

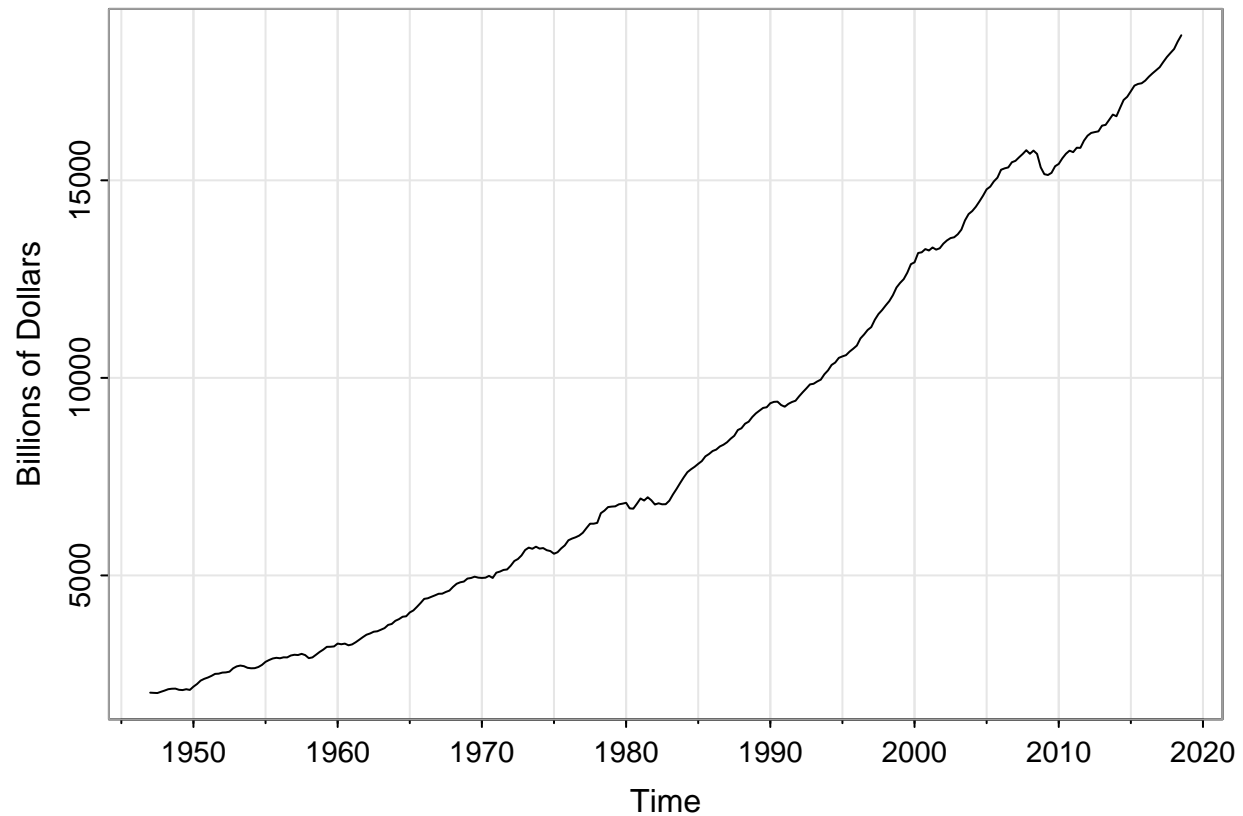
# STAT4870 HW5

Anton Yang

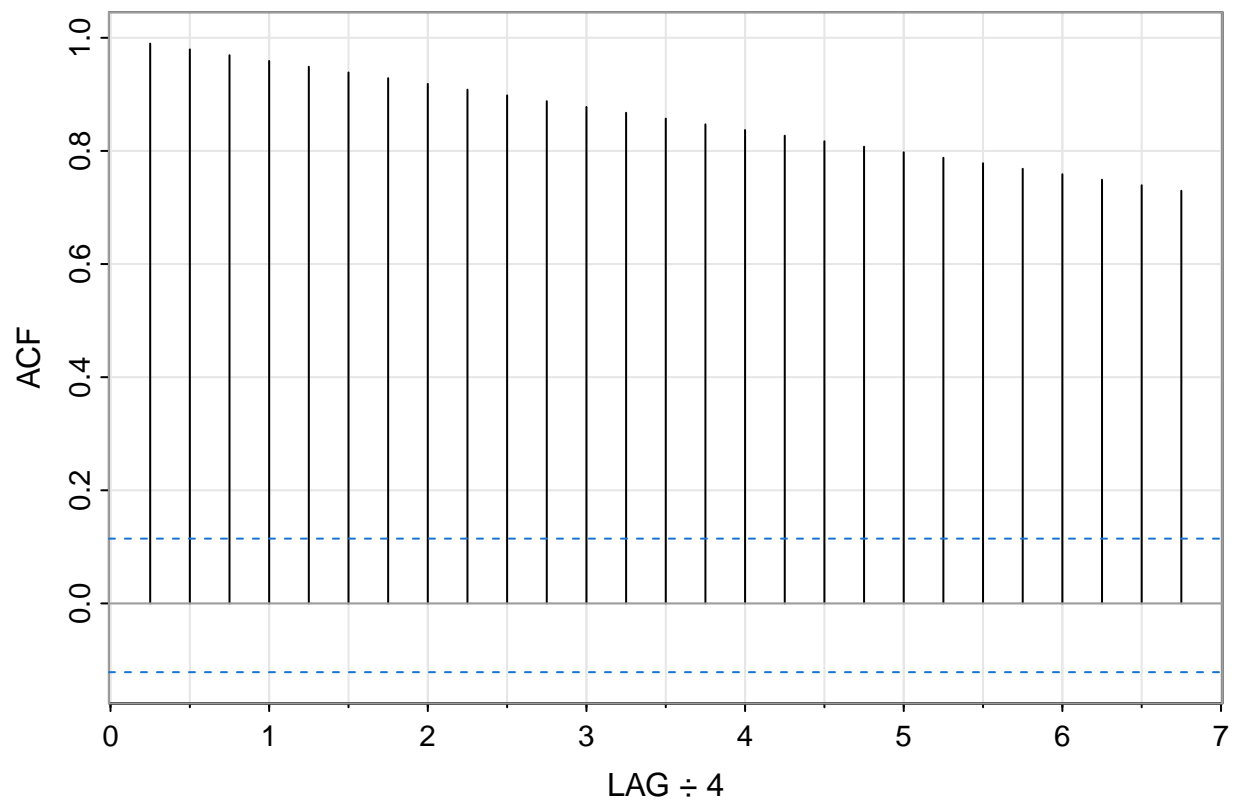
2024-10-28

## Question 5.2

```
library(astsa)
data<-gdp
tsplot(data, ylab = "Billions of Dollars")
```



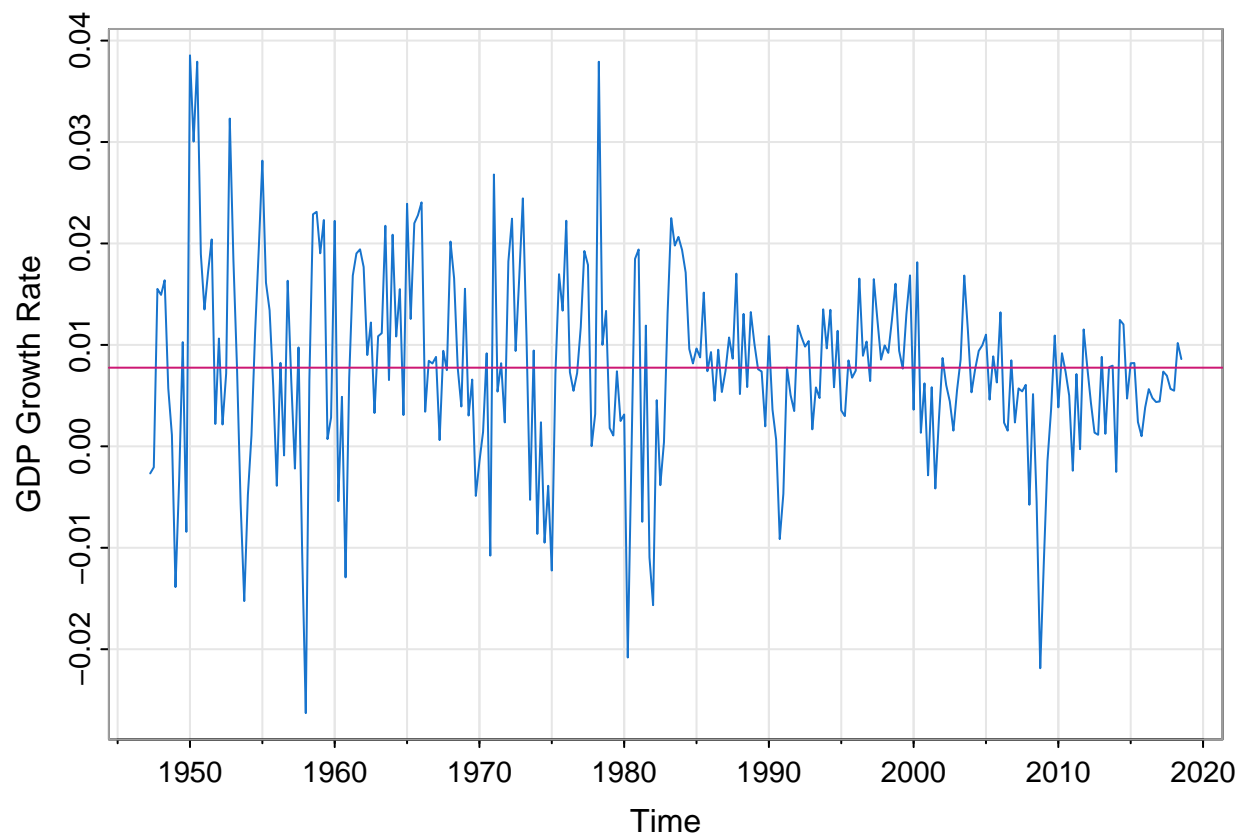
```
acf1(data, main = "")
```



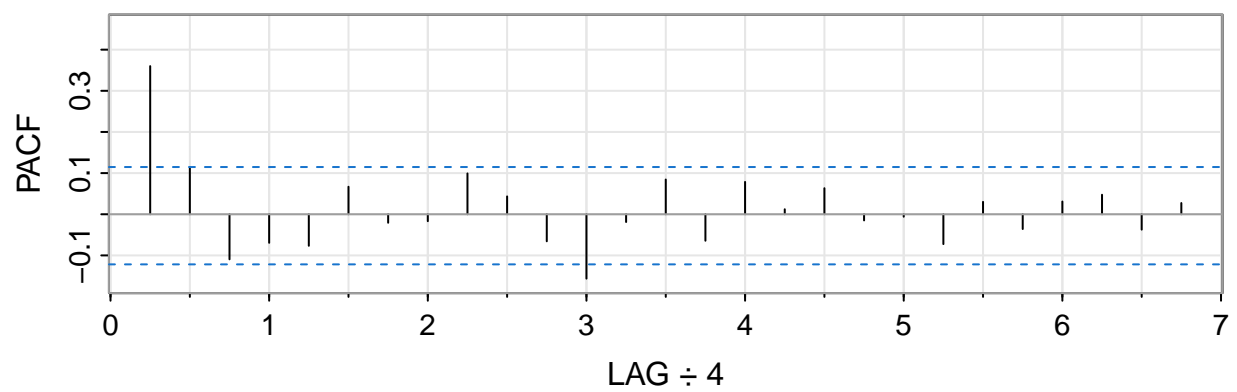
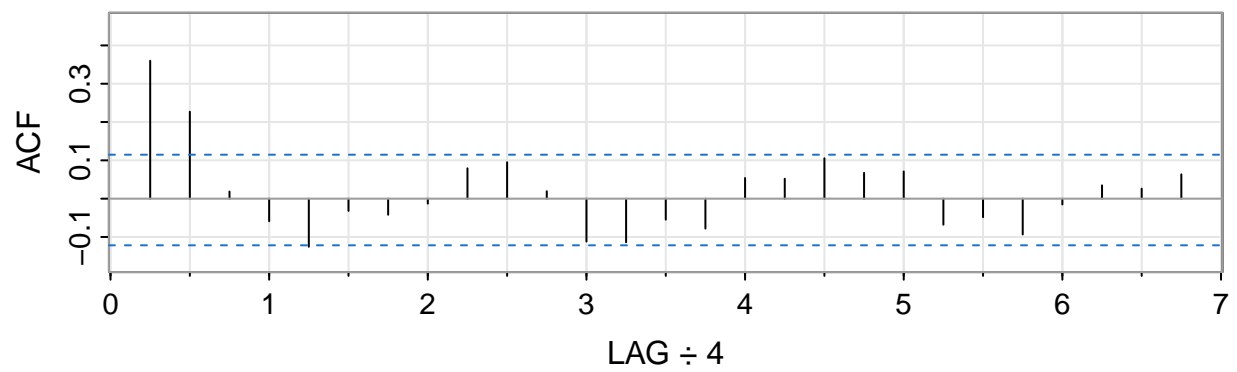
```
## [1] 0.99 0.98 0.97 0.96 0.95 0.94 0.93 0.92 0.91 0.90 0.89 0.88 0.87 0.86 0.85
## [16] 0.84 0.83 0.82 0.81 0.80 0.79 0.78 0.77 0.76 0.75 0.74 0.73
```

GDP data shows US GDP each quarter from each year from 1947 to 2018. First, we plot the GDP, and we can clearly see that there's a positive trend. This mean that this is not stationary.

```
tsplot(diff(log(data)), ylab = "GDP Growth Rate", col = 4)
mean<-mean(diff(log(data)))
abline(h = mean, col = 6)
```



```
acf2(diff(log(gdp)), main = "")
```

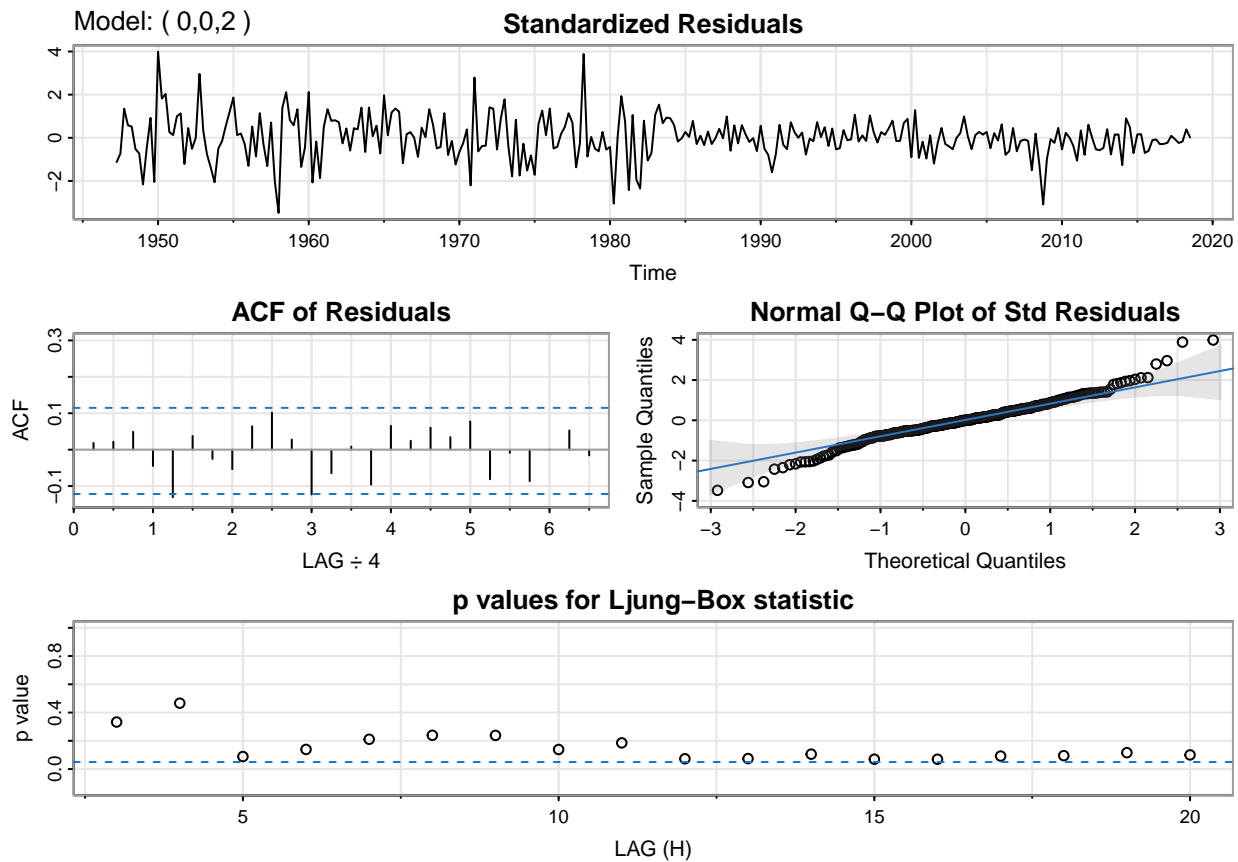


```
##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10] [,11] [,12] [,13]
## ACF  0.36 0.23  0.02 -0.06 -0.13 -0.03 -0.04 -0.01 0.08  0.10  0.02 -0.11 -0.11
## PACF 0.36 0.11 -0.11 -0.07 -0.08  0.07 -0.02 -0.02 0.10  0.04 -0.07 -0.16 -0.02
##      [,14] [,15] [,16] [,17] [,18] [,19] [,20] [,21] [,22] [,23] [,24] [,25]
## ACF  -0.05 -0.08  0.05  0.05  0.11  0.07  0.07 -0.07 -0.05 -0.09 -0.02  0.03
## PACF  0.08 -0.06  0.08  0.01  0.06 -0.02 -0.01 -0.07  0.03 -0.04  0.03  0.05
##      [,26] [,27]
## ACF   0.03  0.06
## PACF -0.04  0.03
```

From the ACF, we can see that the autocorrelation is getting smaller as the lags getting higher. To make it stationary, we'll plot the difference of the log(GDP), and the plot shows more homogeneous data, and we can see that it looks more stationary. We want to check the autocorrelation and the partial autocorrelation of the difference of the log(GDP). By analyzing the sample ACF and PACF, we might feel that the ACF is cutting off at lag 2 and the PACF is tailing off, which suggests an MA(2) model.

```
sarima(diff(log(gdp)), 0, 0, 2)
```

```
## initial value -4.672758
## iter 2 value -4.749239
## iter 3 value -4.750696
## iter 4 value -4.750723
## iter 5 value -4.750724
## iter 6 value -4.750725
## iter 7 value -4.750725
## iter 7 value -4.750725
## iter 7 value -4.750725
## final value -4.750725
## converged
## initial value -4.751078
## iter 2 value -4.751080
## iter 3 value -4.751080
## iter 4 value -4.751081
## iter 5 value -4.751081
## iter 5 value -4.751081
## iter 5 value -4.751081
## final value -4.751081
## converged
## <><><><><><><><><><><><><><>
##
## Coefficients:
##      Estimate      SE t.value p.value
## ma1      0.3070 0.0579  5.2988      0
## ma2      0.2258 0.0547  4.1271      0
## xmean     0.0077 0.0008  9.8631      0
##
## sigma^2 estimated as 7.464598e-05 on 283 degrees of freedom
##
## AIC = -6.636312 AICc = -6.636015 BIC = -6.58518
##
```



From the model, we fit:

$$\hat{x}_t = 0.0077 + 0.303\hat{w}_{t-1} + 0.204\hat{w}_{t-2} + \hat{w}_t$$

where  $\hat{\sigma}_w = 0.00007464598$  is based on 283 degrees of freedom.

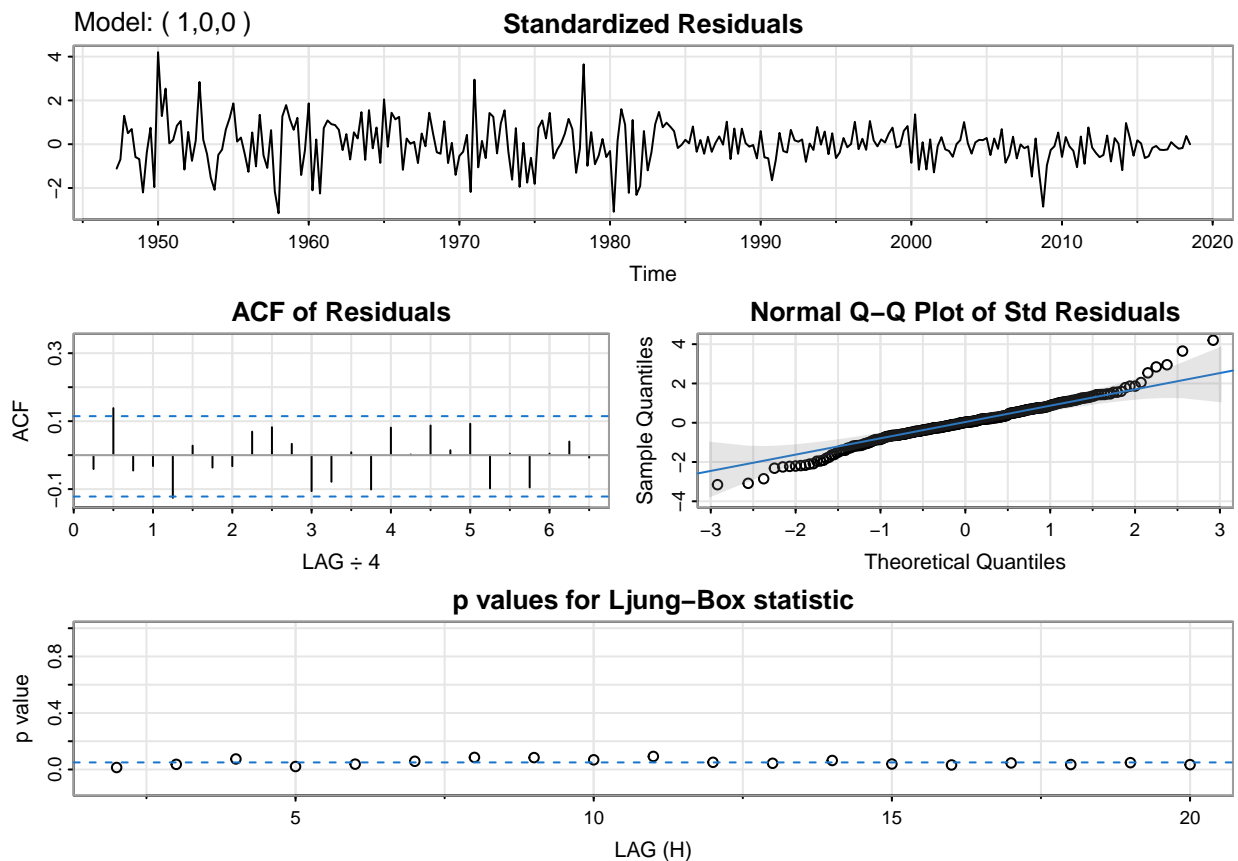
From the model summary, we can see all of the regression coefficients including the constant are very significant to the model.

We can also argue that ACF is tailing off, and PACF cut off at lag 1. Therefore, we want to analyze the AR(1) model.

```
sarima(diff(log(gdp)), 1, 0, 0)
```

```
## initial value -4.673186
## iter 2 value -4.742918
## iter 3 value -4.742921
## iter 4 value -4.742923
## iter 5 value -4.742925
## iter 6 value -4.742925
## iter 6 value -4.742925
## final value -4.742925
## converged
## initial value -4.742229
## iter 2 value -4.742234
## iter 3 value -4.742245
## iter 3 value -4.742245
## iter 3 value -4.742245
## final value -4.742245
```

```
## converged
## <><><><><><><><><><>
##
## Coefficients:
##      Estimate      SE t.value p.value
## ar1      0.3603 0.0551  6.5366      0
## xmean     0.0077 0.0008  9.5915      0
##
## sigma^2 estimated as 7.598489e-05 on 284 degrees of freedom
##
## AIC = -6.625634  AICc = -6.625485  BIC = -6.587284
##
```



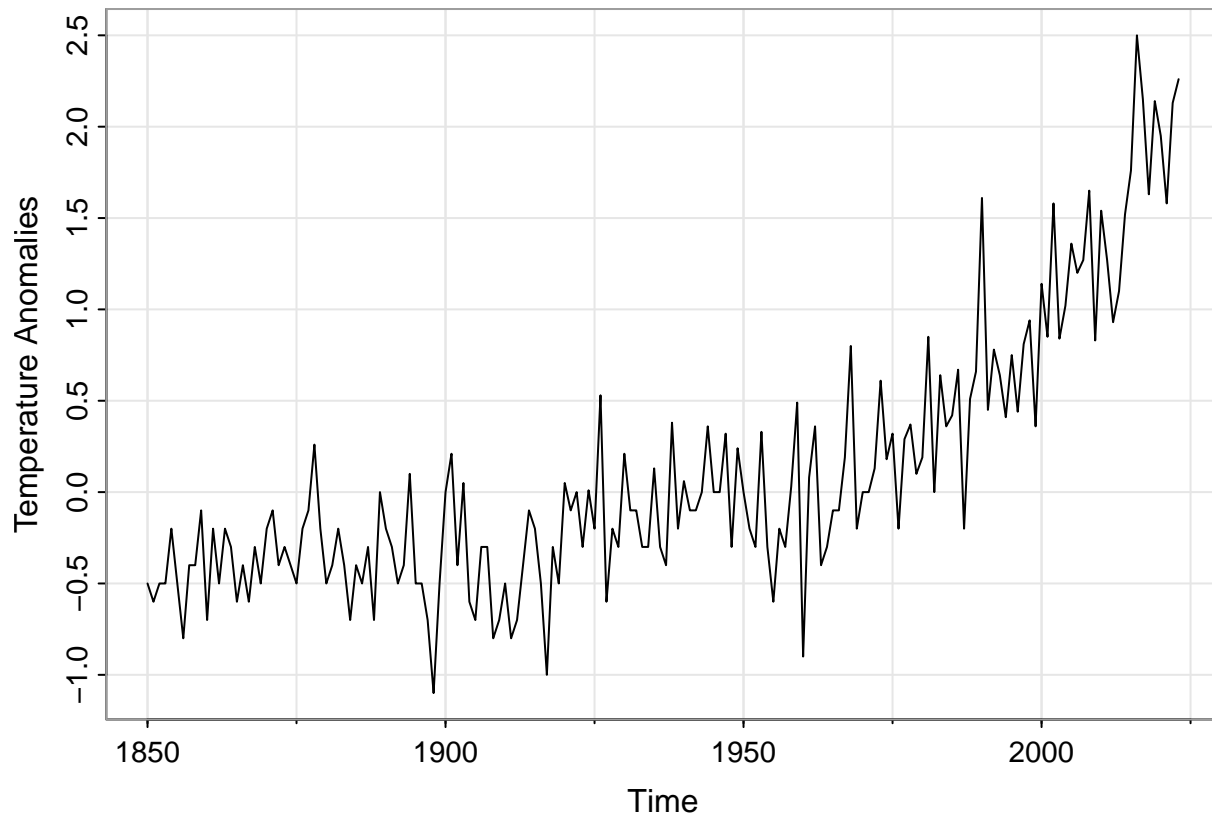
$$\hat{x}_t = 0.0077 + 0.3603\hat{x}_{t-1} + \hat{w}_t$$

We can see that AR(1) model also fits very well with all of the regression coefficients being very significant, and the model has a slightly higher AIC than MA(2), but it has a slightly lower BIC. Since it's better to choose a parsimony model, it might be better to prefer a simpler model AR(1).

However, by investigating the Ljung-Box statistic, we can see that AR(1) clearly looks different than MA(2) model. From the MA(2) model, we can see all the p-values exceed 0.05, so we can feel comfortable not rejecting the null hypothesis that the residuals are white. However, AR(1) have all the p-values of lower than 0.05, so we will reject the null hypothesis that the residuals are white. Therefore, from the diagnostic, we can say that MA(2) could be a better model for GDP data.

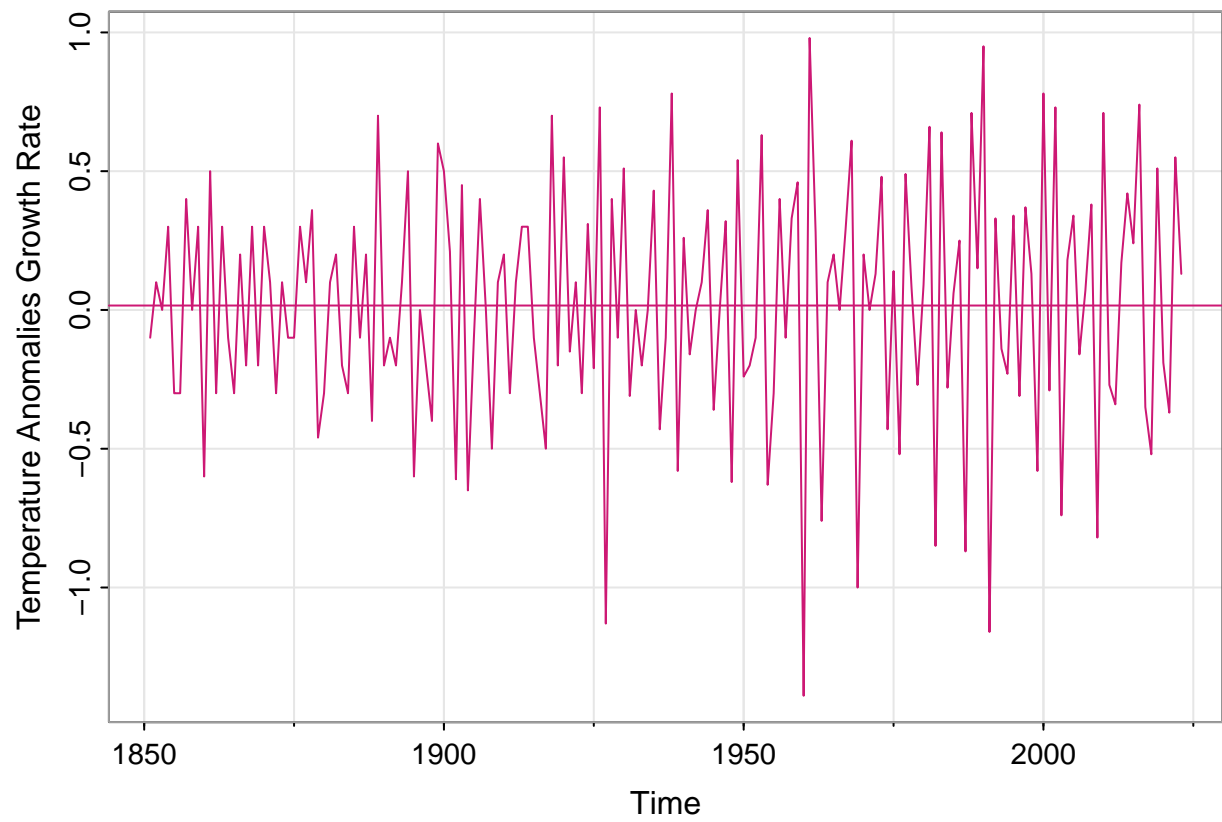
### Question 5.4

```
data<-gtemp_land  
tsplot(data, ylab = "Temperature Anomalies")
```

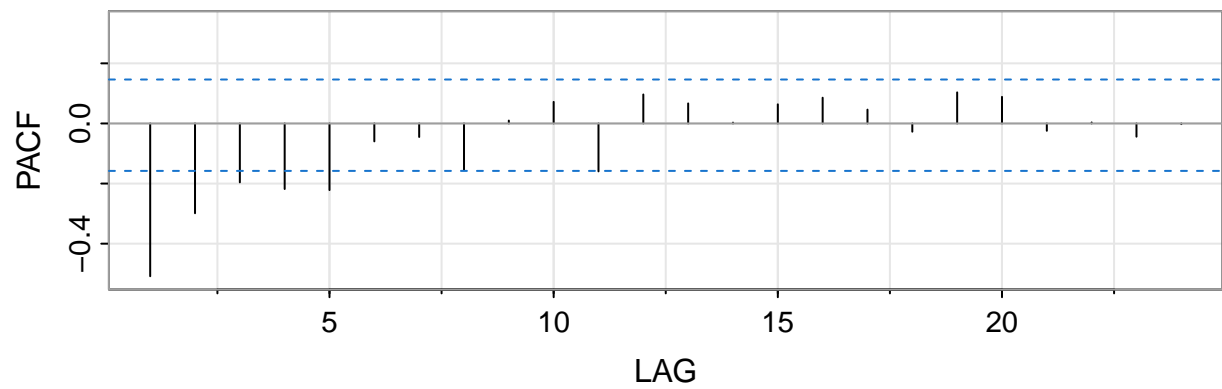
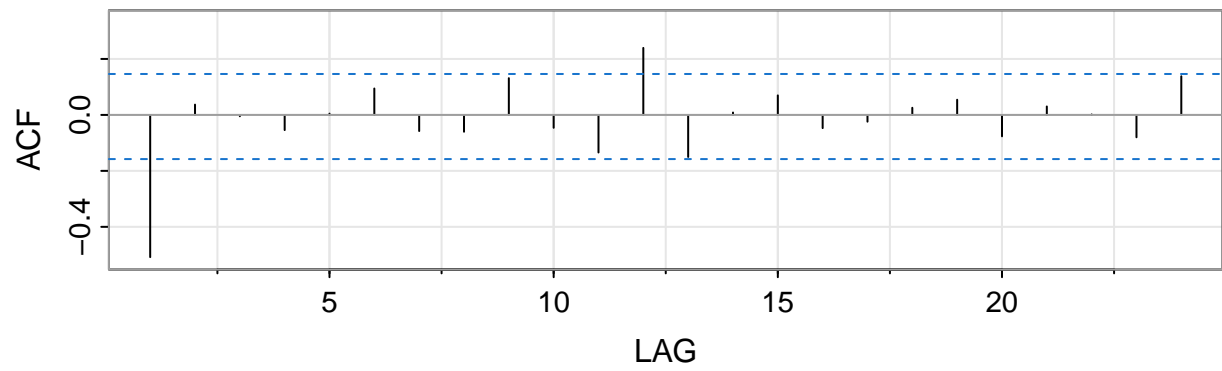


We can clearly see that the temperature anomalies has a positive trend, which means it's not stationary. To make it stationary, we'll take the difference of the gtemp\_land.

```
tsplot(diff(data), col = 6, ylab = "Temperature Anomalies Growth Rate")  
mean<-mean(diff(data))  
abline(h = mean, col = 6)
```



```
acf2(diff(data), main = "")
```





```
##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10] [,11] [,12]
## ACF  -0.51  0.04  0.0 -0.05  0.00  0.09 -0.06 -0.06  0.13 -0.05 -0.13  0.24
## PACF -0.51 -0.30 -0.2 -0.22 -0.22 -0.06 -0.04 -0.15  0.01  0.07 -0.16  0.10
##      [,13] [,14] [,15] [,16] [,17] [,18] [,19] [,20] [,21] [,22] [,23] [,24]
## ACF  -0.15  0.01  0.07 -0.05 -0.02  0.03  0.05 -0.08  0.03    0 -0.08  0.14
## PACF  0.07  0.00  0.06  0.09  0.05 -0.03  0.10  0.09 -0.02    0 -0.04  0.00
```

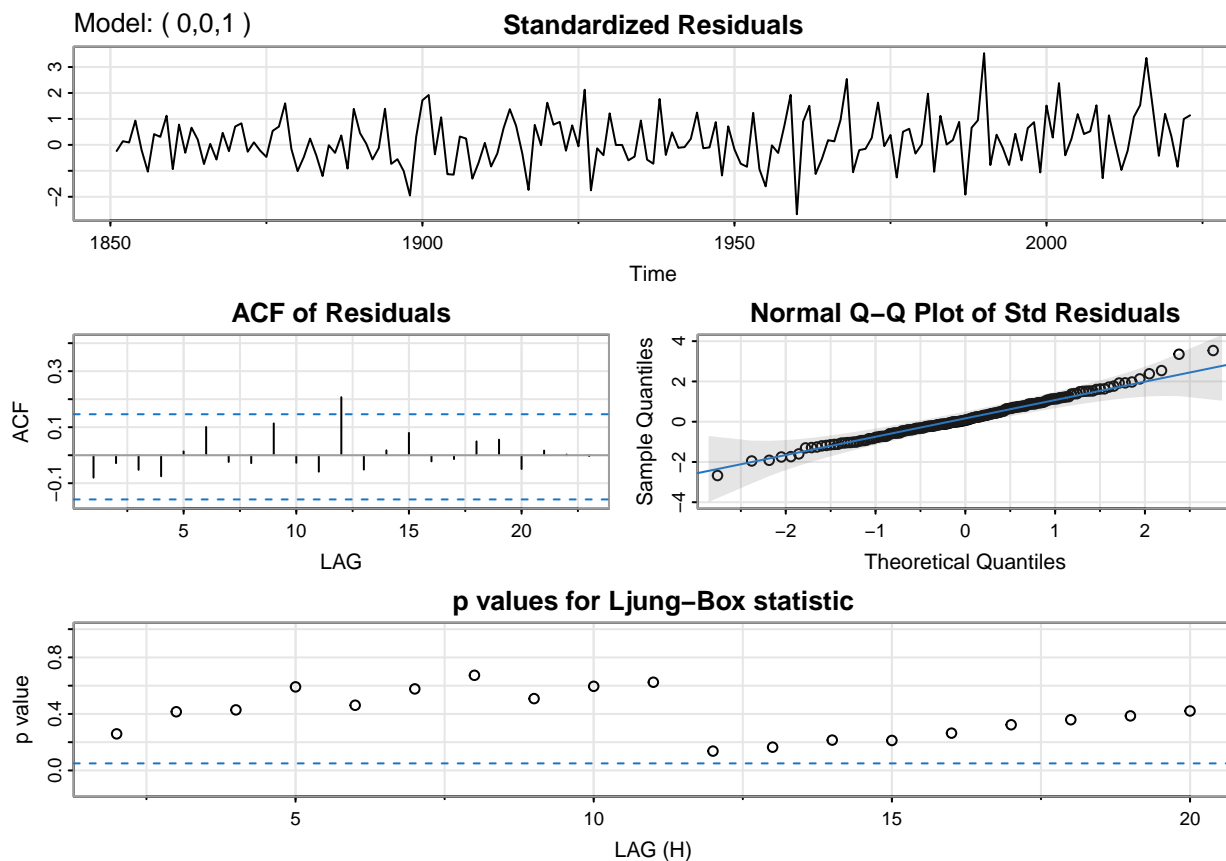
By inspecting the sample ACF and PACF plot, we feel that ACF is cutoff at lag 1 and PACF is tailing off, and this suggest an AR(1) model.

```
sarima(diff(data), 0, 0, 1)
```

```
## initial  value -0.842449
## iter    2 value -1.043995
## iter    3 value -1.086437
## iter    4 value -1.105162
## iter    5 value -1.106533
## iter    6 value -1.107112
## iter    7 value -1.107288
## iter    8 value -1.107352
## iter    9 value -1.107368
## iter   10 value -1.107368
## iter   10 value -1.107368
## iter   10 value -1.107368
## final   value -1.107368
## converged
## initial  value -1.104513
## iter    2 value -1.104517
## iter    3 value -1.104520
## iter    4 value -1.104533
## iter    4 value -1.104533
## iter    4 value -1.104533
## final   value -1.104533
## converged
## <><><><><><><><><><><><><><>
##
## Coefficients:
##      Estimate      SE t.value p.value
## ma1      -0.7974 0.0395 -20.1909  0.0000
## xmean     0.0145 0.0052   2.7905  0.0059
##
## sigma^2 estimated as 0.1091637 on 171 degrees of freedom
##
## AIC = 0.663493  AICc = 0.663901  BIC = 0.7181744
##
```



```
##
## Coefficients:
##      Estimate      SE  t.value p.value
## ma1  -0.7545 0.0423 -17.8414      0
##
## sigma^2 estimated as 0.1133625 on 172 degrees of freedom
##
## AIC = 0.6887024  AICc = 0.6888376  BIC = 0.7251567
##
```



We can see from MA(1) with no intercept, the regression coefficient is still significant, and both AIC and BIC are higher than the original MA(1) model, which suggests that MA(1) with the intercept is a better model.

We can also argue that the model could be ARMA(2,1) where ACF cutoff at lag 1 and PACF cutoff at lag 2.

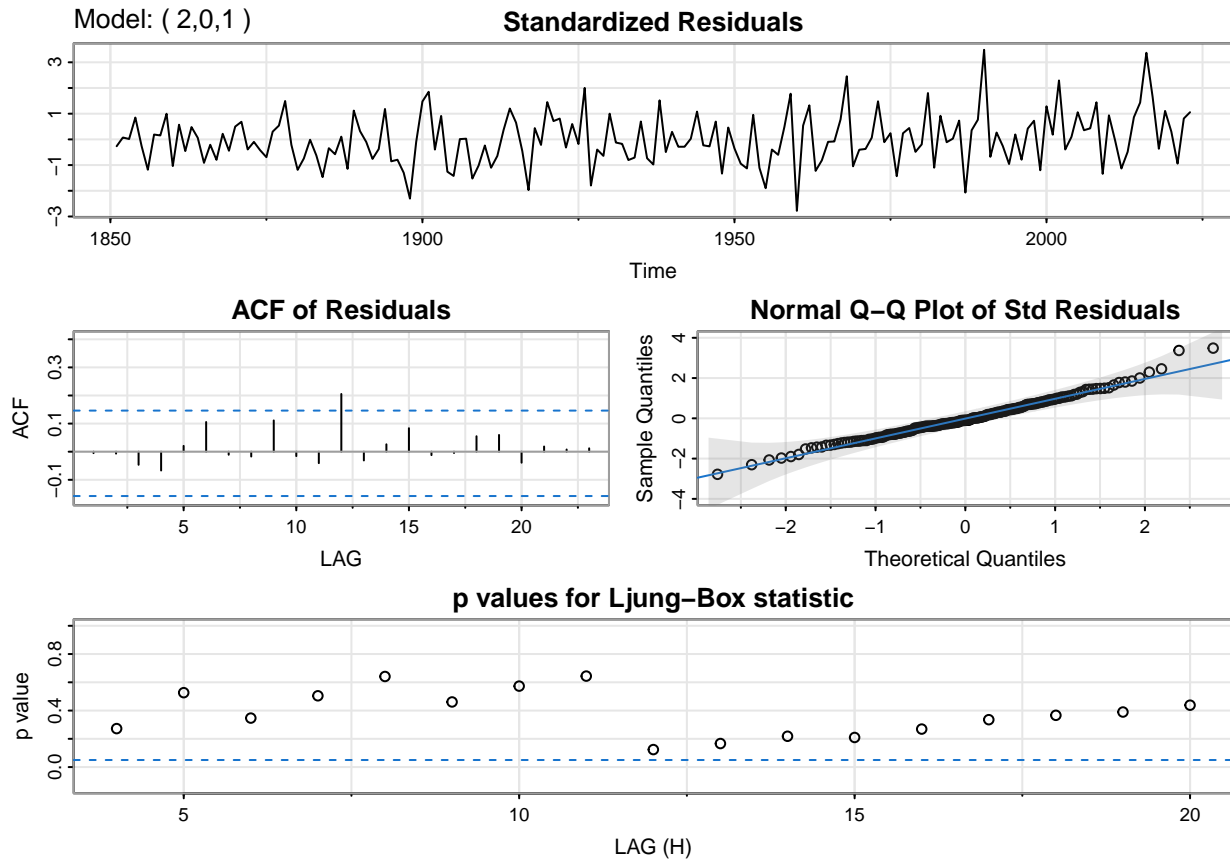
```
sarima(diff(data), 2, 0, 1)
```

```
## initial value -0.836955
## iter 2 value -1.021219
## iter 3 value -1.076671
## iter 4 value -1.078385
## iter 5 value -1.086027
## iter 6 value -1.098045
## iter 7 value -1.101561
## iter 8 value -1.102978
```

```

## iter   9 value -1.103031
## iter  10 value -1.103044
## iter  11 value -1.103052
## iter  12 value -1.103054
## iter  13 value -1.103055
## iter  14 value -1.103055
## iter  15 value -1.103055
## iter  15 value -1.103055
## iter  15 value -1.103055
## final  value -1.103055
## converged
## initial value -1.105579
## iter   2 value -1.105599
## iter   3 value -1.105601
## iter   4 value -1.105602
## iter   5 value -1.105602
## iter   6 value -1.105602
## iter   6 value -1.105602
## iter   6 value -1.105602
## final  value -1.105602
## converged
## <><><><><><><><><><><><><><>
##
## Coefficients:
##      Estimate      SE t.value p.value
## ar1    -0.0579 0.0954  -0.6067  0.5448
## ar2    -0.0156 0.0885  -0.1761  0.8604
## ma1    -0.7776 0.0589 -13.2025  0.0000
## xmean   0.0146 0.0053   2.7502  0.0066
##
## sigma^2 estimated as 0.1089133 on 169 degrees of freedom
##
## AIC = 0.6844762  AICc = 0.6858525  BIC = 0.7756118
##

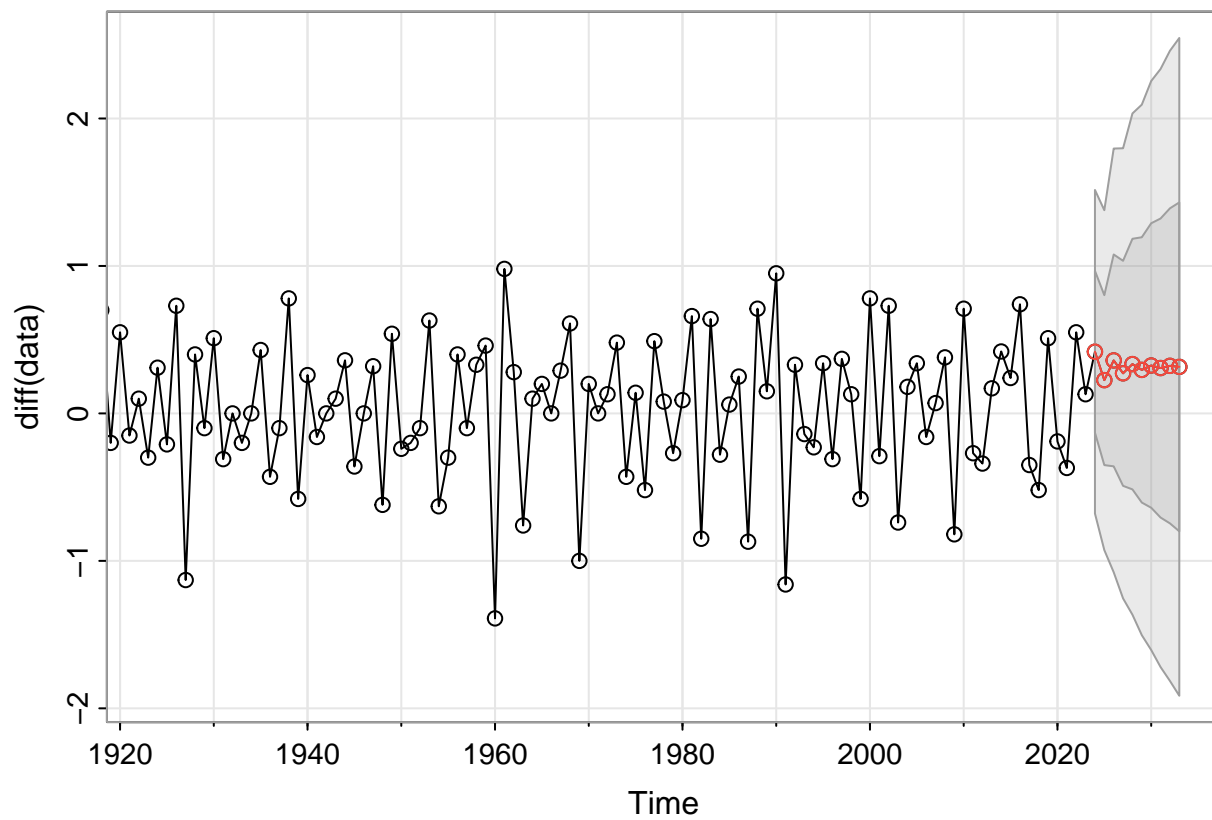
```



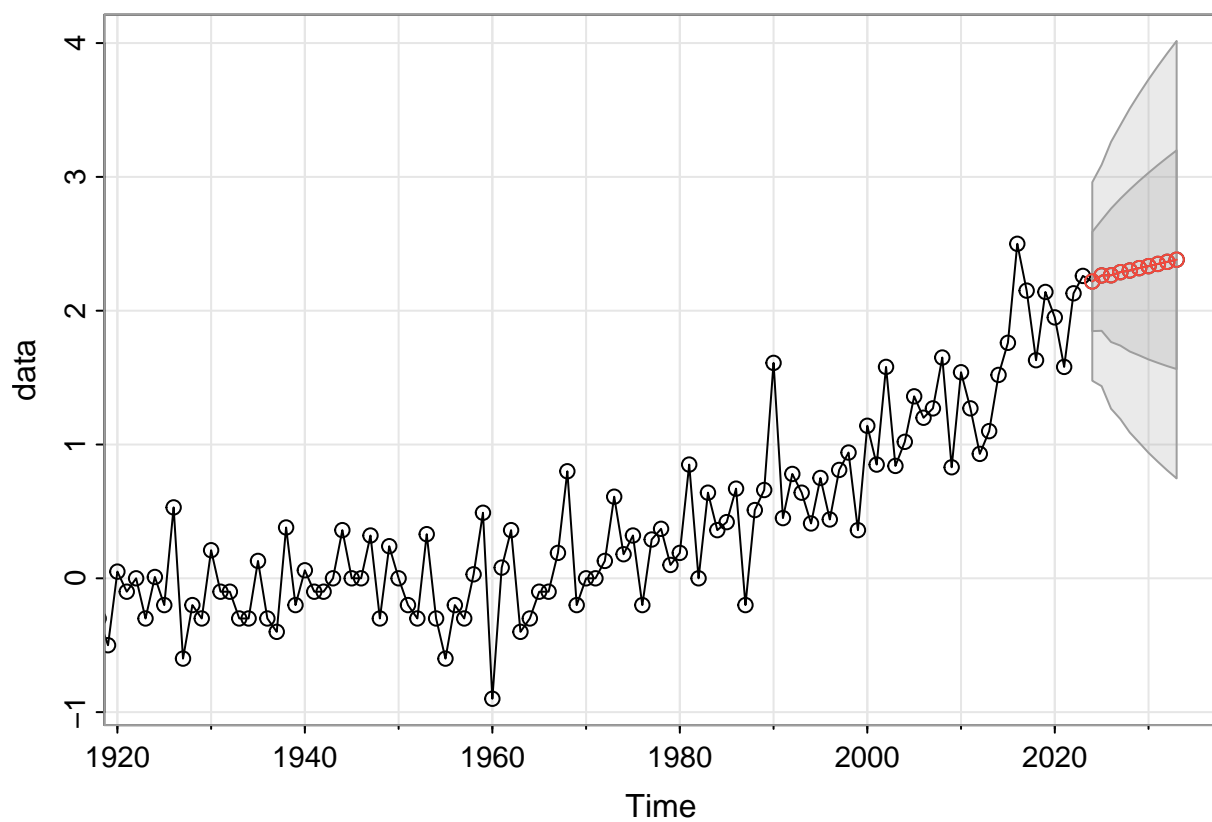
We can see that the some of ARMA(2,1) model's regression coefficients are not significant, which suggests that ARMA(2,1) model is not better than the MA(1) model.

From diagnostic, we can see that from Ljung-Box statistic shows that all lags have a p-value higher than 0.05, which means that we will not reject the null hypothesis of residuals being white noise. From the Q-Q plot, we can see that the observation fits pretty well to theoretical quantile, which means the data is normally distributed. Therefore, the best model for this data is MA(1).

```
predict_diff = sarima.for(diff(data), 10, 1, 1, 0)
```



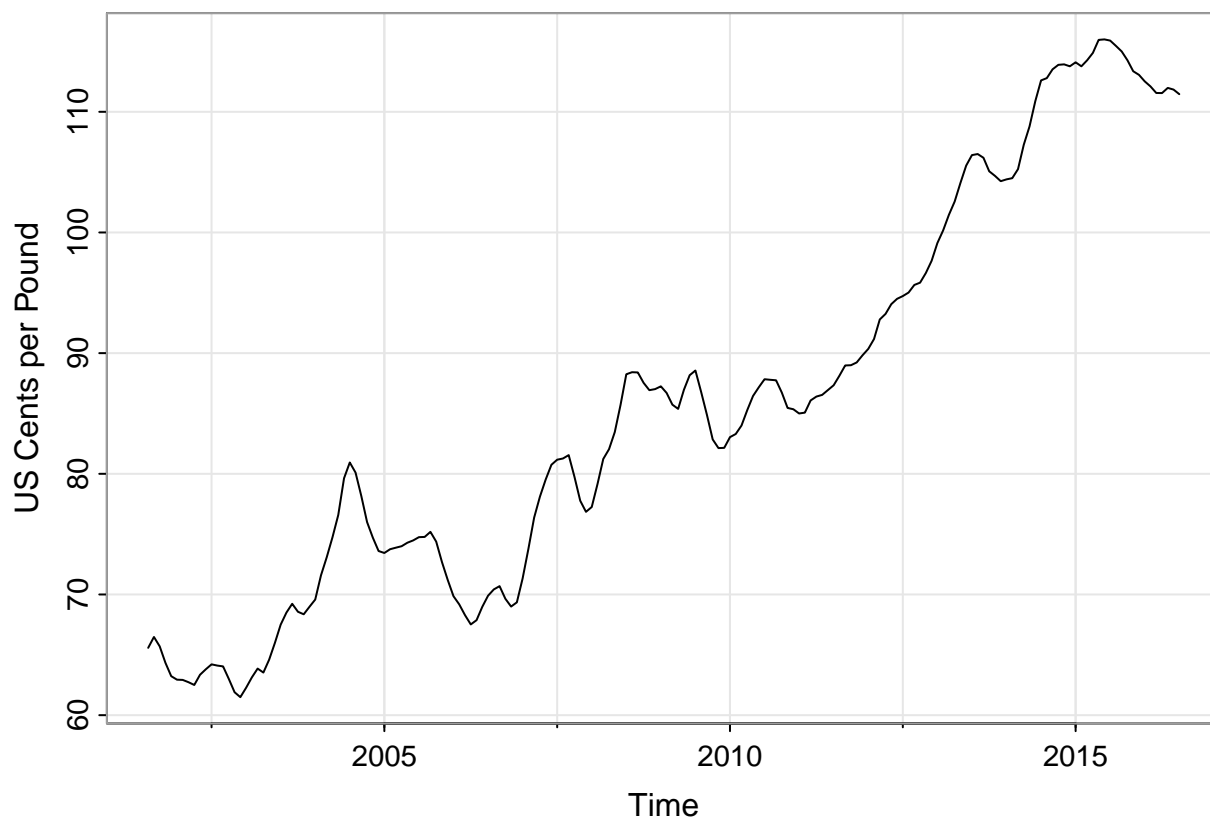
```
predict_data = sarima.for(data, 10, 1, 1, 0)
```



By forecasting with our best model MA(1), the plot shows the continued upward trend of increasing land temperature deviations. This means that as time increase, the variations also increases so we can expect the more fluctuations.

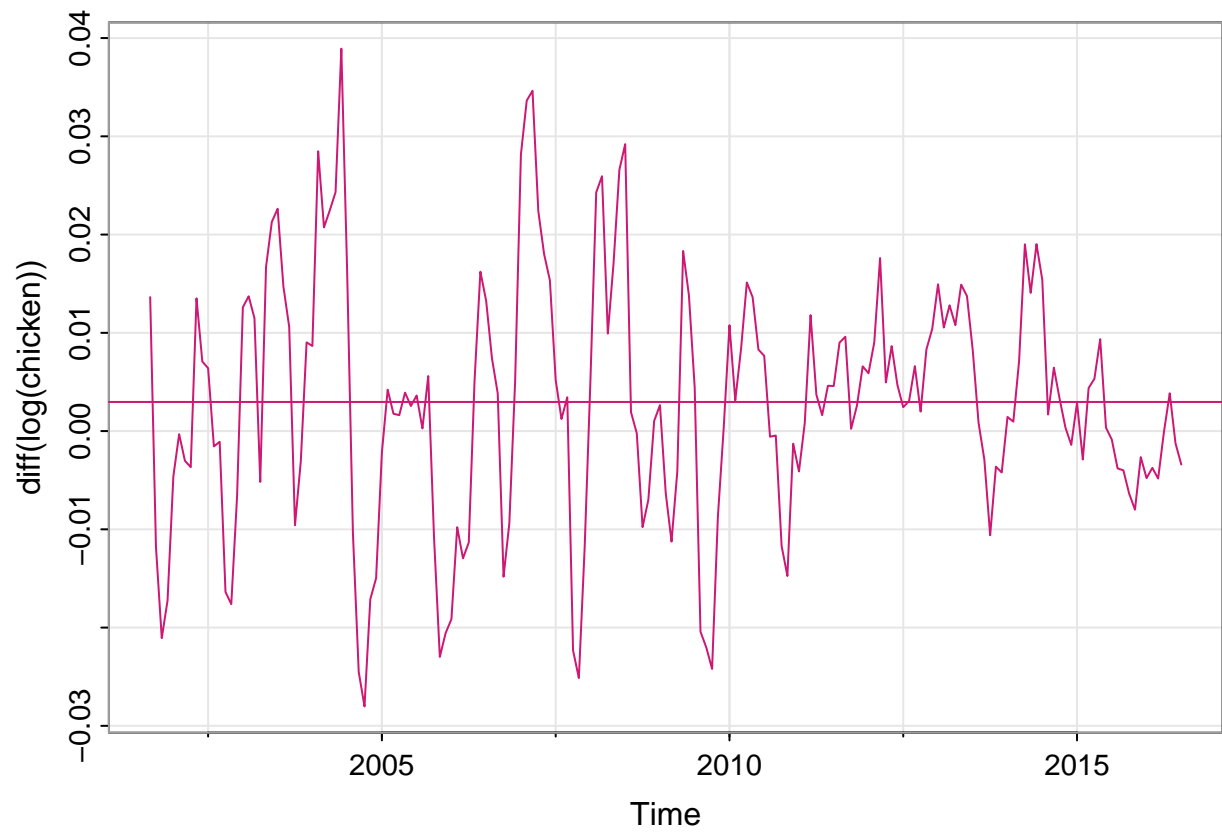
### Question 5.9

```
data<-chicken  
tsplot(data, ylab = "US Cents per Pound")
```



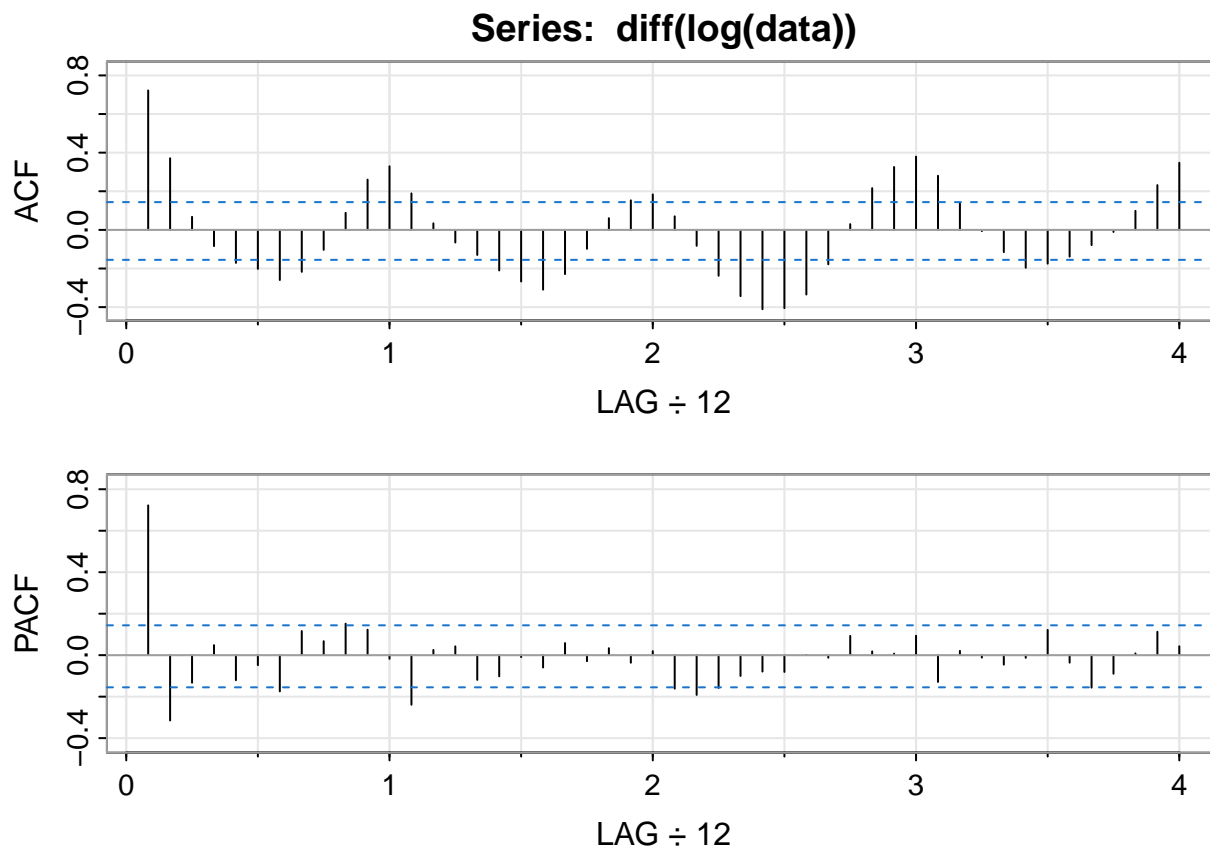
We can see that the Chicken price has an upward trend. This means that this data is not stationary. To make it stationary, we'll take the `diff(log(data))`.

```
tsplot(diff(log(data)), col = 6, ylab = "diff(log(chicken))")  
mean<-mean(diff(log(data)))  
abline(h = mean, col = 6)
```



```
acf2(diff(log(data)))
```





```
##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10] [,11] [,12]
## ACF  0.72  0.37  0.07 -0.08 -0.17 -0.20 -0.26 -0.22 -0.10  0.09  0.26  0.33
## PACF 0.72 -0.31 -0.13  0.05 -0.12 -0.05 -0.17  0.12  0.07  0.15  0.12 -0.02
##      [,13] [,14] [,15] [,16] [,17] [,18] [,19] [,20] [,21] [,22] [,23] [,24]
## ACF   0.19  0.03 -0.07 -0.13 -0.21 -0.27 -0.31 -0.23 -0.10  0.06  0.15  0.18
## PACF -0.24  0.03  0.04 -0.12 -0.10 -0.01 -0.06  0.06 -0.03  0.03 -0.04  0.02
##      [,25] [,26] [,27] [,28] [,29] [,30] [,31] [,32] [,33] [,34] [,35] [,36]
## ACF   0.07 -0.08 -0.24 -0.34 -0.41 -0.41 -0.33 -0.18  0.03  0.22  0.33  0.38
## PACF -0.16 -0.19 -0.16 -0.10 -0.08 -0.08  0.00 -0.01  0.09  0.02  0.01  0.09
##      [,37] [,38] [,39] [,40] [,41] [,42] [,43] [,44] [,45] [,46] [,47] [,48]
## ACF   0.28  0.14 -0.01 -0.12 -0.20 -0.17 -0.14 -0.08 -0.01  0.10  0.23  0.35
## PACF -0.13  0.02 -0.01 -0.05 -0.01  0.12 -0.04 -0.16 -0.09  0.01  0.11  0.04
```

From the sample ACF and PACF plot, we feel that ACF is tailing off and PACF cuts off at lag 1. Therefore, we'll try the model AR(1) model.

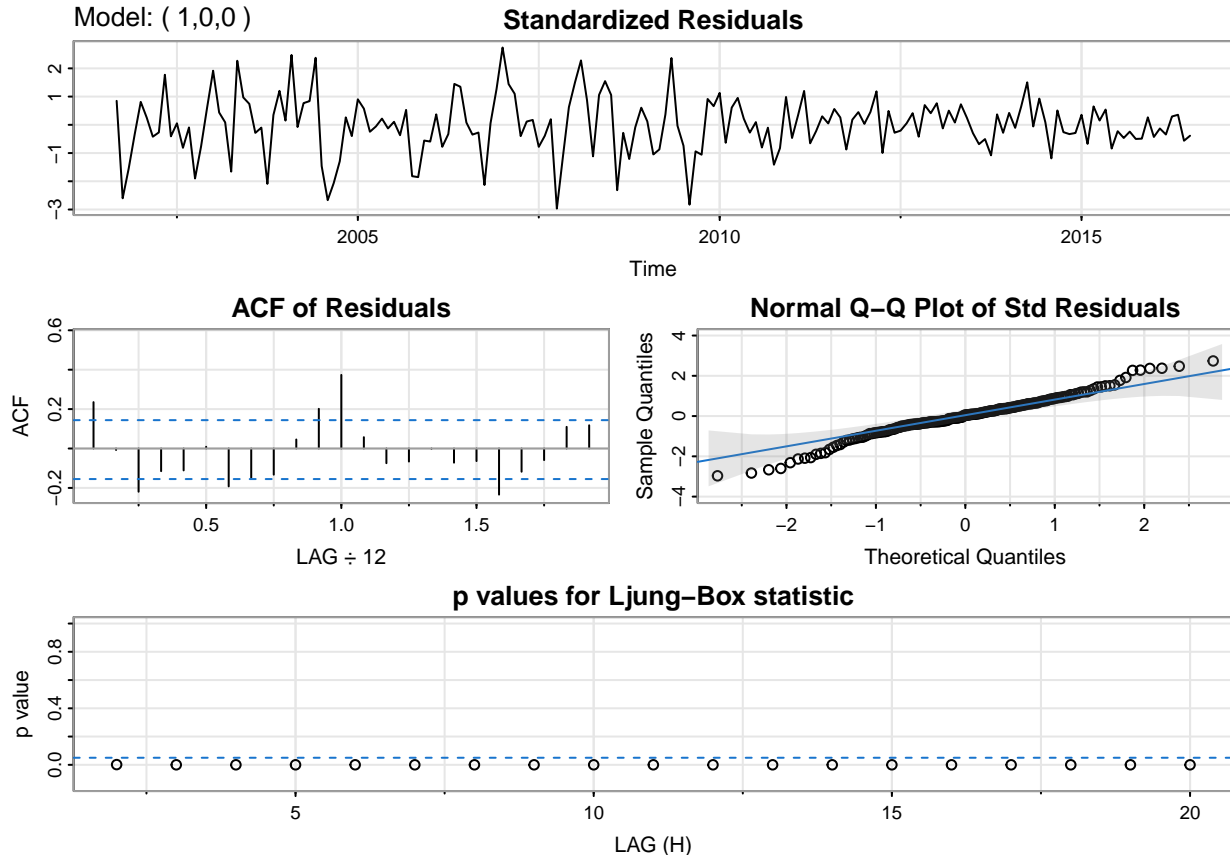
```
sarima(diff(log(data)), 1, 0, 0)
```

```
## initial value -4.378816
## iter 2 value -4.750463
## iter 3 value -4.750469
## iter 4 value -4.750479
## iter 5 value -4.750497
## iter 6 value -4.750505
## iter 7 value -4.750505
## iter 8 value -4.750506
```

```

## iter    9 value -4.750506
## iter   10 value -4.750506
## iter   10 value -4.750506
## iter   10 value -4.750506
## final  value -4.750506
## converged
## initial value -4.749085
## iter    2 value -4.749099
## iter    3 value -4.749155
## iter    4 value -4.749157
## iter    5 value -4.749159
## iter    5 value -4.749159
## iter    5 value -4.749159
## final  value -4.749159
## converged
## <><><><><><><><><><><>
##
## Coefficients:
##      Estimate      SE t.value p.value
## ar1      0.7222 0.0512 14.1052 0.0000
## xmean     0.0030 0.0023  1.3188 0.1889
##
## sigma^2 estimated as 7.466975e-05 on 177 degrees of freedom
##
## AIC = -6.626921  AICc = -6.62654  BIC = -6.573501
##

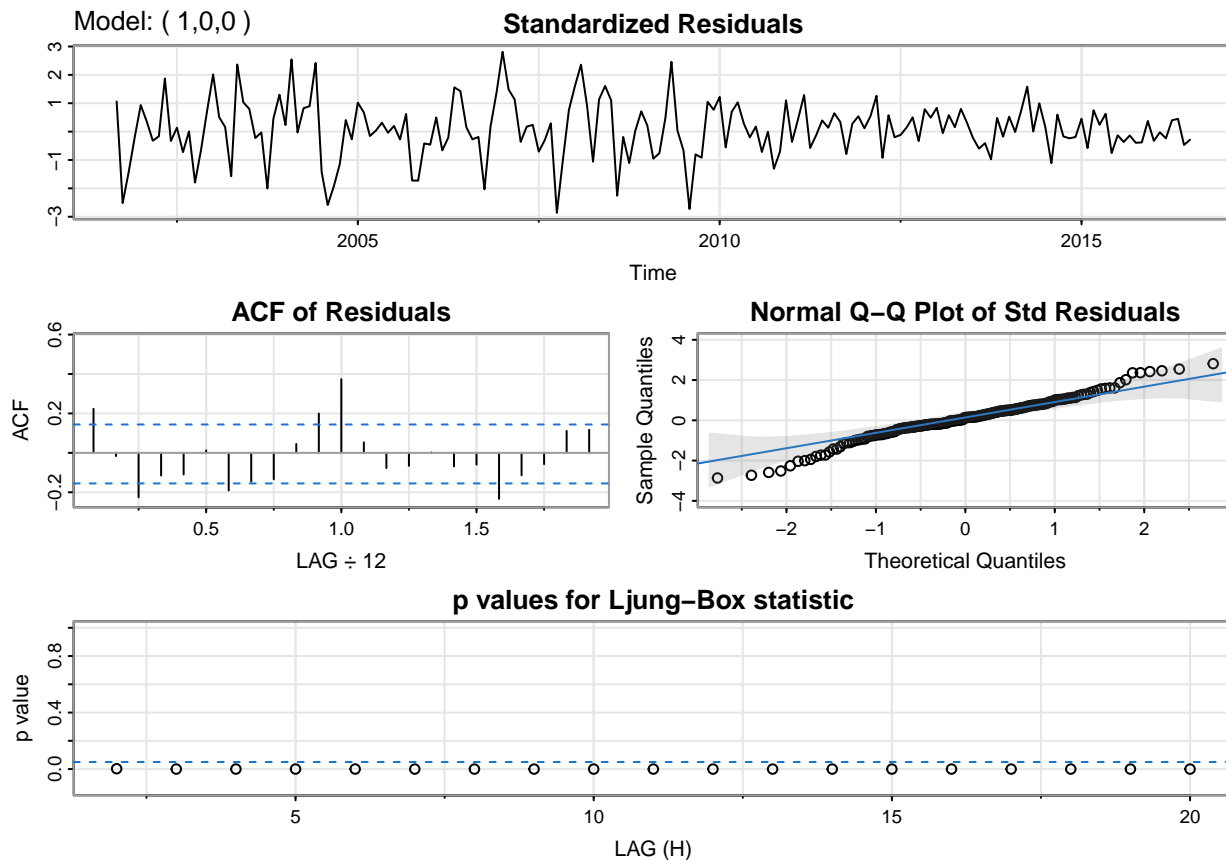
```



We can see that from the model, the intercept is not significant, which suggests the removal of the intercept.

```
sarima(diff(log(data)), 1, 0, 0, no.constant = TRUE)
```

```
## initial value -4.352722
## iter 2 value -4.747087
## iter 3 value -4.747101
## iter 4 value -4.747108
## iter 4 value -4.747108
## final value -4.747108
## converged
## initial value -4.744563
## iter 2 value -4.744563
## iter 3 value -4.744563
## iter 3 value -4.744563
## iter 3 value -4.744563
## final value -4.744563
## converged
## <><><><><><><><><><><><><>
##
## Coefficients:
##      Estimate    SE t.value p.value
## ar1      0.737 0.05 14.7259      0
##
## sigma^2 estimated as 7.533969e-05 on 178 degrees of freedom
##
## AIC = -6.628903  AICc = -6.628777  BIC = -6.59329
##
```



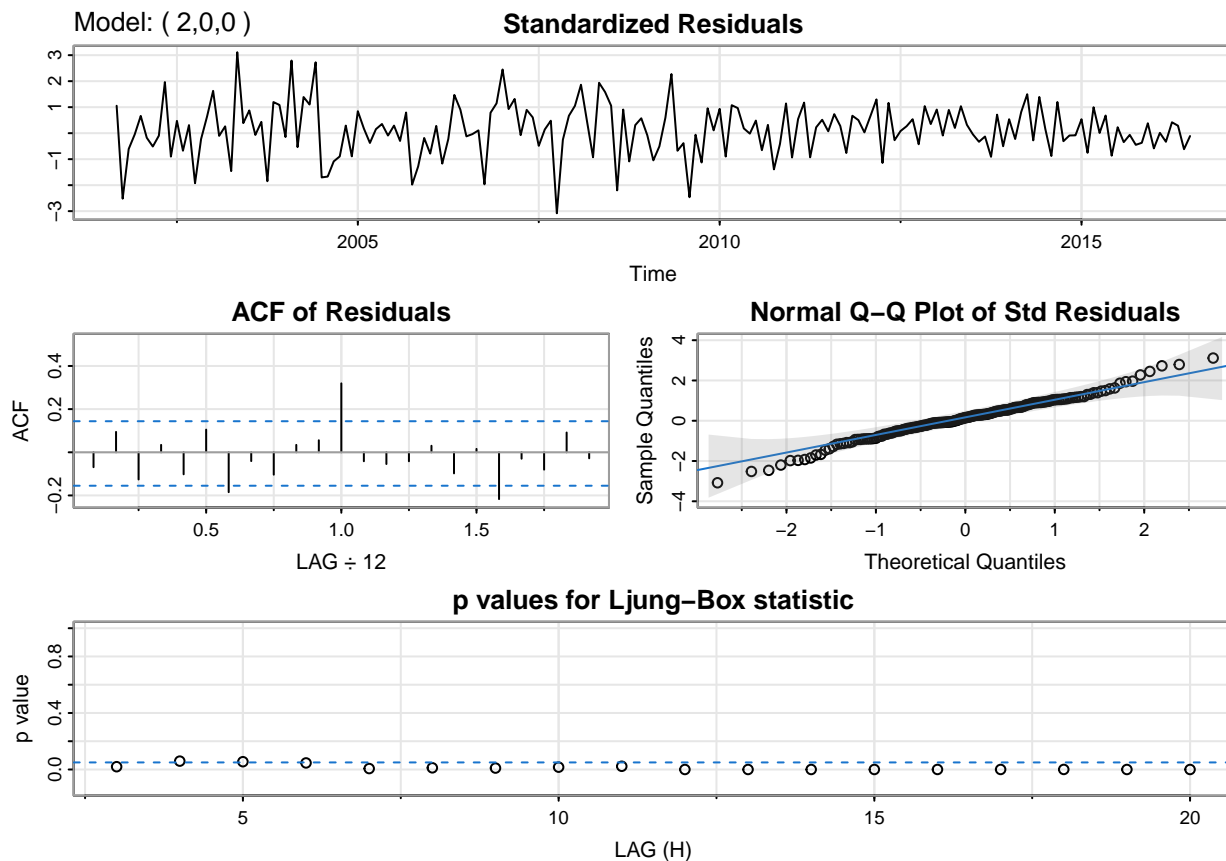
We can see that the regression coefficient remains significant, and both AIC and BIC decreased. Therefore, this suggests that AR(1) model without intercept is better.

We can also see that PACF might be cutting off at lag 2 while ACF is still tailing off. Therefore, we'll try AR(2) model without the intercept.

```
sarima(diff(log(data)), 2, 0, 0, no.constant = TRUE)
```

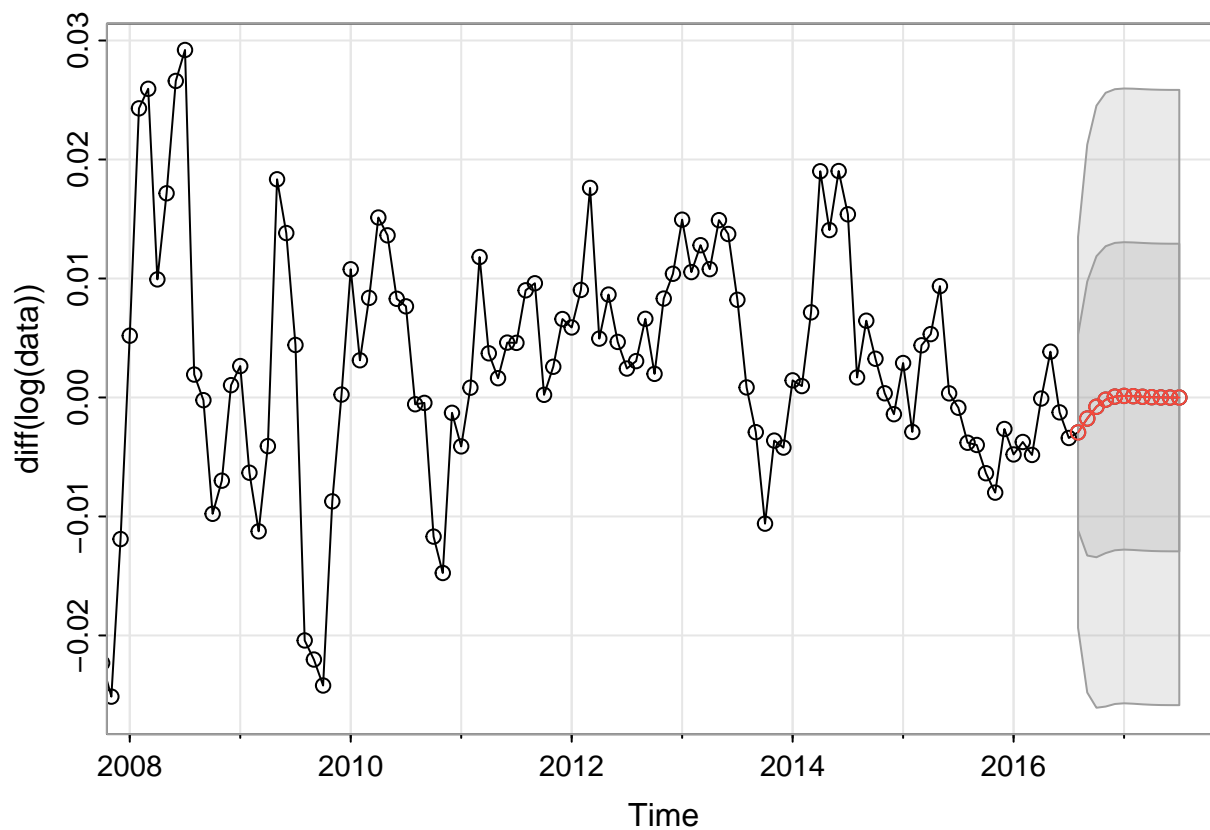
```
## initial value -4.352272
## iter 2 value -4.508697
## iter 3 value -4.731994
## iter 4 value -4.806593
## iter 5 value -4.813483
## iter 6 value -4.818210
## iter 7 value -4.818213
## iter 8 value -4.818214
## iter 9 value -4.818214
## iter 9 value -4.818214
## iter 9 value -4.818214
## final value -4.818214
## converged
## initial value -4.799478
## iter 2 value -4.799584
## iter 3 value -4.799585
## iter 3 value -4.799585
## iter 3 value -4.799585
## final value -4.799585
```

```
## converged
## <><><><><><><><><><>
##
## Coefficients:
##      Estimate      SE t.value p.value
## ar1    0.9830 0.0717 13.7048      0
## ar2   -0.3279 0.0718 -4.5696      0
##
## sigma^2 estimated as 6.739915e-05 on 177 degrees of freedom
##
## AIC = -6.727774  AICc = -6.727393  BIC = -6.674354
##
```

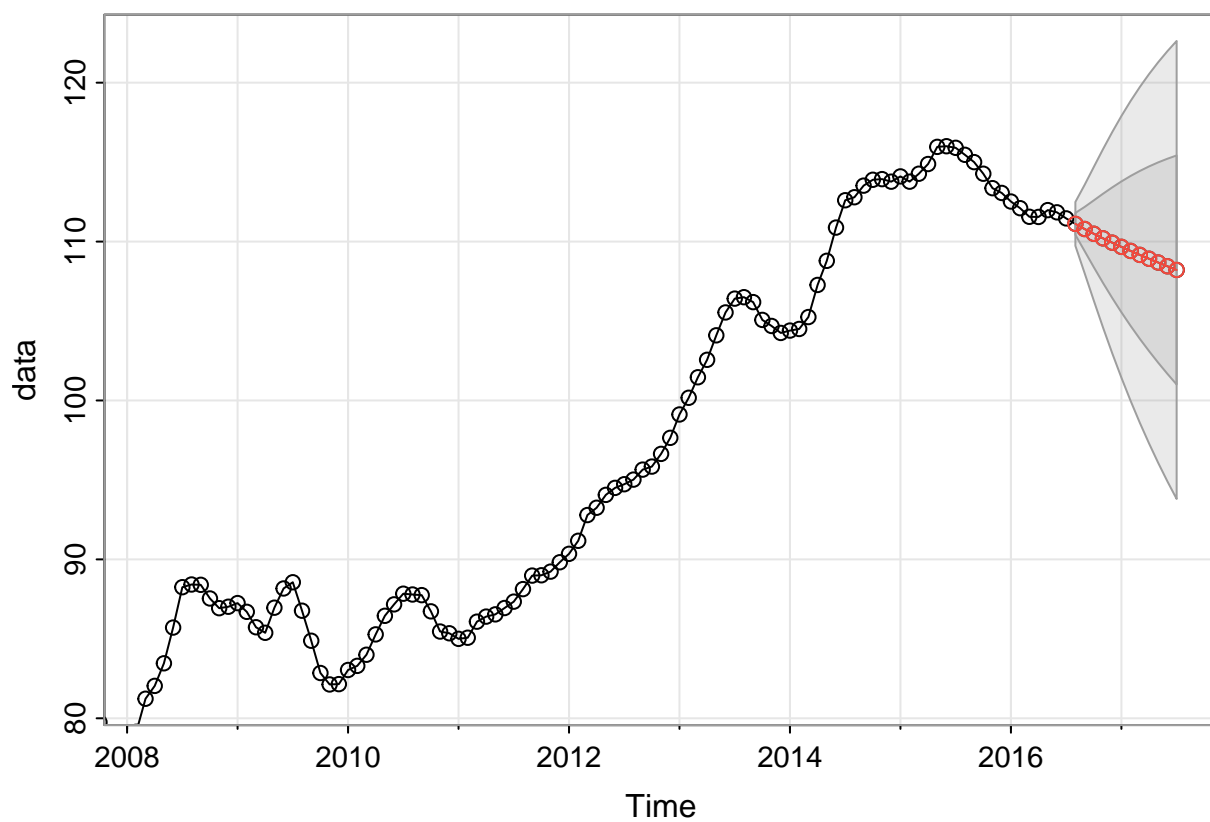


We can see that according to the model summary, both AIC and BIC are lower than the AR(1) model, which means AR(2) are better. Therefore, we'll forecast Chicken data with AR(2) model. However, we can't plot the original data with AR(2) without the intercept, so just for the visualization, we'll plot the chicken data with the AR(2) model with intercept.

```
predict_diff = sarima.for(diff(log(data)), 12, 2, 0, 0, no.constant = TRUE)
```



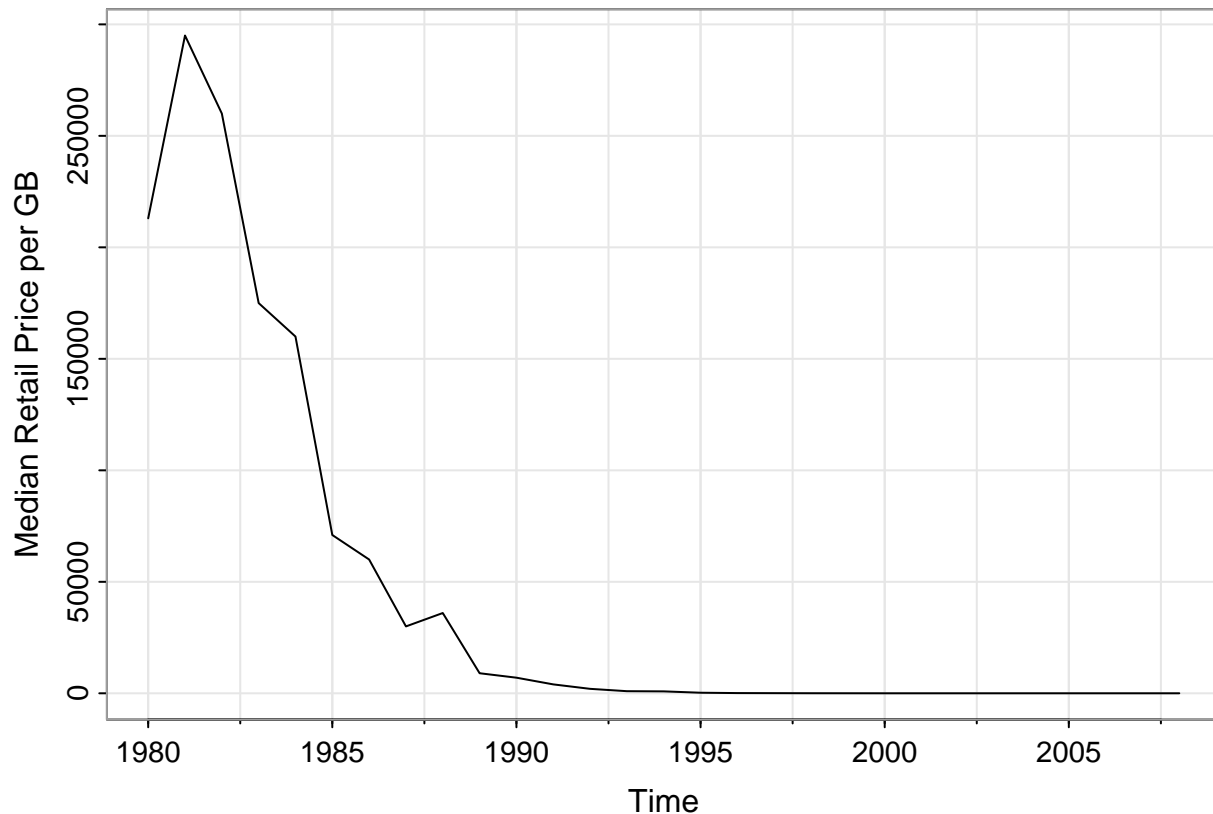
```
predict_data = sarima.for(data, 12, 2, 0, 0)
```



We can see that from the forecast, we can see that it forecast the chicken price will decrease, which doesn't make sense. This mean we have a perfect fit model, but bad prediction.

### Question 5.14a

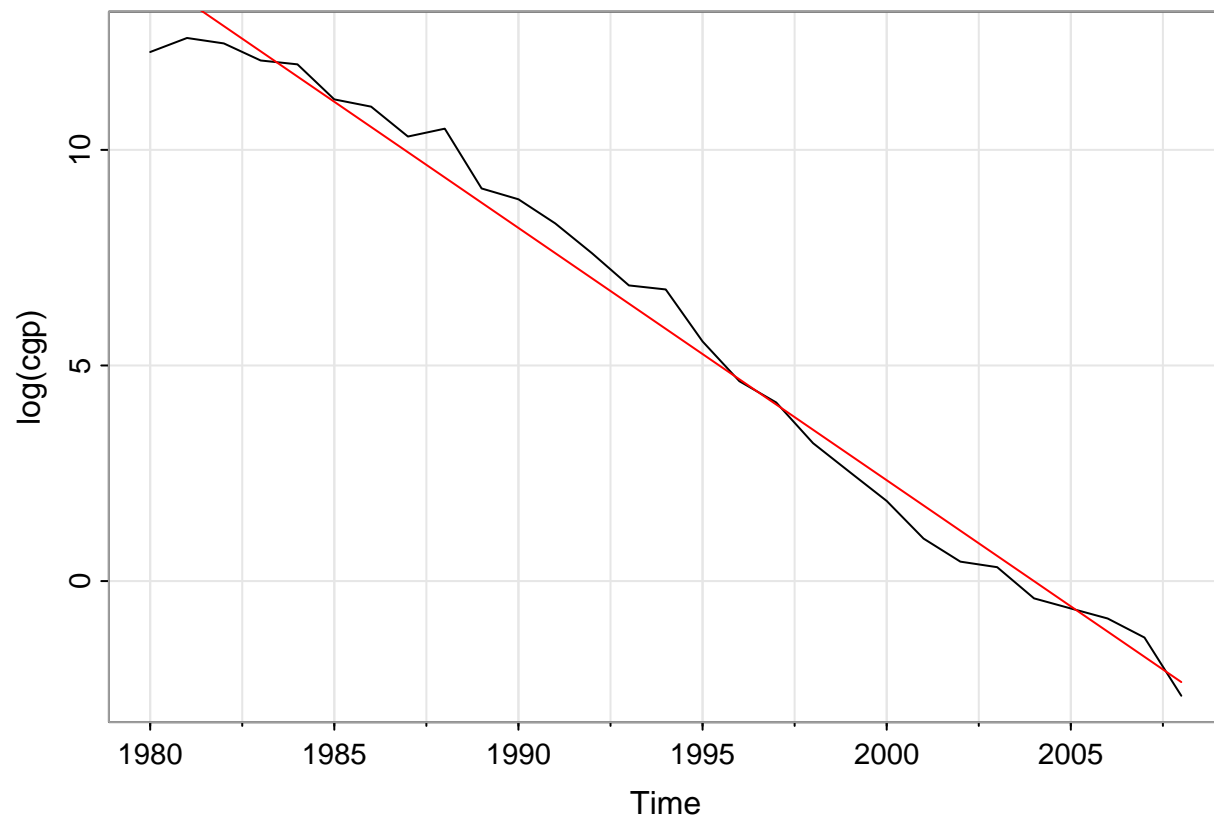
```
data<-cpg  
tsplot(data, ylab = "Median Retail Price per GB")
```



From the plot, we see that the median price have a decreasing trend after 1982. We can see that the median price converge to 0 as the technology is more developed. We can see that the median price is near 0 after 1995.

### Question 5.14b

```
year<-time(data)  
model<-lm(log(data)~year, na.action = NULL)  
tsplot(log(data), ylab = "log(cgp)")  
lines(fitted(model), col = "red")
```

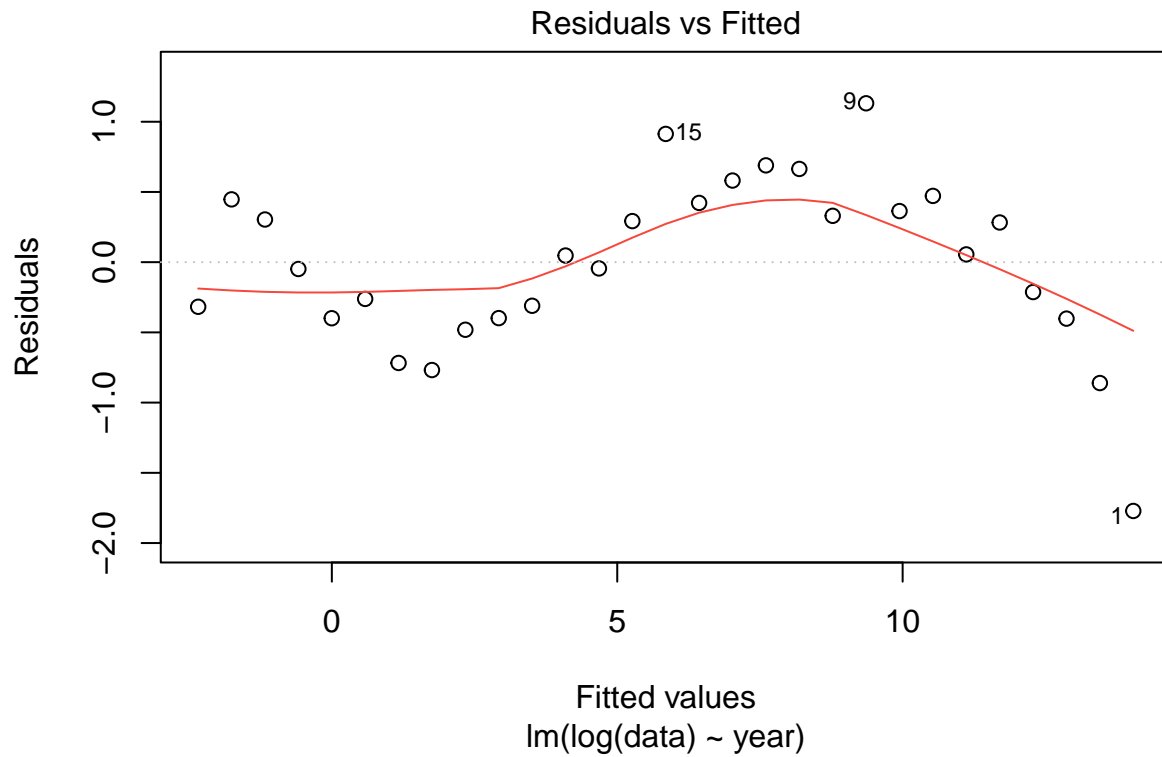


We can see that the fitted model to the plot is a straight line. This means that  $c_t$  behave like  $\alpha e^{1172.4943 - 0.5851x_{year}}$ .

### Question 5.14c

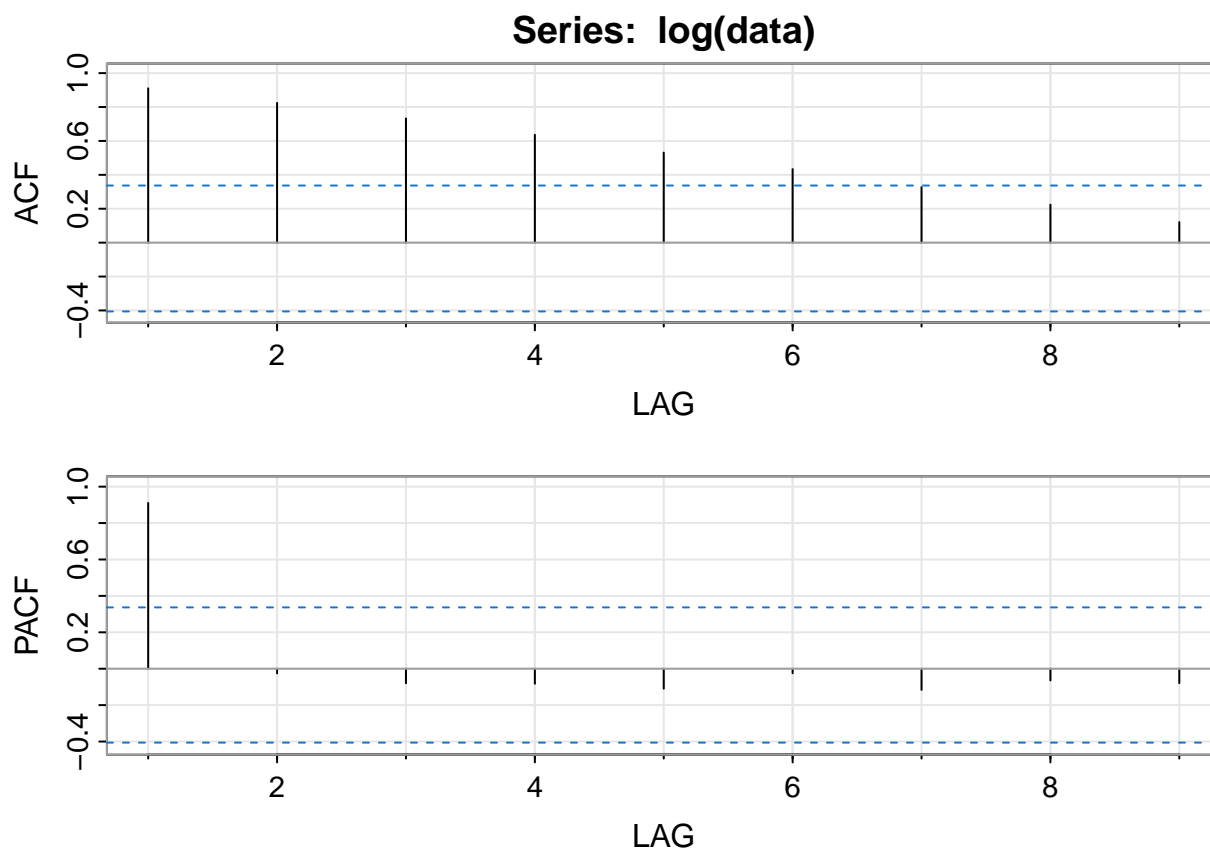
```
plot(model,1)
```





From the residual plot, we can clearly see there's a cyclic pattern in a residuals. This means that there are a pattern where the fitted model do not perform well. This suggests that the linear regression might not be a good model for this data. ## Question 5.14d

```
acf2(log(data))
```



```
##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9]
## ACF 0.91 0.82 0.73 0.64 0.53 0.43 0.33 0.22 0.12
## PACF 0.91 -0.03 -0.08 -0.08 -0.11 -0.02 -0.12 -0.06 -0.08
```

From the ACF and PACF plot, we can see that the ACF is tailing off and the PACF cuts off at lag 1, so we'll use AR(1) model.

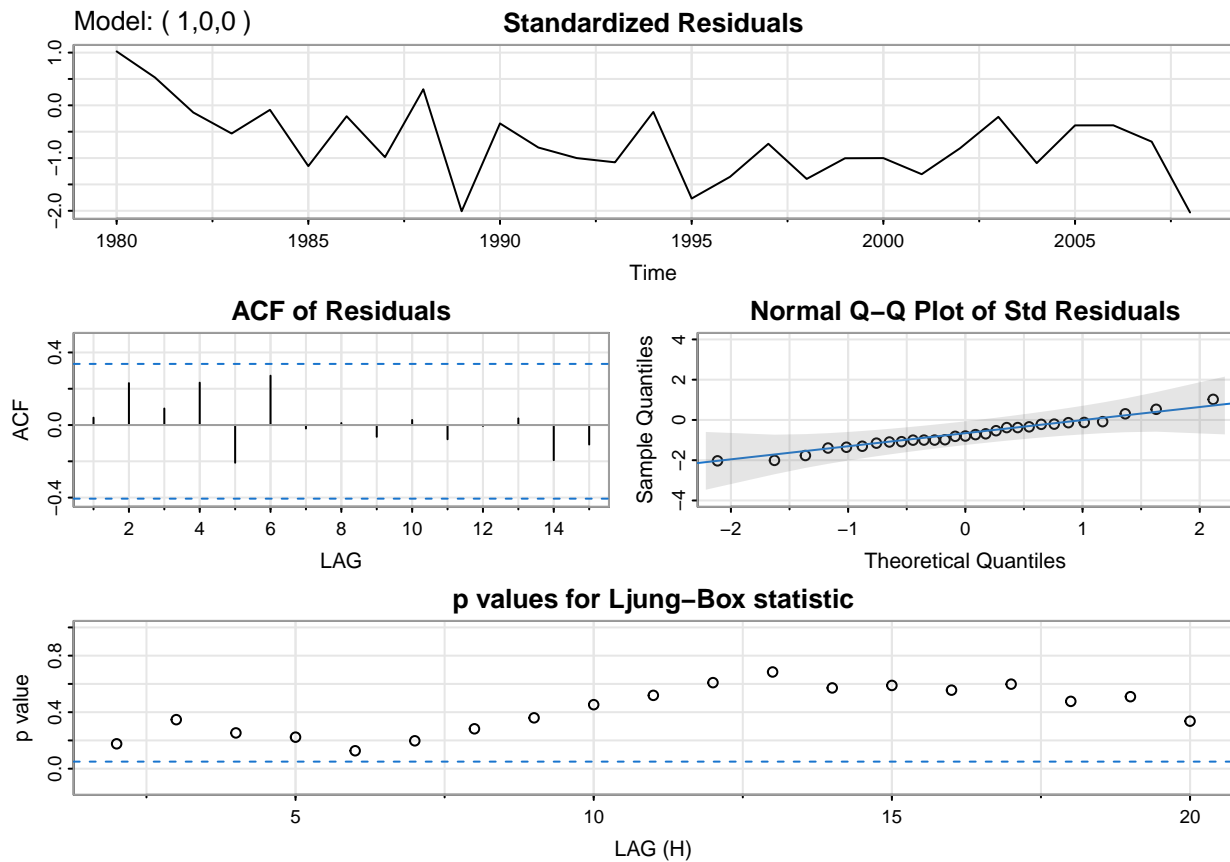
```
sarima(log(data), 1, 0, 0,)
```

```
## initial value 1.583170
## iter 2 value -0.368933
## iter 3 value -0.382228
## iter 4 value -0.399952
## iter 5 value -0.497976
## iter 6 value -0.519338
## iter 7 value -0.589061
## iter 8 value -0.676984
## iter 9 value -0.702665
## iter 10 value -0.756358
## iter 11 value -0.758825
## iter 12 value -0.763442
## iter 13 value -0.772545
## iter 14 value -0.780862
## iter 15 value -0.805718
## iter 16 value -0.806190
## iter 17 value -0.806834
```

```
## iter 18 value -0.808751
## iter 19 value -0.812937
## iter 20 value -0.829659
## iter 21 value -0.830195
## iter 22 value -0.830338
## iter 23 value -0.830514
## iter 24 value -0.831142
## iter 25 value -0.832646
## iter 26 value -0.833581
## iter 27 value -0.833677
## iter 28 value -0.833810
## iter 29 value -0.834230
## iter 30 value -0.834980
## iter 31 value -0.835778
## iter 32 value -0.835833
## iter 33 value -0.836098
## iter 34 value -0.836649
## iter 35 value -0.838422
## iter 36 value -0.839396
## iter 37 value -0.839437
## iter 38 value -0.839529
## iter 39 value -0.839780
## iter 40 value -0.840428
## iter 41 value -0.840904
## iter 42 value -0.840952
## iter 43 value -0.840997
## iter 44 value -0.841193
## iter 45 value -0.841623
## iter 46 value -0.841931
## iter 47 value -0.841983
## iter 48 value -0.842016
## iter 49 value -0.842206
## iter 50 value -0.842596
## iter 51 value -0.842853
## iter 52 value -0.842916
## iter 53 value -0.842941
## iter 54 value -0.843150
## iter 55 value -0.843552
## iter 56 value -0.843783
## iter 57 value -0.843867
## iter 58 value -0.843886
## iter 59 value -0.844146
## iter 60 value -0.844623
## iter 61 value -0.844834
## iter 62 value -0.844965
## iter 63 value -0.844980
## iter 64 value -0.845361
## iter 65 value -0.846054
## iter 66 value -0.846206
## iter 67 value -0.846469
## iter 68 value -0.846480
## iter 69 value -0.847202
## iter 70 value -0.848701
## iter 71 value -0.849008
```

```
## iter 72 value -0.849151
## iter 73 value -0.849158
## iter 74 value -0.849570
## iter 75 value -0.849971
## iter 76 value -0.850366
## iter 77 value -0.850499
## iter 78 value -0.850505
## iter 79 value -0.850920
## iter 80 value -0.851732
## iter 81 value -0.852000
## iter 82 value -0.852059
## iter 83 value -0.852063
## iter 84 value -0.852233
## iter 85 value -0.852446
## iter 86 value -0.852671
## iter 87 value -0.852710
## iter 88 value -0.852713
## iter 89 value -0.852825
## iter 90 value -0.853024
## iter 91 value -0.853143
## iter 92 value -0.853161
## iter 93 value -0.853165
## iter 94 value -0.853219
## iter 95 value -0.853305
## iter 96 value -0.853388
## iter 97 value -0.853400
## iter 98 value -0.853403
## iter 99 value -0.853441
## iter 100 value -0.853512
## final value -0.853512
## stopped after 100 iterations
## initial value 1.595724
## iter 2 value 0.561504
## iter 3 value -0.232372
## iter 4 value -0.276286
## iter 5 value -0.298525
## iter 6 value -0.305311
## iter 7 value -0.307021
## iter 8 value -0.307054
## iter 9 value -0.307239
## iter 10 value -0.307240
## iter 11 value -0.307240
## iter 12 value -0.307240
## iter 12 value -0.307240
## final value -0.307240
## converged
## <><><><><><><><><><>
##
## Coefficients:
##           Estimate      SE t.value p.value
## ar1          0.9956 0.0062 161.3874  0.000
## xmean        4.8760 7.0163  0.6949  0.493
##
## sigma^2 estimated as 0.4594506 on 27 degrees of freedom
```

```
##
## AIC = 2.430294  AICc = 2.446209  BIC = 2.571738
##
```



We can see that the intercept is not significant, so we want to remove the intercept to see if the model fits better.

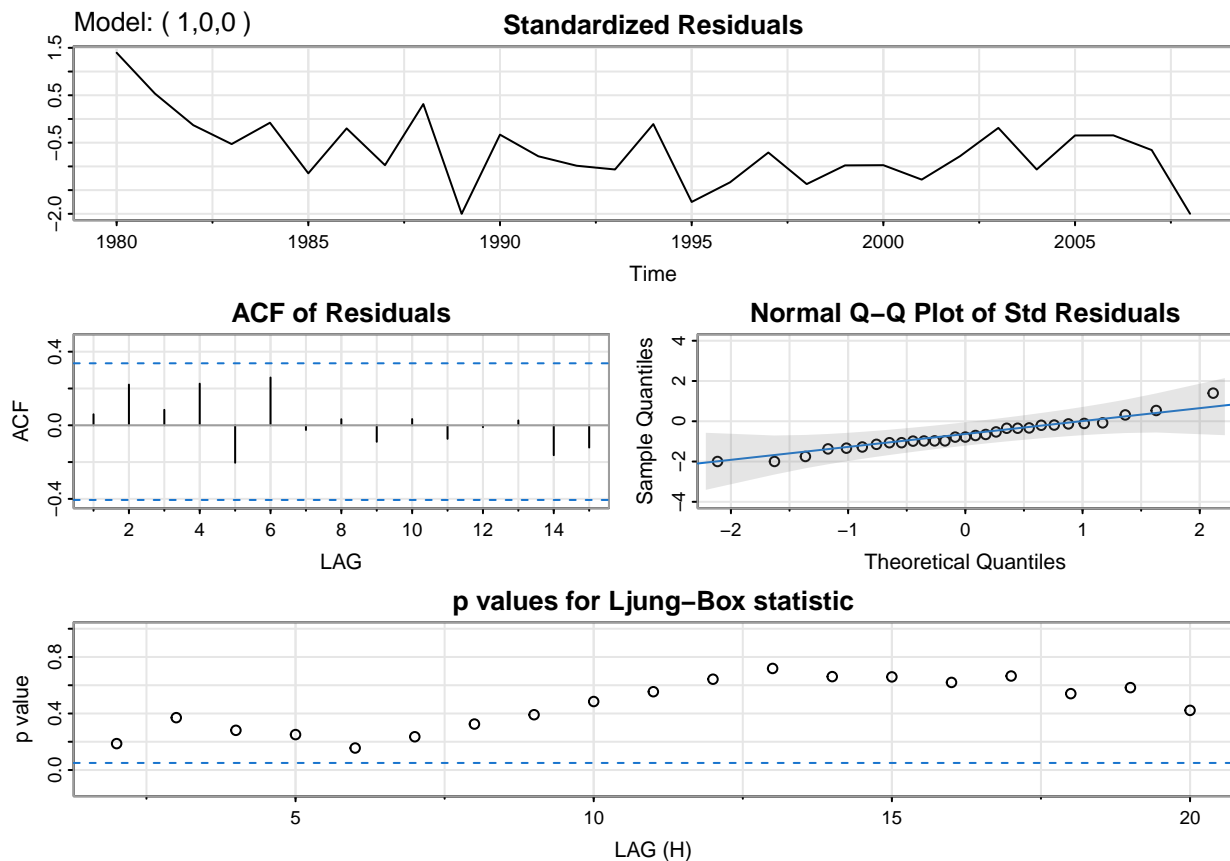
```
sarima(log(data), 1, 0, 0, no.constant = TRUE)
```

```
## initial value 2.005981
## iter 2 value -0.109653
## iter 3 value -0.349967
## iter 4 value -0.519614
## iter 5 value -0.548459
## iter 6 value -0.553046
## iter 7 value -0.553054
## iter 7 value -0.553054
## final value -0.553054
## converged
## initial value -0.077470
## iter 2 value -0.250289
## iter 3 value -0.279653
## iter 4 value -0.295200
## iter 5 value -0.296367
## iter 6 value -0.300260
## iter 7 value -0.300539
```

```

## iter    8 value -0.300542
## iter    9 value -0.300561
## iter   10 value -0.300561
## iter   10 value -0.300561
## final   value -0.300561
## converged
## <><><><><><><><><><>
##
## Coefficients:
##      Estimate      SE t.value p.value
## ar1      0.997 0.0042 239.7262      0
##
## sigma^2 estimated as 0.4594107 on 28 degrees of freedom
##
## AIC = 2.374685  AICc = 2.379794  BIC = 2.468982
##

```



We can see that both AIC and BIC decreases, so AR(1) model without the intercept is a better fitting model.

From the ACF of residuals, we can see that the residuals are more random and closer to white noise. Additionally, from the Ljung-Box statistic, we can see that all p-values are higher than 0.05, so we'll not reject the null hypothesis that the residuals are white noise. Thus, this means that the AR(1) model is better than the linear regression model.