*HW2

2.2 a.) To check whether x_t is (weakly) stationary, we'll check the expected value and covariance.

$$E[X_t] = E[\beta_0 + \beta_1 t + \omega_t]$$
$$= \beta_0 + \beta_1 t$$

$\boxed{\text{We can see that the expected value of } x_t \text{ does depend of time, So } x_t \text{ is not stationary}}$

b.) $y_t = x_t - x_{t-1}$

$$= (\beta_0 + \beta_1 t + \omega_t) - (\beta_0 + \beta_1(t-1) + \omega_{t-1})$$

$$E[y_t] = E[(\beta_0 + \beta_1 t + \omega_t) - (\beta_0 + \beta_1(t-1) + \omega_{t-1})]$$
$$= \beta_1$$

Can simplify y_t

$$y_t = \beta_1 + \omega_t - \omega_{t-1}$$

$$\mathrm{Cov}(y_{t+h}, y_t) = \mathrm{Cov}(\beta_1 + \omega_{t+h} - \omega_{t+h-1}, \beta_1 + \omega_t - \omega_{t-1})$$
$$= \mathrm{Cov}(\beta_1, \beta_1) + \mathrm{Cov}(\beta_1, \omega_t) - \mathrm{Cov}(\beta_1, \omega_{t-1})$$
$$+ \mathrm{Cov}(\omega_{t+h}, \beta_1) + \mathrm{Cov}(\omega_{t+h}, \omega_t) - \mathrm{Cov}(\omega_{t+h}, \omega_{t-1})$$
$$- \mathrm{Cov}(\omega_{t+h-1}, \beta_1) - \mathrm{Cov}(\omega_{t+h-1}, \omega_t) + \mathrm{Cov}(\omega_{t+h-1}, \omega_t))$$

$$= \mathrm{Cov}(\omega_{t+h}, \omega_t) - \mathrm{Cov}(\omega_{t+h}, \omega_{t-1}) - \mathrm{Cov}(\omega_{t+h-1}, \omega_t) + \mathrm{Cov}(\omega_{t+h-1}, \omega_{t-1})$$

This can be split into 3 cases

$$\gamma_y(h) = \begin{cases} 2\sigma_\omega^2 & \text{if } h=0 \\ -\sigma_\omega^2 & \text{if } |h|=1 \\ 0 & \text{if } |h|\geq 2 \end{cases}$$

We can see that all 3 condition does not depend on t

$\boxed{y_t \text{ is stationary}}$

c.) $V_t = \frac{1}{3}(x_{t-1} + x_t + x_{t+1})$

$$V_t = \frac{1}{3}(\beta_0 + \beta_1(t-1) + \omega_{t-1} + \beta_0 + \beta_1 t + \omega_t + \beta_0 + \beta_1(t+1) + \omega_{t+1})$$

$$= \frac{1}{3}(3\beta_0 + 3\beta_1 t + \omega_{t-1} + \omega_t + \omega_{t+1})$$

$$= \beta_0 + \beta_1 t + \frac{1}{3}\omega_{t-1} + \frac{1}{3}\omega_t + \frac{1}{3}\omega_{t+1}$$

$$E[V_t] = E[\beta_0 + \beta_1 t + \frac{1}{3}\omega_{t-1} + \frac{1}{3}\omega_t + \frac{1}{3}\omega_{t+1}]$$
$$= \boxed{\beta_0 + \beta_1 t}$$

2.4 a.) $E[X_t] = E[\phi x_{t-1} + \omega_t]$
$$= \phi E[X_{t-1}]$$

For x_t to be stationary, mean must be a constant $E[X_t] = \mu$

Thus:

$\mu = \phi\mu$ which implies

$$\boxed{\mu = 0}$$

b.) $\gamma_x(0) = \mathrm{Cov}(x_t, x_t)$

$$= \mathrm{Var}(x_t)$$
$$= \mathrm{Var}[\phi x_{t-1} + \omega_t]$$
$$= \phi^2 \mathrm{Var}[x_{t-1}] + 1$$

If x_t is stationary, then $\mathrm{Var}[X_t] = \gamma_x(0)$, is a constant.

$$\gamma_x(0) = \phi^2 \gamma_x(0) + 1$$

$$\Rightarrow \gamma_x(0) - \phi^2 \gamma_x(0) = 1$$
$$\gamma_x(0)(1-\phi^2) = 1$$
$$\Rightarrow \boxed{\gamma_x(0) = \frac{1}{1-\phi^2}}$$

c.) For b.) to make sense

$$\boxed{|\phi| < 1}$$

d.) $\rho_x(1) = \dfrac{\gamma_x(1)}{\gamma_x(0)}$

$$\gamma_x(1) = \mathrm{Cov}(x_{t+1}, x_t)$$
$$= \mathrm{Cov}(\phi x_t + \omega_{t+1}, x_t)$$
$$= \mathrm{Cov}(\phi x_t, x_t)$$
$$= \phi \mathrm{Var}[x_t]$$
$$= \phi \gamma_x(0)$$

$$\rho_x(1) = \frac{\gamma_x(1)}{\gamma_x(0)} = \frac{\phi \gamma_x(0)}{\gamma_x(0)} = \boxed{\phi}$$