Boolean Logic

CS 231

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Proposition/Statement

- A proposition is either true or false – but not both
  - "The sky is blue"
  - "Lisa is a Math major"
  - "x == y"
- Not propositions:
  - "Are you Bob?"
  - "x := 7"

Boolean variables

- We use Boolean variables to refer to propositions
  - Usually denoted with lower case letters starting with p (i.e. p, q, r, s, etc.)
  - A Boolean variable can have one of two values true (T) or false (F)
- A proposition can be…
  - A single variable: p
  - A compound statement: $p \wedge (q \vee \sim r)$

Introduction to Logical Operators

- About a dozen logical operators
  - Similar to algebraic operators + * - /
- In the following examples,
  - p = "It is Tuesday"
  - q = "It is 9/3"

Logical operators: Not

- Negation
- Not switches (negates) the truth value
- Symbol: ~ or $\neg$
- In C/C++ and Java,
  the operand is `!`

p	$\sim p$
T	F
F	T

- $\sim p$ = "It is not Tuesday"

Logical operators: And

- Conjunction
- And is true if both operands are true
- Symbol: $\wedge$
- In C/C++ and Java,
  the operand is `&&`
- $p \wedge q$ = "It is Tuesday and it is 9/3"

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Logical operators: Or

- Disjunction
- Or is true if either operands is true
- Symbol: $\vee$
- In C/C++ and Java,
  the operand is ||

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- $p \vee q$ = "It is Tuesday or
  it is 9/3 (or both)"

Logical operators: Exclusive Or

- Exclusive Or is true if one of the operands are true, but false if both are true
- Symbol: $\oplus$
- Often called XOR
- $p \oplus q \equiv (p \vee q) \wedge \sim(p \wedge q)$
- In Java, the operand is ^ (but not in C/C++)

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

- $p \oplus q$ = "It is Tuesday or it is 9/3, but not both"

Inclusive Or versus Exclusive Or

- Do these sentences mean inclusive or exclusive or?
  - Experience with C++ or Java is required
  - Lunch includes soup or salad
  - To enter the country, you need a passport or a driver's license
  - Publish or perish

Logical Equivalence

- Two statements are logically equivalent if and only if they have identical truth values for all possible substitutions of statement variables
  - $p \equiv q$

Conditional

- A conditional means "if p then q" or "p implies q"
- Symbol: $\rightarrow$
- $p \rightarrow q$ = "If it is Tuesday, then it is 9/3"
- $\boxed{p \longrightarrow q \equiv \sim p \vee q}$

the antecedent

the consequent

p	q	$p \rightarrow q$	$\sim p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Conditional 2

- Let p = "I am elected" and q = "I will lower taxes"
- $p \rightarrow q$ = "If I am elected, then I will lower taxes"

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- The statement doesn't say anything about ~p
- If ~p, then the conditional is true regardless of whether q is true or false

Conditional 3

				Conditional	Inverse	Converse	Contra-positive
p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim p \rightarrow \sim q$	$q \rightarrow p$	$\sim q \rightarrow \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

- The conditional and its contra-positive are equivalent
- So are the inverse and converse

Logical operators: Conditional 4

- Alternate ways of stating a conditional:
  - – p implies q
  - – If p, q
  - – p is sufficient for q
  - – q if p
  - – q whenever p

  - – q is necessary for p (if $\sim q$ then $\sim p$)
  - – p only if q

Bi-conditional

- A bi-conditional means "*p* if and only if *q*"
- Symbol: $\leftrightarrow$
- Alternatively, it means "(if *p* then *q*) and (if *q* then *p*)"
- $p\leftrightarrow q \equiv p\rightarrow q \wedge q\rightarrow \mathrm{p}$
- Note that a bi-conditional has the opposite truth values of the exclusive or

p	q	$p\leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Bi-conditional

- Let p = "You get a grade" and q = "You take this class"
- Then $p \leftrightarrow q$ means "You get a grade if and only if you take this class"
- Alternatively, it means "If you get a grade, then you took (take) this class and if you take (took) this class then you get a grade"

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Boolean operators summary

		not	not	and	or	xor	conditional	bi-conditional
p	q	$\sim p$	$\sim q$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	F	T	T	F	T	T
T	F	F	T	F	T	T	F	F
F	T	T	F	F	T	T	T	F
F	F	T	T	F	F	F	T	T

- Learn what they mean, don't just memorize the table!

Precedence of operators

- Precedence order (from highest to lowest):
  $\sim \wedge \vee \rightarrow \leftrightarrow$
  - The first three are the most important
- Not is *always* performed before any other operation

Translating English Sentences

- Problem:
  - p = "It is below freezing"
  - q = "It is snowing"

- It is below freezing and it is snowing $p \wedge q$
- It is below freezing but not snowing $p \wedge \sim q$
- It is not below freezing and it is not snowing $\sim p \wedge \sim q$
- It is snowing or below freezing (or both) $p \vee q$
- If it is below freezing, it is also snowing $p \rightarrow q$
- It is either below freezing or it is snowing, but it is not snowing if it is below freezing $(p \vee q) \wedge (p \rightarrow \sim q)$
- That it is below freezing is necessary and sufficient for it to be snowing $p \leftrightarrow q$

Tautology and Contradiction

- A tautology **t** is a statement that is always true
  - – $p \vee \sim p$ will always be true (Negation Law)
- A contradiction **c** is a statement that is always false
  - – $p \wedge \sim p$ will always be false (Negation Law)

p	$p \vee \sim p$	$p \wedge \sim p$
T	T	F
F	T	F

DeMorgan's Law

- Probably the most important logical equivalence
- To negate $p \wedge q$ (or $p \vee q$), you "flip" the sign, and negate BOTH p and q
  - Thus, $\sim(p \wedge q) \equiv \sim p \vee \sim q$
  - Thus, $\sim(p \vee q) \equiv \sim p \wedge \sim q$

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F	T	F	F
T	F	F	T	F	T	T	T	F	F
F	T	T	F	F	T	T	T	F	F
F	F	T	T	F	T	T	F	T	T

Logical Equivalences

Communicative	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative	$(p\wedge q)\wedge r \equiv p\wedge(q\wedge r)$	$(p\vee q)\vee r \equiv p\vee(q\vee r)$
Distributive	$p\wedge(q\vee r) \equiv (p\wedge q)\vee(p\wedge r)$	$p\vee(q\wedge r) \equiv (p\vee q)\wedge(p\vee r)$
Identity	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
Negation	$p \vee {\sim}p \equiv \mathbf{t}$	$p \wedge {\sim}p \equiv \mathbf{c}$
Double Negative	${\sim}({\sim}p) \equiv p$	
Idempotent	$p \wedge p \equiv p$	$p \vee p \equiv p$
Universal bound	$p \wedge \mathbf{c} \equiv \mathbf{c}$	$p \vee \mathbf{t} \equiv \mathbf{t}$
De Morgan's	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
Absorption	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Negation of **t** and **c**	${\sim}\mathbf{t} \equiv \mathbf{c}$	${\sim}\mathbf{c} \equiv \mathbf{t}$

How to prove two propositions are equivalent?

- Two methods:
  - Using truth tables
    - Not good for long formulae
    - Should not be your first method to prove logical equivalence!
  - Using the logical equivalences and laws
    - The preferred method

- Example: show that:

$$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$$

Using Truth Tables

$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

p	q	r	$p \to r$	$q \to r$	$(p \to r) \vee (q \to r)$	$p \wedge q$	$(p \wedge q) \to r$
T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F
T	F	T	T	T	T	F	T
T	F	F	F	T	T	F	T
F	T	T	T	T	T	F	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

Using Logical Equivalences

$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$ Original statement

$(\sim p \vee r) \vee (\sim q \vee r) \equiv \sim (p \wedge q) \vee r$ Definition of implication

$(\sim p \vee r) \vee (\sim q \vee r) \equiv (\sim p \vee \sim q) \vee r$ DeMorgan's Law

$\sim p \vee r \vee \sim q \vee r \equiv \sim p \vee \sim q \vee r$ Associativity of Or

$\sim p \vee \sim q \vee r \vee r \equiv \sim p \vee \sim q \vee r$ Re-arranging

$\sim p \vee \sim q \vee r \equiv \sim p \vee \sim q \vee r$ Idempotent Law