Phil 120, Review Notes

Stuff

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Contents

1 References

1.1 Basic Logic Operators

The followings are for A * B, where '*' is an operator, A is top row, B is left column.

$\wedge$	T	F
T	T	F
F	F	F

AND. Conjuction. A $\wedge$ B is true only when both A and B are true.

$\vee$	T	F
T	T	T
F	T	F

OR. Disjunction. A $\vee$ B is true when either A or B, or Both are true.

$\rightarrow$	T	F
T	T	T
F	F	T

IMPLIES. If A then B. A implies B.
A implies B is true when A is true and B is true, or when A is false.
Note: $A \rightarrow B = \neg A \vee B$

$\leftrightarrow$	T	F
T	T	F
F	F	T

IFF, A if and only B. A is logically equivalent to B, two way implication.
A $\leftrightarrow$ B is true exactly when the truth value of A is the same as B.
Note: $A \leftrightarrow B = (A \rightarrow B) \wedge (B \rightarrow A) = (A \wedge B) \vee (\neg A \wedge \neg B)$

1.2 Some Logic Identities

$A \vee \neg A = T$	Excluded Middle, either A or not A must be true.
$\neg(A \wedge \neg A)$	Non-contradiction. It is true that not both A and not A hold at the same time.
$A \rightarrow B, A \implies B$	Modus ponens, to prove. If A implies and B and A is true, then B is true.
$A \rightarrow B, \neg B \implies \neg A$	Modus tollens, to disprove. If the conclusion is false, then the premise must be false also.
$A \vee B, \neg A \implies B$	Disjunctive syllogism. If at least one of A or B is true, then if one of them is false, the other must be true.
$(A \rightarrow B) \iff (\neg B \rightarrow \neg A)$	Contrapositive. Similar to Modus tollens.
$A, \neg A \implies B$	Explosion. From a false premise you can arrive at any conclusion.
$\neg(A \vee B) \iff \neg A \wedge \neg B$ $\neg(A \wedge B) \iff \neg A \vee \neg B$	De Morgan's Law: $\neg$ distributes over $\vee, \wedge$
$A \vee (B \wedge C) \iff (A \vee B) \wedge (A \vee C)$ $A \wedge (B \vee C) \iff (A \wedge B) \vee (A \wedge C)$	Distributability: $\vee$ distrubutes over $\wedge$, and vice versa.

1.3 Tableaux Identities

$\mathfrak{t}\wedge$:	$\begin{array}{c} \mathfrak{t}A \wedge B \\ \vert \\ \mathfrak{t}A \\ \mathfrak{t}B \end{array}$	$\mathfrak{f}\wedge$:	$\begin{array}{cc} \multicolumn{2}{c}{\mathfrak{f}A \wedge B} \\ \diagup & \diagdown \\ \mathfrak{f}A & \mathfrak{f}B \end{array}$
$\mathfrak{t}\vee$:	$\begin{array}{cc} \multicolumn{2}{c}{\mathfrak{t}A \vee B} \\ \diagup & \diagdown \\ \mathfrak{t}A & \mathfrak{t}B \end{array}$	$\mathfrak{f}\vee$:	$\begin{array}{c} \mathfrak{f}A \vee B \\ \vert \\ \mathfrak{f}A \\ \mathfrak{f}B \end{array}$
$\mathfrak{t} \to$:	$\begin{array}{cc} \multicolumn{2}{c}{\mathfrak{t}A \to B} \\ \diagup & \diagdown \\ \mathfrak{t}\neg A & \mathfrak{t}B \end{array}$	$\mathfrak{f} \to$:	$\begin{array}{c} \mathfrak{f}A \to B \\ \vert \\ \mathfrak{f}\neg A \\ \mathfrak{f}B \end{array}$
$\mathfrak{t} \leftrightarrow$:	$\begin{array}{cccc} \multicolumn{4}{c}{\mathfrak{t}A \leftrightarrow B} \\ \diagup & \diagup & \diagdown & \diagdown \\ \mathfrak{t}\neg A & \mathfrak{t}\neg A & \mathfrak{t}\neg B & \mathfrak{t}A \\ \vert & \vert & \vert & \vert \\ \mathfrak{t}\neg B & \mathfrak{t}A & \mathfrak{t}B & \mathfrak{t}B \end{array}$	$\mathfrak{f} \leftrightarrow$:	$\begin{array}{cc} \multicolumn{2}{c}{\mathfrak{f}A \leftrightarrow B} \\ \diagup & \diagdown \\ \mathfrak{f}\neg A & \mathfrak{f}\neg B \\ \vert & \vert \\ \mathfrak{f}B & \mathfrak{f}A \end{array}$
$\mathfrak{t}\forall$:	$\begin{array}{c} \mathfrak{t}\forall x\ Ax \\ \vert \\ \mathfrak{t}A\alpha \end{array}$	$\mathfrak{f}\forall$:	$\begin{array}{c} \mathfrak{f}\forall x\ Ax \\ \vert \\ \mathfrak{f}A\eta \end{array}$
$\mathfrak{t}\exists$:	$\begin{array}{c} \mathfrak{t}\exists x\ Ax \\ \vert \\ \mathfrak{t}A\eta \end{array}$	$\mathfrak{f}\exists$:	$\begin{array}{c} \mathfrak{f}\exists x\ Ax \\ \vert \\ \mathfrak{f}A\alpha \end{array}$
$\mathfrak{t} =$:	$\begin{array}{c} \mathfrak{t}a = b \\ \mathfrak{t}A(a) \\ \vert \\ \mathfrak{t}A(b) \end{array}$	$\mathfrak{f} =$:	$\begin{array}{c} \mathfrak{t}a = b \\ \mathfrak{f}A(a) \\ \vert \\ \mathfrak{f}A(b) \end{array}$

Quantifier & Identity Note:

- α means an arbitrary choice of variable name could used in place of α, might be a good idea to just use a.
- η means a new variable name that has previously unused in the current branch must be used to replace η.
- Use the following to deal with $\neg$:

$$\neg\exists x\ Ax \iff \forall x\ \neg Ax$$

$$\neg\forall x\ Ax \iff \exists x\ \neg Ax$$

$$(\mathfrak{f}\neg(a = b)) \iff (\mathfrak{t}a = b)$$

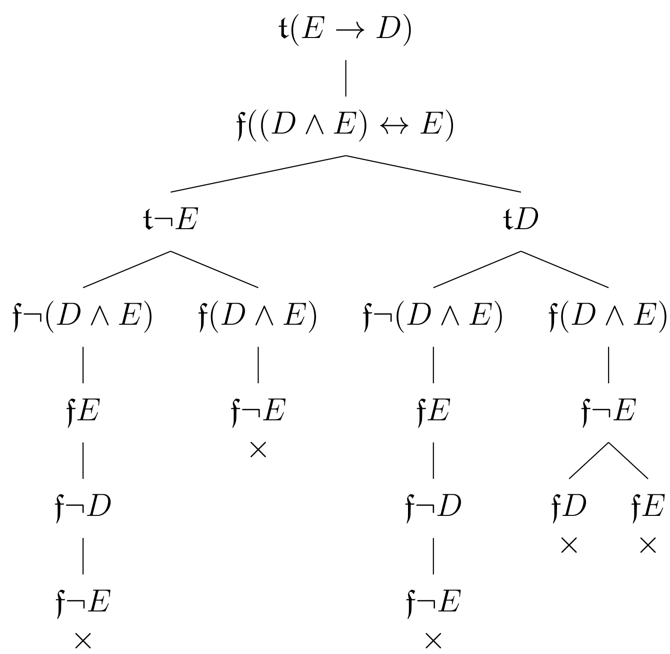
- Identiy closure condidtions: $\mathfrak{f}a = a,\ \mathfrak{t}\neg(a = a)$.

- Remember that $=$ is symmetric, so that $a = b \iff b = a$.
- Branches involving quantifiers are undecidable, so some strategies and luck is involved in closing those branches.
- Quantifier order matters!:
  - Swapping from $\exists x \forall y$ to $\forall y \exists x$ is okay, the reserve is not, in general.

Note:

- Use De Morgan's Law to deal with negations($\neg$), also, $\neg(A \to B) = (\neg B \wedge A)$.
- To prove a consequence, set the premise to true, conclusion to false. This proves that there is no possible counter example, since the negation is never satisfied.
  - If there are atomic branches left open, then those are valid counter examples.
- A branch is closed any of following pairs occurs in a branch: $\{(\mathfrak{f}A, \mathfrak{t}A), (\mathfrak{f}A, \mathfrak{f}\neg A), (\mathfrak{t}A, \mathfrak{t}\neg A)\}$, use a '$\times$' to indicate a closed branch.

Example: Use tableaux to prove $(E \to D) \vdash_1 ((D \wedge E) \leftrightarrow E)$



Since all branches are closed, the negation of the conclusion is never satisfied, thus the relation always holds for all values D and E might take on.

2 Notable Definitions from Part 1

2.1 Consequences

Logical Consequence: $A_1 \dots A_n$ implies B, and B is a consequence of $A_1 \dots A_n$, means when $A_1 \dots A_n$ are all true, then B must be true also.

Case: A case be loosely interpreted a particular combination of values for variables.

Valid Argument: An argument consist of a set of premises and a single conclusion, this argument is *valid* if the conclusion is a *logical consequence* of the premises.

Counter Example: A counter example to an argument is a case where all premises are truth, but the conclusion is false.

Sound Argument: An argument is sound if the premises are true in *all* cases, and the arugment is valid. An argument cannot be sound if its not already valid.

2.2 Language

Syntax: Syntax consist of a basic set of symbols, and a rule set to create more complex words & sentences from symbols. Syntax is not concerned with *meaning* of any symbols or sentences

Semantics: Semantics of a language assigns meaning to a sentence in the language.

Atom, Connectives, Molecules: An atomic sentence is the mostly basic sentence that cannot be reduced further, like 'sky is blue' or 'Bob is eating', atomic sentence do not have connectives. A molecular sentences is made with a number of atomic sentences linked with connectives, like 'Bob is sleeping *or* eating', 'Sun is bright *and* hot'.

2.3 Basics of Set Theory

Set: A set is an arbitrary, unordered *collection* of unique *things*, depending on context, duplicates are usually ignored. 2 sets are equal if they contain indentical items. For example:

$$Food := \{apple, cookie, burger\} = \{apple, apple, apple, burger, cookie\}$$

Membership($\in, \notin$): For any set it is possible to tell if an item belongs in the set. For exmaple:

$$cookie \in Food, dirt \notin Food$$

Which means that 'cookie' is in the set of Food(cookie is a member of Food), but dirt is not.

Set builder notation: A notation used to contruct sets from definitions. For exmaple:

$$L = \{n \in \mathbb{N} : n > 44\}$$

Here the ':' means 'such that', so the set L is the set all natural numbers, n, such that n is larger than 44.

Union($\cup$): The union of 2 sets is a set containing items from either sets:

$$\{1, 3, 7\} \cup \{2, 3, 2\} = \{1, 2, 7, 3\}$$

Intersect($\cap$): The intersection of 2 sets is a set containing items that belongs to both sets:

$$\{1, 3, 7\} \cup \{1, 2, 3, 4\} = \{1, 3\}$$

Subsets($\subseteq$): $A \subseteq B$ if A is contained in B, that is, every item in A is also in B.
Note: $A = B \iff (A \subseteq B) \wedge (B \subseteq A)$.

Proper Subset($\subset$): A is a proper subset of B if A $\subseteq$ B, and B is strictly bigger, that is, contains at least one item A does not.

2.4 Pairs and Relations

Ordered Pair: Unlike sets, ordered pairs/n-tuple are ordered. So $\{a, b\} = \{b, a\}$, however, $\langle a, b\rangle \neq \langle b, a\rangle$. N-tuples contains n ordered items.

Cartesian Product: $A \times B$ is the cartesian product of A and B, which is a set containing all possible ordered pairs $\langle a, b\rangle, a \in A, b \in B$. $\times$ can be applied more than 2 times. For exmaple:

$$\{a, b, c\} \times \{1, 2\} = \{\langle a, 1\rangle, \langle a, 2\rangle, \langle b, 1\rangle, \langle b, 2\rangle, \langle c, 1\rangle, \langle c, 2\rangle\}$$

$$D \times E \times F = \{\langle d_1, e, f_1\rangle, \langle d_1, e, f_2\rangle, \langle d_1, e, f_3\rangle, \langle d_2, e, f_1\rangle, \langle d_2, e, f_2\rangle, \langle d_2, e, f_3\rangle\}$$

$$\text{Where } D = \{d_1, d_2\}, E = \{e\}, F = \{f_1, f_2, f_3\}$$

Relations: A relation $\mathcal{R}$ on sets A and B, is a way to relate elements of A and B. For $a \in A, b \in B$, a and b are in relation $\mathcal{R} \iff \langle a, b\rangle \in \mathcal{R}$, and we can write $a\mathcal{R}b$.
Note: $\mathcal{R} \subseteq A \times B$.

Reflexivity: A relation $\mathcal{R}$ is reflexive when $x\mathcal{R}x$ for all x.

Symmetry: $\mathcal{R}$ is symmetric when $x\mathcal{R}y \iff y\mathcal{R}x$

Transitivity: $\mathcal{R}$ is transitive when $x\mathcal{R}y, y\mathcal{R}z \implies x\mathcal{R}z$.

Equivalence: $\mathcal{R}$ is an equivalence relation if $\mathcal{R}$ is reflexive, symmetric, and transitive.

Function: Like functions in calculus, $f : x \to y$ sends each x to 1 y only, that is, the value of $f(x)$ is not ambiguous.

3 Classical Logics

For logic operators and tableaux references, see Section 1.

3.1 Turntile($\vdash$) vs Double turntile($\models$)

Turntile: '$\vdash$' denotes *syntatic* implication. $A \vdash_1 B$ means with only information from A, it is possible to prove B. Or alternatively, it is possible to obtain B from 'rearranging' symbols of A.

Double turntile: '$\models$', denotes *semantic* implication, or models. $A \models_1 B$ means that B is true whenever A is true.

Notes: A logic system is *sound* if $A \vdash B \implies B$, is *complete* if $A \models B \implies A \vdash B$. Classical logics is sound and complete so there isn't a big difference between the two symbols used.

3.2 Cases

True/False in case: For a particular case c, $c \models_1 A$ means 'A is true in case c', while $c \models_0 A$ means 'A is false in case c'.

Complete/Consistant Cases:

- A case is *complete* if at least 1 of $c \models_1 A, c \models_0 A$ holds.
- A case is *consistant* if at most 1 of $c \models_1 A, c \models_0 A$ holds(so not both).
- All cases in classical logic is both complete and consistant.

Skipping things covered in Section 1 or informally covered in previous sections

3.3 Analytic Tableaux

For example and references on tableaux method of proof, see section 1.3.

Tableaux Method: The tableaux method provides a systematic way to determine if a statement is:

- true / a theorem, if all atomic leaves of the tree are not closed.
- false / a contradiction, if all branches are closed.
- contingent, in which case the remaining unclosed branches determines the truth value of the statement.

In particular, to prove a statement, it is helpful to start with the negation/counter example instead, then try to show the negation is never satisfied. So when proving $A \vdash_1 B$, start with:

$$\mathfrak{t}A$$

$$\mathfrak{f}B$$

Then follow the reference and example in section 1.3, if all branches closes, then that means the condition of A been true, while B is false, is never satisfied, thus no counter example exists, therefore $A \vdash_1 B$.

When there some branches left open, then that branche's nodes together forms a counter example when their truth value is same as their sign($\mathfrak{t}$ or $\mathfrak{f}$).

Sign of Node: For notation in this class, each node of a tableaux is *signed*($\mathfrak{t}$ or $\mathfrak{f}$), which are essentially assuming the truth value of that node.
Note, $\mathfrak{t}$ and $\mathfrak{f}$ signed nodes splits differently, and children nodes have same sgin as their parent. Also:

$$\mathfrak{t}A = \mathfrak{f}\neg A, \mathfrak{f}A = \mathfrak{t}\neg A$$

Skipping chapter 7, doesn't look important.

4 First Order Logic

4.1 Predicates

Unary Predicate: A single variabled *well formed formula*(or a boolean function) that evaluates to either true or false depending on the variable. In this class, a predicate looks like Pa where a is the variable, P is the predicate.

'Denotes': Suppose *Object* has name of *'Name'*, then *'Name'* denotes *Object*, and *Object* is the denotation of *'Name'*

Extension & AntiExtension: The *Extension* is the set of objects(denotations) that makes the predicate true. The anti-extension makes the predicate false. For example, suppose Px is true if x is red:

$$+(P) = \{apple, cherry, stop\ sign, \cdots\},\ -(P) = \{sky, orange, Phil\ 120\ midterm, \cdots\}$$

Domain of Discourse: Domain of Discourse, D, for a predicate is a set of objects applicable to the predicate. So if the predicate is 'x is the first day of the week', the D might be the set of all days in a week. Note: $+(P) \cup -(P) = D$, and $+(P) \cap -(P) = \emptyset$, in order for the predicate to behave classically.

Denotation Function(δ): The denotation function, $\delta(a)$, gets the denotation from the domain of Discourse which a denotes. For a predicate P, Pa is true iff $\delta(a) \in +(P)$

Interpretation: A interpretation(*case*) contains a domain of discourse and a denotation function($\langle D, \delta \rangle$), which is a more formal definition of a *case*, which previous parts have been mentioning.

Interpretations can be used to satisfy statements, or to serve as counter examples, like in A2 Q2:

a) Find an interpretation which satisfy $\Gamma = \{(\neg Ha \vee Hb), (\neg Ha \rightarrow \neg Hb), (\neg Hc \vee Hd)\}$

$$\begin{aligned} &\text{Define } \mathcal{I} = \langle D, \delta \rangle \\ &\text{Where } D = \{o_1, o_2, o_3, o_4\} \\ &\text{and } \delta(a) = o_1, \delta(b) = o_2, \delta(c) = o_3, \delta(d) = o_4 \\ &\text{such that } +(H) = \{o_1, o_2, o_3, o_4\}, -(H) = \{\} \end{aligned}$$

Now, $Ha, Hb, Hc, Hd = T$
$\implies \mathcal{I} \models_1 (\neg Ha \vee Hb),\ \mathcal{I} \models_1 (\neg Ha \rightarrow \neg Hb),\ \mathcal{I} \models_1 (\neg Hc \vee Hd)$
$\implies \mathcal{I} \models_1 \Gamma$

b) Find a counter exmaple to $\{\neg Ga, (Ga \rightarrow Pb), (\neg Pb \rightarrow Ga)\} \models_1 (Ga \vee \neg Pb)$

$$\begin{aligned} &\text{Define } \mathcal{I} = \langle D, \delta \rangle \\ &\text{Where } D = \{o_1, o_2\} \\ &\text{and } \delta(a) = o_1, \delta(b) = o_2 \\ &\text{such that } +(G) = \{\}, -(G) = \{o_1\}, +(P) = \{o_2\}, -(P) = \{\} \end{aligned}$$

So that in this interpreation Ga is false, while Gb is true
$\implies \neg Ga = T,\ (Ga \rightarrow Pb) = T,\ (\neg Pb \rightarrow Ga) = T.$
However $(Ga \vee \neg Pb) = F$, so in this interpretation, the premise is true but the conclusion is false, which makes this a counter example.

4.2 Quantifiers

Motivation: To decribe statements like "Every apple is sweet", "Some cities rains all the time.", which contains quantifiers.

Universal and Existential($\forall, \exists$):

- '$\forall$' translates into 'for all', 'every', 'any', is used to specify any and all of the items in a set:

  $$\forall n \in \mathbb{N}, 2n > n. \text{ Meaning for all natural number n, 2n is larger than n.}$$

- '$\exists$' translates in to 'exists', 'at least one', 'some', is used to specify at least one of an item in a set:

  $$\exists z \in \mathbb{Z} : z^2 = z. \text{ There exist an integer } z\text{, such that } z^2 = z.$$

- Quantifiers are tied to variables within its scope. In $P(z) := \exists x(Px \vee Pz) \wedge (\forall y \neg Py \rightarrow Px)$, $\exists x$ applies to the whole formula, but $\forall y$ only applies to the inner bracket. Here z is a *free* variable, since it is not bound by a quantifier.
- Order matters for quantifiers, for example:

  $$\exists x \forall y \in \mathbb{R}, x < y \text{ is false, since there is no smallest element in } \mathbb{R}$$

  $$\forall y \exists x \in \mathbb{R}, x < y \text{ however holds, since for all } y\text{, there is a } x \text{ that is smaller.}$$

- To prove or find counter example:
  - To prove $\forall x, Px$, it is necessary to show Px holds for any arbitrary x.
  - To disprove/find counter example for $\forall x, Px$, it is sufficient to find a single x such that Px does not hold.
  - To prove $\exists x, Px$, it is sufficient to show there is at least 1 x such that Px holds.
  - To disprove $\exists x, Px$ it is necessary to show Px is not satisfied for all x.

Name vs Variable: Name is refering to a static named reference of a object, while a variable is an unknown. For example : "Bob is short, and for any person, p, p is friends with Bob. x is even shorter than Bob". Here Bob is *named*, while p is a variable bound by quantifier, and x is a free variable since it is not bound by a quantifier.

Valuation Function: For a particular interpretation, a valuation function may be added $\rightarrow \langle D, \delta, v\rangle$. $\delta(a)$ is used to get the object in D that is used in place of names, while $v(x)$ is used to get object to be used in place of *variable*. v is similar to δ, but for variable.

Prof's Definitions for Quantifier: First, for any particular valuation function v, define v' such that $v' {\sim_x} v$, v' returns the same value as v, except for $v'(x)$. Now, universal and Existential quantifier may be define like so:

- $w \models_1^v \forall x Gx \iff$ for any $v' {\sim_x} v, w \models_1^{v'} Gx$.
  So no matter which object in D is put in place of x, the statement holds still.
- $w \models_1^v \exists y Gy \iff$ there is at least 1 $v' {\sim_y} v, w \models_1^{v'} Gy$.
  So at least 1 object exist in D, such that when put in place of y, the statement holds.

Predicate with Multiple Free Variable: A predicate may have more than 1 variable, for example Gab, for the predicate of '$a < b$. Predicate with multiple variable could be written like 'P^n', where n is the number of variable. For a predicate of n variable, then the extension/anti-extension should contain *n-tuples* since the predicate needs n variables.

4.3 Identity

Motivation: To determine if 2 objects are 'Equal', that they behave identically, or have the same properties.

Extention and AntiExtention:

$$+(=) \text{ is } \{\langle o, o\rangle : o \in D\} \text{ (the identity relation, or } Id_D)$$

$$-(=) \text{ is } \{\langle o_1, o_2\rangle : o_1, o_2 \in D, o_1 \neq o_2\}$$

Such that everything is identical to itself, and nothing else. Recall that a equivalence relation is reflexive($x = x$), symmetric($x = y \iff y = x$), and transitive($x = y \wedge y = z \implies x = z$).

First Order Leibniz Law: if $a = b$, then a is interchangable with b:

$$Pa, a = b \models_1 Pb$$

Properties of $\neq$:

- $(a \neq b \iff \neg(a = b))$
- $\forall x, x \neq x$ is always an contradiction, $\neq$ is not reflexive
- $\forall x, y \quad x \neq y \iff y \neq x$, $\neq$ is symmetric
- In general $\exists x, y, z : x \neq y \wedge y \neq z \not\Longrightarrow x \neq z$, $\neq$ is not transitive
- $Pa, a \neq b \not\models_1 Pb$

For tableaux formulas involving quantifier and equality, see the reference section.