The Call Option and The Black-Scholes Formula

Anton Yang

May 4, 2023

Call Option

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Would you accept the offer?

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So what actually is a call option?

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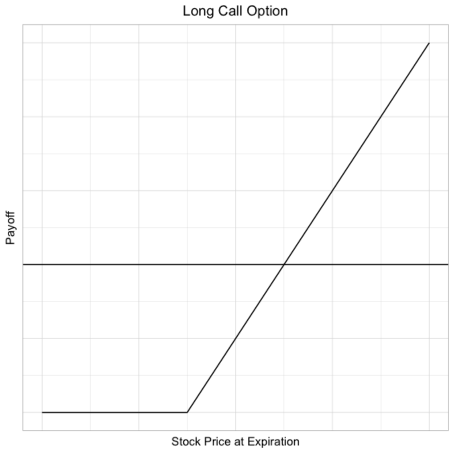
Ex: Suppose you decided to buy a call option at the strike price of \$165 with a premium price of \$5, and Tesla's current stock price is \$160. If the next day, Tesla's stock price increases to \$180. You can exercise the option and get a payoff of \$15 and a profit of \$10 per share.

Call Option

Payoff = max[(S_t-K),0]

Profit=max[(S_t-K),0]-C



The Black-Scholes Formula

$$C(S_0, K, \sigma, r, T, \delta) =$$

$$\frac{1}{\sqrt{2\pi}} \left(Se^{-\delta T} \int_{-\infty}^{\frac{\ln \frac{S}{K} + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}} e^{-\frac{1}{2}x^2} dx - Ke^{-rT} \int_{-\infty}^{\frac{\ln \frac{S}{K} + (r - \delta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}} e^{-\frac{1}{2}x^2} dx \right)$$

S_0 = Stock Price
K = Strike Price
σ = Volatility
r = Interest Rate
T = Time to Expiree (in years)
δ = Dividend Yield

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$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}x^2} dx$$

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Assume that continuously compounded returns, on the stock underlying the call option, from t=0 to t=T, are log-normal distributed with a mean and a variance $((r - \delta \text{-} \frac{1}{2}\sigma^2)T, \sigma^2 T)$

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$$\ln\left(\frac{S_T}{S}\right) \sim \mathcal{N}\left(\left(r-\delta-\frac{1}{2}\sigma^2\right)T, \sigma^2 T\right)$$

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Apply the identity $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$ and shift the mean:

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If $X_1 < X_2$, then $\ln X_1 < \ln X_2$, so $\mathrm{P}(S_T < K) = \mathrm{P}(\ln S_T < K)$.
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The left side of the inequality is Z. The right side:

$$-d_2 = -\frac{\ln S - \ln K + (r - \delta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

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and we already proved that $\mathrm{P}(S_T > K) = N(d_2)$

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$$\mathbb{E}_{x>d}(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{d}^{\infty} x e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

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Then:

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