G. Equity-Linked Insurance and Embedded Options

EQUITY-LINKED INSURANCE

AKA variable annuities, unit-linked policies, segregated fund policies

Primary purpose is investment, with some insurance aspects

Policyholder fund for investments is held separate from insurer's funds

Various features/riders may be available:

- surrender, subject to penalties
- guaranteed minimum maturity benefit
- guaranteed minimum death benefit
- guaranteed minimum income benefit
- guaranteed minimum withdrawal benefit

Management charges, surrender penalties, and cost of guarantees typically depend on F_t

Mortality is a diversifiable risk; surrender is mostly a diversifiable risk; guarantees based on market performance of policyholders' funds are not a diversifiable risk

CASH FLOW NOTATION

P_t – premium

AP_t – allocated premium

UAP_t – unallocated premium

F_t – policyholder fund balance at time t

MC_t – management charge

E_t – expenses

i_t^f – yield rate earned by policyholder

DB_t – death benefit payable at time t

EDB_t – expected additional death benefit payable at time t

CV_t – cash value upon surrender/maturity

ECV_t – expected additional cash value payable at time t

I_t – yield earned by insurer on funds

UK terminology:

- bid-offer spread - % of premium going to insurer's fund
- allocation percentage - % of premium that is allocated to units at the offer price

Premium and expenses usually occur at beginning of period; read policy details for timing of other cashflows

POLICYHOLDER FUND

Cash inflows: AP_t, yield from investment, additional benefits due to guarantees

Cash outflows: MC_t, surrender penalty

INSURER CASH FLOWS

Cash inflows: UAP_t, MC_t, interest on funds, surrender penalty

Cash outflows: expenses, additional benefits due to guarantees

PROFIT TESTING

Project F_t and insurer's cash flows assuming policy is still in force during t-th period

Profit vector, profit signature, and NPV can be determined from there

Profit emerging at time t for a policy in force at time $t-1$:

$$\begin{aligned}\Pr_t = {}_{t-1}V + UAP_t - E_t + I_t + MC_t \\ - EDB_t - ECV_t - E\,{}_tV\end{aligned}$$

USEFUL TIPS

Read policy details closely - lots of variations possible

Diagram of cashflows may help

Use the available spreadsheet to avoid calculation errors and speed up repeated calculations

GMMB & GMDB

Guaranteed Minimum Maturity Benefit

- provides floor for performance by policyholder's investment fund
- pays difference between fund balance at maturity and a guaranteed level

Guaranteed Minimum Death Benefit

- pays difference between fund balance at time of benefit and a guaranteed level

ENHANCING THE GUARANTEES

When investments do well, guarantees lose value, so policyholders may surrender early to get a better deal

Bad outcomes to avoid:

- "lapse and re-entry" - cash in and start a new policy with improved guarantees
- cash in and take the money elsewhere

Solution is to offer upgradeable guarantees:

- Reset option - bumps guarantees to be based on current fund balance, if better for policyholder

  Reset option is available during a given period of time and happens at the request of the policyholder

  Reset would bump out maturity date to minimum contract length, if maturity was originally sooner

  Drawbacks:

  - Bumping maturity dates may make policies linger
  - Strong market performance may lead to many resets at once, concentrating maturity dates
- Step-up/lookback option - bumps guarantees to be based on fund balance on step-up dates, which are specified at regular intervals during policy term
- Rollover option - same as step-up, except on step-up dates where no step-up would occur, in which case more money is deposited into policyholder's fund by insurer to bring it up to the guaranteed minimum

GMIB

Useful for policyholder who wants to buy an annuity using maturity value but is worried about market annuitization rate at that time

Annuitization rate γ is the regular annuity payment that \$1 will purchase

$$\gamma_x = 1/\ddot{a}_x$$

Guaranteed Minimum Income Benefit

- Guarantees an annuitization rate, applied to a benefit base
- Benefit base starts as % of premium, may step-up or grow from there
- If GMIB is exercised, maturity value is surrendered in exchange for the annuity, which is a term or life annuity, as specified in the policy

MC_t may include some additional amount based on benefit base

In real world, policyholder can take some maturity value as cash and annuitize the rest, and annuitization may be available before maturity

GMWB

Guaranteed Minimum Withdrawal Benefit

- Minimum WD amounts are usually a % of premium or benefit base
- WDs can begin after a specified point in time
- WDs come from policyholder fund, while it lasts
- Once fund is exhausted, minimum WDs continue until the end of the term or death, as specified in the policy
- Policyholder fund remains invested during this process, and MC_t continues to be taken while fund is positive

Policyholder still owns fund after WDs begin, so if policyholder exits withdrawal process before fund is empty, policyholder receives remainder of fund

EMBEDDED OPTIONS

GMDBs and GMMBs can be viewed as embedded put options

Liability of GMMB, where G^M is guaranteed minimum:

$$h(n) = \max(0, G^M - F_n)$$

This liability acts like a Black-Scholes put option payoff since F_t moves like stock

Cheaper and more effective to hedge risk as an option than to hold traditional reserve

Complications, relative to European put:

GMMB may expire worthless early

GMDB has random term-to-expiration

MC_t is taken from F_t, so F_t does not exactly move like stock

BLACK-SCHOLES REFRESHER

Assume BS Framework:

Risk-free zero-coupon bonds and stocks are available for purchase in any quantities, long or short, without impacting the market price

No transaction costs for trading

Portfolios are rebalanced continuously

Continuously compounded annual risk-free rate of interest r is constant and the yield curve is flat

Stock price S_t follows a continuous time lognormal process (AKA geometric Brownian motion) with parameters μ and σ (annual volatility)

European put option has:

underlying security S_t, strike price K

payoff of $\max(K - S_T, 0)$ at time T

The Black-Scholes price at time t for the European put is the EPV of the payoff under the risk-neutral measure Q, discounting at the risk-free rate:

$$\mathrm{BSP}(S, K, t, T) = \mathrm{E}_t^Q[e^{-r(T-t)}h(T)]$$

Formula for evaluating BSP(S, K, t, T) available on SOA Formula Sheet

Useful fact: under the Q-measure, the expected value of the stock's price grows at the risk-free rate:

$$\mathrm{E}_t^Q[e^{-r(T-t)}S_T] = S_t$$

GMMB VALUATION

$\pi(0) =$ expected amount required by insurer at issue per contract

$= {}_np_x^{00} * E_0^Q[e^{-rn}h(n)]$

Assume survival to maturity is indep. of F_t and is diversifiable, while using B-S to price non-diversifiable risk associated with F_t

Process:

- Find formula for fund balance F_t
- Find formula for liability $h(n)$, discount at the risk-free rate, and take the expected value under the risk-neutral measure

  At time 0, typically has form

  $\mathrm{E}_0^Q[e^{-rn}\max(G^M - \xi * S_n, 0)]$,

  where ξ is the "expense factor" and G^M is the guarantee amount

  At time t, typically has form

  $\mathrm{E}_t^Q[e^{-r(n-t)}\max(G^M - \xi * S_n, 0)]$,
- Manipulate to get the max into a form of $\max(\text{"stuff"} - S_n, 0)$
- Apply B-S valuation formula and simplify using $S_0 = 1$
- Include probability of survival to maturity

GMMB RESERVING

Usual cash reserving is inefficient when we can hedge embedded option

Value of hedging portfolio = value of GMMB as option

Reserve either by:

Buying hedging assets from third party

Holding replicating portfolio, rebalancing as needed

Hedge is not perfect due to (1) experience not matching assumptions, (2) B-S itself isn't quite right as a model, and (3) other real-world constraints (like discrete rebalancing)

GMDB VALUATION

Similar to GMMB valuation, except summing on all possible payoff times

The time-0 value of payoff $h(t)$ upon death at time t, assuming survival:

$$v(0, t) = \mathrm{E}_0^Q[e^{-rt}h(t)]$$

Time-0 valuation of GMDB, if

payable at moment of death:

$$\pi(0) = \int_0^n v(0, t)\ {}_tp_x\ \mu_{x+t}\, dt$$

payable at end of month of death:

$$\pi(0) = \sum_{j=1}^{12n} v(0, j/12)\ {}_{\frac{j-1}{12}|\frac{1}{12}}q_x$$

GMDB RESERVING

The time-t value of payoff $h(s)$ upon death at time s, where $0 \le t \le s$, assuming survival:

$$v(t, s) = \mathrm{E}_t^Q[e^{-r(s-t)}h(s)]$$

Time-t valuation of GMDB, if

payable at moment of death:

$$\pi(t) = \int_t^n v(t, s)\ {}_{s-t}p_{x+t}\ \mu_{x+s}\, ds$$

$$= \int_0^{n-t} v(t, t+w)\ {}_wp_{x+t}\ \mu_{x+t+w}\, dw$$

payable at end of month of death:

$$\pi(t) = \sum_{k=1}^{12(n-t)} v(t, t+k/12)\ {}_{\frac{k-1}{12}|\frac{1}{12}}q_{x+t}$$

FUNDING THE GUARANTEES

Two approaches:

- Use a portion of initial premium, called a **front-end load**

  Express the guarantee cost as a percentage of the initial premium

  Setting that percentage to get the expected cost of the guarantees at issue, $\pi(0)$, is easy
- Use a portion of the management charges

  Expressed as a **risk premium** at a rate c, where c is a percentage of the policyholder's fund balance

To find c:

Use policy details to find fund balance at time t, which leads to the risk premium received at time t

Then, take the EPV of risk premium at time t, adjust for survival, and sum across all times when risk premium could be paid

Finally, set RP equal to guarantee cost $\pi(0)$ and solve for c

Continuous management charge example:

Suppose no lapses, no expense deduction

$F_t = Pe^{-mt}S_t$ at time t

RP during $(t, t+dt) \approx cPe^{-mt}S_t dt$

EPV $\approx cPe^{-mt}dt$ (using $S_0 = 1$)

Adjust for survival: $cPe^{-mt}\ {}_tp_x\, dt$

RP $= \int_0^n cPe^{-mt}\ {}_tp_x\, dt = cP\bar{a}_{x:\overline{n}|\delta=m}$

$c = \frac{\pi(0)}{P\bar{a}_{x:\overline{n}|\delta=m}}$

Annual management charge example:

Suppose no lapses, no expense deduction

RP $= cF_t = cP(1-m)^t S_t$

EPV $= cP(1-m)^t$ (using $S_0 = 1$)

Adjust for survival and sum:

RP $= \sum_{t=0}^{n-1} {}_tp_x\ cP(1-m)^t = cP\ddot{a}_{x:\overline{n}|i^*}$

where $i^* = m/(1-m)$

$c = \frac{\pi(0)}{P\ddot{a}_{x:\overline{n}|i^*}}$

As c depends on m which depends on c, this approach may not produce solution for c

RISK MANAGEMENT

Using B-S to price guarantee means you should use B-S to hedge guarantee

Not a perfect hedge, but good enough

Rebalancing continuously is not practical - daily for big portfolios, weekly or monthly for smaller

Setting up a time-t Delta hedge for GMMB:

Delta hedge $= \Delta \cdot S_t + B$

hedge value = guarantee cost

$$= \pi(t) = {}_{n-t}p_{x+t}\ v(t,n)$$

$$\to \Delta = {}_{n-t}p_{x+t}\ \frac{d}{dS_t} v(t,n)$$

Value in stocks: ${}_{n-t}p_{x+t} \left(\frac{d}{dS_t} v(t,n)\right) S_t$

Value in bonds: $\pi(t) -$ value in stocks

Protip: $\frac{d}{dS_t} v(t,n)$ when $v(t,n)$ is a B-S valuation is just the coefficient of S_t

Setting up a time-t Delta hedge for GMDB:

Hedge is a mix of bonds of all possible maturities up to n, plus short stocks

$$\pi(t) = \int_0^{n-t} v(t, t+w)\, {}_w p_{x+t}\ \mu_{x+t+w}\, dw$$

Value in stocks:

$$= \int_0^{n-t} S_t \left(\frac{d}{dS_t} v(t, t+w)\right) {}_w p_{x+t}\ \mu_{x+t+w}\, dw$$

Value in w-year bond:

$$\approx \left(v(t,t+w) - S_t \frac{d}{dS_t} v(t,t+w)\right) {}_w p_{x+t}\ \mu_{x+t+w}\, dw$$

PROFIT TESTING GENERALLY

Multiple approaches:

- Deterministic - pick values for unknowns and see how the profit goes
- Stress test - pick values for multiple scenarios and see how the profit goes in each
- Stochastic - use a Monte Carlo simulation multiple times to build up a distribution of profit outcomes

Current syllabus only has us *using* deterministic, even though stochastic is best

When choosing values for deterministic approach, bad choices can be costly

For example, underestimating σ can lead to unexpected losses, and overestimating σ ties up capital unnecessarily

PROFIT TESTING WITH HEDGING

If external hedge used, then there will be no later hedge adjustment costs and no guarantee costs

If internal hedging and (1) rebalancing is continuous and (2) our model assumptions hold, there would be no rebalancing costs and no emerging guarantee costs

If hedging is internal and rebalancing is not continuous:

Profit testing looks the same as before, except must include rebalancing costs

Suppose rebalancing every $1/m$ of a year, and time-t hedge has a value of Υ_t in bonds, $\Psi_t S_t$ in stock

At time $t + \frac{1}{m}$, bonds appreciate by factor $e^{r/m}$ and stocks by factor $S_{t+1/m}/S_t$

$$\to \pi\left(t + \tfrac{1}{m}\right)^- = \Psi_t S_{t+1/m} + \Upsilon_t e^{r/m}$$

Rebalancing cost = new hedge $-$ old hedge, if the policy is still in force

Before maturity (that is, $t + \frac{1}{m} < n$):

$$\text{rebalancing cost} = \pi(t + \tfrac{1}{m}) \cdot {}_{\frac{1}{m}}p_{x+t} - \pi(t + \tfrac{1}{m})^-$$

At maturity (that is, $t + \frac{1}{m} = n$):

No new hedge is needed, so $\pi(n) = 0$

rebalancing cost $= 0 - \pi(n)^-$