Topological Manifolds

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Manifolds

- An m-manifold is a Hausdorff Space X with a countable basis such that each point x of X has a neighborhood that is homeomorphic with an open subset of $\mathbb{R}^m$.
- This means that for every point P in X, there is an open neighborhood U of P and a homeomorphism $f : U \to V$, which maps the set U onto an open set $V \subset \mathbb{R}^m$.

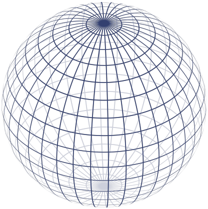

Figure: 2-Manifolds

Regular Spaces

- A topological space X is called regular if points and closed sets can be separated by neighborhoods. That is, for each $x \in X$ and closed set $B \subset X$, there exists disjoint open sets U and V such that U contains x and V contains B.

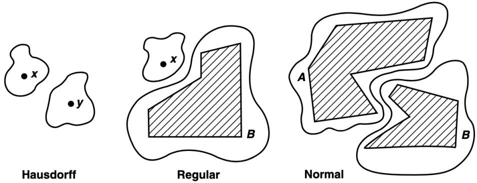

Figure: Separation Axiom's Space

Proof

Prove that every manifold is regular and hence metrizable.

Let X be a m-manifold. Let $x \in X$, let U be a neighborhood of x, and let W be a neighborhood of x, so that there is a homeomorphism $W \to f(W) \subset \mathbb{R}^m$. This means $x \in U \cap W \neq \emptyset$ with $U \cap W \xrightarrow[f|_{U\cap W}]{\sim} f(U \cap W)$, which the restriction $f(|_{U\cap W})^{-1} = (f^{-1})|_{U\cap W}$. This means that $x \subset U \cap W \subset U$. Since X is a manifold, there exists homeomorphism $f : U \to f(U) \subset \mathbb{R}^m$.

Proof (Cont.)

Theorem

Let X be a Hausdorff space. Then X is locally compact if and only if given x in X, and given a neighborhood U of x, there is a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subset U$.

$\mathbb{R}^m$ is a locally compact space since $x \in B_{\epsilon-\delta}(x) \subset B_{\epsilon}(x) \subset U, 0 < \delta < \epsilon$. Therefore, since $\mathbb{R}^m$ is locally compact and X is Hausdorff space, there exists a neighborhood $V \subset f(U)$ of $f(x)$ such that $\bar{V}$ is compact and $f(x) \in \bar{V} \subset f(U)$ by the Theorem.

Proof (Cont.)

Lemma

Let X be a topological space. Let one-point sets in X be closed. X is regular is and only if given a point x of X and a neighborhood U of x, there is a neighborhood V of x such that $\bar{V} \subset U$.

We can see that
$x \in f^{-1}(V) \subset \overline{f^{-1}(V)} \subset f^{-1}(\bar{V}) \subset f^{-1}(f(U)) = U$. Since
$x \in \overline{f^{-1}(V)} \subset U$, X is a regular space by the Lemma.

Proof (Cont.)

Theorem (Urysohn Metrization Theorem)

Every regular space X with a countable basis is metrizable.

By definition of m-manifolds, X is a regular space with countable basis, so X is metrizable by Urysohn Metrization Theorem.