QUANTUM MECHANICS

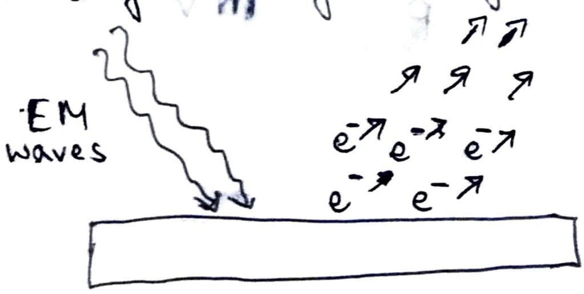
* It describes the smallest things in the universe
  Ex: e^-, p, n
* Classical mechanics does not work on microscopic level.
* Quantum mechanics describes a particle as a wave function as in a distribution in a particular space denoted by. ψ

  Ex: Dual nature of light. which is from Young's double slit experiment. the formation of bright & dark fringes ie... constructive & destructive interference.

→ Photo Electric effect:- (Einstein)

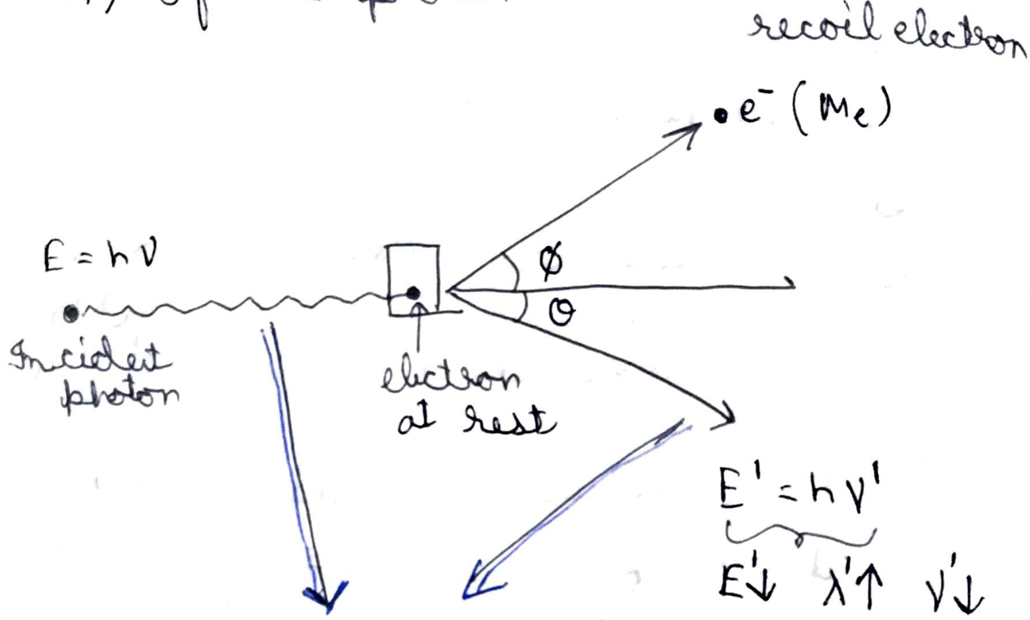
* This ~~effect~~ depends on what kind of light (~~frequency~~ the whole EM spectrum) & also on the type of metal.

- If the metal ~~are~~ are Alkali metal visible light should be incident on the metal to make it ~~eje~~ eject electrons.
- If the metal are non-Alkali metal UV light should be incident on the metal to eject e^-
- If the material is a non-metal then UV light at extreme should be incident ie.... high density UV ligth.



Compton Effect:

It is the scattering of a photon by a charged particle, usually an electon. It results in a decrease of energy (increase in λ) of the photon



The change in wavelength Δλ.

$$\Delta\lambda = \frac{2h}{m_e c}\left(\sin^2 \theta/2\right).$$

$$\lambda - \lambda' = \frac{2h}{m_e c}\left(\sin^2 \theta/2\right).$$

$m_e \rightarrow$ mass of recoil electron ~~Hits~~

Debroglie Hypothesis:

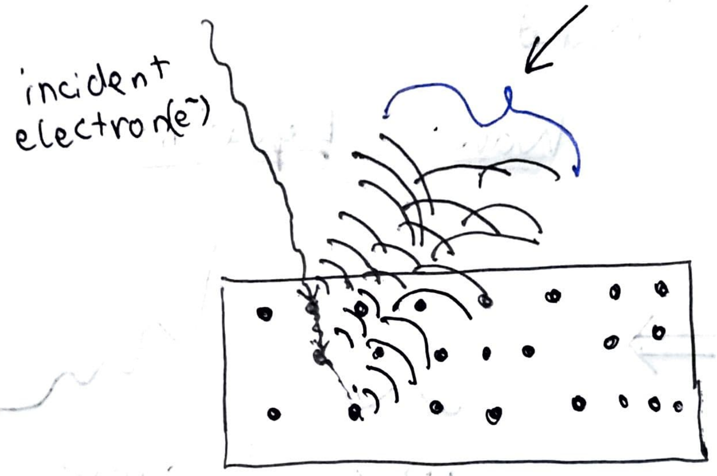
* A wave is associated with every moving particle called matter wave or de-broglie wave.

$$\lambda = \frac{h}{p} \Rightarrow \frac{h}{mv}$$

Davisson & Germer

This experiment was to prove the matter wave as in even an electron has a wave like nature and were able to observe maxima and minima ie... constructive and destructive interference caused due to an electron. [observed diffraction pattern in an e^-]

[observed constructive & destructive interference]



Stern - Gerlach Experiment:

The experiment of Stern & Gerlach gives the idea of space quantisation of electronic orbit. It means only certain discrete orientation of electronic motion are possible. [basically the spin of an electron] ie... the allignment of electron when it is subjected to magnetic field. ↑↓

Heisenberg Uncertainty Principle:

$$\Delta x \cdot \Delta p \geq \frac{h}{4\pi}$$

That is no two properties of the same quantum system can be measured simultaneous-ly.

Wave function: ~~a w~~
Every particle is associated to a wave, and the function describing the same is the wave function.
⇒ Ψ (Psi)

The probability density → $|\Psi|^2 = \Psi_{(r,t)} \Psi^*_{(r,t)}$

↓

a probability to determine where the particle if found.

Schrodinger Wave Equation:

associated wave function.
[[think of this in 3D]]

In simple S.E ⇒ $\hat{H}\Psi = E\Psi$
$\hat{H}$ → Hamiltonian opertor.
⇓
In short it is an operator used for measuring any ~~po~~ property.
$\hat{H}$ being to determine total energy.

$\hat{H} = \boxed{\hat{T}} + \boxed{\hat{V}}$

↓ ↓

K.E P.E

consider for now in x-axis only

wave eqn in x-axis

$$\psi = A \sin \frac{2\pi x}{\lambda} \quad - (2)$$

$$\frac{\partial \psi}{\partial x} = A \frac{2\pi}{\lambda} \cos\left(\frac{2\pi x}{\lambda}\right)$$

$$\frac{\partial^2 \psi}{\partial x^2} = -A\left(\frac{2\pi}{\lambda}\right)^2 \sin\left(\frac{2\pi x}{\lambda}\right) \quad - (3)$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} = -\left(\frac{4\pi^2}{\lambda^2}\right)\psi \quad - (4)$$

de-broglie:

$$\lambda = \frac{h}{mv}$$

$$\therefore (4) \Rightarrow \frac{\partial^2 \psi}{\partial x^2} = -\frac{4\pi^2}{h^2} m^2 v^2 \psi$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{4\pi^2}{h^2} m^2 v^2 \psi = 0 \quad - (5)$$

Total energy:

$$E = K.E + P.E$$

$$= \frac{1}{2} mv^2 + V$$

$$E - V = \frac{1}{2} mv^2$$

$$v^2 = \frac{2(E-V)}{m}$$

in eqn (5)

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{4\pi^2}{h^2} m^{\cancel{2}} \left(\frac{2(E-V)}{\cancel{m}}\right)\psi = 0.$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{8\pi^2 m}{h^2}(E-V)\psi = 0$$

for 3-dimensions:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + \frac{8\pi^2 m}{h^2}[E - V]\psi = 0$$

$V \rightarrow$ potential energy
$E \rightarrow$ full energy.

Time dependent wave equation -

$$\Rightarrow i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi$$

according to wave mechanics.

$$\psi = A e^{-i(\omega t - kx)} \quad -(1)$$

wkt, $\omega = 2\pi\nu$

$= 2\pi \frac{E}{h} \quad (\because E = h\nu)$.

$= \frac{E}{\hbar}$

$\hbar = \frac{h}{2\pi}$

$k = \frac{2\pi}{\lambda}$.

$= 2\pi\left(\frac{p}{h}\right) = \frac{p}{\hbar}$

$\lambda = \frac{h}{p}$.

diff (1) wrt 't'

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar}E \cdot A e^{-i/\hbar(\omega t - kx)}$$

$$= -\frac{i}{\hbar}E\psi$$

$$\Rightarrow E\psi = \frac{\hbar}{-i}\frac{\partial \psi}{\partial t}$$

$-1 \rightarrow i^2$.

$$= (-1)\frac{\hbar}{i}\frac{\partial \psi}{\partial t}$$

$$i^2 \frac{\hbar}{i} \frac{\partial \psi}{\partial t}$$

$$\boxed{E\psi = i\hbar \frac{\partial \psi}{\partial t}}$$

diff ① wrt x

$$\psi = A e^{-i/\hbar (\omega t - px)}$$

$$\frac{\partial \psi}{\partial x} = A\left(\frac{-i}{\hbar}\right) p \, e^{-i/\hbar (\omega t - px)}$$

$$\frac{\partial^2 \psi}{\partial x^2} = A\left(\frac{+ip}{\hbar}\right)^2 e^{-i/\hbar (\omega t - px)}$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{p^2}{\hbar^2} \psi$$

$$\boxed{-\hbar^2 \frac{\partial^2 \psi}{\partial x^2} = p^2 \psi} \Rightarrow \frac{p^2}{\hbar^2} = \frac{1}{\psi} \frac{\partial^2 \psi}{\partial x^2}$$

use hamiltonian.

$$\left(\frac{1}{2} m v^2\right) \times \frac{m}{m}$$

$$\frac{m^2 v^2}{2m} = \frac{(mv)^2}{2m} = \frac{p^2}{2m}$$

$$\hat{H}\psi = E\psi$$

$$E\psi = (\hat{T} + \hat{V})\psi$$

$$E\psi = \left[\frac{1}{2} m v^2 + V(x)\right]\psi$$

$$= \left[\frac{p^2}{2m} + V(x)\right]\psi$$

$$= \left[\frac{\hbar^2}{\psi} \frac{\partial^2 \psi}{\partial x^2} + V(x)\right]\psi$$

$$\boxed{i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi}$$

NORMALISATION OF WAVE FUNCTION

$|\psi|^2 \rightarrow$ gives the probability to find the particle in a particular ~~wo~~ volume element.

$$\int \psi_{(x,y,z)} . \psi^* \, d\tau = 1$$

(r, θ, z)

$d\tau$ is volume element
↓
depends on co-ordinate system.

$$|\psi|^2 = 1$$

if not equal to 1.

$$\int \psi . \psi^* \, d\tau = N$$

$$\Rightarrow \frac{1}{N} \int \psi \psi^* \, d\tau = 1$$

OPERATORS IN QUANTUM MECHANICS.

it is basically used to either measure certain aspects of the quantum states and to perform tasks on the given wave function

$$\boxed{\hat{A}}\,\boxed{\psi} = \boxed{\lambda}\,\psi .$$

eigen value.

operator

eigen function.

QUANTUM MECHANICS → anti commutator

HERMITIAN OPERATOR:

is an operator H where

* $\hat{H} = (\hat{H}^*)^T$

ie $\boxed{\hat{H} = \hat{H}^{\dagger}}$ → dagger

* $\hat{H} = -\hat{H}^{\dagger}$ → anti Hermitian

Ex: ① $H = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$

$H^{\dagger} \Rightarrow \begin{bmatrix} 1 & -i \\ +i & 1 \end{bmatrix}^T \Rightarrow \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$

$\therefore H = H^{\dagger}$

② $\cancel{H = \begin{bmatrix} -1 & -i \\ -i & 1 \end{bmatrix}}$ $\quad H = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

$H^{\dagger} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = -1 \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

$\therefore \hat{H} = -\hat{H}^{\dagger}$

* $\int \psi^* (\hat{A}\psi)\, d\tau = \int (\hat{A}^* \psi^*)\, \psi \, d\tau$

(*) Eigen value of Hermitian operator are real.

$$\hat{H}\,\psi = \boxed{\lambda}\,\psi$$

real no.

(*) Eigen function are orthogonal for two different eigen values.

Eigen values are nothing but the values that define Eigen vectors.

The Eigen vectors are those vectors where there is equillibrium ie... the vector would stay there inspite of any transformation applied &

The eigen vectors vary in scalling if $\lambda = 1$ → these vectors stay put no matter what transformation is done Ex: I transformation on a matrix.

eigen value ((scaling factor))

$$A\vec{v} = \boxed{\lambda}\,\vec{v}$$

$$\Rightarrow A\vec{v} - \lambda I\vec{v} = 0$$

$$(A - \lambda I)\vec{v} = 0$$

$$\begin{bmatrix} \text{Constant (matrix) don't change this matrix} \end{bmatrix} \begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix} = \begin{bmatrix} x_{out} \\ y_{out} \end{bmatrix}$$

Link → // Zach Star (The applications of eigenvectors & values

Hermition → is an operator

Hamiltonian → is an operator that tells total energy of the system

→ consider a hermitian operator.

$$\hat{H} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

find the Eigen values & vector if you dont know.

$$A\vec{V} = \lambda \vec{V}$$

$\vec{V} \to$ Eigen vector

$A \to$ Hermitian

$$\Rightarrow (H - \lambda I)\vec{V} = 0$$

$$\begin{bmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{bmatrix} \vec{V} = 0$$

$\vec{V}$ = cannot be zero.

$$\Rightarrow \det \begin{vmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix}$$

$$\Rightarrow \lambda^2 - 1 = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1$$

$$\therefore \lambda_1 = 1 \quad \& \quad \lambda_2 = -1$$

To find eigen vectors.

$\lambda_1 = 1$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_2 = 0 \qquad \boxed{x_1 = x_2}$$

$$x_1 - x_2 = 0$$

so eigen vectors are

$$\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}, \ldots \begin{pmatrix}-1\\-1\end{pmatrix}$$

rest others are just scaled by a scalar factor

IIIly for $\lambda = -1$

$x_1 = -x_2$

so eigen vectors are

$$\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}2\\-2\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix} \ldots$$

ie. ∴ $\hat{H}\psi = \lambda\psi$

let $\psi = \begin{pmatrix}2\\2\end{pmatrix}$

$$\hat{H} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \quad \text{(operator)}$$

$$\hat{H}\psi = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix} = \boxed{1}\begin{bmatrix}2\\2\end{bmatrix}$$

$\boxed{\lambda}$ eigen vector

is a real number.

* orthogonality:

Two vectors are said to be orthogonal when their corresponding dot product is zero. $\vec{a}\cdot\vec{b} = 0$.

$\psi(\lambda_1) \cdot \psi(\lambda_2)$.

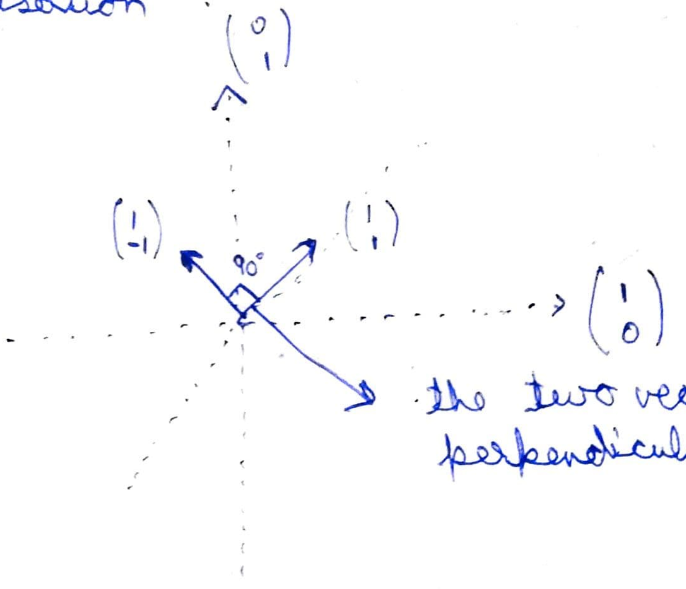
* $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$= 1*1 - 1*1 = 0$

* $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \end{pmatrix}$.

$= 2 - 2 = 0$

visualisation



→ the two vectors are perpendicular.

Postulates of Quantum Mechanics:

1) Any state of a system with n-degree of freedom is described by a wave function $\psi(r, t)$.

2) Every physical observable is associated with a linear $\boxed{\text{hermition operator}}$ ie a measurable quantity can be represented as mathematical operator.

3) The only possible value which a measurement of the observable can give are given by

$$\hat{A}|\psi| = \lambda\psi$$

$\downarrow$

eigen function.

4) The development of time of the wave function ψ of a system is given by Schrödinger Equation.

$$\hat{H}\psi = E\psi$$

5) The average value of an observable 'a' corresponding to the operator '$\hat{A}$' for a system described by wave function ψ is given by.

$$\langle a \rangle = \frac{\int \psi^* \hat{A} \psi \, d\tau}{\int \psi^* \psi \, d\tau}$$

Franck - Hertz Experiment:

It demonstrated the existence of excited states in mercury atoms, helping to confirm the quantum theory which predicted that electrons occupied only discrete, quantized energy states.

$\downarrow$

quantised energy state as in fixed for that particular orbital.

Dirac Notation (Bra, ket)

In quantum mechanics, bra-ket notation is a standard notation for describing quantum state

$| \rangle \rightarrow$ ket vector

$\langle | \rightarrow$ Bra vector

inter conversion b/w Bra & ket

$$| \rangle \longrightarrow \left((| \rangle)^* \right)^T \longrightarrow \langle |$$

ie... $\underset{\text{ket}}{| \rangle} \longrightarrow (| \rangle)^\dagger \longrightarrow \underset{\text{Bra}}{\langle |}$

Ex: $|\psi\rangle = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -i \\ 4 \end{bmatrix}$

$$\Rightarrow \langle \psi | = \begin{bmatrix} 1 & 2 & 0 & +i & 4 \end{bmatrix}$$

multiplication of BRA & KET is called BRACKET called Inner Product

$$\langle \underset{\text{Bra}}{\psi} | \underset{\text{ket}}{\psi} \rangle = \underset{1 \times 5}{\begin{bmatrix} 1 & 2 & 0 & i & 4 \end{bmatrix}} \underset{5 \times 1}{\begin{bmatrix} 1 \\ 2 \\ 0 \\ -i \\ 4 \end{bmatrix}}$$

$= [$ $\quad 1 + 4 + 0 - i^2 + 16$

Bra & ket vector are additive & subtractive.

$$|a\rangle + |b\rangle$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{pmatrix}$$

LINEAR VECTOR SPACE

A linear vector space consists of two sets of elements and two algebraic rules.

→ A set of vectors $\{V_i\}$ and a set of scalars $\{\alpha_i\}$.

→ A rule for vector addition and a rule for scalar multiplication.

1) Addition rules: (vect + vector)

$$\psi_1 + \psi_2 \in \text{vector space}$$

where,

$\psi_1 \in$ vector space

$\psi_2 \in$ vector space.

* Commutativity of Addition

$$\psi_1 + \psi_2 = \psi_2 + \psi_1$$

* Associativity:

$$(\psi_1 + \psi_2) + \psi_3 = \psi_1 + (\psi_2 + \psi_3)$$

* Existence of zero vector

for each vector Ψ, there is a zero vector such that.

$$\vec{0} + \Psi = \Psi + \vec{0} = \Psi$$

* Existence of inverse vector

for each vector Ψ, there exists its additive inverse such that.

$$\Psi + (-\Psi) = 0$$

2) Multiplication Rules: (scalar × vector)

* The product of a scalar with a vector gives another vector. ie...

$\Psi \rightarrow$ vector

$\alpha \rightarrow$ scalar

$$\Psi = a\hat{i} + b\hat{j} + c\hat{k}$$

$$\alpha\Psi = \alpha a\hat{i} + \alpha b\hat{j} + \alpha c\hat{k}$$

$$= A\hat{i} + B\hat{j} + C\hat{k}$$

* Distributivity with respect to addition

$$\alpha_1(\Psi_1 + \Psi_2) = \alpha_1\Psi_1 + \alpha_1\Psi_2$$

* Associativity with respect to multiplication

$$\alpha_1(\alpha_2\Psi_1) = (\alpha_1\alpha_2)\Psi_1$$

* Existence of unitary Scalar & zero scalar.

* $I\psi = \psi I = \psi$

* $0\psi = \psi 0 = 0$

INNER PRODUCT SPACE:

Inner product: (vector × vector)

An inner product is the product of two vector. similar to dot product.

denoted by $\langle , \rangle : V \times V \longrightarrow$ function (F)

$F \in \mathbb{R}, \mathbb{C}$

Axioms of inner products:- $u, v, w \in \vec{V}$

$b, a \in$ Scalars

i) $\langle u, v\rangle = \overline{\langle v, u\rangle}$

value = (value)*

ii) $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ if $\vec{u} = \vec{0}$

iii) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

iv) $\langle au, v\rangle = a\langle u, v\rangle$ & ~~$\langle u, av\rangle = a^*\langle u, v\rangle$~~

v) $\langle au+bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$

If a vector ~~sy~~ space ~~a~~ has inner product and satisfies these axioms they are called INNER ~~VECTOR~~ PRODUCT SPACE

$\langle u, v\rangle \Rightarrow$ $u = (x_1, x_2 \ldots x_n)$

$v = (y_1, y_2 \ldots y_n)$.

* $\langle u, v\rangle = x_1y_1 + x_2y_2 + x_3y_3 \ldots x_ny_n$

if $u, v \in \mathbb{R}$

(*) Existence of unitary Scalar & zero scalar.

(*) $I\psi = \psi I = \psi$

(*) $\hat{0}\psi = \psi\hat{0} = \hat{0}$

INNER PRODUCT SPACE:

Inner product: (vector × vector)

An inner product is the product of two vector. Similar to dot product.

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Axioms of inner products:- $u, v, w \in \vec{V}$

$b, a \in$ Scalars

i) [redacted]

see in next to next page i.e after Hilbert space

[redacted]

If a vector ~~sy~~ space ~~as~~ has inner product and satisfies these axioms they are called INNER PRODUCT SPACE

$\langle u, v\rangle \Rightarrow u = (x_1, x_2 \ldots\ldots x_n)$

$v = (y_1\ y_2 \ldots\ldots y_n).$

(*) $\langle u, v\rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 \ldots\ldots x_n y_n$

if $u, v \in \mathbb{R}$

$\overline{\langle u, v \rangle} = x_1 y_1^* + x_2 y_2^* + x_3 y_3^* \ldots x_n y_n^*$

additional axiom:

$$\langle u, va + wb \rangle$$

$$= a^* \langle u, v \rangle + b^* \langle u, w \rangle.$$

Inner product is similar to dot product but it has to follow these set of axioms.

Dot product: $\vec{a} \cdot \vec{b} = \|\vec{a}\| . \|\vec{b}\| \cos \theta_{ab}$

$$\vec{a} = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}$$
$$\vec{b} = x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k}$$
$$\therefore \vec{a} \cdot \vec{b} = x_1 x_2 + y_1 y_2 + z_1 z_2$$

a space dedicated to dot product as in an extension or additional axioms with dot product.

HILBERT SPACE:

Hilbert space is a space of all quantum wave function

This could be an infinite dimensional space or a finite (we do not still have a theory for everything so, we dont know if there are finite number of dimension space)

SPACE: do not view space litterally as SPACE!!!

It is a collection of information or set of numbers to describe the state or the complete information of the system.

these numbers are the corresponding dimensions

Anology: momentum space (observe that the information pertaining to momentum is described as space).

in quantum mechanics we donot know the complete information meaning the number of dimensions required to describe a system so we use the Hilbert space.

Review back of Axioms of Inner Product

$[\langle a|b\rangle \rightarrow \langle a,b\rangle \rightarrow \langle \Psi\rangle_{ab} = \vec{a}\cdot\vec{b}]$
$\forall\ \vec{a},\vec{b} \in \mathbb{R}$

$(\vec{a})^{*}\cdot\vec{b}$
$\forall\ \vec{a},\vec{b} \in \mathbb{C}$

i) $\langle a|b\rangle = \langle b|a\rangle \quad \vec{a},\vec{b} \in \mathbb{R}$

$\langle a|b\rangle = \langle b|a\rangle^{*} \quad \vec{a},\vec{b} \in \mathbb{C}$

---

ii) $\langle a|a\rangle \geq 0$ 0 when $\vec{a} = \vec{0}$

---

iii) $\langle a, b+c\rangle = \langle a|b\rangle + \langle a|c\rangle$
$\vec{a},\vec{b} \in \mathbb{R}$ & $\mathbb{C}$

$\langle a+b, c\rangle = \cancel{\langle a|b\rangle + \langle c|b\rangle} = \langle c|a\rangle^{*} + \langle c|b\rangle^{*}$
$= \langle a|c\rangle + \langle b|c\rangle$
$\vec{a},\vec{b} \in \mathbb{R}$ & $\mathbb{C}$

$\langle a, \alpha_1 b + \alpha_2 c\rangle = \alpha_1\langle a|b\rangle + \alpha_2\langle a|c\rangle$.

$\langle \alpha_1 a + \alpha_2 b, c\rangle = \alpha_1^{*}\langle a|c\rangle + \alpha_2^{*}\langle b|c\rangle$

---

iv) $\langle a, \alpha_1 b \rangle = \alpha_1^* \langle a, b \rangle$. $\alpha_1 \in \mathbb{C}$

$\langle \alpha_1 a, b \rangle = \alpha_1 \langle a, b \rangle$ $\alpha_1 \in \mathbb{C}$

Hilbert Space = Linear vector space + inner product space.

Properties of hilbert space:

1) A hilbert space is a linear vector space.
2) It has an inner product operation that satisfies certain conditions.

   i) all four properties of inner product.

   ii) $|\psi_2 - \psi_1| = \sqrt{\langle \psi_2 - \psi, \psi_2 - \psi_1 \rangle}$

   (distance b/w 2 vectors in hilbert space

3) Hilbert spaces are separable, so they contain a countable, dense, subset.

Separable: countable, dense subset.

⇓

dense ~~countable~~ meaning consider the set of real numbers. they are dense because there is no gap in the number line.

$\mathbb{R}$ are ~~continuous~~ continuous

countable meaning ~~they are~~ if any irrational number we can find a reational number arbitrarily close.

4) Hilbert spaces are complete (no gaps)
ie... we can find a number that is arbitrarily close to the other number. Explained for a cauchy sequence.

cauchy sequence $\{\psi_i\} = \lim_{m,n \to \infty} |\psi_m - \psi_n| = 0$

$$\lim_{n \to \infty} |\underbrace{\phi}_{\text{element in Hilbert Space}} - \psi_n| = 0$$

Types Of Hilbert Spaces:

a) Finte - Dimensional Hilbert Space:
eg. $\mathbb{R}^n$, $\mathbb{C}^n$: n basis vectors.

- Inner product on $\mathbb{R}^n$: typical dot product

$$\vec{x}_1 = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} ; \quad \vec{x}_2 = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$\vec{x}_1 \cdot \vec{x}_2 = \vec{x}_1^T \times \vec{x}_2 = a_1 b_1 + \dots a_n b_n$$

- Inner product on $\mathbb{C}^n$: complex inner product

ie

$$\vec{x}_1 = \begin{bmatrix} a_1 + b_1 i \\ \vdots \\ a_n + b_n i \end{bmatrix} \quad \vec{x}_2 = \begin{bmatrix} c_1 + d_1 i \\ \vdots \\ c_n + d_n i \end{bmatrix}$$

$$\vec{x}_1 . \vec{x}_2 = (\vec{x}_1^*)^T \times \vec{x}_2$$

$$= \begin{bmatrix} a_1 - b_1 i & \cdots & a_n - b_n i \end{bmatrix} \begin{bmatrix} c_1 + d_1 i \\ \vdots \\ c_n + d_n i \end{bmatrix}$$

b) Infinite - Dimensional Hilbert Space:

e.g.: vector space of complex-valued functions, with inner product:

$$\langle \varphi, \phi \rangle = \int_{-\infty}^{\infty} \varphi^* \phi \, dx$$

φ & ϕ should be square integrable function.

ex: $\int_{-\infty}^{+\infty} |\varphi|^2 dx = \boxed{\text{finite}}$

this value should not end up to ∞ which would mean inner product does not exist.

Properties of bras, kets, braket

1) Every $|\psi\rangle$ (ket) has a $\langle\psi|$ (Bra).

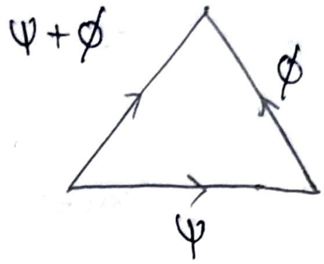
2) constant multiple property: $|a\psi\rangle = a|\psi\rangle$

$$\langle a\psi| = a^* \langle \psi|$$

3) All properties of Inner product (Axiom)

Other properties:

a) Triangle inequality - $\sqrt{\langle \psi + \phi | \psi + \phi \rangle} \leq \sqrt{\langle \psi | \psi \rangle} + \sqrt{\langle \phi | \phi \rangle}$



Note:

$\sqrt{\langle \psi | \psi \rangle}$ is the length in general

$\sqrt{\langle \vec{a} | a \rangle}$

$\Downarrow$

$$\langle a, a \rangle = \vec{a} . \vec{a} \cos \theta$$
$$= \vec{a} . \vec{a} \cos (0)$$
$$= \| \vec{a} \|^2$$

$\downarrow$

square magnitude length

'=' when they are linearly dependent.

b) Schwarz inequality - $[\langle \psi | \phi \rangle]^2 \leq [\langle \psi | \psi \rangle \langle \phi | \phi \rangle]$

they are equal when two ψ & ϕ are linearly dependent

→ two vector lieing on the same line

c) Orthonormality →

$\langle \psi | \phi \rangle = 0$ → orthogonal

&

$\langle \psi | \psi \rangle = 1$;
$\langle \phi | \phi \rangle = 1$ → normalised

$\downarrow$

meaning the vector is a unit vector of a vector of length = 1

PROJECTION OPERATOR:

An operator $\hat{P}$ is said to be a projection operator if it is $\boxed{\text{Hermition}}$ and equal to its own square.

i) $\hat{P} = \hat{P}^{\dagger}$

ii) $\hat{P} = \hat{P}^{2}$

Ex: Identity matrix.

Properties of Projection operator :-

* The sum of two projection operator is generally not a projection operator.

* Two projection operators are orthogonal if their product is zero.

* For a sum of projection operators $\hat{P}_1 + \hat{P}_2 + \dots + \hat{P}_n$ to be a projection operator, it is necessary and sufficient that these projection operators are mutually orthogonal.

TENSOR PRODUCT:

Vector product is a subset of tensor product. ie...

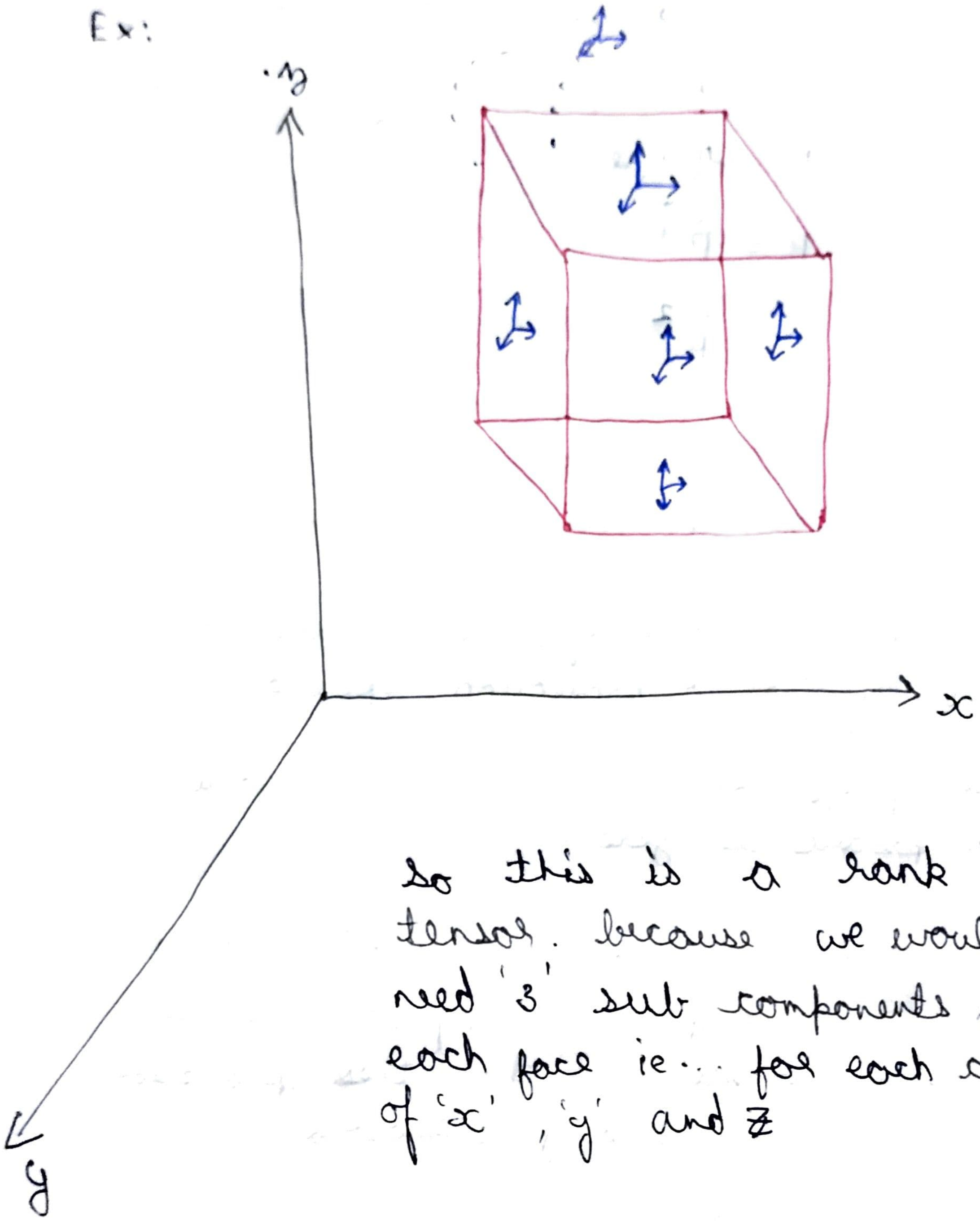
vector product = Tensor product of rank 1

3^1

$\hookrightarrow$ cartesian plane (3-D)

Tensor product is the amount of numbers required to describe one particle component and during tensor product dimension is increased.

Ex:



So this is a rank '3' tensor. because we would need '3' sub components for each face ie... for each component of 'x', 'y' and z

3 basis components

rank 3.

$$= (\text{basis compon})^{\text{rank}}$$

$$= 3^3$$

$$= 27 \text{ dimensions}$$

when it comes to quantum computing we use ~~the~~ vectors with two basis vectors.

$$\boxed{\vec{1}} \quad \begin{pmatrix}0\\1\end{pmatrix} \quad \text{and} \quad \boxed{\vec{0}} \quad \begin{pmatrix}1\\0\end{pmatrix}$$

the vector info isn't 3-D because of the measurement problem so the 2-D basis vector would be easy to analyse.

$\therefore$ basis components = 2.

rank = n.

$$\therefore \boxed{\text{dimension} = 2^n}.$$

Mathematical way of Tensors.

$$\begin{pmatrix}a\\b\end{pmatrix} \otimes \begin{pmatrix}c\\d\end{pmatrix} = \begin{bmatrix} a\begin{pmatrix}c\\d\end{pmatrix} \\ \\ b\begin{pmatrix}c\\d\end{pmatrix} \end{bmatrix} = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}$$

basis vectors = 2.

amount of numbers to describe one basis vector = 2.

$\therefore$ $2^2 = 4$ dimensions.

each state combination on its own in a Hilbert space is a dimension.

since two qubit system are correlated the state of two qubit is taken as a whole.

Ex:

* $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow$ representation of state 0

* $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow$ representation of state 1

1) $|01\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{array}{cl} \begin{pmatrix} 0 \\ \boxed{1} \\ 0 \\ 0 \end{pmatrix} & \begin{array}{l} \to |00\rangle \\ \to |01\rangle \\ \to |10\rangle \\ \to |11\rangle \end{array} \end{array}$$

$2^2 = 4$ dimensions

2) $|11\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{array}{cl} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{array}{l} \to |00\rangle \\ \to |01\rangle \\ \to |10\rangle \\ \to |11\rangle \end{array} \end{array}$$

3) $|100\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ 1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \to |000\rangle \\ \to |001\rangle \\ \to |010\rangle \\ \to |011\rangle \\ \to |100\rangle \\ \to |101\rangle \\ \to |110\rangle \\ \to |111\rangle \end{array}$$

$\boxed{2}^{\boxed{3}} = 8$ dimensions

$2 \to$ basis vector.

$3 \to$ rank.

Some anologies in Classical Computing:

Possible operations possible on a single

- **Identity:**

i/p		o/p
0	$\rightarrow$	0
1	$\rightarrow$	1

$$f(x) = x$$

verification using matrix.

$$I|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$I|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- **Negation**.

$$f(x) = -x.$$

i/p		o/p
0	$\rightarrow$	1
1	$\rightarrow$	0

$$N|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \overset{\vec{0}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \overset{\vec{1}}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$N|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \overset{\vec{1}}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \overset{\vec{0}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

- Constant - 1

$$f(x) = 1$$

constant - 1 matrix.

$$C1 \rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

i/p o/p.

$0 \to 1$

$1 \to 1$

$$C1|0\rangle = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C1|1\rangle = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Constant - 0.

$$f(x) = 0 .$$

constant - 0 matrix

$$C0 \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

NOTE: In classical computing if the o/p and what gate being used ~~is a~~ information is given it is impossible to retrace back the i/p.

Except 'not' gate.

Schrödinger time dependent wave equation:

$$\frac{d}{dt}\left[|\psi(t)\rangle\right] = iH|\psi(t)\rangle$$

Representation of a ~~single~~ quantum System.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

In Q.C. $|0\rangle, |1\rangle$ are the unit vectors like $\hat{i}, \hat{j}$ analogous to cartesian plane.

$$\alpha, \beta \in \mathbb{C}$$

$$|\alpha|^2 + |\beta|^2 = 1$$

$$\Rightarrow r_\alpha e^{i\varphi_\alpha}|0\rangle + r_\beta e^{i\varphi_\beta}|1\rangle$$

now we ~~put~~ multiply this equation with a value having absolute value 1

$$\rightarrow e^{-i\varphi_\alpha}$$

$$\Rightarrow r_\alpha|0\rangle + r_\beta e^{i(\varphi_\beta - \varphi_\alpha)}|1\rangle$$

r_α and r_β are the absolute values of α and β.

$$\equiv r_\alpha^2 + r_\beta^2 = 1$$

$$r_\alpha = \cos\frac{\theta}{2}.$$

$$r_\beta = \sin\frac{\theta}{2}.$$

$$\boxed{\theta \in [0, \pi]}.$$

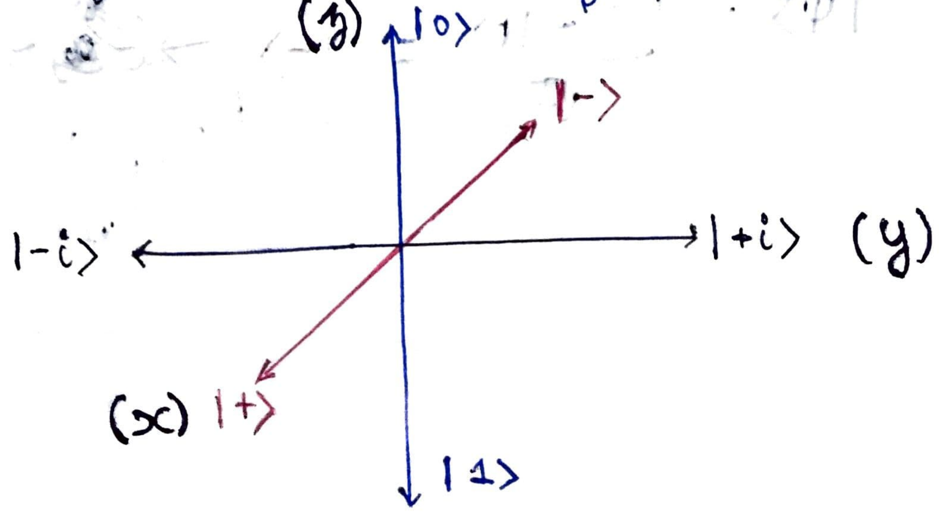
∴ The difference between the phase factors φ_α & φ_β remains the same as 'φ' φ_α & φ_β are component angles

$$\boxed{|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2} e^{i\varphi}|1\rangle}.$$

$$\varphi = \varphi_\beta - \varphi_\alpha.$$

$$\boxed{\varphi \in [0\ 2\pi]}$$

∴ θ & φ are the deciding factors on which the vector lie on a bloch sphere.



All the measurements are done along the z direction so the basis vectors in generic are taken to be $|0\rangle$ & $|1\rangle$

Representation of all the axes with $|0\rangle$ & $|1\rangle$.

$\theta \in [0, \pi]$
$\varphi \in [0, 2\pi]$

1) • $\theta = 0^\circ$; $\varphi = 0^\circ$

$\boxed{|\psi\rangle = |1\rangle} \rightarrow$ +z axis $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

• $\theta = \pi$; $\varphi = 0^\circ$

$\boxed{|\psi\rangle = |0\rangle} \rightarrow$ -z axis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

2) • $\theta = \pi/2$; $\varphi = 0$

$\boxed{|\psi\rangle = \dfrac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle} \rightarrow$ x-axis $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

• $\theta = \pi/2$; $\varphi = \pi$

$\boxed{|\psi\rangle = \dfrac{|0\rangle - |1\rangle}{\sqrt{2}} = |-\rangle} \rightarrow$ x-axis $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

5) $\theta = \pi/2$, $\varphi = \pi/2$

$$\boxed{|\psi\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}} = |+i\rangle} \rightarrow (+y \text{ axis}) \quad \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$$

$\theta = \pi/2$, $\varphi = 3\pi/2$

$$\boxed{|\psi\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}} = |-i\rangle} \rightarrow (-y \text{ axis}) \quad \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}$$

MEASUREMENT OF A QUANTUM SYSTEM.

* In quantum relm. the act of measurement itself changes the state of the system. when measured again.

but !!

if the measuring apparatus is aligned along the quantum system, then the state will not change

* The first measurement changes the state of the system, so that subsequent measurements are deterministic,

and

The result of subsequent measurements is the same as the first measurement.

Mathematical and vector understanding of measurement.

let $\psi \rightarrow$ state of the quantum system.

$M \rightarrow$ state of the apparatus or orientation of apparatus.

⊙ let $|\psi\rangle = |1\rangle$.

$|M\rangle = |1\rangle$.

compute $\langle M|\psi\rangle$ → first measurement

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $\langle M| = (0 \quad 1)$

$\therefore \langle M|\psi\rangle = \langle 1|1\rangle$

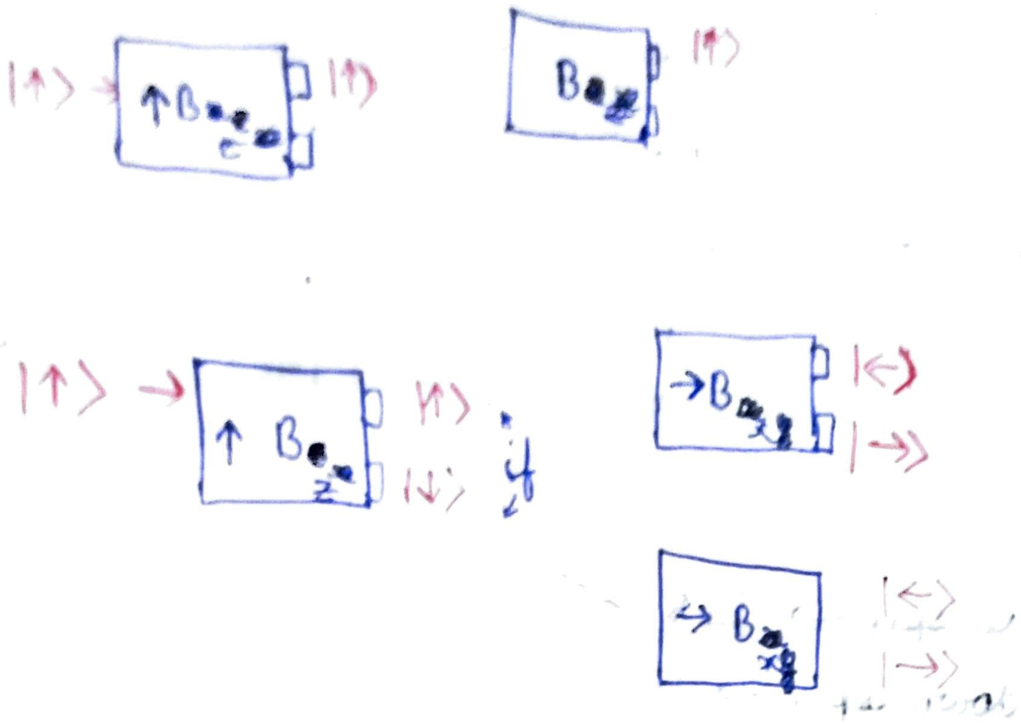
$= (0 \quad 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$= 1$

∴ probability of measuring $|\psi\rangle = |1\rangle$ is '1'

ie...

$$\boxed{\begin{aligned} P(|\psi\rangle = |1\rangle) &= |\langle M|\psi\rangle|^2 \\ &= 1^2 \\ &= 1 \end{aligned}}$$



let $|\psi\rangle = |1\rangle$

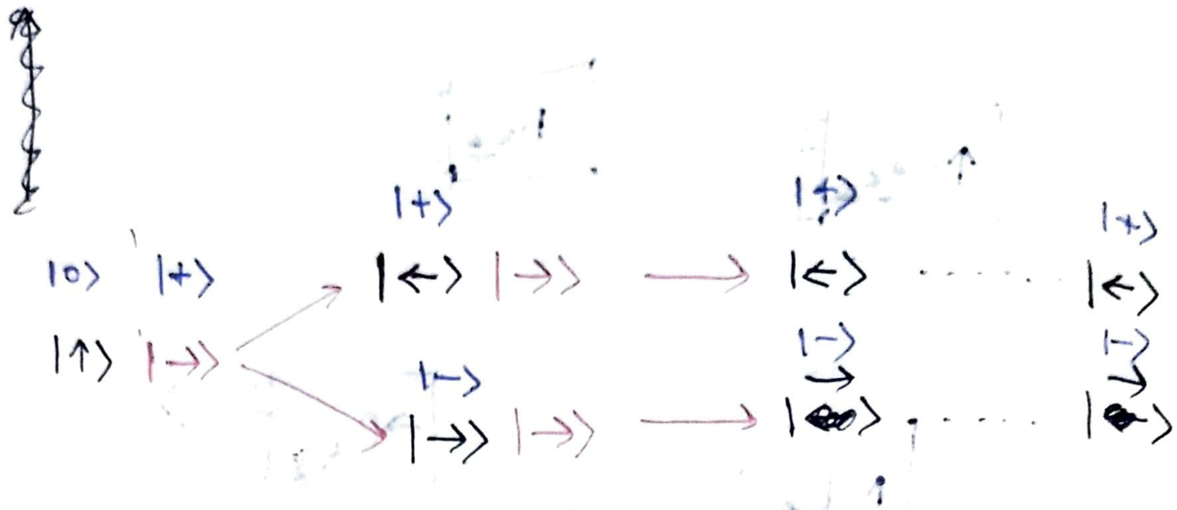
$$|M\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \Rightarrow |+\rangle \text{ +ve x axis}$$

compute $\langle M|\psi\rangle$ → first measurement

$|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$; $\langle M| = \left(1/\sqrt{2} \quad 1/\sqrt{2}\right)$

$\therefore \langle M|\psi\rangle = 1/\sqrt{2}$

$$\boxed{\therefore P(|\psi\rangle = |1\rangle) = |\langle M|\psi\rangle|^2 = 1/2}$$



up spin → $|1\rangle$
down spin → $|0\rangle$
left spin → $|-\rangle$
right spin → $|+\rangle$ } hadamart states (superposition states).

$$a\langle\psi_1|\psi_2\rangle = \langle\psi_1|a|\psi_2\rangle$$

QUANTUM GATES:

1) Pauli's X-gate. [Quantum NOT gate]

It is a quantum gate which flips the orientation of q-bit along $|1\rangle$ or $|0\rangle$ axis

$$|1\rangle \xrightarrow{X-G} |0\rangle$$
$$|0\rangle \xrightarrow{X-G} |1\rangle$$

Pauli's x - gate matrix $\rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$X|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle .$$

$$X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle .$$

$$X|+\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

$$= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$ $\rightarrow$ observe that no change along +y

$$X|-\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}.$$

$$= \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$ $\rightarrow$ observe the no change along -y.

$|||^{ly}$ for $|i\rangle$ & $|-i\rangle$.