MMAN2300 Engineering Mechanics 2 – T2 2022

EQUATION SHEET FOR PART B

Equation of motion for SDOF free vibration

Translational DOF (e.g. x or y)

$$m^*\ddot{\tilde{x}} + c^*\dot{\tilde{x}} + k^*\tilde{x} = 0$$

Rotational DOF (e.g. θ)

$$I^*\ddot{\tilde{\theta}} + c_t^*\dot{\tilde{\theta}} + k_t^*\tilde{\theta} = 0$$

Solution of free vibration equation of motion

- Natural frequency and damping ratio

Translational DOF

$$\omega_n = \sqrt{\frac{k^*}{m^*}}$$

$$\zeta = \frac{c^*}{2m^*\omega_n}$$

Rotational DOF

$$\omega_n = \sqrt{\frac{k_t^*}{I^*}}$$

$$\zeta = \frac{c_t^*}{2I^*\omega_n}$$

- Solution

Case	Vibration type	Formula
$\zeta = 0$	Undamped	$x(t) = A\cos(\omega_n t + \psi)$
$0 < \zeta < 1$	Underdamped	$x(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t + \psi)$ with $\omega_d = \omega_n\sqrt{1-\zeta^2}$
$\zeta = 1$	Critically damped	$x(t) = (A_1 + A_2 t)e^{-\zeta\omega_n t}$
$\zeta > 1$	Overdamped	$x(t) = A_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + A_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$

- Remaining terms (initial conditions)

Case	Formula	
$\zeta = 0$	$A = \sqrt{\tilde{x}(0)^2 + \left(\frac{\dot{\tilde{x}}(0)}{\omega_n}\right)^2}$	$\psi = -\tan^{-1}\left(\frac{\dot{\tilde{x}}(0)}{\omega_n\tilde{x}(0)}\right)$ (*)
$0 < \zeta < 1$	$A = \sqrt{\tilde{x}(0)^2 + \left[\frac{\zeta\omega_n\tilde{x}(0) + \dot{\tilde{x}}(0)}{\omega_d}\right]^2}$	$\psi = -\tan^{-1}\left(\frac{\zeta\omega_n\tilde{x}(0)+\dot{\tilde{x}}(0)}{\omega_d\tilde{x}(0)}\right)$ (**)
$\zeta = 1$	$A_1 = \tilde{x}(0)$	$A_2 = \dot{\tilde{x}}(0) + \omega_n\tilde{x}(0)$
$\zeta > 1$	$A_1 = \frac{\dot{\tilde{x}}(0) + \omega_n\tilde{x}(0)\left(\zeta + \sqrt{\zeta^2-1}\right)}{2\omega_n\sqrt{\zeta^2-1}}$	$A_2 = \frac{\dot{\tilde{x}}(0) + \omega_n\tilde{x}(0)\left(\zeta - \sqrt{\zeta^2-1}\right)}{2\omega_n\sqrt{\zeta^2-1}}$

(*) note that $\psi = -\tan^{-1}\left(\frac{\dot{\tilde{x}}(0)}{\omega_n\tilde{x}(0)}\right) + \pi$ if $\tilde{x}(0) < 0$

(**) note that $\psi = -\tan^{-1}\left(\frac{\zeta\omega_n\tilde{x}(0)+\dot{\tilde{x}}(0)}{\omega_d\tilde{x}(0)}\right) + \pi$ if $\tilde{x}(0) < 0$

Analysis of free vibration solution

- Period in undamped vibration $\tau = \frac{2\pi}{\omega_n}$
- Pseudo-period in damped vibration $\tilde{\tau} = \frac{2\pi}{\omega_d}$
- Frequencies in Hz: $f_n = \frac{\omega_n}{2\pi}$ and $f_d = \frac{\omega_d}{2\pi}$

System identification

Based on peaks at $t_1, \tilde{x}_1$ and $t_{N+1}, \tilde{x}_{N+1}$ with N number of periods in between

- Damped natural frequency $\omega_d = \frac{2\pi N}{t_{N+1} - t_1}$
- Logarithmic decrement $\delta = \frac{1}{N}\log\left(\frac{\tilde{x}_1}{\tilde{x}_{N+1}}\right)$ (note that $\log(\dots)$ represents the natural logarithm)
- Damping ratio $\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}$

Equation of motion for SDOF forced vibration

Translational DOF (e.g. x or y)

$$m^*\ddot{\tilde{x}} + c^*\dot{\tilde{x}} + k^*\tilde{x} = F^*$$

with $F^* = F_0^* \cos(\omega t + \phi_0^*)$ if harmonic force

Rotational DOF (e.g. θ)

$$I^*\ddot{\tilde{\theta}} + c_t^*\dot{\tilde{\theta}} + k_t^*\tilde{\theta} = M^*$$

with $M^* = M_0^* \cos(\omega t + \phi_0^*)$ if harmonic moment

Solution of forced vibration equation of motion

Solution as sum of particular and homogeneous components

$$\tilde{x}(t) = \tilde{x}_h(t) + \tilde{x}_p(t) \quad \text{or} \quad \tilde{\theta}(t) = \tilde{\theta}_h(t) + \tilde{\theta}_p(t)$$

- Particular solution

  In case of translational DOFs (e.g. x or y)

$$\tilde{x}_p(t) = X\cos(\omega t + \phi)$$

$$X = \frac{F_0^*/k^*}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} \quad \text{and} \quad \phi = \phi_0^* - \tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right)$$

  In case of rotational DOF (e.g. θ)

$$\tilde{\theta}_p(t) = \Theta\cos(\omega t + \phi)$$

$$\Theta = \frac{M_0^*/k_t^*}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} \quad \text{and} \quad \phi = \phi_0^* - \tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right)$$

note that for the phase ϕ (for both translational and rotational DOFs) an angle π must be added to the result of the arctangent function in case of negative denominator in the argument.

Analysis of SDOF forced vibration solution

Magnitude of the Frequency Response Function

$$|FRF| = \frac{X}{F_0^*} = \frac{1/k^*}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

Magnification factor

$$MF = \frac{X}{|X(\omega=0)|} = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

Phase difference (angle of the FRF)

$$\Delta\phi = \angle FRF = \phi - \phi_0^* = -\tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)$$

note that for the phase difference an angle π must be added to the result of the arctangent function in case of negative denominator in the argument

Unbalance Force

- Horizontal component: $F_{unb,h} = m_\varepsilon r_\varepsilon \omega^2 \cos(\omega t + \phi_\theta)$
- Vertical component: $F_{unb,v} = m_\varepsilon r_\varepsilon \omega^2 \sin(\omega t + \phi_\theta)$

with $m_\varepsilon r_\varepsilon$ often represented as $m_\varepsilon r_\varepsilon = u_\varepsilon$

2DOF free vibration

- Stability: need to verify that $\frac{\partial^2 V}{\partial x_1^2} > 0$, $\frac{\partial^2 V}{\partial x_2^2} > 0$ and $\frac{\partial^2 V}{\partial x_1^2}\frac{\partial^2 V}{\partial x_2^2} - \left(\frac{\partial^2 V}{\partial x_1 \partial x_2}\right)^2 > 0$

  (x_1 and x_2 represent the general 2 DOFs used to describe the system)
- Eq. of motion $\mathrm{M}\ddot{\mathbf{x}} + \mathrm{C}\dot{\mathbf{x}} + \mathrm{K}\mathbf{x} = \mathbf{0}$
- Natural frequencies found solving:

$$\det(\mathrm{K} - \omega_n^2\mathrm{M}) = 0$$

- Determinant of a 2x2 matrix

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a \cdot d - b \cdot c$$

- Mode shapes $\boldsymbol{\chi} = \begin{bmatrix} 1 \\ \chi \end{bmatrix}$ found solving any row of the system:

$$[\mathrm{K} - \omega_n^2\mathrm{M}]\begin{bmatrix} 1 \\ \chi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2DOF forced vibration

- Eq. of motion $\mathrm{M}\ddot{\mathbf{x}} + \mathrm{C}\dot{\mathbf{x}} + \mathrm{K}\mathbf{x} = \mathbf{F}$ with $\mathbf{F} = \mathbf{F_0}\cos(\omega t)$
- Particular solution $\mathbf{x}_p(t) = \mathbf{X}\cos(\omega t)$, with vector $\mathbf{X}$ found (neglecting damping) as:

$$\mathbf{X} = [\mathrm{K} - \omega^2\mathrm{M}]^{-1}\,\mathbf{F_0}$$

- Inverse of a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Vibration absorbers

- Absorber's frequency $\omega_a = \sqrt{k_a/m_a}$