*HW5

5.17. Show that $x_t = \sum_{j=1}^{\infty} (1-\lambda)\lambda^{j-1} x_{t-j} + \omega_t$ given $\omega_t = \sum_{j=0}^{\infty} \lambda^j y_{t-j}$

Proof.

Given $\omega_t = \sum_{j=0}^{\infty} \lambda^j y_{t-j}$

$$= \sum_{j=0}^{\infty} \lambda^j (x_{t-j} - x_{t-j-1})$$

$$= (x_t - x_{t-1}) + \lambda(x_{t-1} - x_{t-2}) + \lambda^2(x_{t-2} - x_{t-3}) + \lambda^3(x_{t-3} - x_{t-4}) + \dots$$

$$= x_t - x_{t-1} + \lambda x_{t-1} - \lambda x_{t-2} + \lambda^2 x_{t-2} - \lambda^2 x_{t-3} + \lambda^3 x_{t-3} - \lambda^3 x_{t-4} + \dots$$

$$= x_t - (1-\lambda)x_{t-1} - \lambda(1-\lambda)x_{t-2} - \lambda^2(1-\lambda)x_{t-3} - \dots$$

$$= x_t - (1-\lambda)(x_{t-1} + \lambda x_{t-2} + \lambda^2 x_{t-3} + \dots)$$

$$= x_t - (1-\lambda)\left(\sum_{j=1}^{\infty} \lambda^{j-1} x_{t-j}\right)$$

Thus, we can rewrite it as:

$$\boxed{x_t = \sum_{j=1}^{\infty} (1-\lambda)\lambda^{j-1} x_{t-j} + \omega_t \quad (5.24)}$$

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Show that $x_{n+1}^n = \sum_{j=1}^{\infty} (1-\lambda)\lambda^{j-1} x_{n+1-j} = (1-\lambda)x_n + \lambda x_n^{n-1}$

given $x_n^{n-1} = \sum_{j=1}^{\infty} (1-\lambda)\lambda^{j-1} x_{n-j}$

From 5.24:

$$x_{n+1}^n = \sum_{j=1}^{\infty} (1-\lambda)\lambda^{j-1} x_{n+1-j}$$

$$= (1-\lambda)x_n + \sum_{j=2}^{\infty} (1-\lambda)\lambda^{j-1} x_{n+1-j}$$

Let $i = j-1$

$$x_n^{n+1} = (1-\lambda)x_n + \sum_{i=1}^{\infty} (1-\lambda)\lambda^{i} x_{n-i}$$

$$= (1-\lambda)x_n + \lambda \sum_{i=1}^{\infty} (1-\lambda)\lambda^{i-1} x_{n-i}$$

Notice that $x_n^{n-1} = \sum_{j=1}^{\infty} (1-\lambda)\lambda^{j-1} x_{n-j}$, so:

$$\boxed{x_n^{n+1} = (1-\lambda)x_n + \lambda x_n^{n-1}} \quad (5.25)$$

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