Predicates and Quantifiers

CS 231

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Predicates

- Consider $P(x) = x < 5$
  - $P(x)$ has no truth values (x is not given a value)
  - $P(1)$ is true – $1 < 5$ is true
  - $P(10)$ is false – $10 < 5$ is false
- Thus, $P(x)$ will become a statement/proposition when x is given a value

Truth set of predicates

- Let $P(x)$ = "x is a multiple of 5"
  – For what values of x is $P(x)$ true?
- Let $P(x) = x+1 > x$
  – For what values of x is $P(x)$ true?
- Let $P(x) = x + 3$
  – For what values of x is $P(x)$ true?

Multiple variables

- Functions/predicates with multiple variables:
  - $P(x,y) = x + y == 0$
    - $P(1,2)$ is false, $P(1,-1)$ is true
  - $P(x,y,z) = x + y == z$
    - $P(3,4,5)$ is false, $P(1,2,3)$ is true
  - $P(x_1,x_2,x_3 \ldots x_n) = \ldots$

Quantifiers

- A quantifier is "an operator that limits the variables of a predicate"

- Two types:
  - Universal
  - Existential

Universal quantifiers 1

- Represented by an upside-down A: $\forall$
  - It means "for all"
  - Let $P(x) = x+1 > x$

- We can state the following:
  - $\forall x, P(x)$
  - English translation: "for all values of x, P(x) is true"
  - English translation: "for all values of x, x+1>x is true"

Universal quantifiers 2

- But is that always true?
  - $\forall x\ P(x)$
- Let x = the character 'a'
  - Is 'a' +1 > 'a' ?
- Let x = the state of Pennsylvania
  - Is Pennsylvania+1 > Pennsylvania?
- You need to specify your universe!
  - What values x can represent
  - Known as the "domain" of x

Universal quantifiers 3

- $\forall x \in \mathcal{R}$, P(x)
- Let P(x) = x/2 < x
  - $\forall x \in \mathcal{R}$, P(x)?

- To prove that a universal quantification is true, it must be shown for ALL cases – exhaustion
- To prove that a universal quantification is false, it must be shown to be false for only ONE case – counter example

Existential quantification 1

- Represented by an backwards E: $\exists$
  - It means "there exists"
  - Let $P(x) = x+1 > x$

- We can state the following:
  - $\exists x, P(x)$
  - English translation: "there exists (a value of) x such that P(x) is true"
  - English translation: "for at least one value of x, x+1>x is true"

Existential quantification 2

- Note that you still have to specify your domain
  - If the domain we are talking about is all the states in the US, then $\exists x\, P(x)$ is not true

Existential quantification 3

- In order to show an existential quantification is true, you only have to find ONE value

- In order to show an existential quantification is false, you have to show it’s false for ALL values

A note on quantifiers

- $P(x) = x < 1$

- There are two ways to make a predicate (propositional function) into a statement (proposition):
  - Supply it with a value
    - For example, $P(5)$ is false, $P(0)$ is true
  - Provide a quantification
    - $\forall x \in \mathcal{Z}$, $P(x)$ is false and $\exists x \in \mathcal{Z}$, $P(x)$ is true

Universal Conditional Statements

- $\forall x$, if $P(x)$ then $Q(x)$
- $\forall x, P(x) \rightarrow Q(x)$
- $\forall x \in \mathcal{R}$, if $x > 0$ then $x+2 > 0$
- Consider "If a number is positive, then it is not zero"
- Implicit quantification
  - $P(x)$ implies $Q(x) \equiv \forall x, P(x) \rightarrow Q(x)$
  - $P(x)$ iff $Q(x) \equiv \forall x, P(x) \leftrightarrow Q(x)$

Translation from English

- Consider "All cats are black"
  - Let C be the set of all cats
  - $\forall x \in C$, x is black
  - Let B(x) be "x is black": $\forall x \in C, B(x)$
- "Some people are crazy"
  - Let P be the set of all people
  - $\exists x \in P$, x is crazy
  - Let Crazy(x) be "x is crazy": $\exists x \in P, Crazy(x)$

Translation from English

- Consider “Every student in this class has studied compound statements”
- Rephrased: “For every student x in this class, x has studied compound statements”
  - Let C(x) be “ x has studied compound statements”
  - $\forall x, C(x)$, where the domain is “all students in this class”

Equivalent forms

- What if the domain is all people?
  - Let S(x) be "x is a student in this class"
  - Consider: $\forall x\ (S(x) \wedge C(x))$
- $\forall x,\ S(x) \rightarrow C(x)$
- $\forall x,\ C(x)$, where the domain is " all students in this class"
- $\forall x \in U,\ S(x) \rightarrow C(x) \equiv \forall x \in D,\ C(x)$, where D is the domain consisting of the truth set of S(x)

Translating from English

- Consider:
  - “Some students have visited Mexico”
  - “ Every student has visited Canada or Mexico”
- Let:
  - S(x) be “x is a student”
  - M(x) be “x has visited Mexico”
  - C(x) be “x has visited Canada”

Translating from English

- Consider: “Some students have visited Mexico”
- $\exists x \in D, M(x)$ where D = all students
- What if the domain is all people?
  - Consider: $\exists x, (S(x) \rightarrow M(x))$
  - $\exists x, (S(x) \wedge M(x))$

Translating from English

- Consider: “ Every student has visited Canada or Mexico”
- $\forall x \in D, M(x) \vee C(x)$
  - D = all students
- $\forall x, (S(x) \rightarrow (M(x) \vee C(x)))$

Negating quantifications

- Consider the statement:
  - All students in this class have red hair
- What is required to show the statement is false?
  - There exists a student in this class that does NOT have red hair
- To negate a universal quantification:
  - Negate the predicate
  - AND change to an existential quantification
  - Change “All are” to “Some are not”
  - $\sim(\forall x \in D, P(x)) \equiv \exists x \in D, \sim P(x)$

Negating quantifications 2

- Consider the statement:
  - There is a student in this class with red hair
- What is required to show the statement is false?
  - All students in this class do not have red hair
- Thus, to negate an existential quantification:
  - Negate the predicate
  - AND change to a universal quantification
  - $\sim(\exists x \in D, P(x)) \equiv \forall x \in D, \sim P(x)$

More negations

- "No man is an island" – John Donne
  - ∀x, x is not an island
  - ∃x, x is an island
  - Some men are islands
- "All that glitters is not gold" – Chaucer and Shakespeare

Negation of universal conditionals

- $\sim(\forall x \in D, P(x) \rightarrow Q(x)) \equiv \exists x \in D, \sim(P(x) \rightarrow Q(x))$
- $\sim(\forall x \in D, P(x) \rightarrow Q(x)) \equiv \exists x \in D, P(x) \wedge \sim Q(x)$
- For all people x, if x is rich then x is happy
  - There is one person who is rich and is not happy
  - There is one person who is rich but not happy

$\forall$ and $\wedge$

- Given a predicate $P(x)$ and values in the domain $\{x_1, \ldots, x_n\}$

- The universal quantification $\forall x\ P(x)$ implies:

$$P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$$

$\exists$ and $\vee$

- Given a predicate $P(x)$ and values in the domain $\{x_1, \ldots, x_n\}$

- The existential quantification $\exists x\ P(x)$ implies:

$$P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$$

Translating from English

- Translate the statements:
  - “All hummingbirds are richly colored”
  - “No large birds live on honey”
  - “Birds that do not live on honey are dull in color”
  - “Hummingbirds are small”
- Assign our predicates
  - Let $P(x)$ be “x is a hummingbird”
  - Let $Q(x)$ be “x is large”
  - Let $R(x)$ be “x lives on honey”
  - Let $S(x)$ be “x is richly colored”
- Let our domain be all birds

Translating from English

- "All hummingbirds are richly colored"
  - $\forall x, (P(x) \rightarrow S(x))$
- "No large birds live on honey"
  - $\sim\exists x, (Q(x) \wedge R(x))$
  - Alternatively: $\forall x, (\sim Q(x) \vee \sim R(x))$
- "Birds that do not live on honey are dull in color"
  - $\forall x, (\sim R(x) \rightarrow \sim S(x))$
- "Hummingbirds are small"
  - $\forall x, (P(x) \rightarrow \sim Q(x))$