Multiple Quantifiers

CS 231

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Multiple quantifiers

- Let our domain be $\mathcal{R}$
- $\forall x \exists y$ P(x, y)
  - "For all x, there exists a y such that P(x,y)"
  - Example: $\forall x \exists y$ x+y == 0

- $\exists x \forall y$ P(x,y)
  - There exists an x such that for all y P(x,y) is true"
  - Example: $\exists x \forall y$ x*y == 0

Order of quantifiers

- $\exists x \forall y$ and $\forall x \exists y$ are not equivalent!

- $P(x,y) = (x+y == 0)$
  - $\exists x \forall y\ P(x,y)$ is false
  - $\forall x \exists y\ P(x,y)$ is true

Binding variables

- Let P(x,y) be x > y
- Consider: $\forall$x P(x,y)
  - This is not a proposition!
  - What is y?
    - If it's 5, then $\forall$x P(x,y) is false
    - If it's x-1, then $\forall$x P(x,y) is true
- y is a free variable - not "bound" by a quantifier

Binding variables 2

- $(\exists x\ P(x)) \vee Q(x)$
  - The x in Q(x) is not bound; thus not a proposition
- $(\exists x\ P(x)) \vee (\forall x\ Q(x))$
  - Both x values are bound; thus it is a proposition
- $(\exists x\ P(x) \wedge Q(x)) \vee (\forall y\ R(y))$
  - All variables are bound; thus it is a proposition
- $(\exists x\ P(x) \wedge Q(y)) \vee (\forall y\ R(y))$
  - The y in Q(y) is not bound; this not a proposition

Translating between English and quantifiers

- The product of two negative integers is positive
  - $\forall x \forall y\ (x<0) \land (y<0) \rightarrow (xy > 0)$
- The average of two positive integers is positive
  - $\forall x \forall y\ (x>0) \land (y>0) \rightarrow ((x+y)/2 > 0)$

- The difference of two negative integers is not necessarily negative
  - $\exists x \exists y\ (x<0) \land (y<0) \land (x-y \geq 0)$
- The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers
  - $\forall x \forall y\ |x+y| \leq |x| + |y|$

Translating between English and quantifiers

- $\exists x \forall y\ x+y = y$
  - There exists an additive identity for all real numbers
- $\forall x \forall y\ ((x \geq 0) \wedge (y<0)) \rightarrow (x-y > 0)$
  - A non-negative number minus a negative number is greater than zero
- $\exists x \exists y\ ((x \leq 0) \wedge (y \leq 0)) \wedge (x-y > 0)$
  - The difference between two non-positive numbers is not necessarily non-positive (i.e. can be positive)
- $\forall x \forall y\ ((x \neq 0) \wedge (y \neq 0)) \leftrightarrow (xy \neq 0)$
  - The product of two numbers is non-zero if and only if both factors are non-zero

Negating multiple quantifiers

- Recall negation rules for single quantifiers:
  - $\sim(\forall x\ P(x)) \equiv \exists x\ \sim P(x)$
  - $\sim(\exists x\ P(x)) \equiv \forall x\ \sim P(x)$
  - Essentially, you change the quantifiers, and negate what it's quantifying

- Examples:
  - $\sim(\forall x \exists y\ P(x,y))$
    - $\equiv \exists x\ \sim(\exists y\ P(x,y))$
    - $\equiv \exists x \forall y\ \sim P(x,y)$
  - $\sim(\forall x \exists y \forall z\ P(x,y,z))$

    $\equiv \exists x\ \sim(\exists y \forall z\ P(x,y,z))$

    $\equiv \exists x \forall y\ \sim(\forall z\ P(x,y,z))$

    $\equiv \exists x \forall y \exists z\ \sim P(x,y,z)$

Negating multiple quantifiers 2

- Consider $\sim(\forall x \exists y\ P(x,y)) \equiv \exists x \forall y\ {\sim}P(x,y)$
  - The left side is saying "for all x, there exists a y such that P is true"
  - To disprove it (negate it), you need to show that "there exists an x such that for all y, P is false"

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Negation examples

- Rewrite these statements so that the negations only appear within the predicates

a) $\sim(\exists y \exists x\ P(x,y))$
   1. $\forall y \sim(\exists x\ P(x,y))$
   2. $\forall y \forall x \sim P(x,y)$

b) $\sim(\forall x \exists y\ P(x,y))$
   1. $\exists x \sim(\exists y\ P(x,y))$
   2. $\exists x \forall y \sim P(x,y)$

c) $\sim(\exists y\ Q(y) \wedge \forall x \sim R(x,y))$
   1. $\forall y \sim(Q(y) \wedge \forall x \sim R(x,y))$
   2. $\forall y \sim Q(y) \vee \sim(\forall x \sim R(x,y))$
   3. $\forall y \sim Q(y) \vee \exists x\ R(x,y)$

Negation examples

- Negate the following:

a) $\forall x \exists y \forall z\ T(x,y,z)$

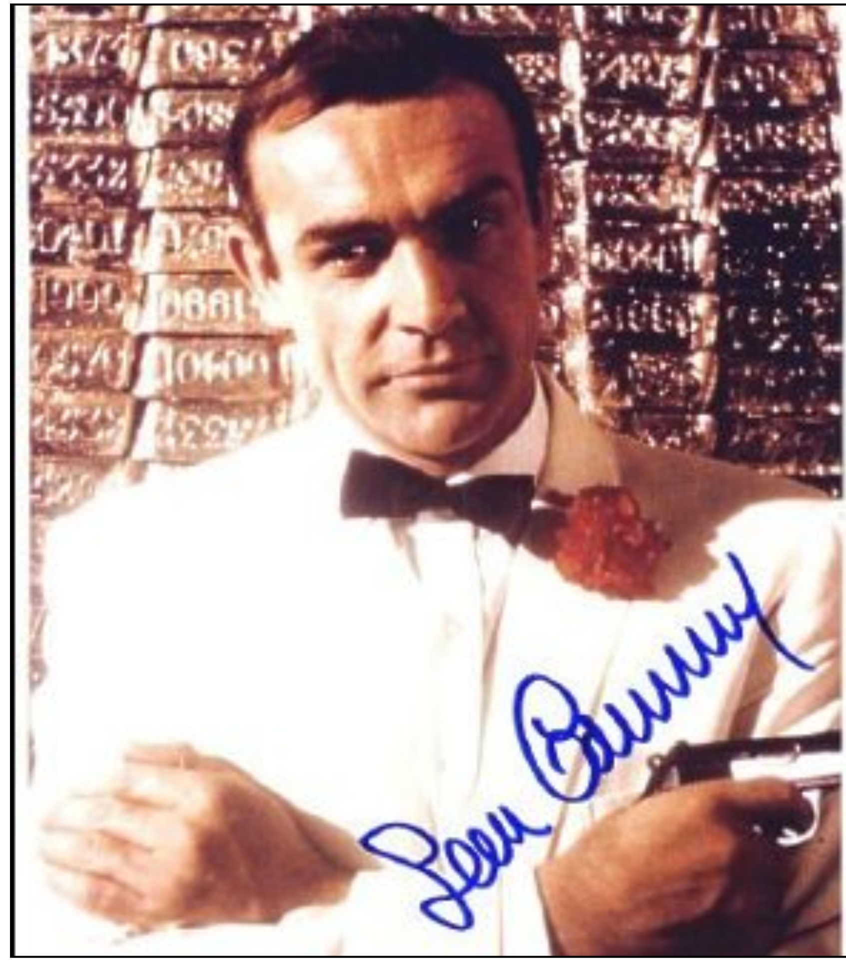
1. $\sim(\forall x \exists y \forall z\ T(x,y,z))$
2. $\exists x \sim(\exists y \forall z\ T(x,y,z))$
3. $\exists x \forall y \sim(\forall z\ T(x,y,z))$
4. $\exists x \forall y \exists z \sim T(x,y,z)$

b) $\forall x \exists y\ P(x,y) \vee \forall x \exists y\ Q(x,y)$

1. $\sim(\forall x \exists y\ P(x,y) \vee \forall x \exists y\ Q(x,y))$
2. $\sim(\forall x \exists y\ P(x,y)) \wedge \sim(\forall x \exists y\ Q(x,y))$
3. $\exists x \sim(\exists y\ P(x,y)) \wedge \exists x \sim(\exists y\ Q(x,y))$
4. $\exists x \forall y \sim P(x,y) \wedge \exists x \forall y \sim Q(x,y)$

Negation

- There is a secret agent who appeals to all women
- Negation?
- For every secret agent there is a woman he doesn't appeal to.
- Common mistake: There is a secret agent who doesn't appeal to all women



Prolog

- A programming language using logic!
- Entering facts (propositions):

```
instructor(xu, cs231).
enrolled(alice, cs231).
enrolled(bob, cs231).
enrolled(claire, cs231).
```

- Extracting data

```
?- enrolled (alice, cs231).
```

Result:

```
yes
```

Prolog 2

- Extracting data

```
?- enrolled(X, cs231).
```

Result:

```
alice
bob
Claire
```

- Entering predicates:

```
teaches(P,S) :- instructor(P,C), enrolled(S,C).
```

- Extracting data

```
?- teaches(X, alice).
```

Result:

```
Xu
```

Arguments with Quantified Statements

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Vacuous Truth

- Presently, all men on the moon are happy.
- $\forall x\ OnTheMoon(x) \rightarrow Happy(x)$
- There is no man on the moon presently.
- $\forall x\ OnTheMoonPresently(x) \rightarrow Happy(x)$
- The statement is vacuously true.
- Presently, all men on the moon are dinosaurs.

Universal Instantiation

- $\forall x \in D, P(x)$
- $x_0 \in D$
- $P(x_0)$
- Example:
- All men are mortal.
- Socrates is a man.
- Socrates is mortal.

Existential Generalization

- $P(x_0)$
- $x_0 \in D$
- $\exists x \in D, P(x)$

Universal Modus Ponens

$$\begin{array}{l} p \\ \underline{p \to q} \\ \therefore q \end{array} \qquad \begin{array}{l} P(a) \\ \underline{\forall x, P(x) \to Q(x)} \\ \therefore Q(a) \end{array}$$

Universal Modus Tollens

$$\begin{array}{ll} \sim q & \sim Q(a) \\ \underline{p \rightarrow q} & \underline{\forall x, P(x) \rightarrow Q(x)} \\ \therefore \sim p & \therefore \sim P(a) \end{array}$$

Universal Transitivity

$$\begin{array}{l} \forall x, P(x) \to Q(x) \\ \underline{\forall x, Q(x) \to R(x)} \\ \therefore \forall x, P(x) \to R(x) \end{array}$$

Example of proof

- Given the hypotheses:
  - "Linda, a student in this class, owns a red convertible."
  - " Everybody who owns a red convertible has gotten at least one speeding ticket"
- Can you conclude: "Somebody in this class has gotten a speeding ticket"?

$C(\text{Linda})$
$R(\text{Linda})$

$\forall x\ (R(x) \rightarrow T(x))$

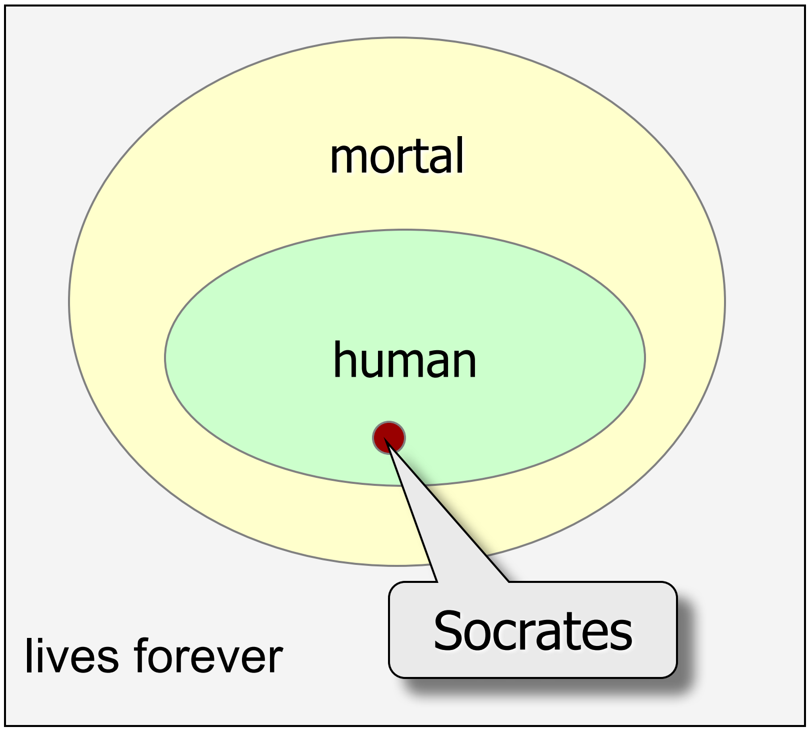
$\exists x\ (C(x) \wedge T(x))$

Example of proof

1.	$\forall x\ (R(x) \rightarrow T(x))$	3rd hypothesis
2.	$R(Linda) \rightarrow T(Linda)$	Universal instantiation using step 1
3.	$R(Linda)$	2nd hypothesis
4.	$T(Linda)$	Modes ponens using steps 2 & 3
5.	$C(Linda)$	1st hypothesis
6.	$C(Linda) \wedge T(Linda)$	Conjunction using steps 4 & 5
7.	$\exists x\ (C(x) \wedge T(x))$	Existential generalization using step 6

Thus, we have shown that "Somebody in this class has gotten a speeding ticket"

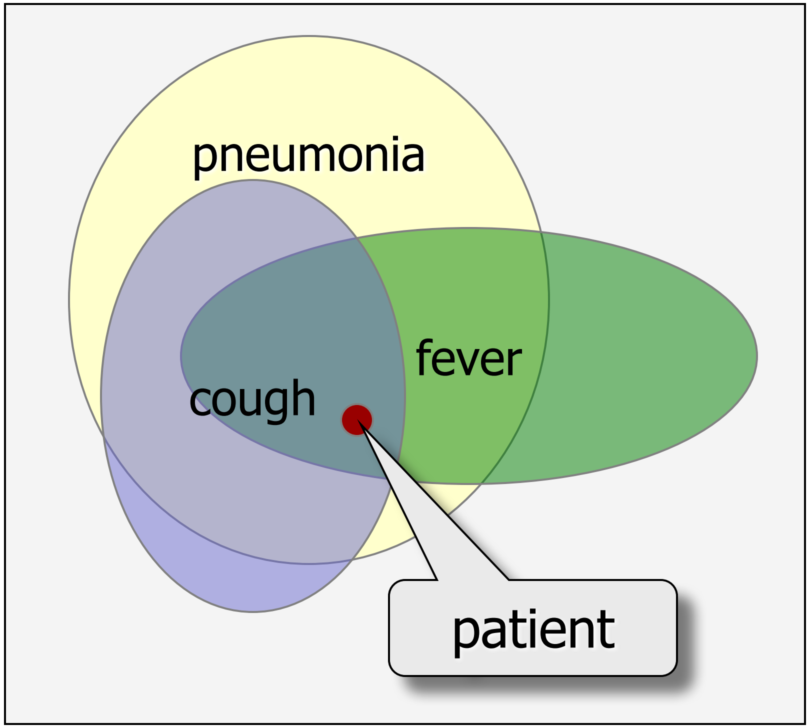
Diagrams for Validity



- To check the validity of an argument
- NOT a proof!

Abduction

- A form of logical inference that goes from observation to a hypothesis that accounts for the reliable data
- The lawn is wet → It rained last night



Common Errors

- Converse error

$$\begin{array}{l} Q(a) \\ \underline{\forall x, P(x) \rightarrow Q(x)} \\ \therefore P(a) \end{array}$$

- Inverse error

$$\begin{array}{l} \sim P(a) \\ \underline{\forall x, P(x) \rightarrow Q(x)} \\ \therefore \sim Q(a) \end{array}$$

Example

- Anyone who grows a money tree is rich
- Bill Gates is rich
- Bill Gates grows a money tree

- Bill Gates does not grow a money tree
- Bill Gates is not rich

- Every Great American City Has At Least One College. Worcester Has Ten.
  - — Highway billboard in Worcester, MA