Review

- $p \rightarrow q \equiv$
- $\sim p \vee q$
- Contrapositive:
- $\sim q \rightarrow \sim p$
- Inverse:
- $\sim p \rightarrow \sim q$
- Converse:
- $q \rightarrow p$
- $p \oplus q \equiv$
- $(p \vee q) \wedge \sim(p \wedge q)$
- $\sim(p \wedge q) \equiv$
- $\sim p \vee \sim q$
- $\sim(p \vee q) \equiv$
- $\sim p \wedge \sim q$
- p is sufficient for q
- $p \rightarrow q$
- p is necessary for q
- $\sim p \rightarrow \sim q$

Valid and Invalid Arguments

CS 231

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Review

- Associative Law:
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- Distributive Law:
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- Absorption Law:
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$

Definitions

- An argument is a sequence of statements (statement forms).
- All statements in an argument except for the last one, are called premises. (assumptions, hypotheses)
- The final statement is the conclusion.
- A valid argument means the conclusion is true if the premises are all true, with all combinations of variable truth values.

Examples

- All Greeks are human and all humans are mortal; therefore, all Greeks are mortal.

- Some men are athletes and some athletes are rich; therefore, some men are rich.

- Some men are swimmers and some swimmers are fish; therefore, some men are fish.

Modus Ponens

$$\begin{array}{l} p \\ \underline{p \to q} \\ \therefore q \end{array}$$

Modus Ponens example

- Assume you are given the following two statements:
  - "you are in this class"
  - "If you are in this class, you are a student"

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$$

- Let *p* = "you are in this class"
- Let *q* = "you are a student"

- By Modus Ponens, you can conclude that you are a student.

Modus Ponens

- Consider $(p \wedge (p \rightarrow q)) \rightarrow q$

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$(p \wedge (p \rightarrow q)) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$$

Modus Tollens

- Assume that we know: $\sim q$ and $p \rightarrow q$
  - Recall that $p \rightarrow q \equiv \sim q \rightarrow \sim p$
- Thus, we know $\sim q$ and $\sim q \rightarrow \sim p$
- We can conclude $\sim p$

$$\begin{array}{l} \sim q \\ \underline{p \rightarrow q} \\ \therefore \sim p \end{array}$$

Modus Tollens example

- Assume you are given the following two statements:
  - "you are not a student"
  - "if you are in this class, you are a student"

$$\begin{array}{r} \sim q \\ \underline{p \rightarrow q} \\ \therefore \sim p \end{array}$$

- Let *p* = "you are in this class"
- Let *q* = "you are a student"

- By Modus Tollens, you can conclude that you are not in this class

Generalization & Specialization

- Generalization: If you know that *p* is true, then $p \vee q$ will ALWAYS be true

$$\frac{p}{\therefore p \vee q} \qquad \frac{q}{\therefore p \vee q}$$

- Specialization: If $p \wedge q$ is true, then *p* will ALWAYS be true

$$\frac{p \wedge q}{\therefore p} \qquad \frac{p \wedge q}{\therefore q}$$

Example of proof

- We have the hypotheses:
  p – "It is not sunny this afternoon and it is colder than yesterday"
  q
  - "We will go swimming only if it is sunny"
  r
  - "If we do not go swimming, then we will take a canoe trip"
  s
  - "If we take a canoe trip, then we will be home by sunset"
  t
- Does this imply that "we will be home by sunset"?

$$\sim p \wedge q$$

$$r \rightarrow p$$

$$\sim r \rightarrow s$$

$$\underline{s \rightarrow t}$$

$$t$$

Example of proof

1.	$\sim p \wedge q$	1st hypothesis
2.	$\sim p$	Specialization using step 1
3.	$r \rightarrow p$	2nd hypothesis
4.	$\sim r$	Modus tollens using steps 2 & 3
5.	$\sim r \rightarrow s$	3rd hypothesis
6.	s	Modus ponens using steps 4 & 5
7.	$s \rightarrow t$	4th hypothesis
8.	t	Modus ponens using steps 6 & 7

$$\frac{p \wedge q}{\therefore p}$$

$$\frac{\begin{array}{c} p \\ p \rightarrow q \end{array}}{\therefore q}$$

$$\frac{\begin{array}{c} \sim q \\ p \rightarrow q \end{array}}{\therefore \sim p}$$

More rules of inference

- Conjunction: if *p* and *q* are true separately, then $p \wedge q$ is true

$$\begin{array}{l} p \\ \underline{q} \\ \therefore p \wedge q \end{array}$$

- Elimination: If $p \vee q$ is true, and *p* is false, then *q* must be true

$$\begin{array}{l} p \vee q \\ \underline{\sim p} \\ \therefore q \end{array} \qquad \begin{array}{l} p \vee q \\ \underline{\sim q} \\ \therefore p \end{array}$$

- Transitivity: If $p \rightarrow q$ is true, and $q \rightarrow r$ is true, then $p \rightarrow r$ must be true

$$\begin{array}{l} p \rightarrow q \\ \underline{q \rightarrow r} \\ \therefore p \rightarrow r \end{array}$$

Even more rules of inference

- Proof by division into cases: if at least one of *p* or *q* is true, then *r* must be true

$$\begin{array}{l} p \lor q \\ p \to r \\ \underline{q \to r} \\ \therefore r \end{array}$$

- Contradiction rule: If $\sim p \to \mathbf{c}$ is true, we can conclude *p* (via the contra-positive)

$$\begin{array}{l} \underline{\sim p \to c} \\ \therefore p \end{array}$$

- Resolution: If $p \lor q$ is true, and $\sim p \lor r$ is true, then $q \lor r$ must be true
  - Not in the textbook

$$\begin{array}{l} p \lor q \\ \underline{\sim p \lor r} \\ \therefore q \lor r \end{array}$$

Example of proof

- Given the hypotheses:
  - "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on"
  - "If the sailing race is held, then the trophy will be awarded"
  - "The trophy was not awarded"
- Can you conclude: "It rained"?

$$(\sim r \lor \sim f) \rightarrow (s \land l)$$

$$s \rightarrow t$$

$$\sim t$$

$$\overline{}$$

$$r$$

Example of proof

1.	$\sim t$	3rd hypothesis
2.	$s \rightarrow t$	2nd hypothesis
3.	$\sim s$	Modus tollens using steps 1 & 2
4.	$(\sim r \vee \sim f) \rightarrow (s \wedge l)$	1st hypothesis
5.	$\sim(s \wedge l) \rightarrow \sim(\sim r \vee \sim f)$	Contrapositive of step 4
6.	$(\sim s \vee \sim l) \rightarrow (r \wedge f)$	DeMorgan's law and double negation law
7.	$\sim s \vee \sim l$	Generalization using step 3
8.	$r \wedge f$	Modus ponens using steps 6 & 7
9.	r	Specialization using step 8

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array} \qquad \begin{array}{l} \underline{p \qquad} \\ \therefore p \vee q \end{array} \qquad \begin{array}{l} \underline{p \wedge q \qquad} \\ \therefore p \end{array} \qquad \begin{array}{l} \sim q \\ \underline{p \rightarrow q} \\ \therefore \sim p \end{array}$$

Fallacy of the converse

~~Modus Badus~~

Fallacy of affirming the conclusion

- Consider the following:

$$\begin{array}{l} q \\ \underline{p \rightarrow q} \\ \therefore p \end{array} \qquad \begin{array}{l} q \\ \underline{\sim q \rightarrow \sim p} \\ \therefore p \end{array}$$

- Is this true?

p	q	$p \rightarrow q$	$q \wedge (p \rightarrow q)$	$(q \wedge (p \rightarrow q)) \rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	$\boxed{\text{F}}$
F	F	T	F	T

Not a valid rule!

Modus Badus example

- Assume you are given the following two statements:
  - "you are a student"
  - "If you are in this class, you are a student"

$$\begin{array}{r} q \\ \underline{p \to q} \\ \therefore p \end{array}$$

- Let *p* = "you are in this class"
- Let *q* = "you are a student"

- It is clearly wrong to conclude that if you are a student, you must be in this class

Fallacy of the inverse

~~Modus Badus~~

Fallacy of denying the hypothesis

- Consider the following:

$$\begin{array}{l} \sim p \\ \underline{p \rightarrow q} \\ \therefore \sim q \end{array}$$

- Is this true?

p	q	$p \rightarrow q$	$\sim p \wedge (p \rightarrow q))$	$(\sim p \wedge (p \rightarrow q)) \rightarrow \sim q$
T	T	T	F	T
T	F	F	F	T
F	T	T	T	$\boxed{\text{F}}$
F	F	T	T	T

Not a valid rule!

Modus Badus example

- Assume you are given the following two statements:
  - "you are not in this class" $\sim p$
  - "if you are in this class, you are a student" $p \rightarrow q$

$$\frac{\begin{array}{r}\sim p \\ p \rightarrow q\end{array}}{\therefore \sim q}$$

- Let p = "you are in this class"
- Let q = "you are a student"

- You CANNOT conclude that you are not a student just because you are not taking Discrete Math