## 11. Free 2DOF and MDOF vibration

So far we have seen SDOF system, which are systems that vibrate only in one way and are fully described by a single coordinate. Reality is however often more complex and shows multiple degrees of freedom. In this section we will first start from a simple example of 2DOF system and then expound the general procedure for a MDOF system.

### *11.1. Finding the equations of motion in a simple 2DOF example*

Let's take the typical example of a mass-spring-damper and repeat it twice as in Figure 35, and assume that $m_1 = m_2 = 2$ kg, $c_1 = c_2 = 1$ Ns/m and $k_1 = k_2 = 200$ N/m. We want to find the system's natural frequencies.

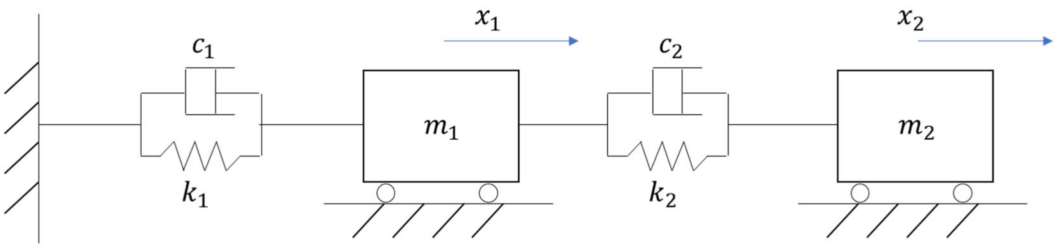



*Figure 35. Simple two-degree-of-freedom system and its FBD.*

1. We start as usual with the FBDs

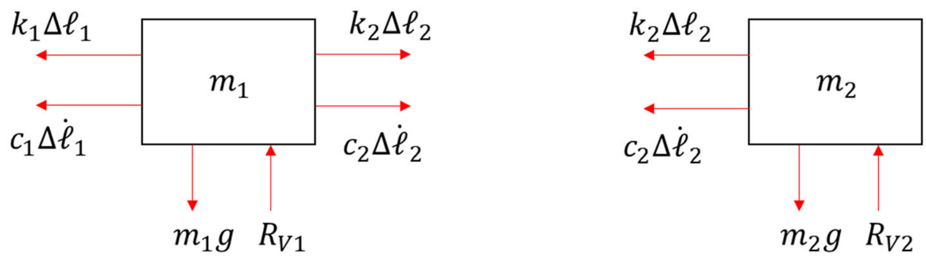



*Figure 36. Free body diagrams for example above*

2. We then write Newton's law for each body

$$\begin{gathered} m_1\ddot{x}_1 = -k_1\Delta\ell_1 - c_1\Delta\dot{\ell}_1 + k_2\Delta\ell_2 + c_2\Delta\dot{\ell}_2 \\ m_1\ddot{x}_2 = -k_2\Delta\ell_2 - c_2\Delta\dot{\ell}_2 \end{gathered} \tag{85}$$

3. There is no reaction force to be removed
4. Kinematics is a key step, as it shows the key dependency of kinematic quantities on <u>two</u> variables ($x_1$ and $x_2$) rather than just one as in SDOF systems. As usual remember the convention of $\Delta\ell$ positive in extension.

$$\begin{gathered} \Delta\ell_1 = x_1 \\ \Delta\ell_2 = x_2 - x_1 \end{gathered} \tag{86}$$

To understand the effect of the two coordinates on the kinematics see the graphical representation of Figure 37.

- Positive $x_1$ and $\dot{x}_1$ make spring/damper 1 longer (forces towards the centre of the spring/damper) and spring/damper 2 shorter (forces outwards from the centre of the spring/damper)
- Positive $x_2$ and $\dot{x}_2$ have no effect on spring 1 and make spring 2 longer (forces towards the centre of the spring/damper)

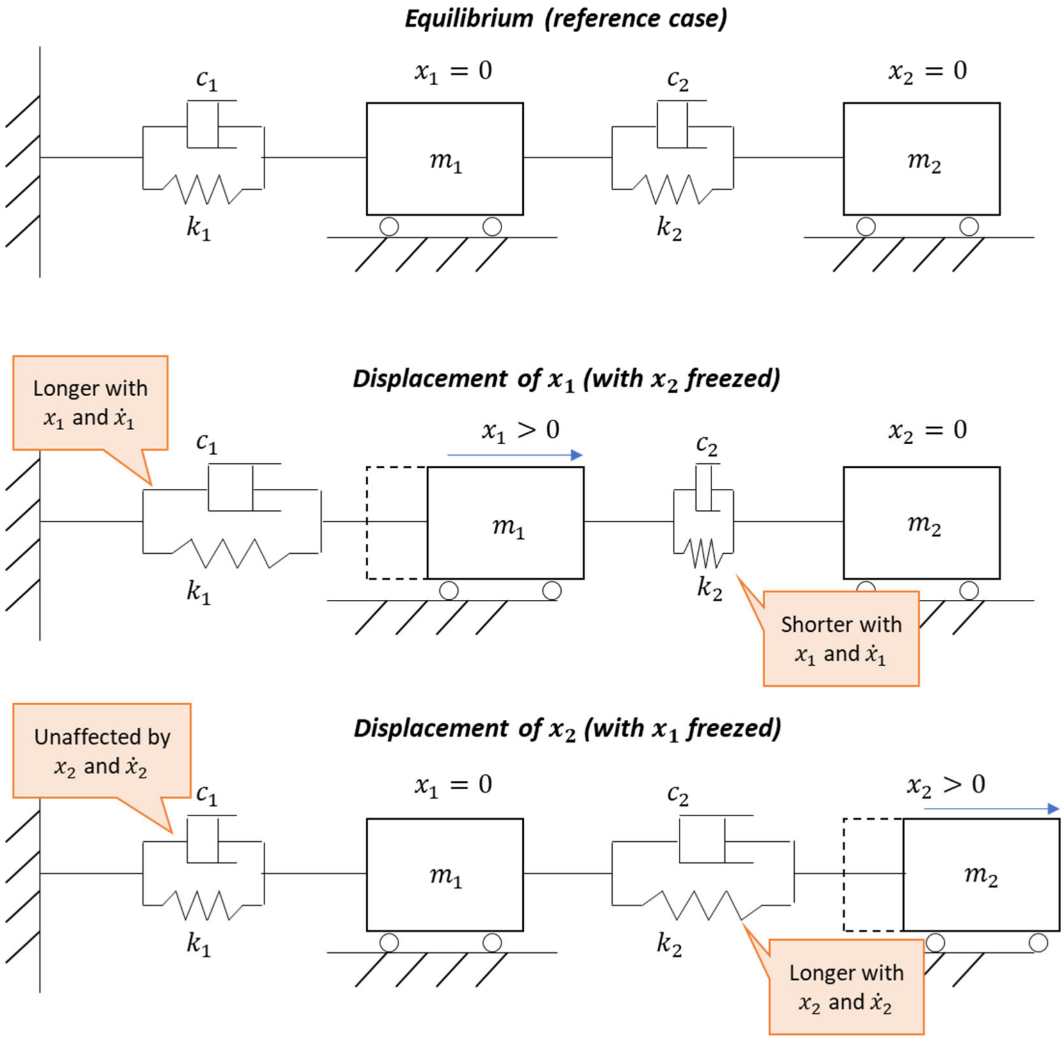



*Figure 37. Effect of displacements (and velocities) on springs (and dampers).*

Now we can rewrite Newton's law for each mass in horizontal direction:

$$\begin{gathered} m_1\ddot{x} = -k_1x_1 - c_1\dot{x}_1 + k_2x_2 - k_2x_1 + c_2\dot{x}_2 - c_2\dot{x}_1 \\ m_2\ddot{x}_2 = -k_2x_2 + k_2x_1 - c_2\dot{x}_2 + c_2\dot{x}_1 \end{gathered} \tag{87}$$

Rearranging:

$$\begin{gathered} m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 + (k_1 + k_2)x_1 - c_2\dot{x}_2 - k_2x_2 = 0 \\ m_2\ddot{x}_2 + k_2x_2 - k_2x_1 + c_2\dot{x}_2 - c_2\dot{x}_1 = 0 \end{gathered} \tag{88}$$

Since $x_1$ and $x_2$ is already a stable equilibrium position and all equations are linear, I do not need to change variables into "tilde"-ones, and the last equations can be rearranged in matrix form remembering the row-column rule for the product of a matrix and a vector:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{89}$$

Which can be re-written in the compact form

$$\mathrm{M}\ddot{\mathbf{x}} + \mathrm{C}\dot{\mathbf{x}} + \mathrm{K}\mathbf{x} = \mathbf{0} \tag{90}$$

where M is the mass matrix

$$\mathrm{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ kg} \tag{91}$$

C is the damping matrix

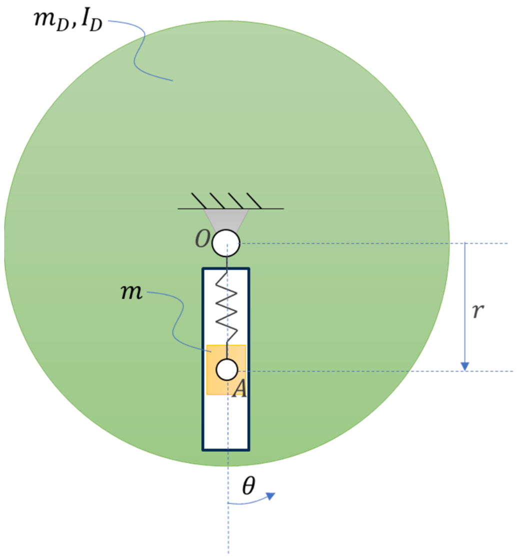
$$\mathrm{C} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \frac{\mathrm{Ns}}{\mathrm{m}} \tag{92}$$

and K is the stiffness matrix, and $\mathbf{x}$ represents the vector of coordinates $\mathbf{x} = [x_1\ x_2]^{\mathrm{T}}$.

$$\mathrm{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} 400 & -200 \\ -200 & 200 \end{bmatrix} \frac{\mathrm{N}}{\mathrm{m}} \tag{93}$$

***Example 8. Initial example***

In the first section of this document, we have seen the following example




$m = 1$ kg
$m_D = 10$ kg
$I_D = 1$ kg·m$^2$
$k = 400$ N/m
$\ell_0 = 5$ cm
$\ell_0$ is the length of the spring when unstretched
Consider $m$ a point mass, with all mass concentrated in point $A$
The centre of mass of disc in its centre $O$
Use $\theta$ and $r$ as d.o.f.

and have derived the equations of motion

$$\begin{cases} m\dot{\theta}^2 r - m\ddot{r} = -mg\cos\theta + kr - k\ell_0 \\ I_D\ddot{\theta} = -m\ddot{\theta}r^2 - 2mr\dot{\theta}\dot{r} - mgr\sin\theta \end{cases}$$

Now we want to find the mass, damping and stiffness matrix, for vibration around a stable equilibrium position (if any).

<u>*Equilibrium*</u>

First of all we find equilibrium by substituting $\ddot{\theta} = 0$, $\ddot{r} = 0$, $\dot{r} = 0$ and $\dot{\theta} = 0$. We then call $\theta = \theta_{eq}$ and $r = r_{eq}$

$$\begin{cases} 0 = -mg\cos\theta_{eq} + kr_{eq} - k\ell_0 \\ 0 = -mgr_{eq}\sin\theta_{eq} \end{cases}$$

The second one gives either

$$r_{eq} = 0 \quad \text{or} \quad \theta_{eq} = 0 \quad \text{or} \quad \theta_{eq} = \pi$$

The case $r_{eq} = 0$, substituted in the first equation, does not lead any feasible $\theta_{eq}$, since

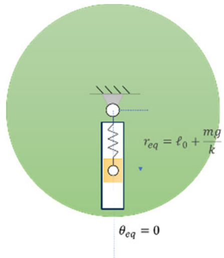
$$\theta_{eq} = \pm\cos^{-1}\left(-\frac{k\ell_0}{mg}\right) = \pm\cos^{-1}(-2.039)$$

and therefore is not an equilibrium position.

Substituting $\theta_{eq} = 0$ and $\theta_{eq} = \pi$ in the first equation we get two equilibrium positions.

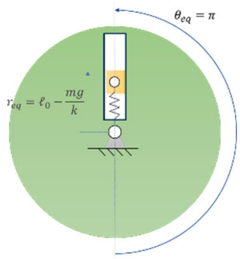
In case of $\theta_{eq} = 0$, the first equation becomes $0 = -mg + kr_{eq} - k\ell_0$, thus giving the equilibrium position:

$$r_{eq} = \ell_0 + \frac{mg}{k} = 0.07452 \text{ m} \quad \text{and} \quad \theta_{eq} = 0$$




In case of $\theta_{eq} = \pi$, the first equation becomes $0 = mg + kr_{eq} - k\ell_0$, thus giving the equilibrium position:

$$r_{eq} = \ell_0 - \frac{mg}{k} = 0.02548 \text{ m} \quad \text{and} \quad \theta_{eq} = \pi$$




*<u>Stability</u>*

The potential energy is the sum of the elastic energy of the spring and the gravitational potential energy of the mass, whose centre of gravity moves with the two dofs.

$$V = \frac{1}{2}k\Delta\ell^2 + mgh_A = \frac{1}{2}k(r - \ell_0)^2 - mr\cos\theta$$

Its first derivatives are:

$$\frac{\partial V}{\partial r} = k(r - \ell_0) - m\cos\theta \quad \text{and} \quad \frac{\partial V}{\partial \theta} = mr\sin\theta$$

The second derivatives are:

$$\frac{\partial^2 V}{\partial r^2} = k = 400\ \frac{\text{N}}{\text{m}}$$

$$\frac{\partial^2 V}{\partial \theta^2} = mr\cos\theta$$

$$\frac{\partial^2 V}{\partial \theta \partial r} = \frac{\partial^2 V}{\partial r \partial \theta} = m\sin\theta$$

Giving a determinant of the Hessian matrix:

$$\det(\mathrm{H}) = \frac{\partial^2 V}{\partial r^2}\frac{\partial^2 V}{\partial \theta^2} - \left[\frac{\partial^2 V}{\partial \theta \partial r}\right]^2 = k\ mr\cos\theta - [m\sin\theta]^2$$

Let’s check stability for our two equilibrium positions:

$$r_{eq} = \ell_0 + \frac{mg}{k} \quad \text{and} \quad \theta_{eq} = 0$$

$$\left.\frac{\partial^2 V}{\partial r^2}\right|_{r_{eq},\theta_{eq}} = k > 0$$

$$\det(\mathrm{H}) = km\left(\ell_0 + \frac{mg}{k}\right) = 29.81 > 0$$

STABLE

$$r_{eq} = \ell_0 - \frac{mg}{k} \quad \text{and} \quad \theta_{eq} = \pi$$

$$\left.\frac{\partial^2 V}{\partial r^2}\right|_{r_{eq},\theta_{eq}} = k > 0$$

$$\det(\mathrm{H}) = km\left(\frac{mg}{k} - \ell_0\right) = -10.19 < 0$$

UNSTABLE

So the only stable equilibrium position is around:

$$r_{eq} = \ell_0 + \frac{mg}{k} \quad \text{and} \quad \theta_{eq} = 0$$

*Linearisation*

First of all we decompose our dofs into equilibrium + vibration:

$$\tilde{r} = r - r_{eq} \quad \text{and} \quad \tilde{\theta} = \theta - \theta_{eq}$$

So we rewrite in our equations of motion $\theta = \theta_{eq} + \tilde{\theta}$ and $r = \tilde{r} + r_{eq}$

$$\begin{cases} m\dot{\tilde{\theta}}^2(\tilde{r} + r_{eq}) - m\ddot{\tilde{r}} = k(\tilde{r} + r_{eq}) - k\ell_0 - mg\cos(\theta_{eq} + \tilde{\theta}) \\ I_D\ddot{\tilde{\theta}} = -m\ddot{\tilde{\theta}}(\tilde{r} + r_{eq})^2 - 2mr\dot{\tilde{\theta}}\dot{\tilde{r}} - mg(\tilde{r} + r_{eq})\sin(\theta_{eq} + \tilde{\theta}) \end{cases}$$

Now we can identify all non-linear term an linearise them:

- $m\dot{\tilde{\theta}}^2(\tilde{r} + r_{eq}) \approx 0$ because it contains only cubic and quadratic terms only
- $mg\cos(\theta_{eq} + \tilde{\theta}) \approx mg[\cos(\theta_{eq}) - \sin(\theta_{eq})\,\tilde{\theta}] = mg$
- $m\ddot{\tilde{\theta}}(\tilde{r} + r_{eq})^2 = m\ddot{\tilde{\theta}}(\tilde{r}^2 + r_{eq}^2 + 2r_{eq}\tilde{r}) \approx m\ddot{\tilde{\theta}}r_{eq}^2 = m\ddot{\tilde{\theta}}\left(\ell_0 + \frac{mg}{k}\right)^2$
- $2mr\dot{\tilde{\theta}}\dot{\tilde{r}} \approx 0$ because it is a cubic term
- $mg(\tilde{r} + r_{eq})\sin(\theta_{eq} + \tilde{\theta}) \approx mg\sin(\theta_{eq})\,\tilde{r} - mg(r_{eq})\cos(\theta_{eq})\,\tilde{\theta} = mg\left(\ell_0 + \frac{mg}{k}\right)\tilde{\theta}$

We can then substitute all non-linear terms with the linearised form:

$$\begin{cases} -m\ddot{\tilde{r}} = k\left(\tilde{r} + \ell_0 + \frac{mg}{k}\right) - k\ell_0 - mg \\ I_D\ddot{\tilde{\theta}} = -m\ddot{\tilde{\theta}}\left(\ell_0 + \frac{mg}{k}\right)^2 - mg\left(\ell_0 + \frac{mg}{k}\right)\tilde{\theta} \end{cases}$$

Cancelling out all equal and opposite terms (all due to equiliubrium), we obtain:

$$\begin{cases} -m\ddot{\tilde{r}} = k\tilde{r} \\ I_D\ddot{\tilde{\theta}} = -m\ddot{\tilde{\theta}}\left(\ell_0 + \frac{mg}{k}\right)^2 - mg\left(\ell_0 + \frac{mg}{k}\right)\tilde{\theta} \end{cases}$$

*Rearrange and identify matrices*

Rearraging:

$$\begin{cases} m\ddot{\tilde{r}} + k\tilde{r} = 0 \\ \left(I_D + \left(\ell_0 + \frac{mg}{k}\right)^2 m\right)\ddot{\tilde{\theta}} + mg\left(\ell_0 + \frac{mg}{k}\right)\tilde{\theta} = 0 \end{cases}$$

We can now rewrite it in matrix form as:

$$\begin{bmatrix} m & 0 \\ 0 & I_D + \left(\ell_0 + \frac{mg}{k}\right)^2 m \end{bmatrix} \begin{bmatrix} \ddot{\tilde{r}} \\ \ddot{\tilde{\theta}} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{r}} \\ \dot{\tilde{\theta}} \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & mg\left(\ell_0 + \frac{mg}{k}\right) \end{bmatrix} \begin{bmatrix} \tilde{r} \\ \tilde{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

thus identifying the mass, damping and stiffness matrices as:

$$\mathrm{M} = \begin{bmatrix} m & 0 \\ 0 & I_D + \left(\ell_0 + \frac{mg}{k}\right)^2 m \end{bmatrix} = \begin{bmatrix} 1\ \mathrm{kg} & 0 \\ 0 & 1.005\ \mathrm{kg \cdot m^2} \end{bmatrix}$$

$$\mathrm{C} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathrm{K} = \begin{bmatrix} k & 0 \\ 0 & mg\left(\ell_0 + \frac{mg}{k}\right) \end{bmatrix} = \begin{bmatrix} 400\,\frac{\mathrm{N}}{\mathrm{m}} & 0 \\ 0 & 0.7311\ \frac{\mathrm{N \cdot m}}{\mathrm{rad}} \end{bmatrix}$$

### *11.2. Natural frequencies and mode shapes for the corresponding undamped example*

The solution of a damped system such as the one found in the previous case is in general complex, but considering that we are often dealing with low-damping systems, it is interesting to study the undamped case. To do that for the example of the previous section, we simplify the system of Figure 35 neglecting damping, and obtaining

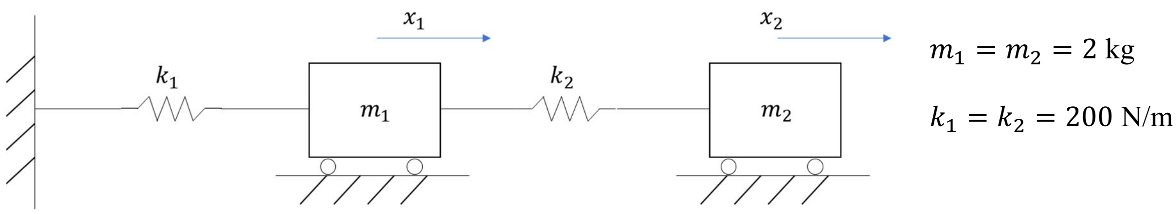



*Figure 38. Simple example of an undamped 2DOF system.*

This system results in the following equations of motion (the same as in the previous section but without damping).

$$\begin{gathered} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0 \\ m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = 0 \end{gathered} \tag{94}$$

Putting them in matrix form, we obtain:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{95}$$

where the mass matrix is

$$\mathrm{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\ \mathrm{kg} \tag{96}$$

and the stiffness matrix K is

$$\mathrm{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} 400 & -200 \\ -200 & 200 \end{bmatrix} \frac{\mathrm{N}}{\mathrm{m}} \tag{97}$$

In general we study the undamped equation of motion

$$\mathrm{M}\ddot{\mathbf{x}} + \mathrm{K}\mathbf{x} = \mathbf{0} \tag{98}$$

which has solution:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A_{n1}\boldsymbol{\chi}_{n1}\cos(\omega_{n1}t + \psi_{n1}) + A_{n2}\boldsymbol{\chi}_{n2}\cos(\omega_{n2}t + \psi_{n2}) \tag{99}$$

In such solution there are three sets of unknowns:

- Natural frequencies $\omega_{n1}$ and $\omega_{n2}$ in rad/s
- Mode shapes $\boldsymbol{\chi}_{n1}$ and $\boldsymbol{\chi}_{n2}$, which are vectors where the first element is 1 and the second is $\chi$
  $$\boldsymbol{\chi}_{n1} = \begin{bmatrix} 1 \\ \chi_{n1} \end{bmatrix} \quad \boldsymbol{\chi}_{n2} = \begin{bmatrix} 1 \\ \chi_{n2} \end{bmatrix} \tag{100}$$
  thus only leaving two scalars ($\chi_{n1}$ and $\chi_{n2}$) to be determined
- The amplitudes $A_{n1}$ and $A_{n2}$ and phases $\psi_{n1}$ and $\psi_{n2}$ (depending on initial conditions)

<u>Natural frequencies</u>

Natural frequencies are found by setting the determinant of the matrix $\mathrm{K} - \omega_n^2\mathrm{M}$ to zero:

$$\det(\mathrm{K} - \omega_n^2\mathrm{M}) = 0 \tag{101}$$

In our example case

$$\det\left(\begin{bmatrix} 400 & -200 \\ -200 & 200 \end{bmatrix} - \omega_n^2 \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = 0 \tag{102}$$

thus

$$\det\left(\begin{bmatrix} 400 - 2\omega_n^2 & -200 \\ -200 & 200 - 2\omega_n^2 \end{bmatrix}\right) = 0 \tag{103}$$

The determinant of a 2x2 matrix is obtained as the product of the diagonal elements minus the product of the extra-diagonal elements, following the rule

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc \tag{104}$$

Thus giving in our case

$$(400 - 2\omega_n^2)(200 - 2\omega_n^2) - 200^2 = 0 \tag{105}$$

which leads to:

$$4\omega_n^4 - 1200\omega_n^2 + 40{,}000 = 0 \tag{106}$$

which is a biquadratic equation with solutions:

$$\omega_n^2 = \frac{1200 \pm \sqrt{1200^2 - 4 \cdot 4 \cdot 40{,}000}}{2 \cdot 4} = \begin{cases} 38.20\,\dfrac{\mathrm{rad}^2}{\mathrm{s}^2} \\ 261.8\,\dfrac{\mathrm{rad}^2}{\mathrm{s}^2} \end{cases} \tag{107}$$

Thus leading to:

$$\omega_n = \begin{cases} \sqrt{63.4} = 6.18\,\dfrac{\mathrm{rad}}{\mathrm{s}} \\ \sqrt{236.6} = 16.18\,\dfrac{\mathrm{rad}}{\mathrm{s}} \end{cases} \tag{108}$$

<u>Mode shapes</u>

The mode shapes are found by solving any of the two equations of the following system

$$[\mathrm{K} - \omega_n^2 \mathrm{M}] \begin{bmatrix} 1 \\ \chi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{109}$$

Which in our example case is:

$$\begin{bmatrix} 400 - 2\omega_n^2 & -200 \\ -200 & 200 - 2\omega_n^2 \end{bmatrix} \begin{bmatrix} 1 \\ \chi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{110}$$

Using (for instance) the first equation (first row of the matrix) we get

$$400 - 2\omega_n^2 - 200\chi = 0 \tag{111}$$

Which gives

$$\chi = \frac{400 - 2\omega_n^2}{200} \tag{112}$$

The second row (second equation) would give us the same result (this is a consequence of setting the determinant of $\mathrm{K} - \omega_n^2\mathrm{M}$ to 0) and in practice we can use either of the two equations/rows.

Substituting the first natural frequency in the equation we found, we get:

$$\chi_{n1} = \frac{400 - 2 \cdot 6.18^2}{200} = 1.618 \tag{113}$$

whereas using the second frequency we get

$$\chi_{n2} = \frac{400 - 2 \cdot 16.18^2}{200} = -0.618 \tag{114}$$

This mams that the two mode shapes will be:

$$\chi_{n1} = \begin{bmatrix} 1 \\ 1.168 \end{bmatrix} \text{ for the natural frequency } \omega_{n1} = 6.18\,\frac{\text{rad}}{\text{s}}$$
$$\chi_{n2} = \begin{bmatrix} 1 \\ -0.618 \end{bmatrix} \text{ for the natural frequency } \omega_{n2} = 16.18\,\frac{\text{rad}}{\text{s}} \tag{115}$$

These mode shapes represent the amount of displacement of $x_2$ with respect to $x_1$ at the two different natural frequencies, i.e. we know that the vibration component at $\omega_{n1}$ will result in a ratio of displacements $\frac{x_2}{x_1} = \chi_{n1}$, and that the component at $\omega_{n2}$ will be characterised by a ratio of displacements $\frac{x_2}{x_1} = \chi_{n2}$. This means that, whereas actual amplitudes (and phases which we have not considered) will depend on initial conditions, the component of vibration with frequency $\omega_{n1}$ will in our case show a displacement of $x_2 = 1.618x_1$ at any time, and the component at $\omega_{n2}$ will show a displacement $x_2 = -0.618x_1$ at any time. This can be represent graphically as follows:

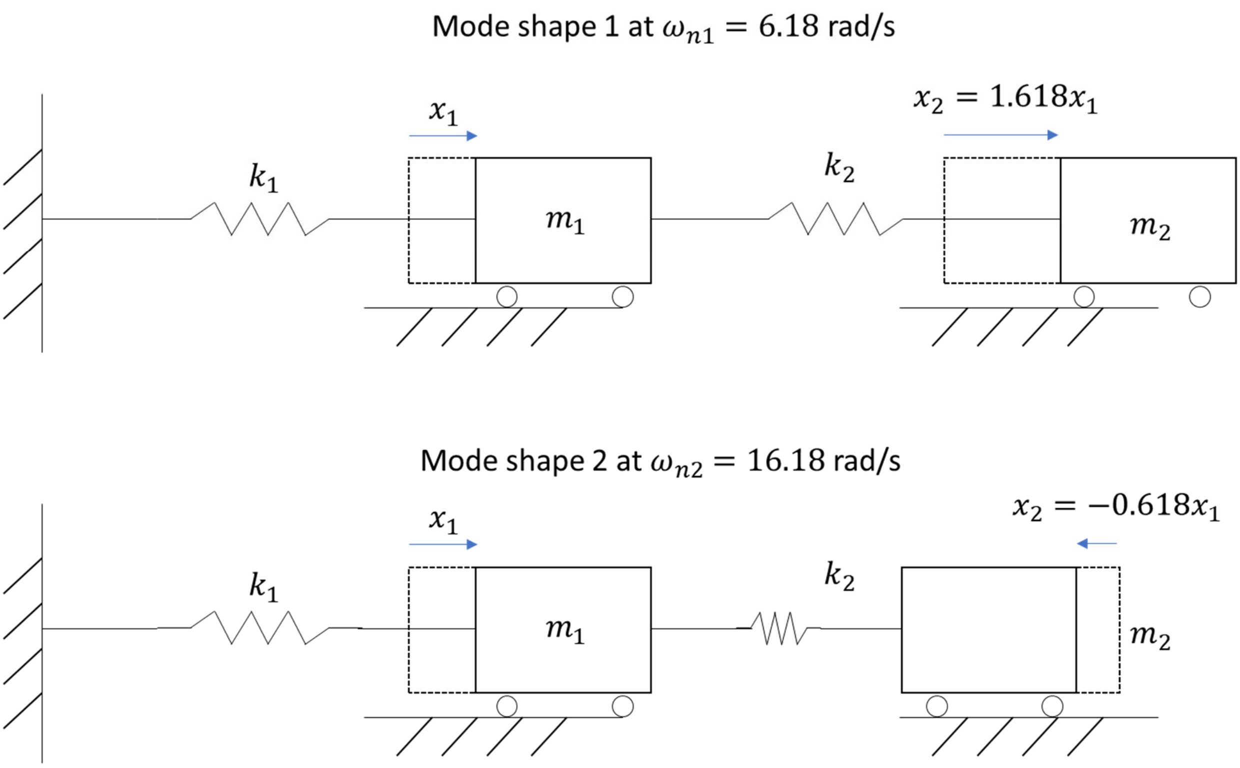



*Figure 39. Graphical representation of mode shapes for the example of this section*

### *11.3. General multi-degree-of-freedom (MDOF) system*

In a general MDOF system I will have $N$ coordinates (2 in a 2DOF) which can move <u>independently from each other</u>. A $N$-DOF system will have $N$ natural frequencies.

In this case I will be able to:

1. Draw the free body diagrams of all bodies in the system
2. Write Newton's law for each (the number of equations could be $\geq N$)
3. Use the excess equations to get rid of internal forces (additional unknowns), and thus obtain $N$ equations all depending only of known system parameters and chosen coordinates
4. Assemble all $N$ resulting equations in matrix form, thus obtaining

$$\mathrm{M}\ddot{\mathbf{x}} + \mathrm{C}\dot{\mathbf{x}} + \mathrm{K}\mathbf{x} = \mathbf{0} \tag{116}$$

with

- M as the mass matrix

$$\mathrm{M} = \begin{bmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,N} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,N} \\ \vdots & \vdots & \cdots & \vdots \\ m_{N,1} & m_{N,2} & \cdots & m_{N,N} \end{bmatrix} \tag{117}$$

- C as the damping matrix

$$\mathrm{C} = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,N} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,N} \\ \vdots & \vdots & \cdots & \vdots \\ c_{N,1} & c_{N,2} & \cdots & c_{N,N} \end{bmatrix} \tag{118}$$

- K as the stiffness matrix

$$\mathrm{K} = \begin{bmatrix} k_{1,1} & k_{1,2} & \cdots & k_{1,N} \\ k_{2,1} & k_{2,2} & \cdots & k_{2,N} \\ \vdots & \vdots & \cdots & \vdots \\ k_{N,1} & k_{N,2} & \cdots & k_{N,N} \end{bmatrix} \tag{119}$$

- and **x** as the vector of coordinates

$$\mathbf{x} = [x_1, x_2, \cdots, x_N]^{\mathrm{T}} \tag{120}$$

By neglecting damping we obtain the undamped natural frequencies of the system setting the determinant of the matrix $\mathrm{K} - \omega_n^2\mathrm{M}$ to zero:

$$\det(\mathrm{K} - \omega_n^2\mathrm{M}) = 0 \tag{121}$$

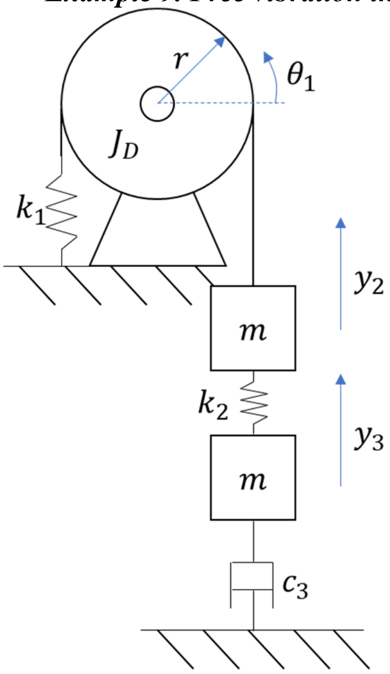
This will lead to an $N$-order algebraic equation in $\omega_n^2$

$$a_M(\omega_n^2)^N + a_{M-1}(\omega_n^2)^{N-1} + \cdots + a_2(\omega_n^2)^2 + a_1(\omega_n^2) + a_0 = 0 \tag{122}$$

Resulting in $N$ $\omega_n^2$ solutions, whose square root will give $N$ different $\omega_{n1}, \dots, \omega_{nN}$ undamped natural frequencies. The corresponding mode shapes are obtained by looking for solution components $x_i = A_i \sin(\omega_n t)$ and using $N-1$ equations from eq. (116) to find the ratios $(A_i/A_1)_{n1}, \dots, (A_i/A_1)_{nM}$.

Each mode shape will therefore consist of a series of values, often collected in a vector $\boldsymbol{\chi}_n$, where the first element is 1 (ratio $x_1/x_1$) and all other elements are the $x_k/x_1$.

***Example 9. Free vibration in a 2DOF system***




The system in figure is composed of two identical masses of mass m = 1 kg and a disc with radius r = 0.5 m and moment of inertia $J_D$ = 2 kg m$^2$ about its centre. A spring of stiffness $k_1$ = 1000 N/m connects one end of the disc to the ground, while the other end is connected to one of the two masses by an inextensible cable. The two masses are connected by a second spring of stiffness $k_2$ = 1500 N/m and the second mass is connected to the ground by a damper with coefficient $c_3$ = 100 Ns/m. All springs are unstretched for null coordinates)

1. Choose a suitable set of coordinates for the description of the system's vibration
2. Find the equation of motion of the system
3. Find the undamped natural frequencies
4. Find the mode-shapes

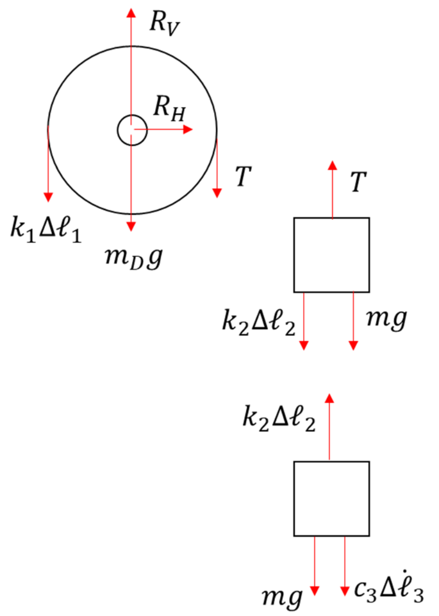
***Solution***

1. Choose a suitable set of coordinates for the description of the system's vibration

The motion $y_3$ of the lower mass is independent from all other coordinates, i.e. the spring and the damper allow for the lower mass to move even if we lock the other mass and the disc. $y_3$ is therefore a good coordinate to describe one of the degrees of freedom of the system.
The motions of the other mass and the disc are instead dependent on each other, i.e. if I lock the upper mass, the disc cannot rotate because of the inextensible cable; and if I lock the disc rotation, the mass cannot move for the same reason. Note that we are assuming the cable must stay in tension all the time and we do not consider motion which would result in a "slack cable". Therefore I can choose any of the two coordinates $\theta_1$ or $y_2$, as they are actually kinematically dependent.
I choose $\theta_1$ and $y_3$ ($y_2$ and $y_3$ would have been a good choice too).
To check if the system has any more DOFs, I lock $\theta_1$ and $y_3$ and see if anything can still move… nothing moves! So the system has 2 DOFs.

2. Find the equation of motion of the system

I start as usual drawing the FBDs (figure on the right).
Then I write Newton's law for:




- Disc rotation about its centre

$$J_D\ddot{\theta}_1 = k_1\Delta\ell_1 r - Tr$$

- Upper mass in vertical direction

$$m\ddot{y}_2 = T - mg - k_2\Delta\ell_2$$

- Lower mass in vertical direction

$$m\ddot{y}_3 = k_2\Delta\ell_2 - mg - c_3\Delta\dot{\ell}_3$$

I have 3 equations in a 2DOF system. I use one of the equations to remove the additional unknown T (the tension force in the cable). From the second equation:

$$T = m\ddot{y}_2 + mg + k_2\Delta\ell_2$$

And then I can substitute it in the first equation, thus having a set of 2 equations, only with kinematic unknowns:

$$J_D\ddot{\theta}_1 = k_1\Delta\ell_1 r - (m\ddot{y}_2 + mg + k_2\Delta\ell_2)r$$
$$m\ddot{y}_3 = k_2\Delta\ell_2 - mg - c_3\Delta\dot{\ell}_3$$

Now I express all as a function of $\theta_1$ and $y_3$:

$$\ddot{y}_2 = \ddot{\theta}_1 r \quad \text{and} \quad y_2 = \theta_1 r$$
$$\Delta\ell_1 = -r\theta_1 \quad \Delta\ell_2 = r\theta_1 - y_3 \quad \Delta\ell_3 = y_3$$

Now I substitute into the two equations:

$$J_D\ddot{\theta}_1 = -k_1 r^2\theta_1 - m\ddot{\theta}_1 r^2 - mgr + rk_2 y_3 - k_2 r^2\theta_1$$
$$m\ddot{y}_3 = -mg - c_3\dot{y}_3 - k_2 y_3 + k_2 r\theta_1$$

I can now apply equilibrium and find

$$0 = -k_1 r^2\theta_{1eq} - mgr + rk_2 y_{3eq} - k_2 r^2\theta_{1,eq}$$
$$0 = -mg - k_2 y_{3,eq} + k_2 r\theta_{1,eq}$$

Substituting $\theta_1 = \theta_{1,eq} + \tilde{\theta}_1$ and $y_3 = y_{3,eq} + \tilde{y}_3$, and using the above equilibrium equations, Ic an get rid of all constants, getting:

$$J_D\ddot{\tilde{\theta}}_1 = -k_1 r^2\tilde{\theta}_1 - m\ddot{\tilde{\theta}}_1 r^2 + rk_2\tilde{y}_3 - k_2 r^2\tilde{\theta}_1$$
$$m\ddot{\tilde{y}}_3 = -c_3\dot{\tilde{y}}_3 - k_2\tilde{y}_3 + k_2 r\tilde{\theta}_1$$

And rearranging I obtain:

$$(J_D + mr^2)\ddot{\tilde{\theta}}_1 + (k_1 + k_2)r^2\tilde{\theta}_1 - k_2 r\tilde{y}_3 = 0$$
$$m\ddot{\tilde{y}}_3 + c_3\dot{\tilde{y}}_3 + k_2\tilde{y}_3 - k_3 r\tilde{\theta} = 0$$

Now I can put all in matrix form:

$$\begin{bmatrix} J_D + mr^2 & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{\tilde{\theta}}_1 \\ \ddot{\tilde{y}}_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c_3 \end{bmatrix}\begin{bmatrix} \dot{\tilde{\theta}}_1 \\ \dot{\tilde{y}}_3 \end{bmatrix} + \begin{bmatrix} (k_1 + k_2)r^2 & -k_2 r \\ -k_2 r & k_2 \end{bmatrix}\begin{bmatrix} \tilde{\theta}_1 \\ \tilde{y}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

And identify:

$$\mathrm{M} = \begin{bmatrix} J_D + mr^2 & 0 \\ 0 & m \end{bmatrix} = \begin{bmatrix} 2.25\ \mathrm{kg\,m^2} & 0 \\ 0 & 1\ \mathrm{kg} \end{bmatrix}$$

$$\mathrm{C} = \begin{bmatrix} 0 & 0 \\ 0 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 100\ \mathrm{Ns/m} \end{bmatrix}$$

$$\mathrm{K} = \begin{bmatrix} (k_1 + k_2)r^2 & -k_2 r \\ -k_2 r & k_2 \end{bmatrix} = \begin{bmatrix} 625\ \mathrm{Nm/rad} & -750\ \mathrm{Nm/m} \\ -750\ \mathrm{N/rad} & 1500\ \mathrm{N/m} \end{bmatrix}$$

Note that in this case units within each row/column of the matrices are different due to the two coordinates being in different units ($\theta_1$ in radians and $y_2$ in metres).

3. Find the undamped natural frequencies.

Neglecting damping and setting the determinant of $\mathrm{K} - \omega_n^2\mathrm{M}$, we get:

$$\det\left(\begin{bmatrix} 625 - 2.25\omega_n^2 & -750 \\ -750 & 1500 - \omega_n^2 \end{bmatrix}\right) = 0$$

$$(625 - 2.25\omega_n^2)(1500 - \omega_n^2) - 750^2 = 0$$

$$2.25\omega_n^4 - 4000\omega_n^2 + 375{,}000 = 0$$

Giving:

$$\omega_n^2 = \frac{4000 \pm \sqrt{4000^2 - 4 \cdot 2.25 \cdot 375{,}000}}{2 \cdot 2.25} = \begin{cases} 99.3\,\dfrac{\mathrm{rad}^2}{\mathrm{s}^2} \\ 1678.5\,\dfrac{\mathrm{rad}^2}{\mathrm{s}^2} \end{cases}$$

And thus the square roots of these solutions will be the undamped natural frequencies of the system:

$$\omega_{n1} = \sqrt{99.3} = 9.96\,\frac{\mathrm{rad}}{\mathrm{s}} \quad \text{and} \quad \omega_{n2} = \sqrt{1678.5} = 40.97\,\frac{\mathrm{rad}}{\mathrm{s}}$$

4. Find the mode shapes

I can use any of the following two equations

$$\begin{bmatrix} 625 - 2.25\omega_n^2 & -750 \\ -750 & 1500 - \omega_n^2 \end{bmatrix}\begin{bmatrix} 1 \\ \chi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For instance I use the second equation (without the damping terms):

$$(1500 - \omega_n^2)\chi - 750 = 0$$

Leading to:

$$\chi = \frac{750}{1500 - \omega_n^2}$$

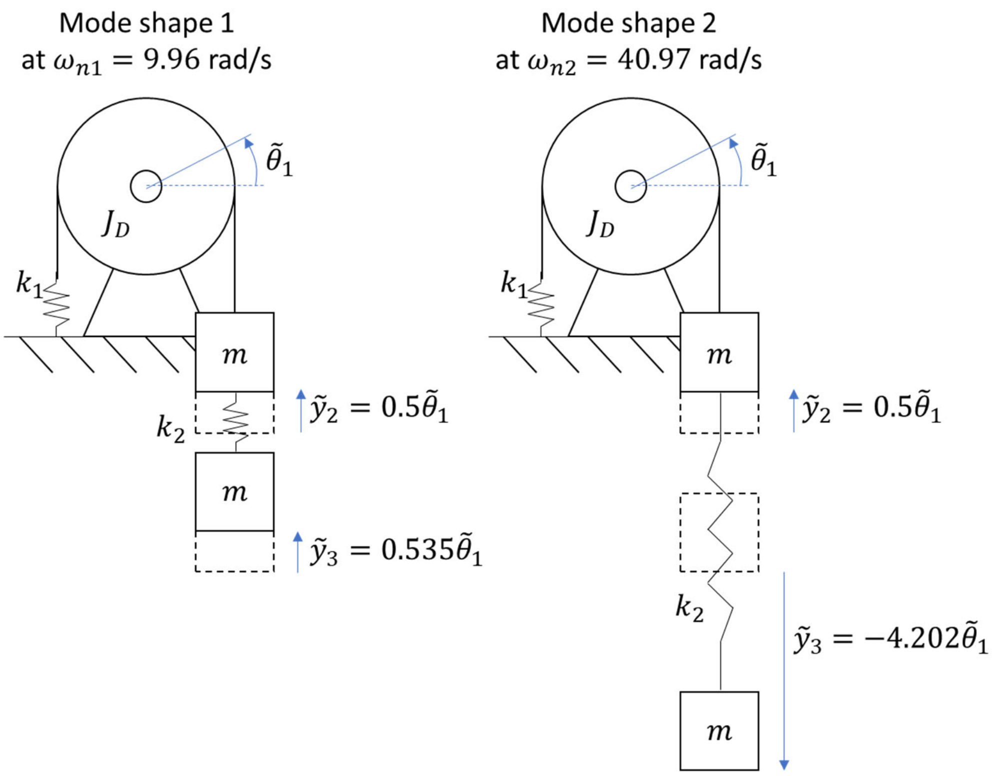
Substituting $\omega_{n1}$ we obtain the displacement ratio of the first mode:

$$\chi_{n1} = \frac{750}{1500 - \omega_{n1}^2} = \frac{750}{1500 - 99.3} = 0.535\,\frac{\mathrm{m}}{\mathrm{rad}} \quad \text{so} \quad \boldsymbol{\chi}_{n1} = \begin{bmatrix} 1 \\ 0.535\,\dfrac{\mathrm{m}}{\mathrm{rad}} \end{bmatrix}$$

Substituting $\omega_{n2}$ we obtain the displacement ratio of the second mode:

$$\chi_{n2} = \frac{750}{1500 - \omega_{n2}^2} = \frac{750}{1500 - 1678.5} = -4.202\,\frac{\mathrm{m}}{\mathrm{rad}} \quad \text{so} \quad \boldsymbol{\chi}_{n2} = \begin{bmatrix} 1 \\ -4.202\,\dfrac{\mathrm{m}}{\mathrm{rad}} \end{bmatrix}$$

A graphical representation (qualitative) is given in the following, considering that for every $\theta$ we have also a displacement $y_2 = \theta_1 r = 0.5\theta_1$ because of a fixed kinematic relationship (due to the cable)



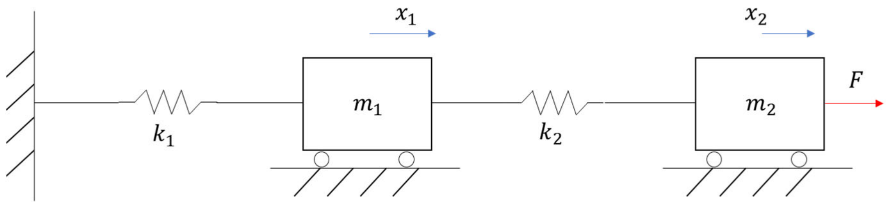

Note that in the first mode the motion of $y_2$ and $y_3$ is in-phase (same sign) and of very similar amplitude, thus resulting in almost no deformation of the spring $k_2$.
In the second mode instead the motion mostly involves $k_2$ given that there is a large motion of $y_3$ (relatively to $y_2$).

## 12. Forced MDOF vibration

We will start with a simple undamped 2DOF example writing the equation of motion and analysing the particular solution. Then we will generalise to undamped MDOF system of any kind. The effect of damping will be discussed only on the transfer function of the system.

### *12.1. Example of forced undamped 2DOF vibrating system*

We will start from the same example as in section 11.2, adding a force $F = F_0 \cos(\omega t)$ on mass $m_2$, with $F_0 = 10$ N and $\omega = 2$ rad/s.




$m_1 = m_2 = 2$ kg

$c_1 = c_2 = 1$ Ns/m

$k_1 = k_2 = 200$ N/m

*Figure 40. Simple 2DOF system with an external force*

The FBDs of each body are shown in Figure 41.

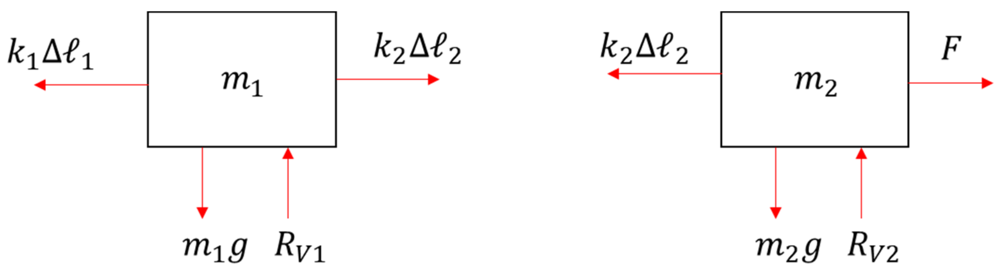



*Figure 41. FBD of the two bodies*

Writing Newton's law in horizontal direction and expressing all kinematic quantities as a function of $x_1$ and $x_2$ we obtain the following equations of motion (see very similar example in section 11.1).

$$\begin{aligned} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0 \\ m_2\ddot{x}_2 + k_2 x_2 - k_2 x_1 = F_0 \cos(\omega t) \end{aligned} \tag{123}$$

Which can be put in matrix form as:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F_0 \end{bmatrix} \cos(\omega t) \tag{124}$$

As for SDOF systems we have particular and homogeneous solution components.

<u>Homogeneous solution</u>
The homogenous solution has the same form of free vibration

$$\mathbf{x}(t) = \begin{bmatrix} x_{1h}(t) \\ x_{2h}(t) \end{bmatrix} = A_{n1h}\boldsymbol{\chi}_{n1}\cos(\omega_{n1}t + \psi_{n1h}) + A_{n2h}\boldsymbol{\chi}_{n2}\cos(\omega_{n2}t + \psi_{n2h}) \tag{125}$$

And is found by studying the free-vibration of the system (i.e. removing the force):

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_{1h} \\ \ddot{x}_{2h} \end{bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_{1h} \\ x_{2h} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{126}$$

We know that this allows us to find the natural frequencies $\omega_n$ by setting the determinant

$$\det(\mathrm{K} - \omega_n^2 \mathrm{M}) = 0 \tag{127}$$

Which in this example leads to the equation

$$[(k_1 + k_2) - \omega_n^2 m][k_2 - \omega_n^2 m] - k_2^2 = 0 \tag{128}$$

Which allows us to find the natural frequencies $\omega_{n1} = 6.18$ rad/s and $\omega_{n2} = 16.18$ rad/s as shown in the example in section 11.2 and the corresponding vibration modes.

<u>Particular solution</u>

We are mostly interested in the particular solution. In practical systems (despite our simplified undamped model) the homogeneous solution fades away after an initial transient, and only the particular solution remains and keeps affecting the system's behaviour in steady-state.

The particular solution is of the form:

$$\begin{aligned} x_{1p}(t) &= X_1 \cos(\omega t) \\ x_{2p}(t) &= X_2 \cos(\omega t) \end{aligned} \tag{129}$$

Or in vectorial form:

$$\mathbf{x}_p(t) = \mathbf{X} \cos(\omega t) \tag{130}$$

with $\mathbf{X} = [X_1 \quad X_2]^{\mathrm{T}}$ Substituting in our equations and dividing all by $\cos(\Omega t)$ we get:

$$[\mathrm{K} - \omega^2 \mathrm{M}]\mathbf{X} = \mathbf{F}_0 \qquad (131) \qquad (132)$$

which gives us the solution

$$\mathbf{X} = [\mathrm{K} - \omega^2 \mathrm{M}]^{-1}\mathbf{F}_0 \tag{133}$$

The calculation of the inverse of a 2x2 matrix is doe following the rule:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{134}$$

In our example therefore:

$$\begin{aligned} [\mathrm{K} - \omega^2 \mathrm{M}]^{-1} &= \begin{bmatrix} (k_1+k_2) - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{bmatrix}^{-1} \\ &= \frac{1}{\left((k_1+k_2) - \omega^2 m_1\right)(k_2 - \omega^2 m_2) - k_2^2}\begin{bmatrix} k_2 - \omega^2 m_2 & k_2 \\ k_2 & (k_1+k_2) - \omega^2 m_1 \end{bmatrix} \end{aligned} \tag{135}$$

Thus leading to:

$$\mathbf{X} = [\mathrm{K} - \omega^2\mathrm{M}]^{-1}\mathbf{F}_0 = [\mathrm{K} - \omega^2\mathrm{M}]^{-1}\begin{bmatrix} 0 \\ F_0 \end{bmatrix} = \frac{1}{\big((k_1+k_2) - \omega^2 m_1\big)(k_2 - \omega^2 m_2) - k_2^2}\begin{bmatrix} k_2 \\ (k_1+k_2) - \omega^2 m_1 \end{bmatrix} F_0 \tag{136}$$

which can be split in $x_1$ and $x_2$ as follows:

$$X_1 = \frac{k_2 F_0}{\big((k_1+k_2) - \omega^2 m_1\big)(k_2 - \omega^2 m_2) - k_2^2}$$
$$X_2 = \frac{\big((k_1+k_2) - \omega^2 m_1\big)F_0}{\big((k_1+k_2) - \omega^2 m_1\big)(k_2 - \omega^2 m_2) - k_2^2} \tag{137}$$

Now given we know $F_0 = 10$ N and $\omega = 2$ rad/s, we can actually obtain the vibration amplitude

$$X_1 = \frac{k_2 F_0}{\big((k_1+k_2) - \omega^2 m_1\big)(k_2 - \omega^2 m_2) - k_2^2} = 0.0567 \text{ m}$$
$$X_2 = \frac{\big((k_1+k_2) - \omega^2 m_1\big)F_0}{\big((k_1+k_2) - \omega^2 m_1\big)(k_2 - \omega^2 m_2) - k_2^2} = 0.1112 \text{ m} \tag{138}$$

Note that $X_1$ and $X_2$ include information about both the amplitude and phase of vibration:

$$|X| = \text{amplitude of vibration} = \text{absolute value of } X$$
$$\Delta\phi = \text{phase with respect to force} = \begin{cases} 0 \text{ (or } -360°\text{) if } X \text{ positive} \\ -180° \text{ if } X \text{ negative} \end{cases} \tag{139}$$

The particular solution can therefore be written as:

$$x_{1p}(t) = |X_1|\cos(\omega t + \phi_F + \Delta\phi_1)$$
$$x_{2p}(t) = |X_2|\cos(\omega t + \phi_F + \Delta\phi_2) \tag{140}$$

### *12.2. Response as a function of frequency for the example 2DOF case*

If we wanted to study the response as a function of frequency, we could take the two expressions of $X_1$ and $X_2$ as function of $\omega$ and divide them by $F_0$, just to represent the system response per unit force (frequency response function or FRF):

$$\frac{X_1}{F_0} = \frac{k_2}{[-\omega^2 m_2 + k_2][-\omega^2 m_1 + (k_1+k_2)] - k_2^2}$$
$$\frac{X_2}{F_0} = \frac{[-\omega^2 m_1 + (k_1+k_2)]}{[-\omega^2 m_2 + k_2][-\omega^2 m_1 + (k_1+k_2)] - k_2^2} \tag{141}$$

These can be plotted (in magnitude $|X|$ and phase $\Delta\phi$) as a function of $\omega$ as follows

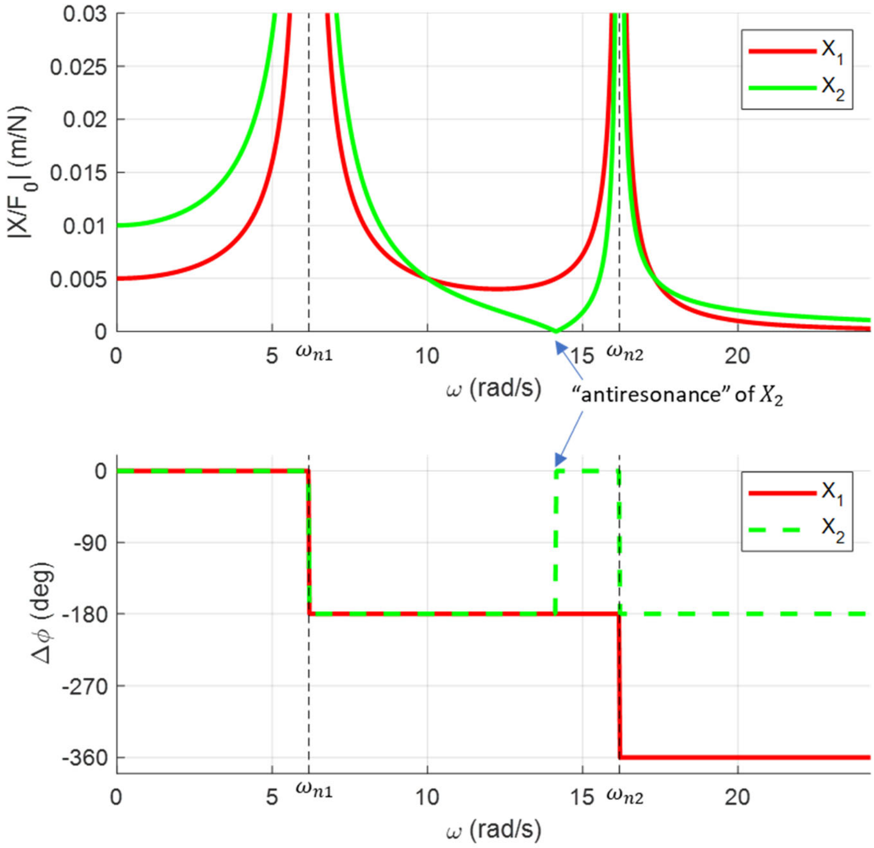



*Figure 42. Frequency response function (in magnitude and phase) for $x_1$ and $x_2$ to a force applied on $m_2$ in the undamped 2DOF example*

Analysing the two formulas we can draw comparisons with the areas identified in the magnification factor for SDOF systems

- The denominator of $X_1$ and $X_2$ is the same, and it is identical to $\det(\mathrm{K}-\omega^2\mathrm{M})$ (compare with eq. (126)). Therefore for $\omega = \omega_{n1}$ and for $\omega = \omega_{n2}$ we will get $\det(\mathrm{K}-\omega^2\mathrm{M}) = 0$. This means that when the force frequency $\omega$ is equal to any of the natural frequencies of the system, the denominator of both $X_1$ and $X_2$ will be zero and the vibration amplitudes $|X_1|$ and $|X_2|$ will tend to infinity, as shown by the vertical asymptotes at 6.18 and 16.18 rad/s in Figure 42. This is the 2DOF equivalent of the SDOF resonant case. In fact in a 2DOF system we have two natural frequencies and therefore two resonance zones for $\omega \approx \omega_{n1}$ and $\omega \approx \omega_{n2}$. It is also worth remembering if the system is forced with $\omega = \omega_{n1}$, the vibration has the same shape as the first mode shape, whereas the system will vibrate as in the second mode shape if $\omega = \omega_{n2}$.
- At low force frequencies $\omega \approx 0 \ll \omega_{n1}$ we get displacement amplitudes which are approximately those obtained with a static force of magnitude $F_0$, i.e. the first spring moves by $F_0/k_1$ and the second by $F_0/k_{12}^{series}$ where $k_{12}^{series}$ is the equivalent stiffness of the two springs in series

$$X_1 \approx \frac{k_2 F_0}{k_2(k_1+k_2)-k_2^2} = \frac{F_0}{k_1} \tag{142}$$

$$X_2 \approx \frac{(k_1 + k_2)F_0}{k_2(k_1 + k_2) - k_2^2} = \frac{k_1 + k_2}{k_1 k_2} F_0 = \frac{F_0}{k_{12}^{series}}$$

- In between the resonances there is in this particular example a specific frequency $\omega = \sqrt{(k_1 + k_2)/m_1} = 14.14$ rad/s where the numerator of $X_2$ is equal to 0. This means that, for $\omega = 14.14$ rad/s,the mass $m_2$ will have no vibration at all, but $m_1$ will still vibrate (its numerator is never 0). This particular frequency is called an antiresonance of the system response in $x_2$. In general, in a 2DOF system, only the degree of freedom where the force is applied will have an antiresonance.
- Finally at very high frequencies $\omega \gg \omega_{n2}$, the denominator grows towards infinity resulting in $X_1$ and $X_2$ going towards 0, thus with very little vibration amplitude.

In terms of phase:

- Crossing a resonance (natural frequency) from left to right (increasing the force frequency), we have a change of phase of $-180°$ for both DOFs, whereas
- Crossing an antiresonance from left to right (increasing the force frequency), we have a change of phase of $+180°$ for the DOFs affected by the anti-resonance (the one where the force is applied).

### *12.3. General undamped MDOF case*

In a general case with $M$ degrees of freedom, we would have a system of $N$ equations resulting in the matrix-form expression

$$\mathrm{M}\ddot{\mathbf{x}} + \mathrm{K}\mathbf{x} = \mathbf{F} \tag{143}$$

where each matrix is $N \times N$ and $\mathbf{F}$ is a force vector with $N$ rows, all empty except for the row corresponding to the degree of freedom where the force $F$ is applied. If the force is harmonic $F = F_0 \cos(\omega t)$

The particular solution of this system will see each $i$-th degree of freedom having an expression

$$x_i(t) = X_i \cos(\omega t) \tag{144}$$

where the vector of amplitudes $\mathbf{X} = [X_1, \dots, X_k, \dots, X_N]^{\mathrm{T}}$ is obtained as:

$$\mathbf{X} = [-\omega^2 \mathrm{M} + \mathrm{K}]^{-1}\, \mathbf{F} \tag{145}$$

i.e. as the matrix-vector product of the inverse of the matrix $[-\omega^2 \mathrm{M} + \mathrm{K}]$ and the force vector.

As an example, the following system (Figure 43), with $m_1 = m_2 = m_3 = m_4 = 2$ kg and $k_1 = k_2 = k_3 = k_4 = 200$ N/m and a force applied to the first mass, has frequency response functions as in Figure 44 (only amplitude is plotted in this case). Note that in this case it is not obvious which DOFs will have antiresonances.

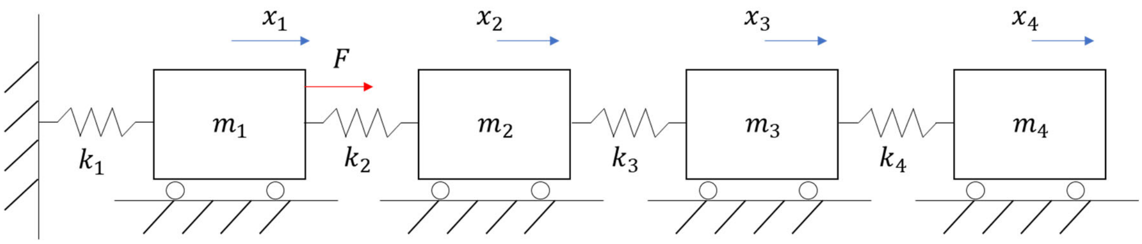



*Figure 43. Example of a forced undamped MDOF (4 DOFs) system.*

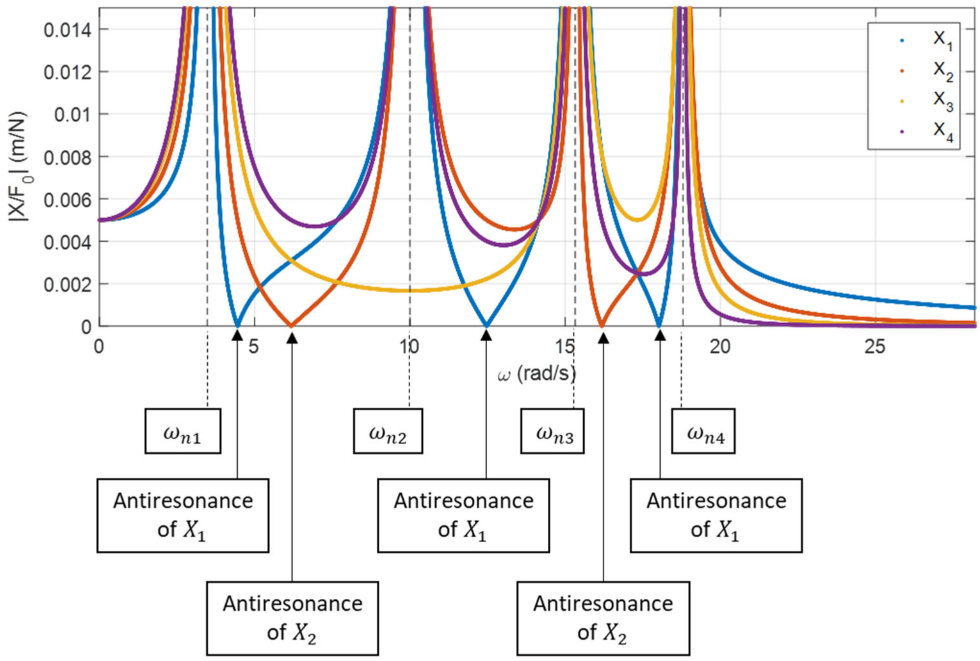



*Figure 44. Amplitude of the frequency response functions for the system of Figure 43.*

### *12.4. Effect of damping*

Treating damping usually requires the use of complex numbers. In a general case with $N$ degrees of freedom, we would have a system of $N$ equations resulting in the matrix-form expression

$$\mathrm{M}\ddot{\mathbf{x}} + \mathrm{C}\dot{\mathbf{x}} + \mathrm{K}\mathbf{x} = \mathbf{F} \tag{146}$$

where each matrix is $N \times N$ and $\mathbf{F}$ is a force vector with $N$ rows, all empty except for the row corresponding to the degree of freedom where the force $F$ is applied. If the force is harmonic $F = F_0 \cos(\omega t)$

The particular solution of this system will see each $i$-th degree of freedom having an expression

$$x_k(t) = |X_i| \cos(\omega t + \Delta\phi_i) \tag{147}$$

where the vector of amplitudes $|\mathbf{X}| = [|X_1|, \dots, |X_i|, \dots, |X_N|]^{\mathrm{T}}$ is obtained using complex numbers as:

$$|\mathbf{X}| = |[-\omega^2\mathrm{M} + j\omega C + \mathrm{K}]^{-1}\mathbf{F}| \tag{148}$$

and the phases $\Delta\boldsymbol{\phi} = [\Delta\phi_1, \dots, \Delta\phi_i, \dots, \Delta\phi_N]^T$ as

$$\Delta\boldsymbol{\phi} = \angle([-\omega^2 \mathrm{M} + j\omega C + \mathrm{K}]^{-1}\mathbf{F}) \tag{149}$$

where the symbol $|z|$ is in this case the absolute value (or modulus) of a complex number, and $\angle(z)$ is the argument of the complex number $z$. These can be obtained from the real and imaginary part of $z$ as:

$$|z| = \sqrt{Re(z)^2 + Im(z)^2} \quad and \quad \angle(z) = \tan^{-1}\left(\frac{Im(z)}{Re(z)}\right) \tag{150}$$

A more qualitative understanding of the effect of damping is obtained by looking at the case of section 0 and adding some damping. The results for an arbitrarily chosen damping matrix is shown in Figure 45, where the magnitude (absolute value) of $X_1$ and $X_2$ is shown (neglecting the sign).

The effect is a "smoothing" of the resonance peaks, which do not reach infinity as in the undamped case. The same however happens for the antiresonance of $X_2$, which does not reach a point at which the response is absolutely equal to 0.

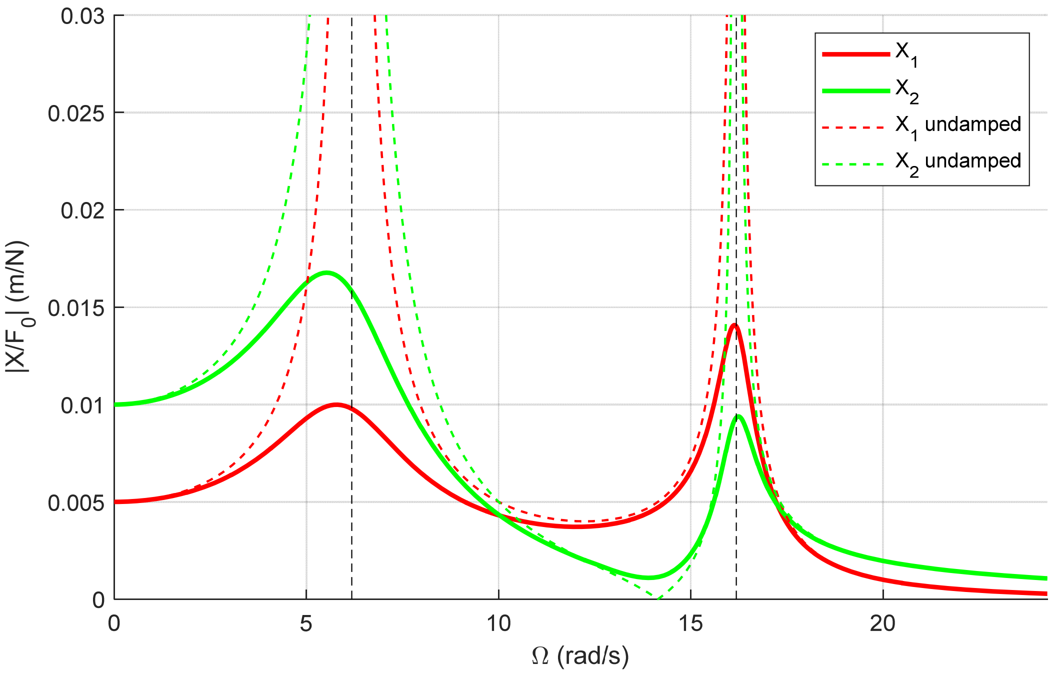
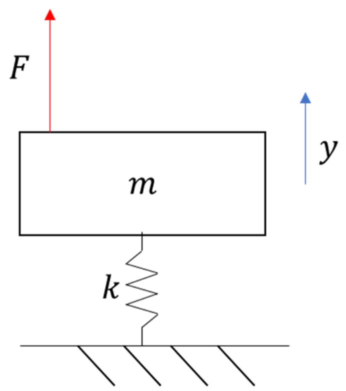



*Figure 45. Frequency response (amplitude) of a 2DOF system with damping (vs the undamped case in dashed line)*

## 13. Vibration absorption

An important application of MDOF (and particularly 2DOF) forced vibration is vibration absorption. In vibration absorption we take advantage of the antiresonance of a transfer function to reduce the vibration of a SDOF system by adding a second mass-spring subsystem.

Let's start with a simple SDOF system $m = 100$ kg and $k = 10$ kN/m, subject to a harmonic excitation $F = F_0 \cos(\omega t)$ with $F_0 = 100$ N and $\omega = 9$ rad/s




The equation of motion of the system for vibrations around equilibrium (static deflection and gravity already cancelled out) is:

$$m\ddot{\tilde{y}} + k\tilde{y} = F \qquad (151)$$

The system has a natural frequency which can be obtained by the corresponding free-vibration problem:

$$m\ddot{\tilde{y}} + k\tilde{y} = 0 \qquad (152)$$

obtained by removing the force.

The natural frequency is $\omega_n = \sqrt{k/m} = \sqrt{10{,}000/100} = 10$ rad/s

The steady state component of the forced vibration will have an amplitude

$$Y = \frac{F/k}{1 - \left(\frac{\omega}{\omega_n}\right)^2} = \frac{100/10{,}000}{1 - \left(\frac{9}{10}\right)^2} = \frac{0.01}{1 - 0.9^2} = 0.0526 \text{ m} \qquad (153)$$

Let's assume this is a large piece of machinery (100 kg) and a vibration of 5 cm is deemed excessive.

Can we reduce the vibration without acting on the system parameters? We can add a second mass-spring (vibration absorber).

The idea of a vibration absorber is the following:

- We start from an original SDOF system (as ours) which is subject to a force of known frequency
- We want to reduce to almost 0 the steady-state vibration of the main system $y$
- To do this we add a second mass-spring (vibration absorber) which makes the system a 2DOF system
- We know that in a 2DOF system the coordinate where the force is applied (in our case the displacement $y$ of the main mas $m$) will have an antiresonance in its frequency response function
- Therefore we choose the parameters of the absorbers ($m_a$ and $k_a$) to have that antiresonance exactly at the frequency of the force, resulting in no vibration of the main system.
- In this way the mass of the original system will not vibrate and all vibration will be absorbed in the added subsystem (the absorber)

So, let's add this absorber:

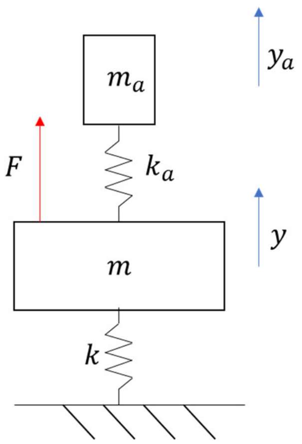



*Figure 46. System with the addition of a vibration absorber.*

The equations of motion will be:

$$\begin{gathered} m\ddot{\tilde{y}} + (k + k_a)\tilde{y} - k_a\tilde{y}_a = F_0\cos(\omega t) \\ m_a\ddot{\tilde{y}}_a + k_a\tilde{y}_a - k_a\tilde{y} = 0 \end{gathered} \tag{154}$$

Now we can substitute the steady state solutions $y = Y\cos(\Omega t)$ and $y_a = Y_a\cos(\omega t)$

$$\begin{gathered} [(k + k_a) - m\omega^2]Y - k_aY_a = F_0 \\ [k_a - \omega^2 m_a]Y_a - k_aY = 0 \end{gathered} \tag{155}$$

Using the second equation we can find:

$$Y_a = \frac{k_a}{k_a - \omega^2 m_a}Y \tag{156}$$

And substitute in the first, obtaining:

$$[(k + k_a) - m\omega^2]Y - \frac{k_a^2}{k_a - \omega^2 m_a}Y = F_0 \tag{157}$$

Rearranging:

$$\frac{[(k + k_a) - m\omega^2](k_a - \omega^2 m_a) - k_a^2}{k_a - \omega^2 m_a}Y = F_0 \tag{158}$$

And therefore the frequency response function will be:

$$\frac{Y}{F_0} = \frac{k_a - \omega^2 m_a}{[(k + k_a) - m\omega^2](k_a - \omega^2 m_a) - k_a^2} \tag{159}$$

The numerator is zero for:

$$\omega = \sqrt{\frac{k_a}{m_a}} \tag{160}$$

So we simply have to choose an absorber whose SDOF natural frequency $\omega_a = \sqrt{k_a/m_a}$ is equal to our force function $\omega$. In our case $\omega = 9$ rad/s, so we want $\omega_a = \sqrt{k_a/m_a} = 9$ rad/s.

Let's assume we want to add an absorber of mass $m_a = 1$ kg, then we have to choose a stiffness that gives us

$$\sqrt{\frac{k_a}{m_a}} = 9\,\frac{\mathrm{rad}}{\mathrm{s}} \tag{161}$$

Which is:

$$k_a = m_a \cdot 9^2 = 1 \cdot 9^2 = 81\ \mathrm{N/m} \tag{162}$$

The new system will have zero steady-state vibration with a force with $\omega = 9$ rad/s.

Be careful however, the system will have two new resonances, since it is now a 2DOF system. Let's calculate the new FRF and plot it:

$$\frac{Y}{F_0} = \frac{k_a - \omega^2 m_a}{[(k + k_a) - m\omega^2](k_a - \omega^2 m_a) - k_a^2} \tag{163}$$

The system will have natural frequencies when the denominator is equal to zero:

$$[(k + k_a) - m\omega_n^2](k_a - \omega_n^2 m_a) - k_a^2 = 0 \tag{164}$$

Substituting the known values of masses and stiffnesses we get:

$$[10{,}081 - 100\omega_n^2](81 - \omega_n^2) - 81^2 = 0 \tag{165}$$

Which can be expanded into

$$100\omega_n^4 - 18181\omega_n^2 + 810{,}000 = 0 \tag{166}$$

Giving solutions:

$$\omega_{n1} = 8.84\,\frac{\mathrm{rad}}{\mathrm{s}} \quad \text{and} \quad \omega_{n2} = 10.18\,\frac{\mathrm{rad}}{\mathrm{s}} \tag{167}$$

The full FRF is plotted in Figure 47. The figure shows how the target of zero vibration at $\omega = 9$ rad/s was achieved. However it also shows a weakness of our choice: the new resonances are very close to the forcing function and a small change in forcing function frequency (which can happen in machines) could cause the system to fall back into resonance. For instance if the force frequency was to drop to 8.84 rad/s we would be operating exactly at the first natural frequency of the system, with obvious terrible consequences.

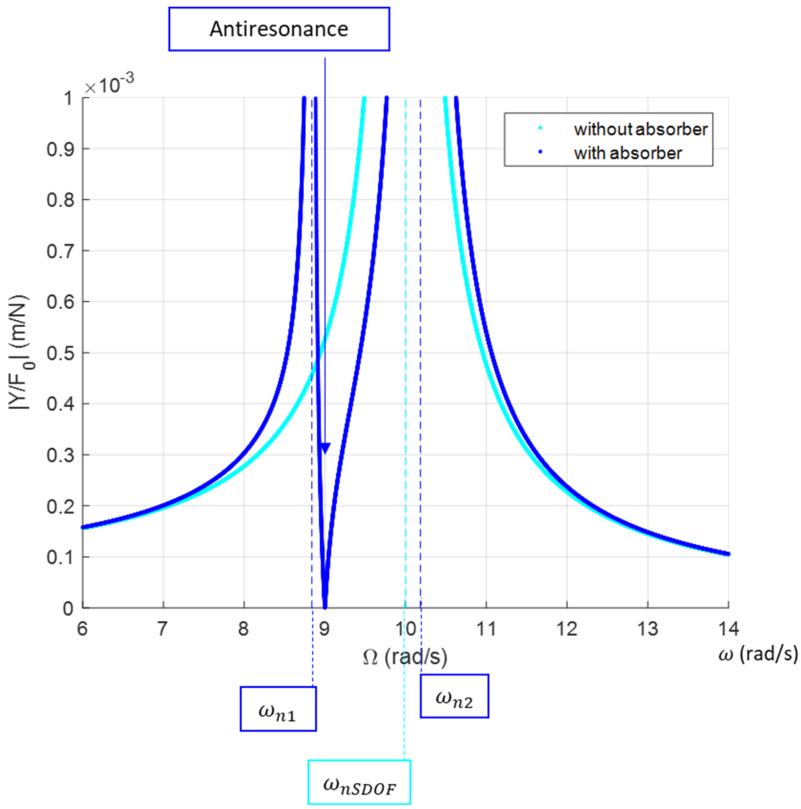



*Figure 47. Frequency response function with an absorber of* $m_a = 1$ *kg*

The fact that the new resonances are close to the original natural frequency of 10 rad/s is a consequence of the fact that our absorber mass is very small ($m_a = 1$ kg) compared to the original mass of the system ($m = 100$ kg). If we choose for instance a mass $m_a = 10$ kg, we get:

$$k_a = m_a \cdot 9^2 = 10 \cdot 9^2 = 810 \text{ N/m} \tag{168}$$

and natural frequencies of the 2DOF system

$$\omega_{n1} = 8.10 \frac{\text{rad}}{\text{s}} \quad \text{and} \quad \omega_{n2} = 11.11 \frac{\text{rad}}{\text{s}} \tag{169}$$

With a transfer function showing a much larger margin between the force frequency and the two system natural frequencies.

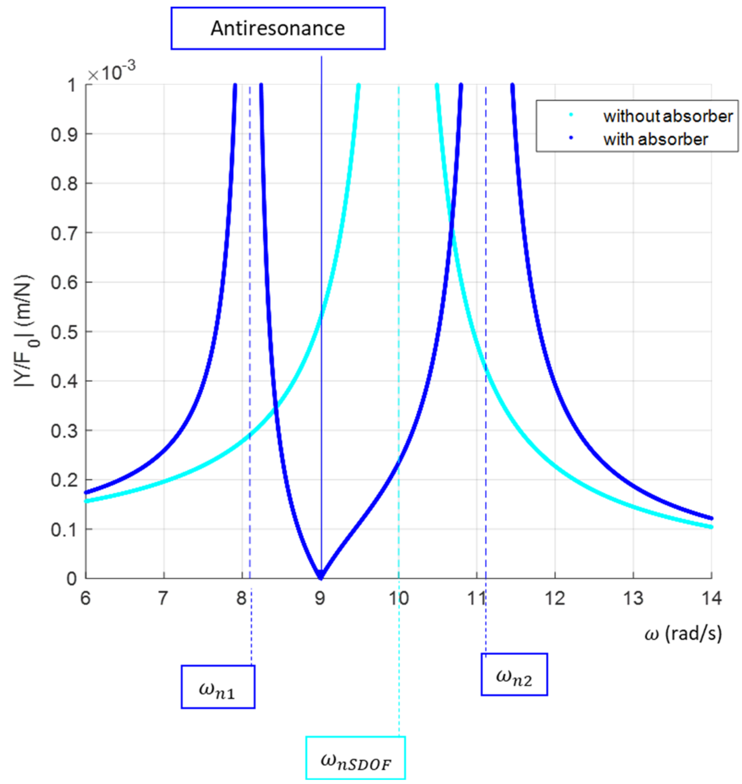



*Figure 48. Frequency response function with an absorber of* $m_a = 10\ kg$