize $\sigma^2(x,y)$ we solve $\dfrac{\partial \sigma^2(x,y)}{\partial w(x,y)} = 0$

for $w(x,y)$, The result is

$$w(x,y) = \frac{\overline{g(x,y)\,\eta(x,y)} - \overline{g}(x,y)\,\overline{\eta}(x,y)}{\overline{\eta^2}(x,y) - \overline{\eta}^2(x,y)} \quad -10$$

To obtain an estimated restored image $\hat{f}(x,y)$. $w(x,y)$ must be computed from eqn 10 and substituted into eqn (9). to determine $\hat{f}(x,y)$

Linear position invariant degradation

The input-output relationship before the restoration (i.e degradation) is expressed as

$$g(x,y) = H[f(x,y)] + \eta(x,y) \quad — (1)$$

The degradation function should follow two most important properties.

1. **Linearity**

Let us assume $\eta(x,y) = 0$ $\therefore$ eqn (1) becomes $g(x,y) = H[f(x,y)]$

A degradation function H is said to be linear if

$$H[a f_1(x,y) + b f_2(x,y)] = aH[f_1(x,y)] + bH[f_2(x,y)] \quad — (2)$$

where a and b are scalars and $f_1(x,y)$ and $f_2(x,y)$ are any two input images

For $a=b=1$, eqn (2) becomes

$$H[f_1(x,y) + f_2(x,y)] = H[f_1(x,y)] + H[f_2(x,y)]$$

This is also called additive property

If $f_2(x,y) = 0$ eqn (2) becomes

$$H[a f_1(x,y)] = aH[f_1(x,y)]$$

This is also called homogeneity property

Thus linear property H possess additivity and homogeneity property.

2. **Position invariant**

An operator having input-output relationship $g(x,y) = H[f(x,y)]$ is said to be position invariant if

$$H[\delta(x-\alpha, y-\beta)] = g(x-\alpha, y-\beta)$$

for any α, β and $f(x,y)$ — (A)

$f(x,y)$ can be expressed in terms of continuous impulse functions

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\,\delta(x-\alpha, y-\beta)\,d\alpha\, d\beta \quad - (3)$$

If $\eta(x,y) = 0$, then eqn (1) becomes

$$g(x,y) = H[f(x,y)] \quad \text{—— } (4)$$

Substituting eqn (3) in (4)

$$g(x,y) = H\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\,\delta(x-\alpha, y-\beta)\,d\alpha\, d\beta\right]$$

If H is a linear operator and extending addition property to integrals, then

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H[f(\alpha,\beta), \delta(x-\alpha, y-\beta)]\,d\alpha\, d\beta$$

Because $f(\alpha,\beta)$ is independent of x and y and using homogeneity property

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\,H[\delta(x-\alpha, y-\beta)]\,d\alpha\, d\beta \quad - (5)$$

The term $H[\delta(x-\alpha, y-\beta)]$ is called — (6)

$$h(x,\alpha,y,\beta) = H[\delta(x-\alpha, y-\beta)]$$

impulse response of H

Substituting eqn (6) in eqn (5)

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\,h(x,\alpha,y,\beta)\,d\alpha\, d\beta \quad \text{—— } (7)$$

If H is position invariant then from eqn (A) becomes

$$H[\delta(x-\alpha, y-\beta)] = h(x-\alpha, y-\beta) \quad \text{—— } (8)$$

(14)

Then eqn (7) reduces to

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\, h(x-\alpha, y-\beta)\, d\alpha\, d\beta \quad \text{--- (9)}$$

In the presence of additive noise, then eqn (7) becomes

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\, h(x,\alpha, y,\beta)\, d\alpha\, d\beta + \eta(x,y) \quad \text{--- (10)}$$

If H is position invariant then

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\, h(x-\alpha, y-\beta)\, d\alpha\, d\beta + \eta(x,y) \quad \text{--- (11)}$$

Using familiar notation for convolution, eqn (11) can be written as

$$g(x,y) = h(x,y) * f(x,y) + \eta(x,y)$$

In frequency domain, above eqn becomes

$$G(u,v) = H(u,v)\, F(u,v) + N(u,v)$$

Estimation of degradation function

In order to restore the image, we need to estimate the degradation function. There are three principal ways to estimate the degradation function

1) observation
2) experimentation
3) Mathematical modelling

(i) Estimation by image observation

Consider an image $g(x,y)$, which is degraded with a known degradation function H. In order to estimate H, consider a small rectangle section of image containing a part of a object and background. In order to reduce the effect of noise, we look for an area in which the signal content is strong (eg.; an area of high contrast). We try to unblur the image manually (eg, by sharpening the sub image) and generate $\hat{f}_s(x,y)$

$g_s(x,y)$ = original sub image

$\hat{f}_s(x,y)$ = Restored version of $g_s(x,y)$

The degradation can be estimated for subimage by

$$\boxed{H_s(u,v) = \frac{G_s(u,v)}{F_s(u,v)}}$$

(ii) Estimation by Experimentation

It is possible to estimate the degradation function accurately if the equipment used to acquire the degraded image is available

First step

Adjust the equipment by varying the system settings such that the image obtained is similar to the degraded image that needs to be restored

<u>Second step</u>

obtain the impulse response of the degradation by imaging an impulse (small dot of light) using same system setting.

An impulse is simulated by a bright dot of light, as bright as possible to reduce the effect of noise to negligible values. Impulse response is given by

$$\boxed{H(u,v) = \frac{G(u,v)}{A}}$$

where $G(u,v)$ = DFT $[g(x,y)]$ = DFT [degraded impulse]

A = constant describing the strength of the impulse.

(iii) <u>Estimation by Modelling</u>

Degradation model based on atmospheric turbulence is given by

$$H(u,v) = e^{-k(u^2+v^2)^{5/6}}$$

where k is constant, that depends on nature of blur

$k = 0.0025$ for severe turbulence

$k = 0.001$ for medium turbulence

$k = 0.00025$ for low turbulence

Blur can be due to motion (camera and object moving with respect to each other). Let an image undergo planar motion $x_0(t)$, and let the camera velocity in x and y direction is $y_0(t)$, then blurred image is obtained by

$$g(x,y) = \int_0^T f(x-x_0(t), y-y_0(t))\,dt \quad — (1)$$

where T = exposure time.

Taking FT of $g(x,y)$

$$G(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, e^{-j2\pi(ux+vy)}\,dx\,dy \quad — (2)$$

for 2 continuous variable t & z w.k.T 2D continuous FT pair is given by

$$F(\mu,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\, e^{-j2\pi(\mu t+\nu z)}\,dt\,dz$$

substituting eqn (1) in eqn (2)

$$G(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_0^T f(x-x_0(t), y-y_0(t))\,dt\right] e^{-j2\pi(ux+vy)}\,dx\,dy$$

Interchanging the order of integration

$$G(u,v) = \int_0^T\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-x_0(t), y-y_0(t))\, e^{-j2\pi(ux+vy)}\,dx\,dy\right]dt \quad — (3)$$

The Term inside the outer bracket is fourier transform of displaced form $f(x-x_0(t), y-y_0(t))$

$\therefore$ eqn (3) becomes

$$G(u,v) = \int_0^T F(u,v)\, e^{-j2\pi[ux_0(t)+vy_0(t)]}\,dt$$

$$= F(u,v)\int_0^T e^{-j2\pi[ux_0(t)+vy_0(t)]}\,dt \quad — (4)$$

w.k.T $G(u,v) = F(u,v)\,H(u,v) \quad — (5)$

and $F(u,v)$ is independent of t

Scanned with CamScanner

comparing eqn 4 and 5

$$H(u,v) = \int_0^T e^{-j2\pi [u x_0(t) + v y_0(t)]} dt$$

Inverse filtering

* The simplest approach to restoration is direct inverse filtering. Inverse filtering is given by

$$\hat{F}(u,v) = \frac{G(u,v)}{H(u,v)}$$

Where $\hat{F}(u,v)$ = Restored image

$G(u,v)$ = Fourier Transform of original image $F(u,v)$

$H(u,v)$ = Degradation function

W.k.t $G(u,v) = H(u,v) \cdot F(u,v) + N(u,v)$

$\therefore$

$$H(u,v) \cdot F(u,v) = G(u,v) - N(u,v)$$

$$F(u,v) = \frac{G(u,v)}{H(u,v)} - \frac{N(u,v)}{H(u,v)}$$

Even though the degradation function $H(u,v)$ is known, the ungraded image $f(x,y)$ can not be obtained completely because

1. The noise component is a random function whose fourier transform $N(u,v)$ is not known
2. In practice $H(u,v)$ can contain numerous zeros

Minimum mean square error (wiener) Filtering

Wiener filter restores the image in the presence of blur as well as noise

The objective is to find an estimate $\hat{f}$ of the corrupted image f such that mean square error between them is minimized

This error measure is given by

$$e^2 = E\left\{(f-\hat{f})^2\right\} \quad -① $$

where E is the expected value of

f is undegraded image [corrupted image]

The solution to eq^n ① in frequency domain is

$$\hat{F}(u,v) = \frac{H^*(u,v)\, S_f\cancel{(u,v)}}{\cancel{S_f(u,v)}\left[|H(u,v)|^2 + \frac{S_\eta(u,v)}{S_f(u,v)}\right]}$$

$$= \frac{H^*(u,v)}{|H(u,v)|^2 + K}$$

$$\frac{S_\eta(u,v)}{S_f(u,v)} = K$$

Multiply and divide by $H(u,v)$

$$= \frac{H^*(u,v)\, H(u,v)}{H(u,v) \cdot |H(u,v)|^2 + K}$$

$$H(u,v) \cdot H^*(u,v) = |H(u,v)|^2$$

$$\hat{F}(u,v) = \frac{|H(u,v)|^2}{H(u,v) \cdot |H(u,v)|^2 + K}$$

$H(u,v)$ = degradation function

$H^*(u,v)$ = complex conjugate of $H(u,v)$

$|H(u,v)|^2 = H^*(u,v)\,H(u,v)$

$S_\eta(u,v) = |N(u,v)|^2$ = Power spectrum of noise

$S_f(u,v) = |F(u,v)|^2$ = Power spectrum of undegraded image.

Wiener filter of the form as in eqn ② is difficult to estimate power spectrum of noise $[S_\eta(u,v)]$ and also power spectrum of undegraded image. $[S_f(u,v)]$

To solve this we can do the approximation

$$\frac{S_\eta(u,v)}{S_f(u,v)} = K$$

∴ eqn ② becomes

$$\hat{F}(u,v) = \left[\frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + K}\right] G(u,v)$$

K is chosen experimentally

Inorder to find $S_\eta(u,v)$ and $S_f(u,v)$ the following ~~Pedal~~ measures are used

(i) Signal to noise ratio (SNR)

$$SNR = \frac{\sum_{u=0}^{M-1} \sum_{v=0}^{N-1} |F(u,v)|^2}{\sum_{u=0}^{M-1} \sum_{v=0}^{N-1} |N(u,v)|^2}$$

* Images with low noise has lower SNR

(ii) Mean square error (MSE)

$$MSE = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left[f(x,y) - \hat{f}(x,y) \right]^2$$

Constrained least square filtering

Wiener filter implementation requires power spectrum of original image and noise to be known or estimation of parameter K is required, which may be wrong. But, for constrained least square filters only mean and variance of noise is needed.

Degradation process is given by

$$g(x,y) = f(x,y) * h(x,y) + n(x,y)$$

$$\boxed{g = Hf + n} \qquad \text{---} \quad (1)$$

H is highly sensitive to noise. H matrix is very large (MN×MN) and its inverse is very sensitive to noise, To solve those type of problem consider second order derivative of image

eg: The Laplacian

i.e $$C = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left[\nabla^2 f(x,y) \right]$$

where $\| g - H\hat{f} \| = \| n \|^2$

where ∇^2 = Laplacian operator

$\hat{f}$ = estimate of undegraded image

In frequency domain, constraint least square filter is given by

(18)

$$\hat{F}(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + \gamma |P(u,v)|^2} \right] G(u,v)$$

where γ should be adjusted

and $P(u,v)$ = 3x3 laplacian mask [FT of $P(x,y)$]

$$P(x,y) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & -4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

If $\gamma = 0$, constraint least square filter = wiener filter