B. Multiple State Models

NOTATION

${}_tp_x^{ij}$ – probability that a subject in state i at time x will be in state j (where j may equal i) at time $x+t$

${}_tp_x^{\overline{ii}}$ – probability that a subject in state i at time x stays in state i continuously until time $x+t$

${}_tp_x^{\overline{ii}} \le {}_tp_x^{ii}$

DISCRETE MARKOV CHAINS

$\mathbf{P}^{(t)}$ is the transition matrix at time t

${}_tp_x^{ij} = ij$ entry of $\mathbf{P}^{(x)} \cdot \mathbf{P}^{(x+1)} \cdots \mathbf{P}^{(x+t-1)}$

${}_tp_x^{\overline{ii}} = p_x^{ii} \cdot p_{x+1}^{ii} \cdots p_{x+t-1}^{ii}$

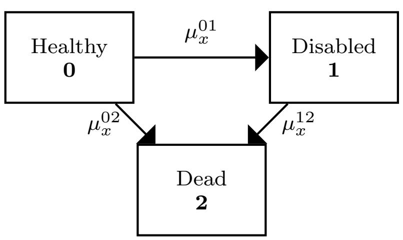
CONTINUOUS MARKOV CHAINS

$\mu_x^{ij} = \lim_{h \to 0} \frac{{}_hp_x^{ij}}{h}$ for $i \ne j$

${}_tp_x^{\overline{ii}} = \exp\left(-\int_0^t \mu_{x+s}^{i\bullet}\,ds\right)$

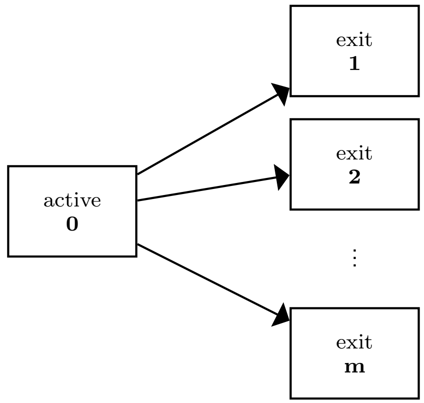
Probability of transition from one state to another depends on the model.

PERMANENT DISABILITY MODEL



${}_tp_x^{01} = \int_0^t {}_sp_x^{\overline{00}}\,\mu_{x+s}^{01}\,{}_{t-s}p_{x+s}^{\overline{11}}\,ds$

MULTIPLE DECREMENT MODEL



If all forces are constant multiples at all times,

${}_tp_x^{0j} = \frac{\mu_x^{0j}}{\mu_x^{0\bullet}}\left(1 - {}_tp_x^{00}\right)$

${}_tp_x^{00} = \exp\left(-\int_0^t \mu_{x+s}^{0\bullet}\,ds\right)$

KOLMOGOROV'S FORWARD EQNS

$$\frac{d}{dt}{}_tp_x^{ij} = \sum_{k=0,k\ne j}\left({}_tp_x^{ik}\,\mu_{x+t}^{kj} - {}_tp_x^{ij}\,\mu_{x+t}^{jk}\right)$$

$$\frac{d}{dt}{}_tp_x^{ij} = \begin{array}{c}\text{prob of}\\ \text{“move into } j\text{”}\\ \text{at time } x+t\end{array} - \begin{array}{c}\text{prob of}\\ \text{“move out of } j\text{”}\\ \text{at time } x+t\end{array}$$

Euler's method turns a continuous Markov Chain into a discrete chain with time increments of h and transition “probabilities”

$${}_hp_{x+t}^{ij} = \begin{cases} h\mu_{x+t}^{ij} & i \ne j \\ 1 - h\mu_{x+t}^{i\bullet} & i = j \end{cases}$$

PREMIUMS

$$\text{benefit premium} = \frac{\text{APV of benefit}}{\text{APV of annuity}}$$

ANNUITIES & INSURANCE

Pay 1 per year continuously while in state j:

$$\bar{a}_x^{ij} = \int_0^\infty e^{-\delta t}\,{}_tp_x^{ij}\,dt$$

If payable at the start of the year:

$$\ddot{a}_x^{ij} = \sum_{k=0}^\infty v^k\,{}_kp_x^{ij}$$

Pay 1 on each future transfer into k:

$$\bar{A}_x^{ik} = \int_0^\infty e^{-\delta t}\sum_{j\ne k}{}_tp_x^{ij}\,\mu_{x+t}^{jk}\,dt$$

CONSTANT FORCE, NO RE-ENTRY

If all transition forces are constant and no re-entry into a state

$$\bar{a}_x^{ii} = \frac{1}{\mu^{i\bullet} + \delta}$$

For the multiple decrement model with constant transitions:

$$\bar{A}_x^{0j} = \frac{\mu^{0j}}{\mu^{0\bullet} + \delta}$$

RESERVES

${}_tV^{(i)}$ – reserve at duration t for a subject in state i at that time

$B_t^{(i)}$ – rate of payment of benefit while the policyholder is in state i

$S_t^{(ij)}$ – lump sum benefit payable instantaneously at time t on transition from state i to state j

General Recursion for h-yearly Cash Flows

$$\left({}_tV^{(j)} + hP_t^{(j)}\right)(1+i)^h = \sum_{k=0}^m {}_hp_{x+t}^{jk}\left(hB_{t+h}^k + S_{t+h}^{(jk)} + {}_{t+h}V^{(k)}\right)$$

Payments at end of period must not depend on intermediate transitions

For benefits paid on transition, replace $S_{t+h}^{(jk)}$ by $S_{t+h}^{(jk)}(1+i)^{h/2}$, the claims acceleration approach.

THIELE'S DIFFERENTIAL EQN

$$\frac{d}{dt}{}_tV^{(i)} = \delta_t\,{}_tV^{(i)} - B_t^{(i)} - \sum_{j=0,j\ne i}^n \mu_{x+t}^{ij}\left(S_t^{(ij)} + {}_tV^{(j)} - {}_tV^{(i)}\right)$$

OTHER INCOME INSURANCE

Disability Income Insurance:
Replace income for individuals who cannot work

Waiting or Elimination Period: time between the beginning of a period of disability and the beginning of the benefit payment

Off Period: required interval for two periods of disability to be considered separately rather than together

Long Term Care Insurance:
Benefit paid while care is required

Critical Illness Insurance:
Lump sum benefit on diagnosis of a specified disease(s) or condition

Chronic Illness Insurance:
Pays a benefit on diagnosis of chronic illness

Hospital Indemnity Insurance (HII):
Lump sum each time the insured is admitted to a hospital

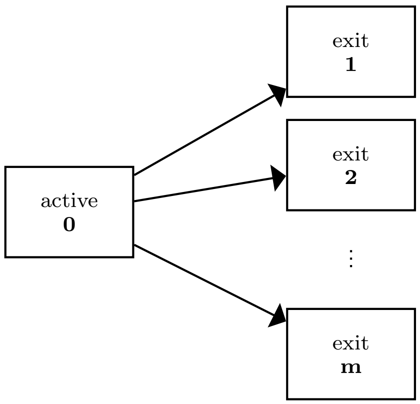
Continuing Care Ret. Communities
residential facilities for seniors with different levels of medical and personal support provided as residents age

Structured Settlements:
Payment schedule agreed between the Injured Party (IP) and Responsible party (RP)

Continuous Sojourn Annuity: payment paid while chain remains in a given state (payments cease upon exit from state)

$$\bar{a}_{x:\overline{n}|}^{\overline{ii}} = \int_0^n v^t\,{}_tp_x^{\overline{ii}}\,dt$$

MULTIPLE DECREMENT MODEL



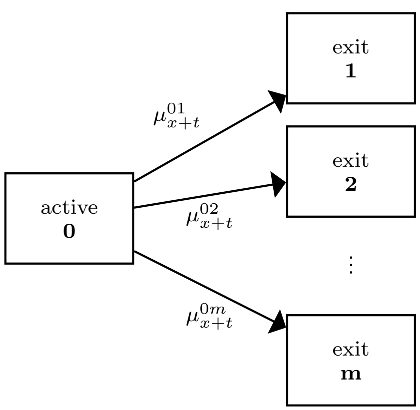
$q_x^{(j)}$ – prob. of decrement due to cause #j

$q_x^{(\tau)}$ – prob. of decrement due to any cause

$${}_tq_x^{(\tau)} = \sum_{j=1}^{m} {}_tq_x^{(j)}$$

$$A_x^{(j)} = \sum_{k=1}^{\infty} v^k\, {}_{k-1}p_x^{(\tau)}\, q_{x+k-1}^{(j)}$$

FORCES OF DECREMENT - 1



0 is redundant so we use:

$$\mu^{0j}_{x+t} = \mu^{(j)}_{x+t}$$

FORCE OF DECREMENT - 2

$$\mu^{(\tau)}_{x+t} = \frac{\frac{d}{dt}\,{}_tq_x^{(\tau)}}{{}_tp_x^{(\tau)}}$$

$$\mu^{(j)}_{x+t} = \frac{\frac{d}{dt}\,{}_tq_x^{(j)}}{{}_tp_x^{(\tau)}}$$

$$\mu^{(\tau)}_{x+t} = \mu^{(1)}_{x+t} + \mu^{(2)}_{x+t} + \cdots + \mu^{(m)}_{x+t}$$

$${}_tp_x^{(\tau)} = \exp\left(-\int_0^t \mu^{(\tau)}_{x+s}\, ds\right)$$

$${}_tq_x^{(j)} = \int_0^t {}_sp_x^{(\tau)} \cdot \mu^{(j)}_{x+s}\, ds$$

$$\Pr(J = j \,|\, T = t) = \frac{\mu^{(j)}_{x+t}}{\mu^{(\tau)}_{x+t}}$$

If all forces are a constant multiple of the total force for all t

$${}_tq_x^{(j)} = \frac{\mu_x^{(j)}}{\mu_x^{(\tau)}}\left(1 - {}_tp_x^{(\tau)}\right)$$

$$\bar{A}_x^{(j)} = \int_0^\infty v^t\, {}_tp_x^{(\tau)}\, \mu^{(j)}_{x+t}\, dt$$

CONSTANT TRANSITION FORCES

$${}_tq_x^{(j)} = \frac{\mu^{(j)}}{\mu^{(\tau)}}\left(1 - {}_tp_x^{(\tau)}\right)$$

$${}_tq_x^{(j)} = \frac{q_x^{(j)}}{q_x^{(\tau)}}\left(1 - \left(p_x^{(\tau)}\right)^t\right)$$

$$\bar{A}_x^{(j)} = \frac{\mu^{(j)}}{\mu^{(\tau)} + \delta}$$

$$\bar{a}_x = \frac{1}{\mu^{(\tau)} + \delta}$$

$$\bar{A}^{(j)}_{\overset{1}{x}:\overline{n|}} = \frac{\mu^{(j)}}{\mu^{(\tau)}+\delta}\left(1 - {}_nE_x^{(\tau)}\right)$$

ABSOLUTE RATE OF DECREMENT

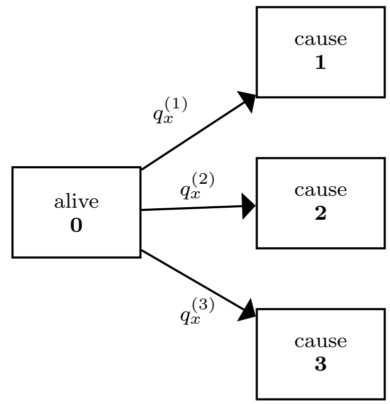
$${}_tq'^{(j)}_x = 1 - {}_tp'^{(j)}_x = 1 - \exp\left(-\int_0^t \mu^{(j)}_{x+s}\, ds\right)$$

$${}_tp_x^{(\tau)} = {}_tp'^{(1)}_x \cdot {}_tp'^{(2)}_x \cdots {}_tp'^{(m)}_x$$

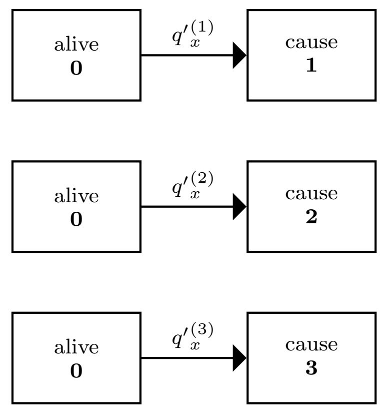
ABSOLUTE RATE OF DECREMENT

Each cause is independent of the other causes.

Probabilities of Decrement



Rates of Decrement



Notice how in the rate of decrement "worlds" there is only one cause of decrement. For the rates of decrement we have 3 tables and for the multiple decrement "world" we have one table with 3 competing decrements.

$${}_tq'^{(j)}_x \geq {}_tq_x^{(j)}$$

MDT vs. AST

Super secret approximation:

$$q'^{(j)}_x = \frac{d_x^{(j)}}{\ell_x^{(\tau)} - \frac{1}{2}\sum_{i \neq j} d_x^{(i)}}$$

Constant Force or UDDMDT:

$$p'^{(j)}_x = \left(p_x^{(\tau)}\right)^{q_x^{(j)}/q_x^{(\tau)}}$$

UDDAST:

For 2 UDDASTs use the midpoint:

$$q_x^{(1)} = q'^{(1)}_x\left(1 - \tfrac{1}{2}q'^{(2)}_x\right)$$

For 3 or more UDDASTs use integration:

$${}_tp'^{(j)}_x \cdot \mu^{(j)}_{x+t} = q'^{(j)}_x \text{ - factor out of integral}$$

For 3 UDDASTs (if you like memorizing):

$$q_x^{(1)} = q'^{(1)}_x\left(1 - \tfrac{1}{2}\left(q'^{(2)}_x + q'^{(3)}_x\right) + \tfrac{1}{3}q'^{(2)}_x\, q'^{(3)}_x\right)$$

Hybrids:

Draw a picture. For 2 competing UDDASTs use the midpoint of the interval.

TRANSITION FORCE ESTIMATION

Maximum Likelihood Estimates:

$$\hat{\mu}_x^{ik} = \frac{D^{ik}}{T^{(i)}} \text{ and } \mathrm{Var}\left[\hat{\mu}_x^{ik}\right] \approx \frac{D^{ik}}{(T^{(i)})^2}$$

where

D^{ik} = # of transitions from State i directly to State k

$T^{(i)}$ = total waiting time in State i

For log-confidence intervals:

$$\mathrm{Var}\left[\log \hat{\mu}_x^{ik}\right] \approx \frac{1}{D^{ik}}$$