CMSC 231: Discrete Mathematics
Practice Exam #1
Fall 2017

- This review exam has *not* been tested for length. The real exam might be shorter or longer.
- You will be able to bring one standard sheet of US-letter-sized paper with notes. The notes may appear on both sides of the sheet.
- No calculators or other electronic devices will be allowed on the exam.
- Several questions ask you to write negations. Simply writing a "not" on the outside of a statement does not qualify as a correct answer; you must use De Morgan's laws to propagate the negation.

Definitions:

- $\mathbb{Z}$ is the set of all integers
- If S is a set, then S^+ is the members of S that are greater than 0.
- A *function* F from a set A to a set B is a relation with domain A and co-domain B that satisfies the following two properties:
    1. For every element x in A, there is an element y in B such that $(x, y) \in F$.
    2. For all elements x in A and y and z in B, if $(x, y) \in F$ and $(x, z) \in F$, then $y = z$.

1. For each pair of sets below, state whether they are equal or not. If you wish, you may include a short explanation of why you think your answer is correct, but a correct answer with no explanation still receives full credit.

   (a) $\{x \in \mathbb{Z} \mid (1 < x < 10) \wedge (x \text{ is prime})\}$
   $\{2, 3, 5, 7\}$
   equal

   (b) $\{1, 2, 3\}$
   $\{x \in \mathbb{Z} \mid 0 < x^2 < 10\}$
   not equal

   (c) $\{\{1\}, \{2\}\}$
   $\{2, 1\}$
   not equal

   (d) Assume $B = \{1, 2, 3\}$ and $C =$ the set of all possible sets of integers.
   $\{x \in C \mid x \subseteq B\}$
   $\{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$
   equal

   (e) Assume $M = \{1, 2, 3\}$ and $N = \{4, 5\}$.
   $\{(x, y) \in (M \times N) \mid (2 < x) \wedge (2 < y)\}$
   $\{(4, 3), (5, 3)\}$
   not equal

2. Each problem below ends with a boxed statement. Indicate whether the boxed statement is true or false. If you wish, you may include a short explanation of why you think your answer is correct, but a correct answer with no explanation still receives full credit.

   You may assume the following definitions:

   $$R = \{1, 2, 3, 4\} \qquad S = \{-2, -1, 0, 1, 2\} \qquad T = \{1, 3, 5, 7, 9\}$$

   (a) Assume $U = \{(x, y) \in (R \times S) \mid x + y = 0\}$. $\boxed{2\ U\ {-2}}$ **true**

   (b) Assume U as above. $\boxed{3\ U\ {-3}}$ **false**

   (c) Assume U as above. $\boxed{2\ U\ 2}$ **false**

   (d) Assume U as above. $\boxed{U \text{ is a function from } R \text{ to } S}$ **false**

   (e) Assume $V = \{(x, y) \in (S \times T) \mid 2x = y\}$. $\boxed{0\ V\ 1}$ **false**

   (f) Assume V as above. $\boxed{1\ V\ 1}$ **false**

   (g) Assume V as above. $\boxed{V \text{ is a function from } S \text{ to } T}$ **false**

   (h) Assume $W = \{(x, y) \in (R \times T) \mid 2x - 1 = y\}$. $\boxed{2\ W\ 3}$ **true**

   (i) Assume W as above. $\boxed{4\ W\ 9}$ **false**

   (j) Assume W as above. $\boxed{W \text{ is a function from } R \text{ to } T}$ **true**

3. Assuming the definitions below, write out each sentence using symbolic notation, with no words.

$$p = \text{“It's raining”} \qquad q = \text{“It's Wednesday”} \qquad r = \text{“It's October”}$$

   (a) It's a rainy Wednesday.
   $\boxed{p \wedge q}$

   (b) It's October, but it's neither Wednesday nor raining out.
   $\boxed{r \wedge \sim q \wedge \sim p}$

   (c) It's raining but not October.
   $\boxed{p \wedge \sim r}$

4. Use De Morgan's laws to write negations for the following statements. Your answers should be complete English sentences.

   (a) I have math class and computer science class today.
   I do not have math class or I do not have computer science class today.

   (b) I love to eat cookies but not peanut butter.
   I do not love to eat cookies or I love to eat peanut butter.

(c) I am wearing red or green socks today.
I am not wearing red socks and I am not wearing green socks today.

5. Give the *converse*, *inverse*, *contrapositive* and *negation* (using De Morgan's laws) of the following:

$$(p \wedge \sim q) \rightarrow r$$

(a) Converse: $\boxed{r \rightarrow (p \wedge \sim q)}$

(b) Inverse: $\boxed{(\sim p \vee q) \rightarrow \sim r}$

(c) Contrapositive: $\boxed{\sim r \rightarrow (\sim p \vee q)}$

(d) Negation: $\boxed{(p \wedge \sim q) \wedge \sim r}$

6. Prove the following logical equivalence, using the rules of Boolean algebra (a truth table is not an acceptable answer):

$$(p \vee q) \wedge \sim p \equiv (q \wedge \sim p)$$

$$\boxed{\begin{aligned} (p \vee q) \wedge \sim p &\equiv (p \wedge \sim p) \vee (q \wedge \sim p) && \textbf{distributive law} \\ &\equiv \mathbf{c} \vee (q \wedge \sim p) && \textbf{negation law} \\ &\equiv (q \wedge \sim p) && \textbf{identity law} \end{aligned}}$$

7. Draw a digital circuit corresponding to the Boolean formula below:

$$(\sim p \wedge q) \vee (p \wedge (q \vee r))$$

8. (a) Add the following binary numbers. Write the result as a 4-bit number, dropping any bits that overflow.

$$\begin{array}{r} 1\,0\,1\,1 \\ +\,0\,1\,1\,0 \\ \hline \end{array}$$

$\boxed{\mathbf{0001}}$

(b) Rewrite the addition problem above in decimal notation, interpreting any number greater than 7_{10} as a two's-complement negative number. (This is 4-bit two's-complement.) If the sum in decimal does not match the sum in binary, explain why.

$\boxed{(-5) + 6 = 1}$

9. For each of the following statements, rewrite the statement formally (using $\forall$ and $\exists$), and write the negation of the statement both in English and in formal mathematics. You may assume these definitions:

$$\begin{aligned} C &= \text{the set of all Bryn Mawr classes} \\ \text{len}(x) &= \text{the meeting length of a class } x\text{, in hours} \\ \text{cs}(x) &= \text{class } x \text{ is a computer science class} \\ \text{math}(x) &= \text{class } x \text{ is a math class} \\ \text{prereq}(x, y) &= \text{class } x \text{ is a prerequisite for class } y \end{aligned}$$

(a) All classes at Bryn Mawr meet for at least 2 hours.

Written formally: $\boxed{\forall (x \in C), \text{len}(x) \geq 2}$

Negated formally: $\boxed{\exists (x \in C), \text{len}(x) < 2}$

Negated informally: **Some class at Bryn Mawr meets for less than 2 hours.**

(b) Every class at Bryn Mawr has at least one prerequisite class at Bryn Mawr.

Written formally: $\boxed{\forall (x \in C), \exists (y \in C), prereq(y, x)}$

Negated formally: $\boxed{\exists (x \in C), \forall (y \in C), {\sim} prereq(y, x)}$

Negated informally: **Some class at Bryn Mawr has no prerequisite class at Bryn Mawr.**

(c) One computer science class at Bryn Mawr has every math class as prerequisites.

Written formally: $\boxed{\exists (x \in C), \text{cs}(x) \wedge \forall (y \in C), \text{math}(y) \rightarrow \text{prereq}(y, x)}$

Negated formally: $\boxed{\forall(x \in C), \text{cs}(x) \rightarrow \exists(y \in C), \text{math}(y) \wedge \sim\text{prereq}(y, x)}$

Negated informally: **Every computer science class at Bryn Mawr has at least one math class that is not its prerequisite.**

(d) There is a math class at Bryn Mawr that is the prerequisite for all computer science courses.

Written formally: $\boxed{\exists(x \in C), \text{math}(c) \wedge \forall(y \in C), \text{cs}(y) \rightarrow \text{prereq}(x, y)}$

Negated formally: $\boxed{\forall(x \in C), \text{math}(c) \rightarrow \exists(y \in C), \text{cs}(y) \wedge \sim\text{prereq}(x, y)}$

Negated informally: **Every math class at Bryn Mawr has at least one computer science class that it is not a prerequisite of.**

Definition: If r is a real number, then r is *rational* iff:

$$\exists(a \in \mathbb{Z}), \exists(b \in \mathbb{Z}), (r = \frac{a}{b}) \land (b \neq 0)$$

10. Prove: The product of any two rational numbers is a rational number.
    See book solution 4.2.15.

Definition: If n is an integer, then n is *odd* iff:

$$\exists(k \in \mathbb{Z}), n = 2k + 1$$

Definition: If x is a real number, then the *ceiling* of x, written $\lceil x \rceil$, is defined as follows:

$$\lceil x \rceil = n \qquad \text{means} \qquad (n - 1 < x \leq n) \wedge (n \in \mathbb{Z})$$

11. Prove: For all odd integers n, $\lceil n/2 \rceil = (n + 1)/2$.

Proof: Let n be any odd integer. *[We must show that* $\lceil \frac{n}{2} \rceil = \frac{n+1}{2}$.*]* By definition of odd, $n = 2k + 1$ for some integer k. The left-hand side of the equation to be proved is

$$\begin{aligned} \left\lceil \frac{n}{2} \right\rceil &= \left\lceil \frac{2k+1}{2} \right\rceil && \text{by substitution} \\ &= \left\lceil k + \frac{1}{2} \right\rceil && \text{by algebra} \\ &= k + 1 && \text{by definition of ceiling because } k \text{ is an integer and } k < k + 1/2 \leq k + 1 \end{aligned}$$

On the other hand, the right-hand side of the equation to be proved is

$$\begin{aligned} \frac{n+1}{2} &= \frac{(2k+1)+1}{2} && \text{by substitution} \\ &= \frac{2k+2}{2} \\ &= \frac{2(k+1)}{2} \\ &= k + 1 && \text{by algebra.} \end{aligned}$$

Thus both the left- and right-hand sides of the equation to be proved equal $k + 1$, and so both are equal to each other. In other words, $\lceil n/2 \rceil = (n + 1)/2$ *[as was to be shown]*.