*HW4

1. MA(1): $x_t = \omega_t + \theta\omega_{t-1}$

$$\rho_x(1) = \frac{\gamma(1)}{\gamma_x(0)}$$

$$\gamma_x(1) = Cov(x_{t+1}, x_t) = Cov(\omega_{t+1} + \theta\omega_t, \omega_t + \theta\omega_{t-1})$$
$$= \theta\sigma_\omega^2$$

$$\gamma_x(0) = Cov(x_t, x_t) = Cov(\omega_t + \theta\omega_{t-1}, \omega_t + \theta\omega_{t-1})$$
$$= \sigma_\omega^2 + \theta^2\sigma_\omega^2 = \sigma_\omega^2(1+\theta^2)$$

$$\rho_x(1) = \frac{\theta\sigma_\omega^2}{\sigma_\omega^2(1+\theta^2)} = \frac{\theta}{1+\theta^2}$$

$$\frac{\partial\rho_x(1)}{\partial\theta} = \frac{(1+\theta^2) - 2\theta^2}{(1+\theta^2)^2} = \frac{1-\theta^2}{(1+\theta^2)^2} = 0$$

$$\Rightarrow 1-\theta^2 = 0 \Rightarrow \theta = \pm 1$$

When $\theta = 1$

$\rho_x(1) = 0.5$

which is a maximum

When $\theta = -1$

$\rho_x(1) = -0.5$

which is a minimum

Thus, this shows that $|\rho_x(1)| \le \frac{1}{2}$

2a.) $x_t = \phi x_{t-1} + \omega_t,\ t = 1, 2, \ldots$

We can rewrite the model as:

$$x_t - \phi x_{t-1} = \omega_t$$
$$(1-\phi B)x_t = \omega_t$$

Let $\phi(B) = (1-\phi B)$

$$\phi(B)x_t = \omega_t$$
$$x_t = \phi^{-1}(B)\,\omega_t$$
$$x_t = \frac{1}{1-\phi B}\omega_t$$

Note $\frac{1}{1-\phi B} = 1 + \phi B + \phi^2 B^2 + \ldots$ by geometric series

$$x_t = (1 + \phi B + \phi^2 B^2 + \ldots)\omega_t$$
$$x_t = \sum_{j=0}^{t} \phi_j \omega_{t-j}$$

This shows that

$$x_t = \sum_{j=0}^{t} \phi^j \omega_{t-j}$$

2b.) $E[x_t] = E\left[\sum_{j=0}^{t} \phi^j \omega_{t-j}\right]$

$$= E[\omega_t + \phi\omega_{t-1} + \phi^2\omega_{t-2} + \ldots]$$
$$= E[\omega_t] + \phi E[\omega_{t-1}] + \phi^2 E[\omega_{t-2}] + \ldots$$
$$= \boxed{0}$$

2c.) $Var[x_t] = Cov(x_t, x_t)$

$$= Cov\left(\sum_{j=0}^{t} \phi^j \omega_{t-j}, \sum_{j=0}^{t} \phi^j \omega_{t-j}\right)$$
$$= \sigma_\omega^2 + \phi^2\sigma_\omega^2 + \phi^4\sigma_\omega^2 + \phi^6\sigma_\omega^2 + \ldots$$
$$= (1 + \phi^2 + \phi^4 + \phi^6 + \ldots)\sigma_\omega^2$$
$$= \left(\sum_{j=0}^{t} \phi^{2j}\right)\sigma_\omega^2$$

By Geometric Series, we can rewrite

$$\left(\sum_{j=0}^{t} \phi^{2j}\right) = \sum_{j=1}^{t+1} \phi^{2(j-1)} = \frac{(1)(1-\phi^{2(t+1)})}{1-\phi^2}$$

$$\boxed{Var[x_t] = \frac{\sigma_\omega^2}{1-\phi^2}\left(1-\phi^{2(t+1)}\right)}$$

2d.) $\gamma_x(h) = Cov(x_{t+h}, x_t) = Cov\left(\sum_{j=0}^{t+h} \phi^j \omega_{t+h-j}, \sum_{j=0}^{t} \phi^j \omega_{t-j}\right)$

$$Cov(\omega_{t+h} + \phi\omega_{t+h-1} + \phi^2\omega_{t+h-2} + \ldots, \omega_t + \phi\omega_{t-1} + \phi^2\omega_{t-2} + \ldots)$$

$$\gamma(h) = \begin{cases} Var[x_t] & h=0 \\ \phi\sigma_\omega^2 + \phi^3\sigma_\omega^2 + \ldots = \phi(\sigma_\omega^2 + \phi^2\sigma_\omega^2 + \ldots) = \phi\, Var[x_t] & h=1 \\ \phi^2\sigma_\omega^2 + \phi^4\sigma_\omega^2 + \ldots = \phi^2(\sigma_\omega^2 + \phi^2\sigma_\omega^2 + \ldots) = \phi^2\, Var[x_t] & h=2 \\ \vdots & \vdots \\ \phi^h\sigma_\omega^2 + \phi^{h+2}\sigma_\omega^2 + \ldots = \phi^h(\sigma_\omega^2 + \phi^2\sigma_\omega^2 + \ldots) = \phi^h\, Var(x_t) & h=h \end{cases}$$

Thus, this shows that

$$\boxed{\gamma_x(h) = Cov(x_{t+h}, x_t) = \phi^h\, Var[x_t]}$$

2e.) $E[x_t] = 0$ shown in part b.

$\gamma(h) = \phi^h\, Var[x_t]$ shown in part d.

$$= \phi^h \frac{\sigma_\omega^2}{1-\phi^2}(1+\phi^{2(t+1)})$$

We can see that the Covariance depends on time t so x_t is not stationary.

2f.) $E[x_t] = 0$

$$\gamma_x(h) = \phi^h \frac{\sigma_\omega^2}{1-\phi^2}(1+\phi^{2(t+1)})$$

As $t \to \infty$:

$$\lim_{t\to\infty} \gamma_x(h) = \lim_{t\to\infty} \phi^h \frac{\sigma_\omega^2}{1-\phi^2}(1+\phi^{2(t+1)})$$
$$= \phi^h \frac{\sigma_\omega^2}{1-\phi^2} \lim_{t\to\infty}(1+\phi^{2(t+1)})$$

We know $|\phi| < 1$ so

$$\lim_{t\to\infty}(1+\phi^{2(t+1)}) = 1$$

Thus

$$\lim_{t\to\infty} \gamma_x(h) = \phi^h \frac{\sigma_\omega^2}{1-\phi^2}$$

We can see that both $E[x_t]$ and $\gamma_x(h)$ are independent of time t when $t \to \infty$, so x_t is asymptotically stationary.

2g.) To simulate n observations of a stationary Gaussian AR(1) model, we can first general n independent and identically distributed $N(0,1)$.

To make the time series stationary, let $x_0 = \frac{\omega_0}{\sqrt{1-\phi^2}}$ and $x_t = \sum_{j=1}^{t} \phi_j \omega_{t-j}$ with $t = 1, 2, \ldots$

Therefore, by making $x_0 = \frac{\omega_0}{\sqrt{1-\phi^2}}$, we could use the results to simulate n observations of a stationary Gaussian AR(1) model.

2h.) $x_0 = \frac{\omega_0}{\sqrt{1-\phi^2}}$

$$E[x_t] = E\left[\frac{\omega_0}{\sqrt{1-\phi^2}} + \phi\omega_1 + \phi^2\omega_2 + \phi^3\omega_3 + \ldots\right]$$
$$= 0$$

$$\gamma_x(h) = Cov(x_{t+h}, x_t) = Cov\left(\sum_{j=0}^{t+h} \phi^j \omega_{t+h-j}, \sum_{j=0}^{t} \phi^j \omega_{t-j}\right)$$
$$= Cov(\omega_{t+h} + \phi\omega_{t+h-1} + \phi^2\omega_{t+h-2} + \ldots, \omega_t + \phi\omega_{t-1} + \phi^2\omega_{t-2} + \ldots)$$

$$\gamma_x(h) = \begin{cases} \sigma_\omega^2 + \phi^2\sigma_\omega^2 + \phi^4\sigma_\omega^2 + \ldots + \phi^{2t}\frac{\sigma_\omega^2}{1-\phi^2} & h=0 \\ \phi\sigma_\omega^2 + \phi^3\sigma_\omega^2 + \phi^5\sigma_\omega^2 + \ldots + \phi^{2t+1}\frac{\sigma_\omega^2}{1-\phi^2} & h=1 \\ \phi^2\sigma_\omega^2 + \phi^4\sigma_\omega^2 + \phi^6\sigma_\omega^2 + \ldots + \phi^{2t+2}\frac{\sigma_\omega^2}{1-\phi^2} & h=2 \\ \vdots & \vdots \\ \phi^h\sigma_\omega^2 + \phi^{h+2}\sigma_\omega^2 + \phi^{h+4}\sigma_\omega^2 + \ldots + \phi^{2t+h}\frac{\sigma_\omega^2}{1-\phi^2} & h=h \end{cases}$$

Thus,

$$\gamma_x(h) = \sigma_\omega^2\phi^h\left(1 + \phi^2 + \phi^4 + \phi^6 + \ldots + \phi^{2t}\frac{1}{1-\phi^2}\right)$$
$$= \sigma_\omega^2\phi^h\left(1 + \phi^2 + \phi^4 + \phi^6 + \ldots + \phi^{2t-2} + \phi^{2t}\frac{1}{1-\phi^2}\right)$$
$$= \sigma_\omega^2\phi^h\left[\sum_{j=0}^{t-1}\phi^{2j} + \phi^{2t}\frac{1}{1-\phi^2}\right]$$
$$= \sigma_\omega^2\phi^h\left[\sum_{j=1}^{t}\phi^{2(j-1)} + \phi^{2t}\frac{1}{1-\phi^2}\right]$$
$$= \sigma_\omega^2\phi^h\left[\frac{(1)(1-\phi^{2(t)})}{1-\phi^2} + \frac{\phi^{2t}}{1-\phi^2}\right]$$
$$= \sigma_\omega^2\left[\frac{1}{1-\phi^2}\right]$$

Thus, we can see that both $E[x_t]$ and $\gamma_x(h)$ are both independent of time t. With $x_0 = \frac{\omega_0}{1-\phi^2}$, it is stationary.