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Massive Spin-2 Fields in Bimetric Theory and Some Implications

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Master Thesis

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Abstract

The General theory of Relativity was first introduced by Albert Einstein. There have been many attempts to unify General Relativity with the Standard Model of Physics and many of these try to do so by modifying General Relativity slightly. One way to do this is to add a mass to the graviton. Such a theory was proposed by Fierz and Pauli. However, a massive gravity theory suffers from the vDvZ discontinuty where taking the masss of the graviton to zero does not reproduce the results of General Relativity exactly. This can, to some extent, be resolved via the Vainshtein mechanism, where General Relativity can be reproduced within a certain radius from a source, called the Vainshtein radius. Another modification that can be imagined, is to add a second metric. However, doing this results in extra degrees of freedom which manifest as a Boulaware Deser ghost. The bimetric action which avoids the Boulaware Deser ghost was first introduced by Hassan and Rosen in 2011. In this theory, only one of the metrics couples to standard model matter to avoid the ghost. In this scenario, the propagating massless and massive spin-2 modes turn out to be linear combinations of the two metrics, just as in neutrino mixings. In this thesis, we review some works which investigate the oscillations between the massless and massive modes and the implications for gravitational waves. In particular we consider the bounds on the parameters of the theory based on the fact that evidence for such oscillations have not been observed by LIGO. We use a new LIGO result to extend these bounds. We also review an investigation which explores the possibility that the dark matter particle could be the massive particle of bimetric gravity.

Abstract

Den allmänna relativitetsteorin introducerades först av Albert Einstein. Många har försökt förena allmän relativitetsteori med partikelfysikens standardmodell och många av dessa försök gör detta genom att lägga till en massa för gravitonen. En sådan teori föreslogs av Fierz och Pauli. Massiv gravitation lider dock av vDvZ-diskontinuiteten där gränsen när gravitonmassan går mot noll inte reproducerar allmän relativitetsteori. Detta kan, till viss del, lösas genom Vainshteinmekanismen, där allmän relativitetsteori kan reproduceras inom ett visst avstånd från källan, kallat Vainshteinradien. En annan modifikation som kan komma på fråga är att lägga till en andra metrik. Att göra detta leder dock till nya frihetsgrader som yttrar sig som ett Boulaware-Deser-spöke. Den bimetriska verkan som undviker Boulaware-Deser-spöket introducerades först av Hassan och Rosen år 2011. I denna teori kopplar enbart en av metrikerna till standardmodellen vilket gör att spöket kan undvikas. I detta scenario visar sig det masslösa och det massiva propagationsegentillstånden vara linjärkombinationer av de två metrikerna i analogi med neutrinoblandning. I detta arbete går vi igenom några arbete som undersöker oscillationerna mellan de två metrikerna och implikationerna för gravitationsvågor. Speciellt kommer vi att betrakta de begränsningar som finns på teoriparametrarna baserat på det faktum att LIGO inte observerat några bevis för sådana oscillationer. Vi använder också nya LIGO-resultat för att utöka dessa begränsningar. Vi diskutera också möjligheten att mörk materia skulle kunna bestå av den massiva gravitonen i bimetrisk gravitation.

Preface

Our current understanding of physics is that there are four fundamental forces in nature: Strong, Weak, Electromagnetism, and Gravitation. While the first three have been unified in the Standard Model, Gravitation is well explained by the General Theory of Relativity. However, the goal of unifying gravitation with the Standard Model has proven elusive. There are several other phenomena that remain unexplained by the Standard Model and General Relativity(GR). For example, studies of galactic rotation curves have shown that most of the matter in the universe does not interact with ordinary matter electromagnetically but interacts only via gravitation. This matter is called dark matter. Another phenomenon is the observed accelerating expansion of the universe which cannot be explained unless we include a component called dark energy to the universe.

One approach to finding a theory that could explain these phenomena is to try and modify GR. One way to do this is to have two metric tensors instead of one. Each metric couples to a different kind of matter and the two metrics interact with each other. Such a bimetric theory that is free of ghosts was first introduced by Hassan and Rosen in 2011 [1] [2] [3].

In this thesis, we review some implications of the bimetric theory, for gravitational waves and dark matter. For gravitational waves in particular, we consider oscillations between the massive and massless mode of the graviton in bimetric theory, which is analogous to mixing among the various neutrino eigenstates.

Unless otherwise indicated, throughout this thesis, we work in natural units i.e., $\hbar = c = 1$. We also follow the metric convention of $(-, +, +, +)$. We use matplotlib for plots. [4]

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Contents

Chapter 1

Introduction to Bimetric Theory

We begin this chapter by recalling some ideas from General Relativity(GR). We then introduce a linear theory of massive gravity, the Fierz-Pauli theory and explain issues faced by it and similar theories before introducing the ghost free bimetric theory which was first introduced by Hassan and Rosen in 2011 [1] [2]. Following [5], we then present a special case of solutions around proportional backgrounds.

1.1 General Theory of Relativity

1.1.1 Introduction to General Relativity

The General Theory of Relativity, first introduced by Albert Einstein is the current most widely accepted theory of gravitation that has passed all tests thrown at it. Referring any standard textbook, say [6], let us recall the following concepts from General Relativity:
The Einstein Hilbert action is give by:

$$S = 8\pi G \int d^4x \sqrt{-\det(g)} R(g) \tag{1.1}$$

The factor $8\pi G$ can also be written in terms of the reduced Planck mass as

$$\frac{1}{M_{\mathrm{Pl}}^2} = 8\pi G$$

where $R(g)$ is the Ricci scalar. Varying this action gives the Einstein Field Equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.2}$$

where

$$T_{\mu\nu} = -\frac{2}{\sqrt{-\det(g)}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} \tag{1.3}$$

In General Relativity, the conservation laws can be expressed as:

$$\nabla^{\mu} T_{\mu\nu} = 0 \tag{1.4}$$

The Bianchi identity is:

$$\nabla_{\rho} R^{\rho}{}_{\mu} = \frac{1}{2}\nabla_{\mu} R \tag{1.5}$$

In terms of the Einstein tensor, this can be written as:

$$\nabla^{\mu} \mathcal{G}_{\mu\nu} = 0 \tag{1.6}$$

The Schwarzchild metric is the unique solution to a spherically symmetric distribution in GR.It is given by:

$$ds^2 = -(1-\frac{r_S}{r})dt^2 + \frac{1}{(1-\frac{r_s}{r})}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{1.7}$$

where $r_S = 2GM$ is the Schwarzchild radius.

1.2 ΛCDM cosmology

Assuming a homogeneous and isotropic universe, a solution to the Einstein Field Equations is given by the Robertson Walker metric.

$$ds^2 = -dt^2 + a^2(t)\Big[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)\Big]. \tag{1.8}$$

In the above expression, a is called the scale factor.
$k = 0$ corresponds to a flat universe, $k = 1$ corresponds to a closed universe, and $k = -1$ corresponds to an open universe.
For a perfect fluid, the energy momentum tensor can be written as:

$$T^{\mu}{}_{\nu} = \mathrm{diag}(-\rho, p, p, p) \tag{1.9}$$

Choosing the equation of state $p = w\rho$, conservation of energy gives:

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \tag{1.10}$$

$$\rho \propto a^{(-3(1+w))}. \tag{1.11}$$

The Friedmann equations are given by:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{1.12}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}. \tag{1.13}$$

The Hubble parameter H is defined as:

$$H = \frac{\dot{a}}{a}. \tag{1.14}$$

The density parameter Ω is defined as:

$$\Omega = \frac{8\pi G}{3H^2}\rho. \tag{1.15}$$

The existence of dark matter can be inferred from the rotation curves of galaxies and other observations which suggests that there is more matter in the universe than is observed electromagnetically. In the ΛCDM(Cold Dark Matter) model,this is accounted for by Cold Dark Matter. The word Cold implies that Dark Matter in this model is non-relativistic. According to the ΛCDM model there are 5 different contributions to Ω: Baryonic matter, cold dark matter, dark energy/cosmological constant, radiation(photons),and neutrinos.
The cosmological redshift z is defined as:

$$1 + z = \frac{a_0}{a} \tag{1.16}$$

where a_0 is the current value of the scale factor and a is the value at an earlier time.
Another useful concept is the conformal time η which is defined as:

$$d\eta = \frac{dt}{a} \tag{1.17}$$

1.2.1 Luminosity distance

The luminosity distance is one measure of the distance of a far away astrophysical object. Knowing that intensity of light(or in our case, Gravitational waves) follows the inverse square law, if we know the intensity of the source, we can estimate a distance to the source based on the intensity observed on earth [7].

1.3 Fierz-Pauli Theory

One obtains the Proca Lagrangian for a massive spin-1 field by adding a mass term to the electromagnetic lagrangian, one can ask if the same procedure can be extended to a spin-2 field [8]. The Fierz-Pauli theory does just that. We start by considering fluctuations around the flat background $\eta_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{\mathrm{Pl}}} + \mathcal{O}(h^2) \tag{1.18}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - \frac{h^{\mu\nu}}{M_{\mathrm{Pl}}} + \mathcal{O}(h^2) \tag{1.19}$$

The Einstein tensor can then be written, to first order in h, as:

$$\mathcal{G}_{\mu\nu} = \mathcal{E}_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} \tag{1.20}$$

where

$$\mathcal{E}^{\alpha\beta}_{\mu\nu} = -\frac{1}{2}(\partial_\sigma\partial^\sigma\delta^\alpha_\mu\delta^\beta_\nu - \delta^\beta_\nu\partial_\mu\partial^\alpha - \delta^\beta_\mu\partial_\nu\partial^\alpha + \eta_{\mu\nu}\partial^\alpha\partial^\beta + \eta^{\alpha\beta}\partial_\mu\partial_\nu - \eta_{\mu\nu}\eta^{\alpha\beta}\partial_\sigma\partial^\sigma) \tag{1.21}$$

We do not derive this here, but derivations can be found in say [6], [9] or [10]. A particularly nice derivation can be found in the appendix of [5] We can write the Lagrangian for this fluctuating field(to quadratic order) as:

$$\mathcal{L} = -\frac{1}{2}\mathcal{E}^{\alpha\beta}_{\mu\nu} h^{\mu\nu} h_{\alpha\beta} + \frac{1}{M_{\mathrm{Pl}}} h_{\mu\nu} T^{\mu\nu} \tag{1.22}$$

To this we can ask what kind of mass terms we can add. The mass term must be scalar (not have any free indices) and depend only on η and h. The only way to construct a scalar out of η and h and having only quadratic terms is either $h_{\mu\nu}h^{\mu\nu}$ or $h^\mu_\mu h^\nu_\nu$. The first term is $\mathrm{tr}(h^2)$ while the second is $(\mathrm{tr}(h))^2$. Therefore, the added mass term would be of the form $\frac{m^2}{2}(ah_{\mu\nu}h^{\mu\nu} + bh^\mu_\mu h^\nu_\nu)$. The spin-2 field $h_{\mu\nu}$ can be defined in terms of a vector field ϕ as:

$$h_{\mu\nu} = h^{'}_{\mu\nu} + \partial_\mu\phi_\nu + \partial_\nu\phi_\mu$$

The vector field can be further written in terms of a scalar field as:

$$\phi_\mu = \phi^{'}_\mu + \partial_\mu\pi \tag{1.23}$$

If we plug this transformation into the massive lagrangian, then the only way to avoid a four-derivate kinetic term in π would be to have $a = -b$ [11].(The usual convention is to take $a = -\frac{1}{4}$ [9]). Thus giving the famed Fierz-Pauli Lagrangian:

$$\mathcal{L}_{\mathrm{FP}} = -\frac{1}{4}h^{\mu\nu}\mathcal{E}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} - \frac{1}{8}m^2_{\mathrm{FP}}(h_{\mu\nu}h^{\mu\nu} - h^\mu_\mu h^\nu_\nu) \tag{1.24}$$

Varying the above lagrangian would give the equations of motion (with $T_{\mu\nu} = 0$):

$$\mathcal{E}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} + \frac{m^2_{\mathrm{FP}}}{2}(h_{\mu\nu} - \eta_{\mu\nu}h^\lambda_\lambda) = 0. \tag{1.25}$$

1.4 Bimetric Theory

1.4.1 Motivation for a new bimetric theory of gravity

Let us first ask ourselves why we would want a new theory of gravity in the first place, especially when General Relativity has withstood several tests. The Standard Model is the current best theory of particle physics and there are several phenomenon the Standard Model cannot explain. The Planck collaboration reports a significant density of cold dark matter in the universe that interacts with normal matter only gravitationally. There is also a significant amount of dark energy that is responsible for the accelerating expansion of the universe. Both these phenomena are unexplained by the standard model. There are several approaches that try to explain these phenomena, for example various supersymmetry theories, string theory etc.

One approach would be to try and modify GR. There are several ways to do this, but these are constrained by Lovelock's theorem. Lovelock's theorem states that the Einstein field equations are the only equations of motion for Lagrangians that depend on the metric tensor and its derivatives up to the second derivative only[12]. One way is to try and add a mass term or another spin-2 field. A linear theory that adds a mass term is the Fierz-Pauli theory. The Fierz-Pauli theory suffers from a discontinuity where taking the mass of the graviton to zero does not reproduce general relativity. The propagator for a massive spin-2 field, such as in Fierz-Pauli theory, as we take the limit of the gravition mass going to zero, is given by[13]:

$$D_{\mu\nu\lambda\sigma}(k) = \frac{1}{2}\frac{\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda} - \frac{2}{3}\eta_{\mu\nu}\eta_{\lambda\sigma}}{k^2 + i\epsilon} \tag{1.26}$$

whereas the propagator for the massless spin 2 field is given by:

$$D_{\mu\nu\lambda\sigma}(k) = \frac{1}{2}\frac{\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda} - \eta_{\mu\nu}\eta_{\lambda\sigma}}{k^2 + i\epsilon} \tag{1.27}$$

The above difference in the $\frac{2}{3}$term is what is often called the van Dam Veltman Zakarov (v-DVZ) discontinuity. Another issue the Fierz-Pauli theory suffers from is the appearance of the Boulaware-Deser ghost at non linear orders. In field theories, a ghost is any kind of kinetic term in the Lagrangian with a negative sign. Such terms are not consistent with a physical theory since they can lead to infinite negative energies.

The de Rham Gabadadze Tolley(dRGT) model [14][15] gets rid of the Boulaware Deser ghost for the special case where the second metric is flat and non-dynamic. Bimetric theory and other modified theories of gravity overcome the v-DVZ discontinuity by using the Vainshtein mechanism where general relativity can be recovered below a certain radius from a source known as the Vainshtein radius [16][17].

1.4.2 Hassan Rosen Bimetric Theory

Hassan and Rosen introduced in 2011, the action which avoids the ghost in bimetric theory while keeping both metrics dynamic[1][5][18].

$$S = m_g^2 \int d^4x\sqrt{-g}R(g) + m_f^2 \int d^4x\sqrt{-f}R(f) - 2m^4 \int \sqrt{-g}d^4x \sum_{n=0}^{4} \beta_n e_n(X) \tag{1.28}$$

where m_g and m_f are the (reduced) Planck masses of the g and $f$metrics respectively. $R(g)$ and $R(f)$ are the Ricci scalars of the g and f metrics respectively. In GR in natural units the reduced Planck mass is given by $\sqrt{\frac{1}{8\pi G}}$ where G is Newton's constant. The matrix X is $\sqrt{g^{-1}f}$ and e_n are the elementary symmetric polynomials. Elementary symmetric polynomilas of a matrix can be written in terms of eigenvalues or in terms of the trace of the matrix. In terms of the trace of the matrix, e_n can be written as:

$$e_0 = 1, e_1 = \mathrm{tr}(X), e_2 = \frac{1}{2}(\mathrm{tr}(X)^2 - \mathrm{tr}(X^2)), e_3 = \frac{1}{6}(\mathrm{tr}(X^3) - 3\mathrm{tr}(X)\mathrm{tr}(X^2) + 2\mathrm{tr}(X^3)),$$
$$e_4 = \det(X) \tag{1.29}$$

Alternately, in terms of Eigenvalues of the matrix, it can be written as:

$$e_0 = 1, e_1 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, e_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_4 + \lambda_4\lambda_1,$$
$$e_3 = \lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3\lambda_4 + \lambda_3\lambda_4\lambda_1, e_4 = \lambda_1\lambda_2\lambda_3\lambda_4 \tag{1.30}$$

Conventionally, the ratio of the Planck masses $\frac{m_f}{m_g}$ is usually written as α . The action remains the same if you perform all the following interchanges: $m_g \iff m_f$, $g \iff f$, and $\beta_n \iff \beta_{4-n}$ [5].
The equations of motion, obtained by varying the action are [5]:

$$R_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} + \frac{m^4}{m_g^2}V^g_{\mu\nu} = \frac{1}{m_g^2}T^g_{\mu\nu} \tag{1.31}$$

$$R_{\mu\nu}(f) - \frac{1}{2}R(f)f_{\mu\nu} + \frac{m^4}{m_f^2}V^f_{\mu\nu} = \frac{1}{m_f^2}T^f_{\mu\nu} \tag{1.32}$$

where:

$$V^g_{\mu\nu} = \sum_{n=0}^{3}(-1)^n\beta_n g_{\mu\lambda}Y^\lambda_{(n)\nu}(\sqrt{g^{-1}f}) \tag{1.33}$$

$$V^f_{\mu\nu} = \sum_{n=0}^{3}(-1)^n\beta_{4-n} f_{\mu\lambda}Y^\lambda_{(n)\nu}(\sqrt{f^{-1}g}) \tag{1.34}$$

and

$$Y_{(n)}(X) = \sum_{r=0}^{n} (-1)^r X^{n-r} e_r(X). \tag{1.35}$$

The Bianchi identities are applicable for each metric, giving:

$$\nabla_g^\mu \mathcal{G}_{\mu\nu}^g = 0, \nabla_f^\mu \mathcal{G}_{\mu\nu}^f = 0, \tag{1.36}$$

where ∇_g^μ and ∇_f^μ are covariant derivatives with respect to the metrics g and f. Taking covariant derivative with respect to g on both sides of (1.31) and with respect to f on both sides of (1.32) and using the Bianchi identities, we get the Bianchi constraints:

$$\nabla_g^\mu V_{\mu\nu}^g = 0, \tag{1.37}$$

$$\nabla_f^\mu V_{\mu\nu}^f = 0. \tag{1.38}$$

Let us do a counting of the degrees of freedom. As the metrics g and f are 4×4 symmetric matrices, there are 10 degrees of freedom from the g metric and 10 from the f metric. Demanding general covariance takes away 8 degrees of freedom, the Bianchi constraints takes away another four degrees of freedom. We are therefore left with 8 degrees of freedom. Bimetric theory as described above, consisting of a massive and massless mode must have 7 degrees of freedom, 5 from the massive mode, and 2 from the massless mode(Recall that any massive particle of spin s has $2s + 1$ degrees of freedom). The extra degree of freedom is the ghost which is removed by the careful choice of the form of action made in bimetric theory[8].

1.4.3 Proportional Background Solutions

A linearisation whereby we can split the theory into massless and massive modes is possible only around proportional backgrounds.

$$\bar{f}_{\mu\nu} = c^2 \bar{g}_{\mu\nu} \tag{1.39}$$

where c is some proportionality constant. These solutions require that $\bar{T}_{\mu\nu}^f = \bar{T}_{\mu\nu}^g$ [5].One can consider fluctuations around the proportional background, usually defined and normalised as[5]:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{\delta g_{\mu\nu}}{m_g}, \tag{1.40}$$

$$f_{\mu\nu} = \bar{f}_{\mu\nu} + c\frac{\delta f_{\mu\nu}}{m_f}. \tag{1.41}$$

The $\frac{1}{m_g}$ serves as a normalisation constant. We get the equations of motion in proportional backgrounds:

$$\bar{R}_{\mu\nu}(g) - \frac{1}{2}\bar{R}(g)\bar{g}_{\mu\nu} + \Lambda_g \bar{g}_{\mu\nu} = \frac{1}{m_g^2} T_{\mu\nu}^g \tag{1.42}$$

$$\bar{R}_{\mu\nu}(g) - \frac{1}{2}\bar{R}(g)\bar{g}_{\mu\nu} + \Lambda_f \bar{g}_{\mu\nu} = \frac{1}{m_f^2} T^f_{\mu\nu} \tag{1.43}$$

where:

$$\Lambda_g = \frac{m^4}{m_g^2}(\beta_0 + 3c\beta_1 + 3c^2\beta_2 + c^3\beta_3), \Lambda_f = \frac{m^4}{m_f^2 c}(c\beta_1 + 3c^2\beta_2 + 3c^3\beta_3 + c^4\beta_4) \tag{1.44}$$

Subtracting (1.43) from (1.42), we get:

$$(\Lambda_g - \Lambda_f)\bar{g}_{\mu\nu} = \frac{1}{m_g^2} T^g_{\mu\nu} - \frac{1}{m_f^2} T^f_{\mu\nu} \tag{1.45}$$

If the right hand side of (1.42) has any term that contains $\bar{g}_{\mu\nu}$ then that term can be brought to the left hand side and absorbed into the cosmological constant term by suitably redefining β_0. Similarly, any term containing $\bar{g}_{\mu\nu}$ in (1.43) can be brought to the left hand side by suitably redefining β_4. Therefore, we can always write (1.45) such that its right hand side has no terms proportional to $\bar{g}_{\mu\nu}$. Having done this, for localisable sources, [5] demands that both sides of (1.45) must be zero independently,

$$\Lambda_g = \Lambda_f \tag{1.46}$$

and

$$m_f^2 T^g_{\mu\nu} = m_g^2 T^f_{\mu\nu} \tag{1.47}$$

Under proportional backgrounds we can see the theory as consisting of a massive mode δM and a massless mode δG

The explicit relation between the mass eigenstates and the interaction eigenstates would be:

$$\delta G_{\mu\nu} = \frac{1}{\sqrt{c^2\alpha^2 + 1}}(\delta g_{\mu\nu} + c\alpha\delta f_{\mu\nu}) \tag{1.48}$$

$$\delta M_{\mu\nu} = \frac{1}{\sqrt{c^2\alpha^2 + 1}}(\delta f_{\mu\nu} - c\alpha\delta g_{\mu\nu}) \tag{1.49}$$

Considering fluctuations around the proportional background, we can get the equations for the fluctuations as [5]:

$$\overline{\mathcal{E}}^{\rho\sigma}_{\mu\nu}\delta G_{\rho\sigma} + \Lambda_g \delta G_{\mu\nu} = \frac{\delta T^g_{\mu\nu} + c^2 \delta T^f_{\mu\nu}}{m_g\sqrt{c^2\alpha^2 + 1}} \tag{1.50}$$

$$\overline{\mathcal{E}}^{\rho\sigma}_{\mu\nu}\delta M_{\rho\sigma} + \Lambda_g \delta M_{\mu\nu} + \frac{m_{FP}^2}{2}(\delta M_{\mu\nu} - \bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\delta M_{\rho\sigma}) = c\frac{\delta T^f_{\mu\nu} - \alpha^2 \delta T^g_{\mu\nu}}{m_f\sqrt{c^2\alpha^2 + 1}} \tag{1.51}$$

where

$$m_{\mathrm{FP}}^2 = m^4(c\beta_1 + 2c^2\beta_2 + c^3\beta_3)\left(\frac{1}{c^2 m_f^2} + \frac{1}{m_g^2}\right) \tag{1.52}$$

is the Fierz-Pauli mass of the massive mode, and where $\bar{\mathcal{E}}$ is defined, by action on a fluctuation δh of an example background field $\bar{h}(h = \bar{h} + \delta h)$ as:

$$\begin{aligned}\bar{\mathcal{E}}^{\rho\sigma}_{\mu\nu}\delta h_{\rho\sigma} = -\frac{1}{2}\Big[&\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu}\bar{\nabla}^2 + \bar{h}^{\rho\sigma}\bar{\nabla}_{\mu}\bar{\nabla}_{\nu} - \delta^{\rho}_{\mu}\bar{\nabla}^{\sigma}\bar{\nabla}_{\nu} \\ &- \delta^{\rho}_{\nu}\bar{\nabla}^{\sigma}\bar{\nabla}_{\mu} - \bar{h}_{\mu\nu}\bar{h}^{\rho\sigma}\bar{\nabla}^2 + \bar{g}_{\mu\nu}\bar{\nabla}^{\rho}\bar{\nabla}^{\sigma} + \bar{h}_{\mu\nu}\bar{R}^{\rho\sigma} + \delta^{\rho}_{\mu}\delta^{\sigma}_{\nu}\bar{R}\Big]\delta h_{\rho\sigma},\end{aligned} \tag{1.53}$$

where $\bar{\nabla}$ is the covariant(torsion free and metric compatible) derivative for the background metric $\bar{h}$.

Note that the constant c can be set to 1 without any loss of generality. (In proportional backgrounds, c is fixed by (1.46) as $\alpha^2\beta_3 c^4 + (3\alpha^2\beta_2 - \beta_4)c^3 + 3(\alpha^2\beta_1 - \beta_3)c^2 + (\alpha^2\beta_0 - 3\beta_2)c - \beta_1 = 0$ [5]. Therefore, fixing c to 1 would simply put a constraint on the beta parameters, ie., $\alpha^2\beta_3 + (3\alpha^2\beta_2 - \beta_4) + 3(\alpha^2\beta_1 - \beta_3) + (\alpha^2\beta_0 - 3\beta_2) - \beta_1 = 0$.)

Chapter 2

Gravitational Wave Oscillations in Bimetric Theory

2.1 Introduction

In this section we shall look at a parallel between neutrino oscillations and oscillations between the mass and interaction eigenstates of the graviton in bimetric theory.

There are three flavours of neutrinos, electron, muon, and tau neutrinos. However when neutrinos are emitted in the Sun, they do not travel to the earth in these flavour eigenstates but rather in mass eigenstates, which constitute a linear combination of different flavour eigenstates. Thus it is possible for neutrinos to be emitted in the Sun in one flavour eigenstate, travel between the sun and the earth as mass eigenstates, and therefore be observed in the Earth in a mixture of flavour eigenstates different from the original flavour. Thus if the Sun emits only electron neutrinos, and a detector on Earth can only detect them, the detector on Earth would detect less electron neutrinos than expected. This is what is known as the solar neutrino problem.

The first theoretical work on neutrino oscillations was done by Pontecorvo, followed by Maki, Nakagawa, and Sakata introducing the idea of mass and flavour eigenstates [19]. Experimentally, the Brookhaven experiment discovered the muon neutrino, and observations by the Super Kamiokande observatory in Japan provided the first evidence for neutrino oscillations. For a detailed historical review, see [20].

Bimetric theory consists of two metrics g and f which couple to two different sets of matter fields seperately. In proportional backgrounds or cosmological backgrounds gravitons travel as either a massive or a massless mode, which conisist of a linear combination of fluctuations δg and δ.

Much like neutrino oscillations, we can think of g and f as being interaction eigenstates while the massive and massless graviton are seen as the mass eignenstates. This allows us to use a lot of the framework for neutrino oscillations in bimetric theory. Following [21], [22], and [23], we shall use this analysis to derive some exclusion regions for the parameters of bimetric theory. We then extend the bounds in [21] and [22] using the newer observation of the binary black hole merger GW190521 by LIGO. Early work in gravitational wave oscillations in bimetric theory was first done in [24] and [25]. However, both these works were before the first detection of gravitational waves by LIGO [26].

As a first step to the goal of looking at gravitational wave oscillations, we do a review of two flavour oscillations for a particle that obeys the Schrodinger equation.

2.2 Two flavour oscillations

Following [27], let us consider a field Ψ with two flavours a and b. ie., Ψ can be written as $\Psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}$. Let the field Ψ evolve at rest according to the time dependent Schrodinger equation as follows:

$$-i\frac{\partial \Psi}{\partial t} = \begin{pmatrix} \mu_a & \tau \\ \tau & \mu_b \end{pmatrix} \Psi \tag{2.1}$$

If τ is zero, the flavours develop independently, ie., the system of equations decouple. In this case, μ_a and μ_b are the masses of the two flavours a and b, and the mass and flavour eigenstates coincide.

If τ is non-zero, it does not make sense to talk of the mass of the flavors a and b. To talk about mass in this scenario, we need to rewrite the above equation in such a way that we have a diagonal matrix. To do this, we consider the unitary matrix U given by:

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{2.2}$$

and

$$U^\dagger = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2.3}$$

where:

$$\tan(2\theta) = \frac{2\tau}{\mu_b - \mu_a}. \tag{2.4}$$

Multiplying both sides of (2.1) by $U^\dagger$ and recalling $U^\dagger U = UU^\dagger = I$

$$-iU^\dagger \frac{\partial \Psi}{\partial t} = U^\dagger \begin{pmatrix} \mu_a & \tau \\ \tau & \mu_b \end{pmatrix} UU^\dagger \Psi \tag{2.5}$$

giving:

$$-iU^\dagger \frac{\partial \Psi}{\partial t} = \begin{pmatrix} m_- & 0 \\ 0 & m_+ \end{pmatrix} U^\dagger \Psi \tag{2.6}$$

where

$$m_\mp = \frac{\mu_a + \mu_b}{2} \mp \sqrt{\left(\frac{\mu_a - \mu_b}{2}\right)^2 + \tau^2} \tag{2.7}$$

In this form, it is evident that the states represented by $U^\dagger \Psi$ can be seen as the mass eigenstates. The matrices U and $U^\dagger$ can be used to shift between the mass and flavour eigenstates.

$$\begin{pmatrix} \psi_{m_1} \\ \psi_{m_2} \end{pmatrix} = U^\dagger \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} \tag{2.8}$$

If instead of at rest, the particle is in motion, then one simply multiplies a boost factor of γ to the matrix $\begin{pmatrix} \mu_a & \tau \\ \tau & \mu_b \end{pmatrix}$. However, in general such a transformation is valid only if both eigenstates have the same momentum which need not be the case depending on how the states are produced[27].

2.3 Wave Packets

The wave packet is another useful concept that can be used to model oscillations. The wave packet for a wave travelling in the z direction is described by the following wave function:

$$\Psi(t, z) = \int a(k) e^{i(kz - E(k)t)} dk \tag{2.9}$$

where $E(k) = \sqrt{k^2 + m^2}$. Here $a(k)$ can be any momentum distribution function. One choice would be to have $a^2(k)$ follow a Gaussian distribution with standard deviation σ [27]. We can model the propagation of the particle as one wave packet propagating for each mass eigenstate. The different wave packets will move with different velocities. After some time, the different wave packets will then seperate and can be observed seperately.

2.4 Cosmologies in bimetric theory

We repeat here results from [28]. They start with the following ansatz for the metrics:

$$g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) dx^2, \tag{2.10}$$

$$f_{\mu\nu} dx^\mu dx^\nu = -X^2(t) dt^2 + Y^2(t) dx^2, \tag{2.11}$$

$$dx^2 = \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2(\theta) d\phi^2). \tag{2.12}$$

The Friedmann equations are obtained for the g sector and the f sector, assuming that only g couples with matter:

$$-3\left(\frac{\dot{a}}{a}\right)^2 - 3\frac{k}{a^2} + m^2\left[\beta_0 + 3\beta_1\frac{Y}{a} + 3\beta_2\frac{Y^2}{a^2} + \beta_3\frac{Y^3}{a^3}\right] = \frac{1}{m_g^2}T_0^0, \tag{2.13}$$

$$-3\left(\frac{\dot{a}}{Y}\right)^2 - 3\frac{k}{Y^2} + \frac{m_{\mathrm{FP}}^2 m_g^2}{m_f^2}\left[\beta_4 + 3\beta_3\frac{a}{Y} + 3\beta_2\frac{a^2}{Y^2} + \beta_1\frac{a^3}{Y^3}\right] = 0 \tag{2.14}$$

Let us also introduce the Hubble parameters for both metrics, H for the g metric, given by $H = \frac{\dot{a}}{a}$ and J for the f metric, given by $J = \frac{\dot{Y}}{Y}$. We also introduce the ratio of scale factors y, given by $y = \frac{Y}{a}$. The Bianchi constraints give [29](assuming $y \neq 0$):

$$(\dot{Y} - \dot{a}X)(\beta_1 + 2\beta_2 y + 3\beta_3 y^2) = 0 \tag{2.15}$$

There exists two branches of the solution to the above equation. One of these is one where y is a constant over time, coming from the constant βs being chosen such that the the second term in brackets in the above equation is zero. This is called the algebraic branch and leads to certain instabilities at late times [29].
The condition for the second branch, called the dynamical branch, is $X(t) = \frac{\dot{Y}}{\dot{a}} = \frac{dY}{da}$ [28]. This is the branch we shall work in.
In analysing late time phenomena, which is what we will do in the next section when we will look at gravitational waves propagating from binary black hole mergers and being observed by LIGO, we make an approximation to set $\frac{Y}{a}$ to be a constant and setting this constant to 1. While setting $\frac{Y}{a}$to constant is a good late time approximation to the modified Friedmann equations, setting this constant as one requires us to put a constraint on some of the β parameters[23] . We also set $X = 1$. How good these approximations are depend on the redshift z. There is a fair range of redshifts for the LIGO observations we will be looking at. Therefore, we must define what we mean by late times.

We use here the criterion for late times of Platscher[23]

$$\frac{\rho(\eta)}{M_{\mathrm{Pl}}^{*2} m_{\mathrm{FP}}^2} \ll 1 \tag{2.16}$$

as the condition for $X \simeq 1$ to be valid. In the above expression M_{Pl}^* is the physical planck mass which we approximate as the planck mass of the g sector for the calculations done below. Let us start by considering $\rho(\eta) = \rho^0_{\mathrm{matter}}$ where ρ^0_{matter} is the energy density of matter in the universe today.
The Planck collaboration gets a result of $\Omega_{\mathrm{matter}} = 0.315 \pm 0.007$ [30] including both SM and dark matter. Recalling (1.15) we obtain

$$\rho^0_{\mathrm{matter}} = \frac{3H^2}{8\pi G}\Omega_{\mathrm{matter}}, \tag{2.17}$$

giving $\rho^0_{\text{matter}} = 1.26 \times 10^{-11}\text{eV}^4$. Assuming that the universe consists of only matter, let us try and find what a critical redshift in the above scenario would be:

$$\frac{\rho(z_c)}{m^2_{\text{FP}}M^2_{\text{Pl}}} = \frac{1.26 \times 10^{-11}\text{eV}^4(1+z_*)^3}{m^2_{\text{FP}}M^2_{\text{Pl}}} = 1. \tag{2.18}$$

Taking $M_{\text{Pl}} \simeq 2.4 \times 10^{27}\text{eV}$ giving:

$$1 + z_* = m^{\frac{2}{3}}_{\text{FP}}\Big(\frac{(2.4 \times 10^{27})^2}{1.26 \times 10^{-11}}\Big)^{\frac{1}{3}}, \tag{2.19}$$

$$1 + z_* = m^{\frac{2}{3}}_{\text{FP}}(7.7 \times 10^{21}), \tag{2.20}$$

for say $m_{\text{FP}} = 10^{-23}$, this gives:

$$1 + z_* = 3.57 \times 10^6, \tag{2.21}$$

which is fairly large and shows that we can write $1+z_* \simeq z_*$. At the time represented by this redshift it would be radiation which is dominant, and so we must check the above criterion for radiation to get the correct value of the critical redshift.
Looking at the radiation component, as radiation is the component whose energy density decays at the highest rate as the universe expands ($\rho \propto a^{-4}$), we can take ρ_0 to be the energy density of the cosmic microwave background as observed today. We use a value of $\rho_0 \simeq 2 \times 10^{-15}\text{eV}^4$ [31] and

$$\frac{\rho(z_c)}{m^2_{\text{FP}}M^2_{\text{Pl}}} = \frac{2 \times 10^{-15}\text{eV}^4 z^4_*}{m^2_{\text{FP}}M^2_{\text{Pl}}} = 1. \tag{2.22}$$

Taking $M_{\text{Pl}} = 2.4 \times 10^{27}\text{eV}$ leads to:

$$z_* = \Big(5.36 \times \frac{m_{\text{FP}}}{10^{-34}\text{eV}}\Big)^{\frac{1}{2}} \tag{2.23}$$

Platscher [23] gets a slightly different value

$$z_* = 13 \times \Big(\frac{m_{\text{FP}}}{10^{-33}\text{eV}}\Big)^{\frac{1}{2}}. \tag{2.24}$$

For ranges we are considering, $m_{\text{FP}} = 10^{-22} - 10^{-23}\text{eV}$, this gives a rather large value of z_*.

2.5 Bimetric Oscillations

2.5.1 In proportional backgrounds

For $\delta T^f_{\mu\nu} = \delta T^g_{\mu\nu} = 0$, the linearised bimetric equations of motion in proportional backgrounds read [5]:

$$\bar{\mathcal{E}}^{\rho\sigma}_{\mu\nu}\delta G_{\rho\sigma} + \Lambda_g \delta G_{\mu\nu} = 0 \tag{2.25}$$

$$\bar{\mathcal{E}}^{\rho\sigma}_{\mu\nu}\delta M_{\rho\sigma} + \Lambda_g \delta M_{\mu\nu} + \frac{m^2_{FP}}{2}(\delta M_{\mu\nu} - \bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\delta M_{\rho\sigma}) = 0 \tag{2.26}$$

And in terms of δg and δf they read(where tr is the trace operator):

$$\begin{aligned}\bar{\mathcal{E}}^{\rho\sigma}_{\mu\nu}\delta g_{\rho\sigma} + \Lambda_g \delta g_{\mu\nu} = \delta f_{\mu\nu}[\frac{m^4 B}{2m_f m_g}] - \\ \frac{m^4 B}{2m_g m_f}\eta_{\mu\nu}\mathrm{tr}(\delta f) - \frac{m^4 B m_f c}{m_g^2}\delta g_{\mu\nu} + \frac{c m^4 B}{2m_g^2}\eta_{\mu\nu}\mathrm{tr}(\delta g)\end{aligned} \tag{2.27}$$

$$\begin{aligned}\bar{\mathcal{E}}^{\rho\sigma}_{\mu\nu}\delta f_{\rho\sigma} + \Lambda_f \delta f_{\mu\nu} = -\frac{m^4 B}{2cm_f^2}\delta f_{\mu\nu} + \\ \frac{m^4 B}{2m_f m_g}\delta g_{\mu\nu} + \frac{m^4 B}{2cm_f^2}\eta_{\mu\nu}\mathrm{tr}(\delta f) - \frac{m^4 B}{2m_f m_g}\eta_{\mu\nu}\mathrm{tr}(\delta g)\end{aligned} \tag{2.28}$$

where

$$B = \beta_1 + 2c\beta_2 + c^2\beta_3 \tag{2.29}$$

To make the analogy with neutrino oscillations more explicit, we can call $\mathcal{E} + \Lambda$ as some operator $\mathcal{O}$ and rewrite the above equations in a matrix form as:
In terms of interaction eigenstates:

$$\mathcal{O}\begin{pmatrix}\delta g \\ \delta f\end{pmatrix} = \begin{bmatrix} \frac{-m^4 B c m_f}{m_g^2} + \frac{c m^4 B}{2m_g^2}\eta_{\mu\nu}\mathrm{tr} & \frac{m^4 B}{2m_f m_g} - \frac{m^4 B}{2m_f m_g}\eta_{\mu\nu}\mathrm{tr} \\ \frac{m^4 B}{2m_f m_g} - \frac{m^4 B}{2m_f m_g}\eta_{\mu\nu}\mathrm{tr} & -\frac{m^4 B}{2cm_f^2} + \frac{m^4 B}{2cm_f^2}\eta_{\mu\nu}\mathrm{tr}\end{bmatrix}\begin{pmatrix}\delta g \\ \delta f\end{pmatrix} \tag{2.30}$$

where tr is the trace operator. In terms of δG and δM the above equations read:

$$\mathcal{O}\begin{pmatrix}\delta G \\ \delta M\end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{m^2_{FP}}{2} + \frac{m^2_{FP}}{2}\eta_{\mu\nu}tr\end{bmatrix}\begin{pmatrix}\delta G \\ \delta M\end{pmatrix} \tag{2.31}$$

Writing in this form more clearly shows that δG is massless. Comparison with (2.27) and (2.28) makes the meaning of the indices in (2.31) and (2.30) more clear. $\eta_{\mu\nu}$ represents the metric tensor of the Minkowski metric.

Follwing [23], one can look at just the transverse traceless components. Let us rewrite the expression for $\bar{\mathcal{E}}^{\rho\sigma}_{\mu\nu}$ here:

$$\bar{\mathcal{E}}^{\rho\sigma}_{\mu\nu} = -\frac{1}{2}\Big[\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu}\bar{\nabla}^2 + \bar{h}^{\rho\sigma}\bar{\nabla}_{\mu}\bar{\nabla}_{\nu} - \delta^{\rho}_{\mu}\bar{\nabla}^{\sigma}\bar{\nabla}_{\nu} - \delta^{\rho}_{\nu}\bar{\nabla}^{\sigma}\bar{\nabla}_{\mu} - \bar{h}_{\mu\nu}\bar{h}^{\rho\sigma}\bar{\nabla}^2 + \bar{h}_{\mu\nu}\bar{\nabla}^{\rho}\bar{\nabla}^{\sigma} + \bar{h}_{\mu\nu}\bar{R}^{\rho\sigma} + \delta^{\rho}_{\mu}\delta^{\sigma}_{\nu}\bar{R}\Big] \quad (2.32)$$

In a flat background, $\bar{\nabla} = \partial$, and $\bar{\nabla}_{\mu}$ and $\bar{\nabla}_{\nu}$ commute and in cartesian coordiantes $\bar{R} = \bar{R}^{\rho\sigma} = 0$

Let us act this operator on $\delta G_{\rho\sigma}$. Traceless implies:

$$\eta^{\rho\sigma}\delta G_{\rho\sigma} = 0 \quad (2.33)$$

getting rid of the second and fifth terms in (2.32)
To say the gauge is transverse implies:

$$\partial^i \delta G^{TT}_{ij} = 0 \quad (2.34)$$

and

$$\delta G_{0i} = \delta G_{00} = 0 \quad (2.35)$$

getting rid of the third, fourth, and sixth terms, leaving only the first term which equals $-\frac{1}{2}\Box$ where:

$$\Box = -\partial_t^2 + \nabla^2 \quad (2.36)$$

Therefore one can rewrite (2.25) and (2.26) as:

$$\frac{1}{2}\Box\delta G^{TT}_{\mu\nu} - \Lambda_g \delta G^{TT}_{\mu\nu} = 0 \quad (2.37)$$

and

$$\frac{1}{2}\Box\delta M^{TT}_{\mu\nu} - \Lambda_f \delta M^{TT}_{\mu\nu} - \frac{m^2_{FP}}{2}\delta M^{TT}_{\mu\nu} = 0 \quad (2.38)$$

For ease of analysis, we neglect Λ_g and Λ_f. This is justifiable considering the fact that the measured value of Λ is quite small. A typical value of the wave number k would be much larger than this.

2.5.2 In a cosmological background

Platscher [23] derives the equations of motion in a cosmological background:

$$\Box\delta G^{\mathrm{TT}}_{\mu\nu} + 2\mathcal{H}\delta M^{'\mathrm{TT}}_{\mu\nu} = 0 \quad (2.39)$$

$$\Box\delta M^{\mathrm{TT}} + 2\mathcal{H}\delta M^{'\mathrm{TT}} - \frac{a^2(\eta)m^2_{\mathrm{FP}}}{y^2}\delta M^{\mathrm{TT}} = 0 \quad (2.40)$$

We can move to Fourier space using

$$\delta g(\eta, x) = \int \frac{d^4k}{2\pi^4} e^{-i\omega\eta + i\vec{k}.\vec{x}} \hat{\delta g}(\omega, \vec{k}) \tag{2.41}$$

We find that:

$$\omega^2 \delta\hat{G}^{TT} - k^2 \delta\hat{G}^{TT} - 2\mathcal{H} i\omega \delta\hat{G}^{TT} = 0 \tag{2.42}$$

and

$$\omega^2 \delta\hat{M}^{TT} - k^2 \delta\hat{M}^{TT} - 2\mathcal{H} i\omega \delta\hat{M}^{TT} - a^2(\eta)\frac{m_{\rm FP}^2}{y^2}\delta\hat{M}^{TT} = 0 \tag{2.43}$$

To get plane wave solutions, the approximations $y = 1$ and $\mathcal{H} < k$ have to be made.[23] Another approximation to be made is to set $a = 1$ for all times. Typical values of redshifts observed by LIGO is less than 1. The observation with the highest redshift we shall use in this thesis has a redshift of $z = 0.82$ which would correspond to $a(\eta) \simeq 0.55$ which is fairly close to 1. This reduces the above equations to [23]:

$$-\omega^2 \delta\hat{G}^{TT} + k^2 \delta\hat{G}^{TT} = 0 \tag{2.44}$$

$$-\omega^2 \delta\hat{M}^{TT} + k^2 \delta\hat{M}^{TT} + m_{\rm FP}^2 \delta\hat{M}^{TT} = 0 \tag{2.45}$$

which are the same as that of a proportional background with Λ neglected which we derived earlier. We get the solutions:

$$\delta G \propto \exp(-i\omega_0 t + i\vec{k}.\vec{x}) \tag{2.46}$$

and

$$\delta M \propto \exp(-i\sqrt{\omega_0^2 + m_{\rm FP}^2}t + i\vec{k}.\vec{x}) \simeq \exp(-i\omega_0 t + i\vec{k}.\vec{x})\exp(-i\frac{m_{\rm FP}^2 t}{2\omega_0}) \tag{2.47}$$

For convenience we write $\frac{m_{\rm FP}^2}{2\omega_0} = \delta\omega$. Referring to [5], we see the relation between fluctuations in mass and flavour eigenstates in proportional backgrounds as:

$$\begin{pmatrix} \delta G \\ \delta M \end{pmatrix} = \frac{1}{\sqrt{c^2\alpha^2 + 1}} \begin{pmatrix} 1 & c\alpha \\ -c\alpha & 1 \end{pmatrix} \begin{pmatrix} \delta g \\ \delta f \end{pmatrix}$$

where α is the ratio of the Planck masses of the f and g metrics. It is straightforward to set $\cos(\theta) = \frac{1}{\sqrt{c^2\alpha^2+1}}$ and $\sin(\theta) = \frac{c\alpha}{\sqrt{c^2\alpha^2+1}}$. Platscher [23] uses a slightly different definition of θ, ie., $\sin^2(\theta) = \frac{M_{\rm eff}^2}{m_g^2}, \cos^2(\theta) = \frac{M_{\rm eff}^2}{m_f^2}$ where $M_{\rm eff}^2 = \frac{m_g^2 m_f^2}{m_g^2 + m_f^2}$. In proportional backgrounds with c = 1, the two definitions coincide.

If we start off with a state where, initially $\begin{pmatrix}\delta g\\ \delta f\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$. At $t = 0$

$$\delta G = \cos(\theta)$$

$$\delta M = -\sin(\theta)$$

then after a time t, the state becomes, in terms of the propogation modes:

$$\delta G = \cos(\theta)\exp(-i\omega_0 t)\exp(ikx),$$

$$\delta M = \sin(\theta)\exp(-i\omega_0 t)\exp(-i\delta\omega t)\exp(ikx).$$

converting this back to the interaction states using:

$$\begin{pmatrix}\delta g\\ \delta f\end{pmatrix} = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}\delta G\\ \delta M\end{pmatrix}$$

we find:

$$\delta g = [\cos^2(\theta)\exp(-i\omega_0 t) + \sin^2(\theta)\exp(-i\omega_0 t)\exp(-i\delta\omega t)]\exp(ikx), \tag{2.48}$$

$$\delta f = [\sin(\theta)\cos(\theta)[\exp(-i\omega_0 t) - \exp(-i\omega_0 t)\exp(-i\delta\omega t)]]\exp(ikx) \tag{2.49}$$

looking at the real part of eqns (2.48) and (2.49):

$$\mathrm{Re}(\delta g) = \cos^2(\theta)\cos(\omega_0 t) + \sin^2(\theta)\cos(\omega_0 t + \delta\omega t) \tag{2.50}$$

$$\mathrm{Re}(\delta f) = \sin(\theta)\cos(\theta)[\cos(\omega_0 t) - \cos(\omega_0 t + \delta\omega t)] \tag{2.51}$$

If the initial state was created by the merger of two binary black holes, we are interested in what state would be observed on earth by say LIGO. Platscher[23] suggests two ways to proceed both of which give the same result. One is to look at $\delta g \delta g^*$

$$\delta g\delta g^* = \cos^4(\theta) + 2\cos^2\theta\sin^2\theta\cos(\delta\omega t) + \sin^4\theta \tag{2.52}$$

and similarly $\delta f \delta f^*$

$$\delta f\delta f^* = 2\sin^2(\theta)\cos^2(\theta)[1 - \cos(\delta\omega t)] \tag{2.53}$$

What is observed in a gravitational wave detector is of course proportional to $\delta g\delta g^*$. In equation (2.50), if θ and ω_0 are held fixed, the most difference from the emitted value of δg occurs when $\delta\omega t = \pi$. Calling this time T_*, we obtain [23]:

$$T_* = \frac{\pi}{\delta\omega} = \frac{2\pi\omega_0}{m_{\mathrm{FP}}^2} \tag{2.54}$$

As an example, for $m_{\mathrm{FP}} = 10^{-19}$ and $\omega_0 = 2\pi \times 30$ we get

$$T_* = 1.62 \times 10^5 \text{years}$$

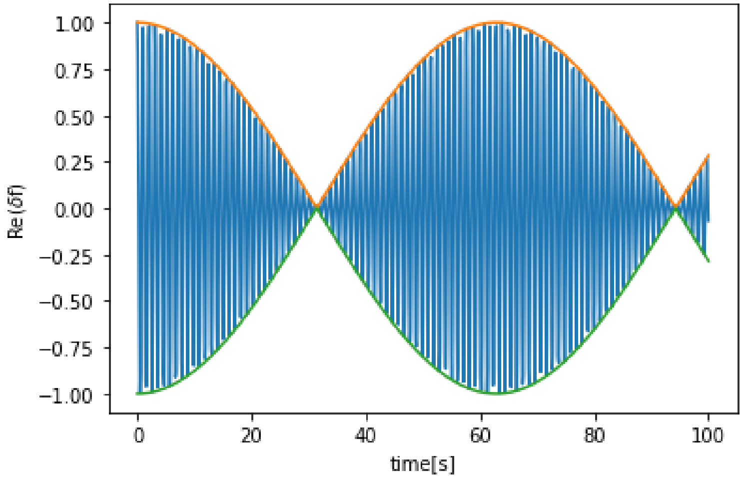

Figure 2.1. real part of δg with envelope $\pm\sqrt{\delta g^2}$. Following [23], for illustrative purposes we have chosen $\omega_0 = 6\,\mathrm{rad/s}, \delta\omega = 0.1\,\mathrm{rad/s}, \theta = \frac{\pi}{4}$

The second method is to begin by looking at $\mathrm{Re}(\delta g)^2$:

$$(\mathrm{Re}(\delta g))^2 = \cos^4(\theta)\cos^2(\omega_0 t) + \sin^4(\theta)\cos^2(\omega_0 t + \delta\omega t) + 2\cos^2(\theta)\sin^2(\theta)\cos(\omega_0 t)\cos(\omega_0 t + \delta\omega t) \tag{2.55}$$

We can now average the above expression over a time T, basically averaging out the effect of the faster oscillation.

$$\langle\cos^2(\omega_0 t)\rangle = \langle\sin^2(\omega_0 t)\rangle = \frac{1}{2} \tag{2.56}$$

$$\langle\sin(\delta\omega t)\rangle \simeq \sin(\delta\omega T) \tag{2.57}$$

$$\langle\cos(\delta\omega t\rangle \simeq \cos(\delta\omega T) \tag{2.58}$$

giving:

$$s(\delta\omega T)^2 = \cos^4(\theta)[1 + \tan^4(\theta) + 2\tan^2(\theta)\cos(\delta\omega T)] \tag{2.59}$$

We observe that this is the same expression as given by (2.52).

Now, if z is defined as $1 + z = \frac{a(t_0)}{a(t)}$, for a universe dominated by a cosmological constant, $H = \text{const.}$ and $a(t) = e^{Ht}$, $t = -\frac{1}{H}\log(1+z)^2$, we can get[21]:

$$s^2 = \cos^4\theta[1 + \tan^4\theta + 2\tan^2\theta\cos\frac{m_{\mathrm{FP}}^2\log(1+z)}{2\omega_0 H_0}]. \tag{2.60}$$

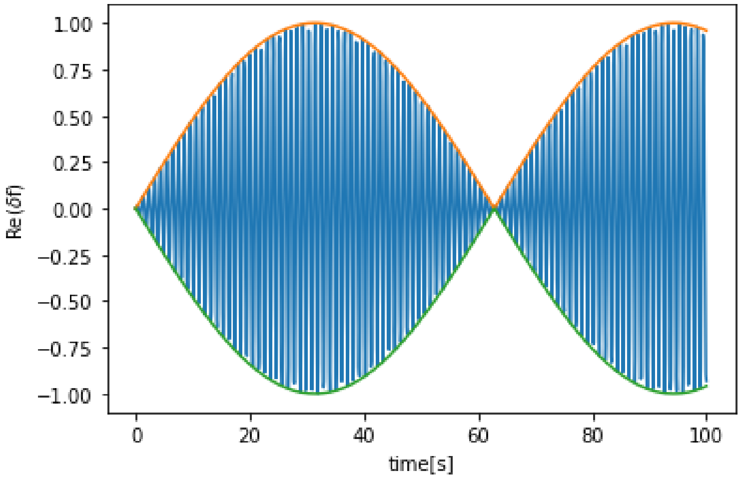

Figure 2.2. Real part of δf with envelope $\pm\sqrt{\delta f^2}$. As before we have chosen $\omega_0 = 6\,\mathrm{rad/s}, \delta\omega = 0.1\,\mathrm{rad/s}, \theta = \frac{\pi}{4}$.

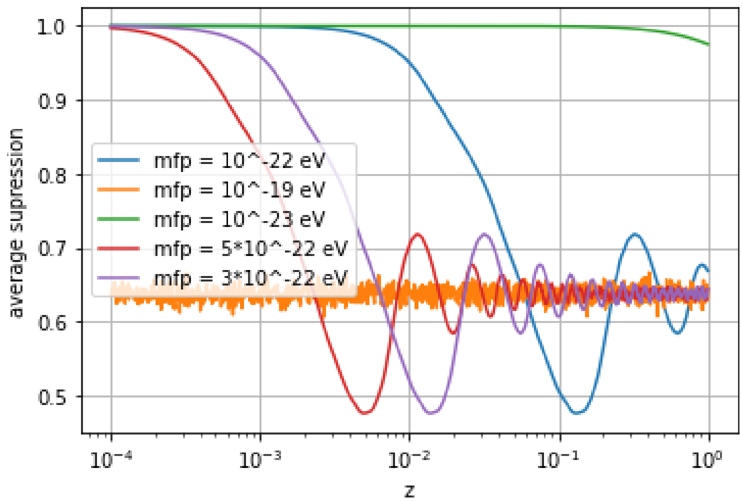

Figure 2.3. Average suppression factor (2.60) as a function of redshift for various Fierz-Pauli masses. $\theta = \frac{\pi}{4}$ for all cases Averaging has been done from 20 Hz to 300 Hz. Note that the results differ somewhat from those in [21], partly because the results are very sensitive to the values of input parameters.

2.6 Decoherence regime

The expresssion (2.48) can be written in terms of wave packets as:

$$\delta g(t,\vec{x}) = \cos^2(\theta)\int \frac{d^3k}{(2\pi^3)}\frac{\exp(-i\omega_1(k)t + i\vec{k}.\vec{x})}{2\omega_1(\vec{k})}\delta G(\omega_1(\vec{k}),\vec{k}) + \\ \sin^2(\theta)\int \frac{d^3k}{(2\pi^3)}\frac{\exp\left(-i\omega_2(k)t + i\vec{k}.\vec{x}\right)}{2\omega_2(\vec{k})}\delta M(\omega_1(\vec{k}),\vec{k}), \quad (2.61)$$

where $\omega_1(k) = k = \omega_0$ and $\omega_2(k) = \sqrt{k^2 + m_{\mathrm{FP}}^2} \simeq \omega_0 + \delta\omega$. The first term in (2.61) represents the massless mode and the second term represents the massive mode. The massive mode δM has a group velocity:

$$\frac{d\omega_2}{dk} = \frac{k}{\omega_2} \simeq 1 - \frac{m_{\mathrm{FP}}^2}{2\omega_0^2} \quad (2.62)$$

One can say that the wave is in the decoherence regime when, due to their difference in velocities, the two wavepackets have completeley seperated. ie.,

$$\frac{m_{\mathrm{FP}}^2}{2\omega_0^2} \times T_{\mathrm{coh}} > \sigma \quad (2.63)$$

where σ is the width of the wave packet and T_{coh} is the time that has been taken until this decoherence is acheived. We can introduce a decoherence length $L \simeq T_{\mathrm{coh}}$, where L is the distance travelled as the massless mode travels at the speed of light. If we fix $T_{\mathrm{coh}} \simeq L$ and ω_0, we can find a lower bound on m_{FP} to enter the decoherence regime:

$$m_{\mathrm{FP}}^2 > \frac{2\sigma\omega_0^2}{L}. \quad (2.64)$$

Any gravitational wave detector couples to g. Both δG and δM contain components of g. If the waves are in the decoherence regime, one would expect a sufficiently sensitive detector on Earth to detect both signals, seperated in time. The second signal can be viewed as an echo of the first.

LIGO detects gravitational waves by the principle of Michelson interferometry. The detector consists of two arms arranged in an L shape. As the gravitational wave passes, it changes the length of the arms, which is the length between two test masses. The difference in lengths of the arms is measured by the difference in the propagation time of light travelling along the two arms.[26]

We can use observations by LIGO to quickly come up with bounds based on the above analysis. One limit on the bounds derived this way is the assumption that the waves are in the decoherence regime. Since δG is suppressed by a factor of $\cos^2(\theta)$ and δM by a factor of $\sin^2(\theta)$, LIGO may observe only one of them if either $\sin(\theta)$ or $\cos(\theta)$ is too small. LIGO uses a factor called the Signal to Noise ratio(SNR). Although the exact expression for the signal to noise ratio is complicated, for a heuristic argument in this section we view it as simply being the ratio of a measure of signal strength and noise. More details about SNR can be found in [32].

$$(\text{Signal}) = (\text{Noise}) \times \text{SNR} \tag{2.65}$$

Since the decoherence suppresses the signal by $\sin^2(\theta)$, as a criterion for detection we follow [22] and [23], where they use the criterion that if $\sin^2(\theta) \times \text{SNR} < 1$, the modified signal cannot be observed, ie., if $\text{SNR} < \frac{1}{\sin^2(\theta)}$ So if $\sin^2(\theta) > \frac{1}{\text{SNR}}$, massive mode of the modified signal can be detected by LIGO. Similarly, if $\cos^2(\theta) > \frac{1}{\text{SNR}}$, the massless mode can be detected by LIGO. Therefore, it is only in this regime (ie., both $\sin^2(\theta)$ and $\cos^2(\theta)$ are greater than $\frac{1}{\text{SNR}}$)that we can come up with meaningful exclusion regions for the parameters of the bimetric theory from non-observation of echoes by LIGO, as it only in this regime when both δG and δM can be detected by LIGO.

We also demand that we are in the decoherence regime, therefore we can only exclude Fierz Pauli masses such that

$$m_{\text{FP}}^2 > \frac{2\sigma\omega_0^2}{L},$$

where L can be taken as the luminosity distance to the source. For the purpose of calculating the bounds, we take σ to be a rough timescale of the gravitational wave as can be observed from the LIGO data. We can take ω_0 as the minimum frequency observed in the gravitational wave, which can sometimes be the minimum frequency LIGO is capable of observing above background noise at the time of observation.

Table 2.1 illustrates some of the mass exclusion bounds. Note that for the mass exclusion bounds, our central values differ slightly from those in [22].

GW150914 is the original first observation of gravitational waves by LIGO [26].

GW170817 [33] is interesting for having a large signal to noise ratio and thus allowing us to exclude a larger range of angles. It is also interesting for being the first observed neutron star merger.

GW190521 is from the third observing run and reported in GWTC-2.[34]. GW190521 extends the bounds from [22] very well since it has a large luminosity distance. GW190521 is also interesting from an astrophysical point of view since it is the first observed merger resulting in an intermediate mass black hole, and as such its results have also been reported in a seperate paper([35]).

If the signal length in seconds is given by t, the frequency in Hertz is given by ν and the luminosity distance is given by L(in megaparsecs), then $m_{\text{FP}}^2 = \frac{2\times t\times(2\pi\nu)^2\times(6.58\times10^{-16})}{L\times10^6\times3.086\times10^{16}\times(5.07\times10^6)}$. Note that we assume here that the luminosity distance as observed by LIGO is not modified by bimetric theory.

Note also that this analysis does not exclude large values of m_{FP}. In their plot of exclusion regions due to non-observation of echoes, [22] places the upper bound for regions obtained from the binary black hole merger observations at about 10^{-20}eV.

For large m_{FP}, the massive mode would not reach the earth in a reasonable amount of time for the echo to be observable. We present now a rough analysis of this reasoning. Recall that the group velocity of the massive mode is $\frac{k}{\omega_2} = \frac{\omega_0}{\sqrt{\omega_0^2+m_{\mathrm{FP}}^2}}$. Therefore, for a total distance of L travelled, time taken by the massless mode equals L whereas the massive mode takes time $\frac{L}{\omega_0}\sqrt{\omega_0^2+m_{\mathrm{FP}}^2}$, and therefore, the time difference between the massive and massless modes as observed on earth would be $L\times\left[\frac{\sqrt{\omega_0^2+m_{\mathrm{FP}}^2}}{\omega_0}-1\right]$. If we say that LIGO cannot observe echoes seperated by a time t, then an upper limit on the exclusion region of m_{FP} that can be obtained by this analysis would be given by $m_{\mathrm{FP}}^2 = \omega_0^2\left[\frac{t^2}{L^2}+2\frac{t}{L}\right]$ Taking for example, $\frac{t}{L} \simeq 10^{-9}$, $\omega_0 \simeq 10^{-12}\mathrm{eV}$, we get $m_{\mathrm{FP}} \simeq 10^{-17}\mathrm{eV}$.

Name	luminosity distance(Mpc)	min frequency	time scale	SNR	mass exclsn lower bnd(eV)	excluded angle ranges	redshift	sources
GW190521	5300^{+2400}_{-2600}	30	0.1	14.7	7.5×10^{-23}	$0.264 < \theta < 1.307$	$0.82^{+0.28}_{-0.34}$	[35]
GW150914	440	35	0.1	24	3.04×10^{-22}	$0.206 < \theta < 1.365$	0.09	[36][37]
GW170817	40	40	10	32.4	1.15×10^{-20}	$0.177 < \theta < 1.394$	0.01	[23] [37] [33]
GW170104	990	35	0.1	13	2.02×10^{-22}	$0.28 < \theta < 1.29$	0.18	[23] [37] [38]

Table 2.1. Summary of selected exclusion regions. Note that the uncertainities are not shown

2.7 Signal modulation

For clarity, we repeat here (2.60):

$$s^2 = \cos^4\theta\Big[1 + \tan^4\theta + 2\tan^2\theta\cos\frac{m_{\rm FP}^2\log(1+z)}{2\omega_0 H_0}\Big] \tag{2.66}$$

This is the square of the suppression of the signal for frequency ω_0. If $h(t)$ is the strain in the detector as expected by GR, and $h_{\rm bi}$ is the strain in the detector expected due to bimetric oscillations as described above, and if $\tilde{h}$ denotes the fourier transform of h, then:

$$\tilde{h}_{\rm bi}(\omega) = s(\omega)\tilde{h}(\omega) \tag{2.67}$$

$h_{bi}(t)$ is obtained by taking the Fourier transform of $\tilde{h}_{bi}(\omega)$ giving [23]:

$$h_{\rm bi}(t) = \int d\omega e^{i\omega t} s(\omega)\tilde{h}(\omega) \tag{2.68}$$

$$h_{\rm bi}(t) = \int d\omega e^{i\omega t} s(\omega)\int \frac{d\tau}{2\pi}e^{-i\omega\tau}h(\tau) = \int d\tau\tilde{s}(t-\tau)h(\tau) \tag{2.69}$$

The last equality can be seen as following from the convolution theorem. As long as the waves have not yet decohered, one would expect to see a modified signal $h_{\rm bi}(t)$ as opposed to $h(t)$. We can obtain $h(t)$ from numerical relativity or other models, and $h_{\rm bi}(t)$ can be obtained from $h(t)$ from the convolution procedure as shown above. Max et. al[21] compares $h_{\rm bi}(t)$ to $h(t)$ and uses a χ^2 analysis to get exclusion regions.

Note that it is assumed, in this section, and the previous, that the physics of emission of gravitational waves due to the binary black hole/neutron star merger is unmodified from that of GR due to the Vainshtein mechanism [23]. That is, any suppression or modulation of the gravitational wave occurs during propagation and not at creation.

2.8 Solar system bounds for graviton mass

A lot of work has been done in finding bounds for the mass of the graviton based on solar system tests. Talmadge et al [39] uses a model where the mass of the graviton causes the Netwonian potential of gravity to be modified to a Yukawa like potential of

$$V = -\frac{GM}{r} e^{-\frac{r}{\lambda}}, \tag{2.70}$$

where λ is the Compton wavelength. [22] modifies the graviton mass bounds obtained to get a bound on m_{FP} that depends on θ as:

$$m_{\mathrm{FP}} = \sin(\theta) \times (7.2 \times 10^{-23})\mathrm{eV}. \tag{2.71}$$

For bimetric theory [40] derives the following modified potential:

$$V(r) = \frac{1}{m_g^2(1+\alpha^2 c^2)} \Big(\frac{1}{r} + \frac{4\alpha^2 c^2}{3} \frac{e^{-m_{\mathrm{FP}} r}}{r} \Big) \tag{2.72}$$

In the above expression $\frac{1}{m_g^2(1+\alpha^2 c^2)}$ is a modified Newton's constant. For fixed α $m_{\mathrm{FP}} > \frac{1}{l}$ wtih $l \simeq 1\mathrm{AU}$ would be the requirement for GR to be recovered without a Vainshtein mechanism. GR can also be recovered by having $\alpha c \ll 1$. Without the Vainshtein mechanism, we require that [40]:

$$\frac{4\alpha^2}{3} e^{-m_{\mathrm{FP}} r} < 10^{-9}. \tag{2.73}$$

With $c = 1$, in terms of θ, this becomes:

$$\frac{4}{3} \tan^2(\theta) e^{-m_{\mathrm{FP}} r} < 10^{-9} \tag{2.74}$$

In bimetric theory, for very small values of m_{FP}, the Vainshtein mechanism comes into action and the potential V is the same as in GR. However these values of m_{FP} are much smaller than our exclusion bounds. Plots for these regions can be found in [40]. Outside this regime, [40] finds that for $10^{-24} \lesssim m_{\mathrm{FP}} \lesssim 10^{-16}\mathrm{eV}$ constrain α to be $\alpha \lesssim 10^{-5}$(ie., $\tan(\theta) \lesssim 10^{-5}$).

[41] plots a compilation of m_{FP} vs $c\alpha$ exclusion regions that can be obtained from various tests within the solar system.

The Vainshtein radius is given by

$$r_V = (\frac{r_S}{m_{\mathrm{FP}}^2})^{\frac{1}{3}} \tag{2.75}$$

where the modified Schwarzchild radius is given by:

$$r_S = \frac{(1+\alpha^2)M}{M_{\mathrm{Pl}}^2} \tag{2.76}$$

Note that the Vainshtein radius is different for different modified theories of gravity [16].

2.9 Gravitational waves from the early universe

Another idea, suggested by [42], relates to the possibility we will explore in the next chapter that the massive spin-2 field could be the dark matter particle. However, unlike the work by Babichev et al [43] which we review in the next chapter, which considers the massive graviton as dark matter generated by the freeze-in mechanism, [42] considers the massive graviton as dark matter generated by preheating after inflation. If this is the case, analysis of gravitational waves from this era, if detected, could provide information about the massive graviton.

2.10 Scope for future work

The exclusion bounds on echoes derived above are derived under the assumption that no echoes are present in the LIGO data. Possible future work could involve looking at the LIGO data and searching for echoes [23]. The χ^2 analysis of [23] can also be done for newer LIGO observations, especially those reported in GWTC-2 [34] by comparing the waveform of mergers in bimetric gravity and in GR.

Chapter 3

Massive spin 2 particles as dark matter

3.1 Introduction

In this chapter we briefly review part of the work by Babichev et al.[43][44] which explore the possibility that the massive spin 2 field δM of bimetric theory could account for dark matter. We explicitly derive the vertex factor for the interaction of the massive spin 2 field with Standard Model(SM) matter starting from an action for the minimal coupling of the metric g to a Standard Model scalar field ϕ and outline the account from [43] on production of the bimetric massive spin 2 field via the freeze-in mechanism.

3.2 Vertex terms from the bimetric action

[43] writes the bimetric action in a slightly different form as:

$$S = m_g^2 \int d^4x\sqrt{-g}R_g + m_f^2 \int d^4x\sqrt{-f}R_f - 2m^2 m_g^2\sqrt{-g}\int d^4xV, \tag{3.1}$$

where $m^2 = \alpha^2 m_g^2$ and,

$$V(X, \beta_n) = \sum_{n=0}^{4} \beta_n e_n(X) \tag{3.2}$$

Again, as before, let us expand around a flat background:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{\rm Pl}}, \tag{3.3}$$

$$f_{\mu\nu} = \eta_{\mu\nu} + \frac{l_{\mu\nu}}{M_{\rm Pl}}, \tag{3.4}$$

and further:

$$h_{\mu\nu} = \delta G_{\mu\nu} - \alpha \delta M_{\mu\nu}, \tag{3.5}$$

$$l_{\mu\nu} = \delta G_{\mu\nu} + \frac{1}{\alpha}\delta M_{\mu\nu}. \tag{3.6}$$

We now repeat here a general argument from [43].

They define a new background metric:

$$G_{\mu\nu} = \eta_{\mu\nu} + \frac{\delta G_{\mu\nu}}{M_{\rm Pl}}. \tag{3.7}$$

The fluctuations around the flat background can be then written as:

$$g_{\mu\nu} = G_{\mu\nu} - \frac{\alpha \delta M_{\mu\nu}}{M_{\rm Pl}}, \tag{3.8}$$

$$f_{\mu\nu} = G_{\mu\nu} + \frac{\delta M_{\mu\nu}}{\alpha M_{\rm Pl}}. \tag{3.9}$$

Let us now look at different possible fluctuations in δM of (3.1). If we take δM to be zero, then we see that the first two terms in the action, (3.1), result in an Einstein–Hilbert-like term in $G_{\mu\nu}$ and the last term results in a cosmological constant like term. (Since in this case $g^{-1}f = 1$ and thus the e_ns are just the elementary symmetric polynomials of the identity matrix.)
Let us now look at first order fluctuations in δM, ie., terms of the form $G\delta M$ etc. Any term we get with $\delta M_{\mu\nu}$ on expanding $\sqrt{g}R(g)$ will have $\delta M_{\mu\nu}$ multiplied by $\frac{-\alpha}{M_{\rm Pl}}$. Any term we get with $\delta M_{\mu\nu}$ on expanding $\sqrt{f}R(f)$ will have $\delta M_{\mu\nu}$ multiplied by $\frac{1}{M_{\rm Pl}\alpha}$, but $\sqrt{f}R(f)$ is itself multiplied by α^2 as $m_f^2 = \alpha^2 m_g^2$. Therefore, terms which are linear in δM cancel between the first two (Einstein–Hilbert-like) terms of (3.1).

Now, for the third term, we can Taylor expand it as:

$$m^2\sqrt{-g}V = m^2\Big[\sqrt{-g}V\Big]_{f=g=G} + m^2\Big[\frac{\partial(\sqrt{-g}V)}{\partial g^{\mu\nu}}\frac{\partial g^{\mu\nu}}{\partial \delta M_{\rho\sigma}} + \frac{\partial(\sqrt{-g}V)}{\partial f^{\mu\nu}}\frac{\partial f^{\mu\nu}}{\partial \delta M_{\rho\sigma}}\Big]_{f=g=G}\delta M_{\rho\sigma}, \tag{3.10}$$

However, using an expression for Λs for a background of $f_{\mu\nu} = g_{\mu\nu} = \bar{g}_{\mu\nu}$ [43],

$$\Lambda_g \bar{g}_{\mu\nu} = \Big[\frac{-2m^2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}V)}{\partial g^{\mu\nu}}\Big]_{f=g=\bar{g}}, \quad \Lambda_f \bar{g}_{\mu\nu} = \Big[\frac{-2m^2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}V)}{\partial f^{\mu\nu}}\Big]_{f=g=\bar{g}}, \tag{3.11}$$

as well as

$$m^2[\sqrt{-g}V]_{f=g=\bar{g}} = (\Lambda_g + \alpha^2\Lambda_f)\sqrt{-\bar{g}} = (1+\alpha^2)\sqrt{-\bar{g}}, \tag{3.12}$$

(3.10) can be rewritten as:

$$m^2\sqrt{-g}V = (1+\alpha^2)\Lambda\sqrt{G} + \frac{\alpha}{2m_{Pl}}(\Lambda_f - \Lambda_g)\sqrt{G}G^{\mu\nu}\delta M_{\mu\nu} \tag{3.13}$$

now using the fact that $\Lambda_f = \Lambda_g = \Lambda$ we get:

$$m^2\sqrt{-g}V = (1+\alpha^2)\Lambda\sqrt{G} \tag{3.14}$$

This shows that the $m^2\sqrt{-g}V$ term in the action does not give rise to any terms of the form $G\delta M$. Since we have already shown that such terms originating from the Einstein–Hilbert-like terms in the action will cancel each other, we can conclude that there will not be any $G\delta M$ terms upon expanding. ie., there will be no decay of the massive graviton into massless gravitons [43].Thus such terms are not included in the Feynman diagram. (Figure 3.1)

3.2.1 Validity of Perturbative Expansion

We can get some bounds by looking at when the perturbative expansion would be valid[43]. A general vertex from the interaction part of the action is of the form:

$$h^k l^n \sim \sum_{s=0}^{k}\sum_{r=0}^{n} \frac{\alpha^{s-r}}{M_{\mathrm{Pl}}^{k+n}}\delta G^{k+n-s-r}\delta M^{s+r} \tag{3.15}$$

If we take $\delta G \sim \delta M \sim E$ and drop numerical factors to get an order of magnitude estimate in terms of α:

$$h^k l^n \sim \sum_{s=0}^{k} \frac{E^{k+n}}{M_{\mathrm{Pl}}^{k+n}}\left(\alpha^{-n} + \alpha^{-n+1} + ... + \alpha^{k-1} + \alpha^k\right) \tag{3.16}$$

Ignoring numerical factors, a general vertex with m powers will have the form

$$V_m \sim \sum_{k=0}^{m} h^k l^{m-k} \sim \left(\frac{E}{\alpha M_{\mathrm{Pl}}}\right)^m \left(1 + \alpha + ... + \alpha^{2m-1} + \alpha^{2m}\right) \tag{3.17}$$

For the above expression to converge, and thus for the perturbative expansion in α to be valid, we require $\alpha < 1$ and $E < \alpha M_{\mathrm{Pl}}$ [43][44].

3.3 Coupling to matter

In general, only one of the two fields f or g can couple to a given matter field at the same time. A minimal coupling of say, g, to a standard model scalar field ϕ would look like:

$$S = \frac{1}{2}\int d^4x\sqrt{-\det(g)}[g^{\mu\nu}(\partial_\mu\phi\partial_\nu\phi) - m^2\phi^2] \tag{3.18}$$

To first order:

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}/M_{\mathrm{Pl}} \tag{3.19}$$

$$\sqrt{(-\det g)} = 1 + \frac{1}{2}\eta^{\mu\nu}h_{\mu\nu}\frac{1}{M_{\mathrm{Pl}}} + \mathcal{O}(h^2) \tag{3.20}$$

giving:

$$S = \int d^4x \frac{1}{2}(1 + \frac{1}{2}\eta^{\rho\lambda}h_{\rho\lambda})\Big[(\eta^{\mu\nu} - \frac{h^{\mu\nu}}{M_{\mathrm{Pl}}})(\partial_\mu\phi\partial_\nu\phi) - m^2\phi^2\Big] \tag{3.21}$$

$$S = \int d^4x \frac{1}{2}(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2) - \frac{h^{\mu\nu}}{2M_{\mathrm{Pl}}}(\partial_\mu\phi\partial_\nu\phi) + \frac{1}{4M_{\mathrm{Pl}}}\eta^{\rho\lambda}\eta^{\mu\nu}h_{\rho\lambda}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4M_{\mathrm{Pl}}}\eta^{\rho\lambda}h_{\rho\lambda}m^2\phi^2 \tag{3.22}$$

The first grouping of terms does not interact with gravity Let's call the second grouping (a) and have a closer look at it. Using

$$h_{\mu\nu} = \delta G_{\mu\nu} - \alpha\delta M_{\mu\nu}$$

we get:

$$\frac{h^{\mu\nu}}{2M_{\mathrm{Pl}}}(\partial_\mu\phi\partial_\nu\phi) = \frac{1}{2M_{Pl}}(\delta G^{\mu\nu} - \alpha\delta M^{\mu\nu})\partial_\mu\phi\partial_\nu\phi$$

Now, looking at the third which we shall call (b).

$$\frac{1}{4M_{\mathrm{Pl}}}\eta^{\rho\lambda}\eta^{\mu\nu}h_{\rho\lambda}\partial_\mu\phi\partial_\nu\phi = \frac{1}{4}\eta^{\rho\lambda}\eta^{\mu\nu}\frac{1}{M_{Pl}}(\delta G_{\rho\lambda} - \alpha\delta M_{\rho\lambda}))\partial_\mu\phi\partial_\nu\phi$$

Now, looking at the fourth term, which we shall call (c), we get

$$\frac{1}{4M_{\mathrm{Pl}}}\eta^{\rho\lambda}h_{\rho\lambda}m^2\phi^2 = -\frac{1}{4}\eta^{\rho\lambda}\frac{1}{M_{Pl}}(\delta G_{\rho\lambda} - \alpha\delta M_{\rho\lambda})m^2\phi^2$$

combining all terms:

$$S = S_{\mathrm{free}} + \int d^4x\Big[\frac{-\delta G_{\mu\nu}}{2M_{\mathrm{Pl}}}\partial_\mu\phi\partial_\nu\phi + \frac{1}{4M_{\mathrm{Pl}}}\eta^{\rho\lambda}\eta^{\mu\nu}\delta G_{\rho\lambda}\partial_\mu\phi\partial_\nu\phi - \frac{\eta^{\rho\lambda}}{4M_{\mathrm{Pl}}}\delta G_{\rho\lambda}m^2\phi^2 + \alpha\frac{\delta M_{\mu\nu}}{2M_{\mathrm{Pl}}}\partial_\mu\phi\partial_\nu\phi - \alpha\frac{\eta^{\rho\lambda}\eta^{\mu\nu}}{4M_{\mathrm{Pl}}}\delta M_{\rho\lambda}\partial_\mu\phi\partial_\nu\phi + \frac{\alpha}{4M_{\mathrm{Pl}}}\eta^{\rho\lambda}\delta M_{\rho\lambda}m^2\phi^2\Big] \tag{3.23}$$

The instructive thing to note above is that the terms representing the vertices between the massive mode of the graviton and the scalar matter is suppressed by a factor of α.(compared to the coupling between the massless mode and scalar matter)

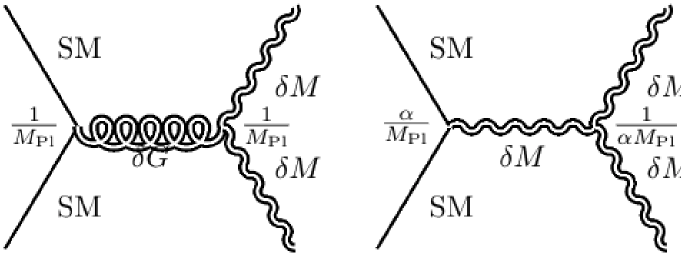

Figure 3.1. Feynman diagrams for production of δM from standard model mattter.

3.4 The massive mode as dark matter

There are multiple mechanisms to explain the abundance of dark matter in the universe. According to the freeze-out mechanism, dark matter was abundant in the early universe and in thermal equilibirum with standard model matter. However, as the universe expands and the Hubble rate increases, and the temperature of the universe decreases, the Hubble rate dominates over the interaction of dark matter and standard model matter, and dark matter no longer decays into standard model matter.
The Freeze-in mechanism consists of a massive particle which is more massive than other particles. The abundance of this particle is also very small at early times. This massive particle is produced from standard model matter with which it interacts feebly, and the production of this particle dominates as the temperature becomes lower than the mass of the particle[45].

[43] looks at the freeze-in mechanism as a potential generation mechanism for dark matter comprised of the massive spin 2 field of bimetric gravity.

[43] proposes that for very small α and large m_{FP} the massive particle of bimetric gravity is a good candidate to be this FIMP(Feebly Interacting Massive Particle). The production of δM occur via two s-channel Feynman diagrams, with one being mediated by δG and another being mediated by δM. The vertex factor between SM and δG is of course $\frac{1}{M_{\mathrm{Pl}}}$. As we have derived earlier, between SM and δM, the vertex factor is proportional to $\frac{\alpha}{M_{\mathrm{Pl}}}$. [43] derives the largest vertex factor for δM^3 to be proportional to $\frac{1}{\alpha M_{\mathrm{Pl}}}$.

From the consideration that the perturbative expansion be valid, as well as experimental considerations [43] places a limit on the mass as:

$$1\ \mathrm{TeV} \lesssim m_{\mathrm{FP}} \lesssim 66\ \mathrm{TeV} \tag{3.24}$$

Chapter 4

Summary and conclusions

In this thesis we began by motivating and outlining the ghost free bimetric theory of gravity as proposed by Hassan and Rosen.

Then, following work in [21][22][23], we outlined the phenomenon of gravitational wave oscillations that would occur in bimetric gravity. We extend the lower bound on an exclusion region for the Fierz Pauli mass that can be obtained this way using the non observation of echoes from the binary black hole merger GW190521. It is found that this particular non observation extends the lower limit of the mass exclusion bound in[22] to around 7.5×10^{-23}eV for mixing angle θ between 0.26 and 1.3. This region overlaps with the exclusion region reported in [40].

We also briefly present some results from a paper by Babichev et.al[43] which explores the massive spin 2 field of bimetric theory as a candidate for dark matter.

Appendix A

List of constants

(Reduced)Planck mass	2.4×10^{27}eV
Hubble constant H	70km$(\mathrm{s}^{-1})(\mathrm{Mpc})^{-1} = 1.5 \times 10^{-33}$eV
Energy density from CMB	$2 \times 10^{-15}\mathrm{eV}^4$
1 parsec	3.086×10^{16}m
Newtonian Gravitational Constant G	$6.7 \times 10^{-57}\mathrm{eV}^{-2}$

Table A.1. List of selected constants

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