

Abdur Rehman, Shahid Mubeen\*, Rabia Safdar, and Naeem Sadiq

# Properties of $k$ -beta function with several variables

**Abstract:** In this paper, we discuss some properties of beta function of several variables which are the extension of beta function of two variables. We define  $k$ -beta function of several variables and derive some properties of this function which are the extension of  $k$ -beta function of two variables, recently defined by Diaz and Pariguan [4]. Also, we extend the formula  $\Gamma_k(2z)$  proved by Kokologiannaki [5] via properties of  $k$ -beta function.

**Keywords:**  $k$ -Gamma function,  $k$ -Beta function, Several variables

**MSC:** 33B15, 33B20, 26B35

DOI 10.1515/math-2015-0030

Received May 26, 2014; accepted February 3, 2015.

## 1 Introduction

The beta function or Eulerian integral of the first kind with two variables is defined by

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad \operatorname{Re}(x) > 0, \operatorname{Re}(y) > 0. \quad (1)$$

The Euler gamma function or Euler integral of the second kind is given by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad \operatorname{Re}(x) > 0. \quad (2)$$

The beta function in terms of gamma function is defined in [8] as

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \operatorname{Re}(x) > 0, \operatorname{Re}(y) > 0. \quad (3)$$

Also, the authors [2] proved some properties of beta function by a simple change of variables as

$$\beta(x, y) = 2 \int_0^{\pi/2} (\cos\theta)^{2x-1} (\sin\theta)^{2y-1} d\theta \quad (4)$$

$$\beta(x, y) = \int_0^\infty \frac{t^{x-1}}{(t+1)^{x+y}} dt. \quad (5)$$

**Abdur Rehman:** Department of Mathematics, University of Sargodha, Sargodha, Pakistan, E-mail: abdurrehman2007@hotmail.com

**\*Corresponding Author: Shahid Mubeen:** Department of Mathematics, University of Sargodha, Sargodha, Pakistan, E-mail: smjhanda@gmail.com

**Rabia Safdar:** Department of Mathematics, G.C.University, Faisalabad, Pakistan, E-mail: rabia.safdar1109@gmail.com

**Naeem Sadiq:** Department of Mathematics, G.C.University, Faisalabad, Pakistan, E-mail: naeemsadiq-008@yahoo.com

The collection of most important properties of the beta function of two and more variables is given in [1–3]. The definition of beta function is extended to three or more variables in [3]. Let  $p = (p_1, p_2, \dots, p_n) \in \mathbb{C}^n$ ,  $p_i \neq \{0, -1, -2, \dots\}$ ,  $n \geq 2$ . The beta function of  $n$  variables is defined as

$$\beta(p) = \beta(p_1, p_2, \dots, p_n) = \frac{\Gamma(p_1)\Gamma(p_2)\dots\Gamma(p_n)}{\Gamma(p_1 + p_2 + \dots + p_n)}. \quad (6)$$

For  $n = 2$ , the classical beta function is represented by integral (1) over a unit interval  $0 \leq t \leq 1$ . If  $n = 3$ , we integrate over a triangular region in  $(x, y)$  plane with vertices  $(0, 0)$ ,  $(1, 0)$  and  $(0, 1)$ . This is the set  $\{(x, y), x \geq 0, y \geq 0, x + y \leq 1\}$  and is called the standard simplex in  $\mathbb{R}^2$ . The points  $(x, y)$  of the simplex are (1-1) in correspondence with the triples  $(x, y, 1 - x - y)$  of non-negative weights with unit sum. The standard simplex in  $\mathbb{R}^3$  is  $\{(x, y, z), x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1\}$  which is a solid tetrahedron with vertices  $(0, 0, 0)$ ,  $(1, 0, 0)$ ,  $(0, 1, 0)$  and  $(0, 0, 1)$ . In general, we denote the standard simplex in  $\mathbb{R}^n$  by

$$E = E_n = \{(x_1, x_2, \dots, x_n) : x_i \geq 0, \sum x_i \leq 1, i = 1, 2, \dots, n\}. \quad (7)$$

The points  $(x_1, x_2, \dots, x_n)$  of the simplex are (1-1) in correspondence with the  $(n + 1)$ -tuples  $(x_1, x_2, \dots, x_n, 1 - x_1 - x_2 - \dots - x_n)$  of non-negative weights with unit sum. If  $n$  is not mentioned, we write  $E$  in place of  $E_n$ . The interior of  $E$  is denoted by

$$\text{int}(E) = \{(x_1, x_2, \dots, x_n) : x_i > 0, \sum x_i < 1, i = 1, 2, \dots, n\}. \quad (8)$$

Let  $p \in \mathbb{C}^n$ ,  $p_i \neq 0$ ,  $|\arg p_i| < \frac{\pi}{2}$  and  $n \geq 2$ . If  $E = E_{n-1}$  be the standard simplex in  $\mathbb{R}^{n-1}$ , then integral form of beta function of  $n$  variables defined by the relation (9) and is proved in [3].

$$\beta(p) = \beta(p_1, p_2, \dots, p_n) = \int_E x_1^{p_1-1} \dots x_{n-1}^{p_{n-1}-1} (1 - x_1 - \dots - x_{n-1})^{p_n-1} dx_1 \dots dx_{n-1}. \quad (9)$$

## 2 Properties of beta function of several variables

**Proposition 2.1.** *If  $\text{Re}(x) > 0$ ,  $\text{Re}(x + y) > 0$ ,  $\text{Re}(x + y + z) > 0$  and  $\text{Re}(w) > 0$ , then beta function of four variables can be written as*

$$\beta(x, y)\beta(x + y, z)\beta(x + y + z, w) = \beta(x, y, z, w) \quad (10)$$

*Proof.* From the relation (3), we see that for three variables we have

$$\beta(x + y, z) = \frac{\Gamma(x + y)\Gamma(z)}{\Gamma(x + y + z)}, \quad \text{Re}(x + y) > 0, \quad \text{Re}(z) > 0, \quad (11)$$

and for four variables we have

$$\beta(x + y + z, w) = \frac{\Gamma(x + y + z)\Gamma(w)}{\Gamma(x + y + z + w)}, \quad \text{Re}(x + y + z) > 0, \quad \text{Re}(w) > 0. \quad (12)$$

Multiplying equations (3), (11) and (12), we get

$$\begin{aligned} \beta(x, y)\beta(x + y, z)\beta(x + y + z, w) &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} \frac{\Gamma(x + y)\Gamma(z)}{\Gamma(x + y + z)} \frac{\Gamma(x + y + z)\Gamma(w)}{\Gamma(x + y + z + w)} \\ &\Rightarrow \beta(x, y)\beta(x + y, z)\beta(x + y + z, w) = \frac{\Gamma(x)\Gamma(y)\Gamma(z)\Gamma(w)}{\Gamma(x + y + z + w)}. \end{aligned} \quad (13)$$

Comparing the relation (13) with definition of beta function of several variables (6) implies the required result.  $\square$

**Remarks.** *The relation (10) can be extended up to  $n$  variables which will be the representation for beta function of several variables. If there are only two variables, we get the classical definition of beta function.*

**Theorem 2.2.** *If there are  $n$  variables, then there exists a relation between gamma function and beta function of several variables as*

$$\Gamma\left(\frac{1}{n}\right) = \sqrt[n]{\beta\left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)} \quad (14)$$

*Proof.* Using  $p_1 = p_2 = \dots = p_n = \frac{1}{n}$  in the relation (6), we have

$$\beta\left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right) = \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{n}\right)\dots\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}\right)} = \frac{\left[\Gamma\left(\frac{1}{n}\right)\right]^n}{\Gamma(1)},$$

which implies

$$\Gamma\left(\frac{1}{n}\right) = \sqrt[n]{\beta\left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)}.$$

□

**Corollary 2.3.** *If  $n = 2$  .i.e, there are only two variables we have a classical result (See [2]).*

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\beta\left(\frac{1}{2}, \frac{1}{2}\right)} = \sqrt{\pi}.$$

**Theorem 2.4.** *If  $\operatorname{Re}(x) > 0$ ,  $\operatorname{Re}(x + y) > 0$ ,  $\operatorname{Re}(x + y + z) > 0$  and  $\operatorname{Re}(w) > 0$ , then we have the following properties of beta function with three and four variables.*

$$\beta(x, y, z) = 2^2 \int_0^{\pi/2} \int_0^{\pi/2} (\cos \theta)^{2(x+y)-2} (\sin \theta)^{2(y+z)-2} d^2 \theta \quad (15)$$

and

$$\beta(x, y, z, w) = 2^3 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} (\cos \theta)^{2(x+y+z)-3} (\sin \theta)^{2(y+z+w)-3} d^3 \theta. \quad (16)$$

*Proof.* From equation (9), we see that the integral form of beta function for three variables is

$$\beta(x, y, z) = \int_E p^{x-1} q^{y-1} (1 - p - q)^{z-1} dp dq,$$

taking  $v = 1 - p$ , the above equation takes the form

$$\beta(x, y, z) = \int p^{x-1} dp \int_0^v q^{y-1} (v - q)^{z-1} dq.$$

Setting  $q = vt$  implies  $dq = v dt$  and limits of integration becomes 0 to 1. Thus we have

$$\beta(x, y, z) = \int p^{x-1} dp \int_0^1 v^{y+z-1} q^{y-1} (1 - q)^{z-1} dt = \int p^{x-1} v^{y+z-1} dp \beta(y, z).$$

Using the similar reasons, we conclude that

$$\beta(x, y, z) = \beta(y, z) \int_0^1 p^{x-1} (1 - p)^{y+z-1} dp = \beta(y, z) \beta(x, y + z).$$

Applying the relation (4) on right hand side of above equation, we get

$$\beta(x, y, z) = 2 \int_0^{\pi/2} (\cos \theta)^{2y-1} (\sin \theta)^{2z-1} d\theta \int_0^{\pi/2} (\cos \theta)^{2x-1} (\sin \theta)^{2y+2z-1} d\theta$$

$$\Rightarrow \beta(x, y, z) = 2^2 \int_0^{\pi/2} \int_0^{\pi/2} (\cos \theta)^{2(x+y)-2} (\sin \theta)^{2(y+z)-2} d^2 \theta.$$

Similarly, we can prove the relation (16).  $\square$

**Remarks.** We can extend the results of the Theorem 2.4 up to  $n$  variables and that can be expressed as

$$\beta(x_1, x_2, \dots, x_n) = 2^{n-1} \int_0^{\pi/2} \int_0^{\pi/2} \dots \int_0^{\pi/2} (\cos \theta)^{2(x_1+x_2+\dots+x_{n-1})-(n-1)} (\sin \theta)^{2(x_2+2x_3+\dots+(n-1)x_n)-(n-1)} d^{(n-1)} \theta, \quad (17)$$

which can be proved like the Theorem 2.4.

**Proposition 2.5.** If  $m_1, m_2, \dots, m_n \in \mathbb{N}$  and  $\alpha \neq 0$ , then we have the following properties of beta function with  $n$  variables.

$$\beta(\alpha, m_1 + 1, m_2 + 1, \dots, m_n + 1) = \frac{m_1! m_2! \dots m_n!}{(\alpha)_{((m_1+m_2+\dots+m_n)+n)}} \quad (18)$$

where  $(\alpha)_n = \alpha(\alpha + 1) \dots (\alpha + n - 1)$ , is the Pochhammer's symbol.

$$\beta(m_1 + 1, m_2 + 1, m_3 + 1, \dots, m_n + 1) = \frac{m_1 m_2 m_3 \dots m_n \beta(m_1, m_2, m_3, \dots, m_n)}{(m_1 + \dots + m_n)(m_1 + \dots + m_n + 1) \dots (m_1 + \dots + m_n + n - 1)}, \quad (19)$$

and

$$\beta(m_1 + 1, m_2, \dots, m_n) + \beta(m_1, m_2 + 1, \dots, m_n) + \dots + \beta(m_1, m_2, \dots, m_n + 1) = \beta(m_1, m_2, \dots, m_n). \quad (20)$$

*Proof.* Using the definition of beta function of several variables (6), we have

$$\beta(\alpha, m_1 + 1, m_2 + 1, \dots, m_n + 1) = \frac{\Gamma(\alpha) \Gamma(m_1 + 1) \Gamma(m_2 + 1) \dots \Gamma(m_n + 1)}{\Gamma(\alpha + m_1 + m_2 + \dots + m_n + n)}.$$

Using  $\Gamma(n + 1) = n!$  and the property of Pochhammer's symbol  $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$  (See [12]), in the above equation, we find the required relation (18). To prove the relation (19), we use the same definition along with the property of classical gamma function  $\Gamma(n + 1) = n\Gamma(n)$  in the numerator and denominator to conclude the desired result. For (20), we use the relation (6) and proceed as

$$\begin{aligned} & \beta(m_1 + 1, m_2, \dots, m_n) + \beta(m_1, m_2 + 1, \dots, m_n) + \dots + \beta(m_1, m_2, \dots, m_n + 1) = \\ & \frac{\Gamma(m_1 + 1) \Gamma(m_2) \dots \Gamma(m_n)}{\Gamma(m_1 + \dots + m_n + 1)} + \frac{\Gamma(m_1) \Gamma(m_2 + 1) \dots \Gamma(m_n)}{\Gamma(m_1 + \dots + m_n + 1)} + \dots + \frac{\Gamma(m_1) \Gamma(m_2) \dots \Gamma(m_n + 1)}{\Gamma(m_1 + \dots + m_n + 1)}, \end{aligned}$$

by using  $\Gamma(n + 1) = n\Gamma(n)$  in numerator as well as in denominator, we get

$$\begin{aligned} & = \frac{m_1 \Gamma(m_1) \Gamma(m_2) \dots \Gamma(m_n)}{\Gamma(m_1 + \dots + m_n + 1)} + \frac{\Gamma(m_1) m_2 \Gamma(m_2) \dots \Gamma(m_n)}{\Gamma(m_1 + \dots + m_n + 1)} + \dots + \frac{\Gamma(m_1) \Gamma(m_2) \dots m_n \Gamma(m_n)}{\Gamma(m_1 + \dots + m_n + 1)} \\ & = \frac{\Gamma(m_1) \Gamma(m_2) \dots \Gamma(m_n) (m_1 + m_2 + \dots + m_n)}{(m_1 + m_2 + \dots + m_n) \Gamma(m_1 + m_2 + \dots + m_n)} = \beta(m_1, m_2, \dots, m_n). \end{aligned}$$

$\square$

**Corollary 2.6.** If 1 is added to any one of the variables, then following results hold ( only three variables are provided here).

$$p\beta(p, q, r + 1) = r\beta(p + 1, q, r)$$

$$q\beta(p + 1, q, r) = p\beta(p, q + 1, r)$$

and

$$q\beta(p, q, r + 1) = r\beta(p, q + 1, r).$$

*Proof.* Just use the definition (6) along with the result  $\Gamma(n+1) = n\Gamma(n)$ .  $\square$

### 3 Main results properties of $k$ -beta function

Recently, Diaz and Pariguan [4] introduced the generalized  $k$ -gamma function as

$$\Gamma_k(x) = \lim_{n \rightarrow \infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \quad k > 0, x \in \mathbb{C} \setminus k\mathbb{Z}^- \quad (21)$$

and also gave the properties of said function. The  $\Gamma_k$  is one parameter of deformation of the classical gamma function such that  $\Gamma_k \rightarrow \Gamma$  as  $k \rightarrow 1$ . The  $\Gamma_k$  is based on the repeated appearance of the expression of the following form

$$\alpha(\alpha+k)(\alpha+2k)(\alpha+3k)\dots(\alpha+(n-1)k). \quad (22)$$

The function of the variable  $\alpha$  given by the statement (22), denoted by  $(\alpha)_{n,k}$ , is called the Pochhammer  $k$ -symbol. We obtain the usual Pochhammer symbol  $(\alpha)_n$  by taking  $k = 1$ . The definition given in (21), is the generalization of  $\Gamma(x)$  and the integral form of  $\Gamma_k$  is given by

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt, \quad \operatorname{Re}(x) > 0. \quad (23)$$

From (23), we can easily show that

$$\Gamma_k(x) = k^{\frac{x}{k}-1} \Gamma\left(\frac{x}{k}\right). \quad (24)$$

The same authors defined the  $k$ -beta function as

$$\beta_k(x, y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \quad \operatorname{Re}(x) > 0, \operatorname{Re}(y) > 0 \quad (25)$$

and the integral form of  $\beta_k(x, y)$  is

$$\beta_k(x, y) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1} (1-t)^{\frac{y}{k}-1} dt. \quad (26)$$

From the definition of  $\beta_k(x, y)$  given in (25) and (26), we can easily prove that

$$\beta_k(x, y) = \frac{1}{k} \beta\left(\frac{x}{k}, \frac{y}{k}\right). \quad (27)$$

Also, the researchers [5–9] have worked on the generalized  $k$ -gamma and  $k$ -beta functions and discussed the following properties:

$$\Gamma_k(x+k) = x\Gamma_k(x) \quad (28)$$

$$(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)} \quad (29)$$

$$\Gamma_k(k) = 1, \quad k > 0 \quad (30)$$

$$\Gamma_k\left((2n+1)\frac{k}{2}\right) = k^{\frac{2n-1}{2}} \frac{(2n)!\sqrt{\pi}}{2^n n!}, \quad k > 0, n \in \mathbb{N}. \quad (31)$$

In [2], it is proved that gamma function  $\Gamma(z)$  is analytic on  $\mathbb{C}$  except the poles at  $z = 0, -1, -2, \dots$  and the residue at  $z = -n$  is equal to  $\frac{(-1)^n}{n!}$ ,  $n = 0, 1, 2, \dots$ . Recently, Mubeen and Rehman [11] proved that for  $k > 0$ , the function  $\Gamma_k(x)$  is analytic on  $\mathbb{C}$ , except the single poles at  $x = 0, -k, -2k, \dots$  and the residue at  $x = -nk$  is  $\frac{1}{(-1)^n k^n n!}$ .

Using (25) and (27), we see that, for  $x, y > 0$  and  $k > 0$ , the following properties of  $k$ -beta function of two variables are satisfied by authors (see [5, 6] and [10]):

$$\beta_k(x+k, y) = \frac{x}{x+y} \beta_k(x, y) \quad (32)$$

$$\beta_k(x, y+k) = \frac{y}{x+y} \beta_k(x, y) \quad (33)$$

$$\beta_k(xk, yk) = \frac{1}{k} \beta_k(x, y) \quad (34)$$

$$\beta_k(nk, nk) = \frac{[(n-1)!]^2}{k(2n-1)!}, \quad n \in \mathbb{N} \quad (35)$$

$$\beta_k(x, k) = \frac{1}{x}, \quad \beta_k(k, y) = \frac{1}{y}. \quad (36)$$

**Remarks.** When  $k \rightarrow 1$ ,  $\beta_k(x, y) \rightarrow \beta(x, y)$ .

Now, we define  $k$ -beta function of several variables and prove some properties of the said function. With the help of this definition, we extend some previous results which will help us to compute the values of  $\beta_k$  function at particular points.

**Definition 3.1.** Let  $p = (p_1, p_2, \dots, p_n) \in \mathbb{C}^n$ ,  $p_i \neq \{0, -1, -2, \dots\}$ ,  $n \geq 2$ ,  $k > 0$ . The  $k$ -beta function of  $n$  variables is defined as

$$\beta_k(p) = \beta_k(p_1, p_2, \dots, p_n) = \frac{\Gamma_k(p_1)\Gamma_k(p_2)\dots\Gamma_k(p_n)}{\Gamma_k(p_1 + p_2 + \dots + p_n)}. \quad (37)$$

**Remarks.** For  $n = 2$ , we obtain  $k$ -beta function represented by (25) and if  $k = 1$ , we have the classical beta function.

**Lemma 3.2.** Let  $\lambda$  and  $\mu$  be the positive  $\sigma$ -finite or a complex measure on a measure space  $X$  and  $T$  respectively with  $|\lambda|$  and  $|\mu|$  be the corresponding total variation. Assume that the function  $f : X \times T \rightarrow \mathbb{C}$  is measurable on the product space  $X \times T$  with respect to the product  $\sigma$ -algebra. If the integral

$$\int_X \int_T |f(x, t)| d|\lambda|(x) d|\mu|(t)$$

is finite when evaluated as an iterated integral in one order or the other, then the value of

$$\int_X \int_T f(x, t) d\lambda(x) d\mu(t)$$

is independent whether it is evaluated as a double integral or an iterated integral in either order.

**Remarks.** The above Lemma is the Fubini's Theorem. For more complete and precise detail of Fubini's Theorem for positive measure (see Rudin 1974 [13]).

**Proposition 3.3.** Let  $p \in \mathbb{C}^n \setminus \{0\}$ ,  $n \geq 2$ ,  $k > 0$  and let  $E = E_{n-1}$  be the standard simplex in  $\mathbb{R}^{n-1}$ , then integral form of  $k$ -beta function of  $n$  variables is defined by

$$\beta_k(p) = \beta_k(p_1, \dots, p_n) = \frac{1}{k^{n-1}} \int_E x_1^{\frac{p_1}{k}-1} \dots x_{n-1}^{\frac{p_{n-1}}{k}-1} (1 - x_1 - \dots - x_{n-1})^{\frac{p_n}{k}-1} dx_1 \dots dx_{n-1}. \quad (38)$$

*Proof.* We denote the integral by  $I_{n,k} = \beta_k(p_1, \dots, p_n)$ . Thus  $I_{2,k} = \beta_k(p_1, p_2)$  which is equivalent to (26). For  $n \geq 3$ , first we take the case when  $p \in \mathbb{R}^n$ , where  $\mathbb{R}$  is the set of positive real numbers. The integrand is positive and the integral may be evaluated as an iterated integral by Lemma 3.2. Integrating over  $x_{n-1}$  and defining  $v = 1 - x_1 - x_2 - \dots - x_{n-2}$ , we see that

$$I_{n,k} = \int_{E_{n-1}} x_1^{\frac{p_1}{k}-1} \dots x_{n-2}^{\frac{p_{n-2}}{k}-1} dx_1 \dots dx_{n-2} \cdot \int_0^v x_{n-1}^{\frac{p_{n-1}}{k}-1} (v - x_{n-1})^{\frac{p_n}{k}-1} dx_{n-1}. \quad (39)$$

The region of integration for outer integral is

$$\{(x_1, \dots, x_{n-2}) : x_1 \geq 0, \dots, x_{n-2} \geq 0, 1 - x_1 - \dots - x_{n-2} \geq 0\} = E_{n-2},$$

putting  $x_{n-1} = vt$ , the inner integral becomes

$$v^{\frac{p_{n-1}+p_n}{k}-1} \int_0^1 t^{\frac{p_{n-1}}{k}-1} (1-t)^{\frac{p_n}{k}-1} dt = v^{\frac{p_{n-1}+p_n}{k}-1} \cdot k\beta_k(p_{n-1}, p_n), \quad (40)$$

and using the above result in equation (39), we have

$$\begin{aligned} I_{n,k} &= \int_{E_{n-2}} x_1^{\frac{p_1}{k}-1} \dots x_{n-2}^{\frac{p_{n-2}}{k}-1} dx_1 \dots dx_{n-2} v^{\frac{p_{n-1}+p_n}{k}-1} \cdot k\beta_k(p_{n-1}, p_n) \\ &= k\beta_k(p_{n-1}, p_n) \int_{E_{n-2}} x_1^{\frac{p_1}{k}-1} \dots x_{n-2}^{\frac{p_{n-2}}{k}-1} (1 - x_1 - \dots - x_{n-2})^{\frac{p_{n-1}+p_n}{k}-1} dx_1 \dots dx_{n-2} \\ &= I_{n-1,k}(p_1, \dots, p_{n-2}, p_{n-1} + p_n) k\beta_k(p_{n-1}, p_n). \end{aligned} \quad (41)$$

Now we make the inductive assumption that  $I_{n-1,k}$  is the  $k$ -beta function of  $n-1$  variables. So, by the relations (41) and (37), we get

$$I_{n,k} = k^{n-2} \beta_k(p_1, \dots, p_{n-2}, p_{n-1} + p_n) \cdot k\beta_k(p_{n-1}, p_n) = k^{n-1} \beta_k(p_1, \dots, p_n),$$

which is the relation (38) for  $k > 0$ ,  $n \geq 3$  and  $p \in \mathbb{R}^n$ . The same proof holds for the case when  $p \in \mathbb{C}^n \setminus \{0\}$  if the integrand of the equation (39) can be replaced by the absolute values, simply each  $p_i$  by  $\operatorname{Re}(p_i)$ ,  $i = 1, 2, \dots, n$ . The iterated integral of the absolute value is finite by the preceding proof for the real case and Lemma 3.2, ensures that the relation (39) is valid for complex case.

Now, we prove the symmetry of all arguments. According to the relation (37),  $\beta_k(p_1, \dots, p_n)$  is symmetric in  $p_1, \dots, p_n$  and it is worth-while to verify directly that the integral on R.H.S of the equation (38) has the same property. Symmetry in  $p_1, \dots, p_{n-1}$  is made evident by simply relabeling the variables of integration, so it suffices to prove symmetry in  $p_1$  and  $p_n$ . We define

$$x_n = 1 - x_1 - \dots - x_{n-1} \quad (42)$$

and change the variables from  $x_1, \dots, x_{n-1}$  to  $x_2, \dots, x_n$ . The jacobian determinant of the transformation has unit magnitude and the region of integration changes from

$$\{(x_1, \dots, x_{n-1}) : x_1 \geq 0, \dots, x_{n-1} \geq 0, x_1 + \dots + x_{n-1} \leq 1\} \quad (43)$$

to

$$\{(x_2, \dots, x_n) : x_2 \geq 0, \dots, x_n \geq 0, x_2 + \dots + x_n \leq 1\}. \quad (44)$$

The last two inequalities in (44) results from the last and first in (43). Thus the region of integration is again  $E$  in terms of the new variables and integral on R.H.S of the equation (38) becomes

$$\frac{1}{k^{n-1}} \int_E (1 - x_2 - \dots - x_n)^{\frac{p_1}{k}-1} x_2^{\frac{p_2}{k}-1} \dots x_n^{\frac{p_n}{k}-1} dx_2 \dots dx_n. \quad (45)$$

Relabeling  $x_n$  as  $x_1$  changes it to the original form with  $p_1$  and  $p_n$  interchanged. Hence the original integral is symmetric in  $p_1$  and  $p_n$ . We may think of the variables  $p_1, \dots, p_n$  as being attached to the vertices of the simplex  $E = E_{n-1}$  as follows:  $\frac{p_1}{k}$  to the vertex  $(1, 0, \dots, 0)$ ,  $\frac{p_2}{k}$  to  $(0, 1, \dots, 0), \dots, \frac{p_{n-1}}{k}$  to  $(0, 0, \dots, 1)$  and  $\frac{p_n}{k}$  to the  $(0, 0, \dots, 0)$ . Also, the vertex at the origin is at an equal footing with all other vertices. The points  $(x_1, \dots, x_{n-1})$  of  $E_{n-1}$  are in (1-1) correspondence with  $n$ -tuples  $(x_1, \dots, x_{n-1}, 1 - x_1 - \dots - x_{n-1})$  of non negative weights with unit sum and the last of these weights is the quantity  $x_n$  defined in (42) which has the value unity at the vertex  $(0, 0, \dots, 0)$ . The complete symmetry in  $1, \dots, n$  can be exhibited by writing the  $n - 1$ -fold integral in the relation (38) as  $n$ -fold integral with a Dirac delta function in the integrand

$$\beta_k(p) = \frac{1}{k^{n-1}} \int_0^1 \int_0^1 \dots \int_0^1 x_1^{\frac{p_1}{k}-1} \dots x_n^{\frac{p_n}{k}-1} \delta(1 - x_1 - \dots - x_n) dx_1 \dots dx_n. \quad (46)$$

Now we discuss some properties of  $k$ -beta function of several variables.

**Proposition 3.4.** For  $\operatorname{Re}(x), \operatorname{Re}(y) > 0$  and  $k > 0$ , the  $k$ -beta function of two variables satisfy the following properties:

- (i)  $\beta_k(x, y) = \int_0^\infty \frac{s^{\frac{x}{k}-1}}{(s+1)^{\frac{x+y}{k}}} ds$
- (ii)  $\beta_k(x, y) = \frac{2}{k} \int_0^{\pi/2} (\cos \phi)^{\frac{2x}{k}-1} (\sin \phi)^{\frac{2y}{k}-1} d\phi$
- (iii)  $k(b-a)^{\frac{x+y}{k}-1} \beta_k(x, y) = \int_a^b (s-a)^{\frac{x}{k}-1} (b-s)^{\frac{y}{k}-1} ds.$

*Proof.* By changing the variables as  $t = \frac{s}{s+1}$ ,  $t = \cos^2 \phi$  or  $t = \sin^2 \phi$  and  $t = \frac{s-a}{b-a}$  respectively in the relation (26), we have (i), (ii) and (iii).  $\square$

**Corollary 3.5.** For  $k > 0$ , we have  $\beta_k(\frac{k}{2}, \frac{k}{2}) = \frac{\pi}{k}$  and hence the values of  $k$ -beta function can be computed at particular points.

*Proof.* By Proposition 3.4, putting  $x = y = \frac{k}{2}$ , we have

$$\beta_k\left(\frac{k}{2}, \frac{k}{2}\right) = \frac{2}{k} \int_0^{\pi/2} (\cos \phi)^0 (\sin \phi)^0 d\phi = \frac{2}{k} \cdot \frac{\pi}{2} = \frac{\pi}{k}. \quad (47)$$

Now, we can compute the values of  $k$ -beta function from equation (47) by using  $k = 1, 2, 3, \dots, n$  as

$$\begin{aligned} \beta_1\left(\frac{1}{2}, \frac{1}{2}\right) &= \pi \\ \beta_2(1, 1) &= \frac{\pi}{2} \\ \beta_3\left(\frac{3}{2}, \frac{3}{2}\right) &= \frac{\pi}{3} \\ &\vdots \\ \beta_n\left(\frac{n}{2}, \frac{n}{2}\right) &= \frac{\pi}{n} \\ \beta_{2n}(n, n) &= \frac{\pi}{2n}. \quad \square \end{aligned}$$

**Corollary 3.6.** Using the relation (25) on left hand side of the relation (47), we get

$$\frac{\Gamma_k(\frac{k}{2})\Gamma_k(\frac{k}{2})}{\Gamma_k(k)} = \frac{\pi}{k} \Rightarrow \Gamma_k\left(\frac{k}{2}\right) = \sqrt{\frac{\pi}{k}},$$

which is a conclusion of the relation (31).



**Remarks.** From the corollary 3.5 it seems that  $k$  is any natural number. However, we can compute the values of  $k$ -beta function for all real numbers  $k > 0$ . From the above results, if  $k = 1$ , we conclude an important result [2].

$$\left(\beta_1\left(\frac{1}{2}, \frac{1}{2}\right)\right)^{\frac{1}{2}} = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \quad (48)$$

**Proposition 3.7.** If  $k > 0$ ,  $\operatorname{Re}(x) > 0$ ,  $\operatorname{Re}(x + y) > 0$  and  $\operatorname{Re}(z) > 0$ , then  $k$ -beta function of three variables can be written as

$$\beta_k(x, y)\beta_k(x + y, z) = \beta_k(x, y, z). \quad (49)$$

*Proof.* From the relation (25), we see that for three variables

$$\beta_k(x + y, z) = \frac{\Gamma_k(x + y)\Gamma_k(z)}{\Gamma_k(x + y + z)}, \quad \operatorname{Re}(x + y) > 0, \operatorname{Re}(z) > 0. \quad (50)$$

Multiplying equations (25) and (50) we have

$$\beta_k(x, y)\beta_k(x + y, z) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x + y)} \frac{\Gamma_k(x + y)\Gamma_k(z)}{\Gamma_k(x + y + z)} = \frac{\Gamma_k(x)\Gamma_k(y)\Gamma_k(z)}{\Gamma_k(x + y + z)}. \quad (51)$$

Comparing the relation (51) with definition of  $k$ -beta function of several variables (37) implies the desired proof.  $\square$

**Remarks.** The relation (49) can be extended up to  $n$  variables. Here, we introduced the new notation for beta function of several variables in right hand side of relation (49). If there are only two variables, we get the definition of  $k$ -beta function defined in [4] and if  $k = 1$ , the classical one.

**Theorem 3.8.** If there are  $n$  variables, then  $k$ -gamma and  $k$ -beta functions are related as

$$\Gamma_k\left(\frac{k}{n}\right) = \sqrt[n]{\beta_k\left(\frac{k}{n}, \frac{k}{n}, \dots, \frac{k}{n}\right)} \quad (52)$$

*Proof.* Using  $p_1 = p_2 = \dots = p_n = \frac{k}{n}$  in the definition (37), we have

$$\beta_k\left(\frac{k}{n}, \frac{k}{n}, \dots, \frac{k}{n}\right) = \frac{\Gamma_k\left(\frac{k}{n}\right)\Gamma_k\left(\frac{k}{n}\right)\dots\Gamma_k\left(\frac{k}{n}\right)}{\Gamma_k\left(\frac{k}{n} + \frac{k}{n} + \dots + \frac{k}{n}\right)} = \frac{\left[\Gamma_k\left(\frac{k}{n}\right)\right]^n}{\Gamma_k(k)},$$

and using  $\Gamma_k(k) = 1$ , we have

$$\Gamma_k\left(\frac{k}{n}\right) = \sqrt[n]{\beta_k\left(\frac{k}{n}, \frac{k}{n}, \dots, \frac{k}{n}\right)} \quad \square$$

**Corollary 3.9.** If  $k = 1, n = 2$  .i.e, there are only two variables we have a classical result

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\beta\left(\frac{1}{2}, \frac{1}{2}\right)} = \sqrt{\pi}$$

**Theorem 3.10.** If  $\operatorname{Re}(x) > 0$ ,  $\operatorname{Re}(x + y) > 0$ ,  $\operatorname{Re}(x + y + z) > 0$  and  $\operatorname{Re}(w) > 0$ , then we have the following properties of  $k$ -beta function with three and four variables.

$$\beta_k(x, y, z) = \left(\frac{2}{k}\right)^2 \int_0^{\pi/2} \int_0^{\pi/2} (\cos \theta)^{\frac{2(x+y)}{k}-2} (\sin \theta)^{\frac{2(y+z)}{k}-2} d^2\theta \quad (53)$$

and

$$\beta_k(x, y, z, w) = \left(\frac{2}{k}\right)^3 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} (\cos \theta)^{\frac{2(x+y+z)}{k}-3} (\sin \theta)^{\frac{2(y+z+3w)}{k}-3} d^3\theta. \quad (54)$$

*Proof.* From Proposition 3.3, we see that the integral form of  $k$ -beta function of three variables is

$$\beta_k(x, y, z) = \frac{1}{k^2} \int_E p^{\frac{x}{k}-1} q^{\frac{y}{k}-1} (1-p-q)^{\frac{z}{k}-1} dp dq.$$

Setting  $v = 1 - p$ , the above equation takes the form

$$\beta_k(x, y, z) = \frac{1}{k^2} \int p^{\frac{x}{k}-1} dp \int_0^v q^{\frac{y}{k}-1} (v-q)^{\frac{z}{k}-1} dq,$$

putting  $q = vt$  implies  $dq = v dt$  and limit of integration becomes 0 to 1. Thus we have

$$\beta_k(x, y, z) = \frac{1}{k} \int p^{\frac{x}{k}-1} dp \frac{1}{k} \int_0^1 v^{\frac{y+z}{k}-1} q^{\frac{y}{k}-1} (1-q)^{\frac{z}{k}-1} dt = \frac{1}{k} \int p^{\frac{x}{k}-1} v^{\frac{y+z}{k}-1} dp \beta_k(y, z)$$

and by similar arguments, we conclude that

$$\beta_k(x, y, z) = \beta_k(y, z) \frac{1}{k} \int_0^1 p^{\frac{x}{k}-1} (1-p)^{\frac{y+z}{k}-1} dp = \beta_k(y, z) \beta_k(x, y+z).$$

Using the Proposition 3.4 on right hand side of above equation, we have

$$\begin{aligned} \beta_k(x, y, z) &= \frac{2}{k} \int_0^{\pi/2} (\cos \theta)^{\frac{2x}{k}-1} (\sin \theta)^{\frac{2y+2z}{k}-1} d\theta \cdot \frac{2}{k} \int_0^{\pi/2} (\cos \theta)^{\frac{2y}{k}-1} (\sin \theta)^{\frac{2z}{k}-1} d\theta \\ \Rightarrow \beta_k(x, y, z) &= \left(\frac{2}{k}\right)^2 \int_0^{\pi/2} \int_0^{\pi/2} (\cos \theta)^{\frac{2(x+y)}{k}-2} (\sin \theta)^{\frac{2(y+z)}{k}-2} d^2\theta. \end{aligned}$$

Similarly, we can prove the relation (54). □

**Remarks.** We can extend the results of Theorem 3.10 up to  $n$  variables and can be expressed as

$$\begin{aligned} \beta_k(x_1, x_2, \dots, x_n) &= \left(\frac{2}{k}\right)^{n-1} \int_0^{\pi/2} \int_0^{\pi/2} \dots \int_0^{\pi/2} \\ &(\cos \theta)^{\frac{2(x_1+x_2+\dots+x_{n-1})}{k}-(n-1)} (\sin \theta)^{\frac{2(x_n+2x_3+\dots+(n-1)x_n)}{k}-(n-1)} d^{(n-1)}\theta \end{aligned} \quad (55)$$

which can be proved like the Theorem 3.10.

**Proposition 3.11.** If  $m_1, m_2, \dots, m_n \in \mathbb{N}, k > 0$  and  $\alpha \neq 0$ , then we have the following properties of the  $k$ -beta function with  $n$  variables.

$$\beta_k(\alpha, m_1 + k, m_2 + k, \dots, m_n + k) = \frac{m_1 m_2 \dots m_n}{(\alpha)_{n,k}} \beta_k(m_1, m_2, \dots, m_n, \alpha + nk) \quad (56)$$

where  $(\alpha)_{n,k}$  is the Pochhammer  $k$ -symbol defined in the relation (22),

$$\beta_k(m_1 + k, m_2 + k, m_3 + k, \dots, m_n + k) =$$

$$\frac{m_1 m_2 m_3 \dots m_n}{(m_1 + \dots + m_n)(m_1 + \dots + m_n + k) \dots (m_1 + \dots + m_n + (n-1)k)} \beta_k(m_1, m_2, \dots, m_n) \quad (57)$$

and

$$\beta_k(m_1 + k, m_2, \dots, m_n) + \beta_k(m_1, m_2 + k, \dots, m_n) + \dots + \beta_k(m_1, m_2, \dots, m_n + k) = \beta_k(m_1, m_2, \dots, m_n) \quad (58)$$

*Proof.* Using the definition of  $k$ -beta function of several variables (37), we have

$$\beta_k(\alpha, m_1 + k, m_2 + k, \dots, m_n + k) = \frac{\Gamma_k(\alpha)\Gamma_k(m_1 + k)\Gamma_k(m_2 + k)\dots\Gamma_k(m_n + k)}{\Gamma_k(\alpha + m_1 + m_2 + \dots + m_n + nk)}.$$

Use of  $\Gamma_k(x + k) = x\Gamma_k(x)$  in R.H.S of the above relation gives

$$= \frac{m_1 m_2 \dots m_n \Gamma_k(\alpha)\Gamma_k(m_1)\Gamma_k(m_2)\dots\Gamma_k(m_n)}{\Gamma_k(\alpha + m_1 + m_2 + \dots + m_n + nk)}.$$

Multiplying the numerator and denominator by  $\Gamma_k(\alpha + nk)$  implies

$$= \frac{m_1 m_2 \dots m_n \Gamma_k(\alpha)\Gamma_k(m_1)\Gamma_k(m_2)\dots\Gamma_k(m_n) \cdot \Gamma_k(\alpha + nk)}{\Gamma_k(\alpha + m_1 + m_2 + \dots + m_n + nk)\Gamma_k(\alpha + nk)},$$

and by the property of Pochhammer  $k$ - symbol,  $(\alpha)_{n,k} = \frac{\Gamma_k(\alpha + nk)}{\Gamma_k(\alpha)}$ , the above equation creates the desired result (56). To prove the relation (57), we use the definition (37) and the relation (28) in the numerator and denominator, and we conclude the required result. For (58), we use the definition (3.17) and proceed as

$$\begin{aligned} & \beta_k(m_1 + k, m_2, \dots, m_n) + \beta_k(m_1, m_2 + k, \dots, m_n) + \dots + \beta_k(m_1, m_2, \dots, m_n + k) = \\ & \frac{\Gamma_k(m_1 + k)\dots\Gamma_k(m_n)}{\Gamma_k(m_1 + \dots + m_n + k)} + \frac{\Gamma_k(m_1)\Gamma_k(m_2 + k)\dots\Gamma_k(m_n)}{\Gamma_k(m_1 + \dots + m_n + k)} + \dots + \frac{\Gamma_k(m_1)\dots\Gamma_k(m_n + k)}{\Gamma_k(m_1 + \dots + m_n + k)}, \end{aligned}$$

and using the relation (28), we obtain the required relation (58).  $\square$

**Corollary 3.12.** *If  $k$  is added to any one of the variables, then following results hold (only three variables are provided here).*

$$p\beta_k(p, q, r + k) = r\beta_k(p + k, q, r)$$

$$q\beta_k(p + k, q, r) = p\beta_k(p, q + k, r)$$

and

$$q\beta_k(p, q, r + k) = r\beta_k(p, q + k, r).$$

*Proof.* Just use the definition 37 along with the result  $\Gamma_k(x + k) = x\Gamma_k(x)$ .  $\square$

Here, we give some properties of  $k$ -gamma function which will be used to prove some results involving  $k$ -beta function of several variables.

**Lemma 3.13.** *If  $r - 1 \in \mathbb{N}$ , then we have*

$$\Gamma\left(\frac{1}{r}\right)\Gamma\left(\frac{2}{r}\right)\Gamma\left(\frac{3}{r}\right)\dots\Gamma\left(\frac{r-1}{r}\right) = r^{-\frac{1}{2}}(2\pi)^{\frac{(r-1)}{2}}. \quad (59)$$

**Proposition 3.14.** *If  $r - 1 \in \mathbb{N}, k > 0$ , then we have*

$$\Gamma_k\left(\frac{k}{r}\right)\Gamma_k\left(\frac{2k}{r}\right)\Gamma_k\left(\frac{3k}{r}\right)\dots\Gamma_k\left(\frac{(r-1)k}{r}\right) = k^{\frac{(1-r)}{2}}r^{-\frac{1}{2}}(2\pi)^{\frac{(r-1)}{2}} \quad (60)$$

*Proof.* Replacing  $x$  by  $\frac{k}{r}, \frac{2k}{r}, \dots, \frac{(r-1)k}{r}$  and  $\frac{rk}{r}$  in the relation (24) respectively, we have

$$\begin{aligned} \Gamma_k\left(\frac{k}{r}\right) &= k^{\frac{1}{r}-1}\Gamma\left(\frac{1}{r}\right) \quad , \quad \Gamma_k\left(\frac{2k}{r}\right) = k^{\frac{2}{r}-1}\Gamma\left(\frac{2}{r}\right) \\ \Gamma_k\left(\frac{(r-1)k}{r}\right) &= k^{\frac{r-1}{r}-1}\Gamma\left(\frac{r-1}{r}\right) \quad \dots \quad \Gamma_k\left(\frac{rk}{r}\right) = k^{1-1}\Gamma(1) = 1. \end{aligned}$$

Multiplying all above equations and applying the Lemma 3.13, we get

$$\begin{aligned} \Gamma_k\left(\frac{k}{r}\right)\Gamma_k\left(\frac{2k}{r}\right)\dots\Gamma_k\left(\frac{(r-1)k}{r}\right) &= k^{\frac{1}{r} + \frac{2}{r} + \dots + \frac{r-1}{r} - (r-1)}\Gamma\left(\frac{1}{r}\right)\Gamma\left(\frac{2}{r}\right)\dots\Gamma\left(\frac{r-1}{r}\right) \\ &= k^{\frac{1+2+3+\dots+(r-1)}{r} - (r-1)}r^{-\frac{1}{2}}(2\pi)^{\frac{(r-1)}{2}} = k^{\frac{(1-r)}{2}}r^{-\frac{1}{2}}(2\pi)^{\frac{(r-1)}{2}}. \end{aligned} \quad \square$$

**Lemma 3.15.** If  $r - 1 \in \mathbb{N}$ ,  $k > 0$ , and  $r \in \mathbb{C} \setminus \{0, -k, -2k, \dots\}$ , then we have

$$\Gamma_k(rx) = r^{\frac{rx}{k} - \frac{1}{2}} k^{\frac{(r-1)}{2}} (2\pi)^{\frac{(1-r)}{2}} \Gamma_k(x) \Gamma_k(x + \frac{k}{r}) \Gamma_k(x + \frac{2k}{r}) \dots \Gamma_k(x + \frac{(r-1)k}{r}).$$

**Remarks.** The above Lemma 3.15 is the  $k$ -analogue of Gauss multiplication theorem. If we use  $r = 2$ , we have  $k$ -analogue of Legendre duplication formula proved in [5]. Also, if  $k = 1$ , we have the classical Gauss multiplication and Legendre duplication theorems [12].

**Theorem 3.16.** If  $z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ , then  $k$ -beta function of  $n$  variables,  $n \geq 2$  satisfies the identity

$$n^{\frac{nz}{k}-1} \beta_k(z, z, \dots, z) = \beta_k\left(z, \frac{k}{n}\right) \beta_k\left(z, \frac{2k}{n}\right) \dots \beta_k\left(z, \frac{(n-1)k}{n}\right). \quad (61)$$

*Proof.* Using the relation (25) on R.H.S of above equation (61), we have

$$\begin{aligned} \beta_k\left(z, \frac{k}{n}\right) \dots \beta_k\left(z, \frac{(n-1)k}{n}\right) &= \frac{\Gamma_k(z) \Gamma_k(\frac{k}{n})}{\Gamma_k(z + \frac{k}{n})} \frac{\Gamma_k(z) \Gamma_k(\frac{2k}{n})}{\Gamma_k(z + \frac{2k}{n})} \dots \frac{\Gamma_k(z) \Gamma_k(\frac{(n-1)k}{n})}{\Gamma_k(z + \frac{(n-1)k}{n})} \\ &= \frac{(\Gamma_k(z))^n \Gamma_k(\frac{k}{n}) \Gamma_k(\frac{2k}{n}) \dots \Gamma_k(\frac{(n-1)k}{n})}{\Gamma_k(z) \Gamma_k(z + \frac{k}{n}) \Gamma_k(z + \frac{2k}{n}) \dots \Gamma_k(z + \frac{(n-1)k}{n})}. \end{aligned}$$

By the Proposition 3.14 and Lemma 3.15, we obtain the required proof.  $\square$

Lately, Kokologianaki [5] worked on properties and inequalities of generalized  $k$ -gamma,  $k$ -beta and zeta functions and proved that the function  $\Gamma_k(x)$  satisfies the identity

$$\Gamma_k(2x) = \sqrt{\frac{k}{\pi}} 2^{\frac{x}{k}} \Gamma_k(x) \Gamma_k(x + \frac{k}{2}),$$

for  $x \in \mathbb{C}$  and  $\operatorname{Re}(x) > 0$ . Here, we extend this result.

**Theorem 3.17.** For  $x \in \mathbb{C}$  and  $\operatorname{Re}(x) > 0$ , and  $n \in \mathbb{N}$ , prove that

$$\Gamma_k(2^n x) = \sqrt{\frac{k}{\pi}} 2^{\frac{2^n x}{k}-1} \Gamma_k(2^{n-1} x) \Gamma_k(2^{n-1} x + \frac{k}{2}). \quad (62)$$

*Proof.* The integral form of  $k$ -beta function in equation (26) is

$$\beta_k(x, y) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1} (1-t)^{\frac{y}{k}-1} dt.$$

Replacing  $x$  and  $y$  by  $2^{n-1}x$ , in the above equation, we have

$$\beta_k(2^{n-1}x, 2^{n-1}x) = \frac{1}{k} \int_0^1 t^{\frac{2^{n-1}x}{k}-1} (1-t)^{\frac{2^{n-1}x}{k}-1} dt, \quad (63)$$

and changing the variable as  $t = \frac{1+s}{2}$  which implies that  $dt = \frac{1}{2}ds$ . Equation (63) gives

$$\beta_k(2^{n-1}x, 2^{n-1}x) = \frac{2}{k} \int_0^1 \left(\frac{1-s^2}{2}\right)^{\frac{2^{n-1}x}{k}-1} \frac{1}{2} ds.$$

By setting  $s^2 = v \Rightarrow 2sds = dv$ , we have

$$\beta_k(2^{n-1}x, 2^{n-1}x) = \frac{1}{k} \frac{1}{2^{\frac{2^n x}{k}-1}} \int_0^1 v^{\frac{1}{2}-1} (1-v)^{\frac{2^{n-1}x}{k}-1} dv = \frac{1}{2^{\frac{2^n x}{k}-1}} \frac{1}{k} \beta\left(\frac{2^{n-1}x}{k}, \frac{1}{2}\right).$$

We obtain the following expression by using equation (34)

$$\beta_k(2^{n-1}x, 2^{n-1}x) = \frac{1}{2^{\frac{2^n x}{k}-1}} \beta_k(2^{n-1}x, \frac{k}{2}). \quad (64)$$

By applying the definition of  $k$ -beta function as given in equation (25) on both sides of equation (64), we have

$$\frac{\Gamma_k(2^{n-1}x) \cdot \Gamma_k(2^{n-1}x)}{\Gamma_k(2^n x)} = \frac{1}{2^{\frac{2^n x}{k}-1}} \frac{\Gamma_k(2^{n-1}x) \cdot \Gamma_k(\frac{k}{2})}{\Gamma_k(2^{n-1}x + \frac{k}{2})},$$

which implies

$$\Gamma_k(2^n x) = \frac{2^{\frac{2^n x}{k}-1} \Gamma_k(2^{n-1}x) \cdot \Gamma_k(2^{n-1}x + \frac{k}{2})}{\Gamma_k(\frac{k}{2})}.$$

By using the Corollary 3.5, we get

$$\Gamma_k(2^n x) = \sqrt{\frac{k}{\pi}} 2^{\frac{2^n x}{k}-1} \Gamma_k(2^{n-1}x) \Gamma_k(2^{n-1}x + \frac{k}{2}). \quad \square$$

**Remarks.** The Theorem 3.17 is the generalization of the Legendre's duplication formula of  $\Gamma(x)$  in [12]. By using  $n = 1$ , we find a formula  $\Gamma_k(2x)$  which is proved in [5]. Further, if we use  $k = 1$  and  $n = 0$  in the result obtained, we get the classical gamma function and the product form of the above Theorem is given by

$$\Gamma_k(2^n x) = \left(\frac{k}{\pi}\right)^{\frac{n}{2}} \prod_{i=1}^n 2^{\frac{2^i x}{k}-1} \Gamma_k(x) \Gamma_k(2^{i-1}x + \frac{k}{2}), n \in \mathbb{N}.$$

**Acknowledgement:** The authors would like to express profound gratitude to referees for deeper review of this paper and the referee's useful suggestions that led to an improved presentation of the paper.

## References

- [1] Anderson G.D., Vamanmurthy M.K., Vuorinen M.K., Conformal Invariants, Inequalities and Quasiconformal Maps, Wiley, New York, 1997
- [2] Andrews G.E., Askey R., Roy R., Special Functions Encyclopedia of Mathematics and its Application 71, Cambridge University Press, 1999
- [3] Carlson B.C., Special Functions of Applied Mathematics, Academic Press, New York, 1977
- [4] Diaz R., Pariguan E., On hypergeometric functions and  $k$ -Pochhammer symbol, Divulgaciones Mathematics, 2007, 15(2), 179-192
- [5] Kokologiannaki C.G., Properties and inequalities of generalized  $k$ -gamma, beta and zeta functions, International Journal of Contemp. Math. Sciences, 2010, 5(14), 653-660
- [6] Kokologiannaki C.G., Krasniqi V., Some properties of  $k$ -gamma function, LE MATHEMATICS, 2013, LXVIII, 13-22
- [7] Krasniqi V., A limit for the  $k$ -gamma and  $k$ -beta function, Int. Math. Forum, 2010, 5(33), 1613-1617
- [8] Mansoor M., Determining the  $k$ -generalized gamma function  $\Gamma_k(x)$ , by functional equations, International Journal Contemp. Math. Sciences, 2009, 4(21), 1037-1042
- [9] Mubeen S., Habibullah G.M., An integral representation of some  $k$ -hypergeometric functions, Int. Math. Forum, 2012, 7(4), 203-207
- [10] Mubeen S., Habibullah G.M.,  $k$ -Fractional integrals and applications, International Journal of Mathematics and Science, 2012, 7(2), 89-94
- [11] Mubeen S., Rehman A., Shaheen F., Properties of  $k$ -gamma,  $k$ -beta and  $k$ -psi functions, Bothalia Journal, 2014, 4, 371-379
- [12] Rainville E.D., Special Functions, The Macmillan Company, New York(USA), 1960
- [13] Rudin W., Real and Complex Analysis, 2nd edition McGraw-Hill, New York, 1974