

Conditional Probability

CS231

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Boy or Girl?

- A couple has two children, one of them is a girl. What is the probability that the other one is also a girl? Assuming 50/50 chances of conceiving boys and girls.

Conditional Probability

- Let A and B be events in a sample space S . If $P(A) \neq 0$, then the *conditional probability* of B given A , $P(B|A)$ is:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A) = \frac{P(A \cap B)}{P(B|A)}$$

$$P(A \cap B) = P(B|A) \times P(A)$$

Example

- Two cards are drawn from a well-shuffled deck. What is the probability that:
 - both are kings?
 - second draw is a king?
- $A = 1\text{st draw is king}, B = 2\text{nd draw is king}$
- $P(A) = 4/52, P(\bar{A}) = 48/52$
- $P(B|A) = 3/51, P(B|\bar{A}) = 4/51$
- $P(A \cap B) = 4/52 \times 3/51 = 12/2652$
- $P(A \cap B) + P(\bar{A} \cap B) = 4/52 \times 3/51 + 48/52 \times 4/51$

Example

- If the experiment of drawing a pair is repeated over time, what would be the expected value of the number of kings?
- 2 kings: $P(A \cap B) = 4/52 \times 3/51 = 12/2652$
- 1 king: $P(\bar{A} \cap B) + P(A \cap \bar{B}) = 48/52 \times 4/51 + 4/52 \times 48/51 = 384/2652$
- Expected value of # of kings: $2 \times 12/2652 + 1 \times 384/2652 \approx 0.154$

Example

- 5% of manufactured components are defective in general.
- The method for screening out defective items is not totally reliable. The test rejects good parts as defective in 1% of the cases and accepts defective parts as good ones in 10% of the cases.
- Given that the test indicates that an item is good, what is the probability that this item is, in fact, defective?

Definitions

- T = A component cleared the test (tested good)
- D = A component is defective
- \bar{T} = A component did not clear the test (tested defective)
- G = A component is good ($G = \bar{D}$)
- Want to solve: $P(D|T)$

$$P(D|T)$$

- $P(D) = 0.05, P(G) = 0.95$
- $P(\bar{T}|G) = 0.01$ (false positive) $P(T|G) = 0.99$
- $P(T \cap G) = P(T|G) \times P(G) = 0.99 \times 0.95 = 0.9405$
- $P(T|D) = 0.1$ (false negative)
- $P(T \cap D) = P(T|D) \times P(D) = 0.1 \times 0.05 = 0.005$
- $T = (T \cap G) \cup (T \cap D)$
- $P(T) = 0.9405 + 0.005 = 0.9455$
- $P(D|T) = \frac{P(T \cap D)}{P(T)} = 0.005/0.9455 = 0.0052882$

Medical Screening

- 1% of population suffer from a certain disease.
- The method for screening is not totally reliable. The test reports false positive in 5% of the cases and false negative in 10% of the cases.
- Given that a person has a negative test result, what is the probability that this person is, in fact, sick?
- Given that a person has a positive test result, what is the probability that this person is, in fact, sick?

Definitions

- T = A person cleared the test (negative)
- S = A person is sick
- \bar{T} = A person did not clear the test (positive)
- H = A person is healthy ($H = \bar{S}$)
- Want to solve: $P(S|T)$ and $P(S|\bar{T})$

$P(S|T)$

- $P(S) = 0.01, P(H) = 0.99$
- $P(\bar{T}|H) = 0.05$ (false positive) $P(T|H) = 0.95$
- $P(T \cap H) = P(T|H) \times P(H) = 0.95 \times 0.99 = 0.9405$
- $P(T|S) = 0.1$ (false negative)
- $P(T \cap S) = P(T|S) \times P(S) = 0.1 \times 0.01 = 0.001$
- $T = (T \cap H) \cup (T \cap S)$
- $P(T) = 0.9405 + 0.001 = 0.9415$
- $P(S|T) = \frac{P(T \cap S)}{P(T)} = 0.001/0.9415 \approx 0.001$

Bayes' Theorem

- If E_1, E_2, \dots, E_n are mutually exclusive and exhaustive events in a sample space, the total probability of any event F is:

$$P(F) = \sum_{i=1}^n P(F|E_i)P(E_i)$$

- For any event E and F with $P(F) \neq 0$:

$$P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|\bar{E})P(\bar{E})}$$

$P(S|T)$ and $P(S|\bar{T})$ with Bayes'

- $P(S) = 0.01$, $P(H) = P(\bar{S}) = 0.99$
- $P(\bar{T}|H) = P(\bar{T}|\bar{S}) = 0.05$, $P(T|H) = P(T|\bar{S}) = 0.95$
- $P(T|S) = 0.1$, $P(\bar{T}|S) = 0.9$

$$\begin{aligned} P(S|T) &= \frac{P(T|S)P(S)}{P(T|S)P(S) + P(T|\bar{S})P(\bar{S})} \\ &= \frac{0.1 \times 0.01}{0.1 \times 0.01 + 0.95 \times 0.99} = \frac{0.001}{0.001 + 0.9405} \approx 0.001 \end{aligned}$$

$$\begin{aligned} P(S|\bar{T}) &= \frac{P(\bar{T}|S)P(S)}{P(\bar{T}|S)P(S) + P(\bar{T}|\bar{S})P(\bar{S})} \\ &= \frac{0.9 \times 0.01}{0.9 \times 0.01 + 0.05 \times 0.99} = \frac{0.009}{0.009 + 0.0495} \approx 0.1538 \end{aligned}$$

Bayes' Theorem

- If E_1, E_2, \dots, E_n , are mutually exclusive and exhaustive events in a sample space, given any F with $P(F) \neq 0$,

$$P(E_k|F) = \frac{P(F|E_k)P(E_k)}{\sum_{i=1}^n P(F|E_i)P(E_i)} = \frac{P(F|E_k)P(E_k)}{P(F)}$$

- $P(\theta)$ is also known as prior
- $P(\theta|X)$ is the posterior probability after observing X and obtaining $P(X|\theta)$

Monty Hall Revisited

- $A/B/C$ = prize is behind door A/B/C
- $M_A/M_B/M_C$ = Monty opens door A/B/C
- You choose door A and Monty opens a door revealing no prize, say door B
- $P(M_B|A) = 1/2, P(M_B|B) = 0, P(M_B|C) = 1$
- $P(M_B) = 1/2, P(A) = P(B) = P(C) = 1/3$
- $P(A|M_B) = \frac{P(M_B|A)P(A)}{P(M_B)} = 1/2 \times 1/3 / 1/2 = 1/3$
- $P(C|M_B) = \frac{P(M_B|C)P(C)}{P(M_B)} = 1 \times 1/3 / 1/2 = 2/3$
- Exact same analysis holds for M_C

Bayes' Ratio

- When there are three events, A , B and C and the comparative posterior probabilities are of interest, consider the ratio:

$$\frac{P(A|C)}{P(B|C)} = \frac{P(C|A)}{P(C|B)} \times \frac{P(A)}{P(B)}$$

Example

- Two bags, one contains 70 red and 30 blue balls, and the other 30 red and 70 blue balls.
- Choose one bag randomly and draw with replacement.
- 8 red and 4 blue balls are drawn in 12 tries.
- What is the probability that it was the predominantly red bag that was chosen?

Solution

- A = selecting the 1st bag, B = selecting the 2nd bag, C = getting the draws we did
- $P(C|A) = (7/10)^8 \times (3/10)^4 \times C(12,8)$
- $P(C|B) = (7/10)^4 \times (3/10)^8 \times C(12,8)$
- $P(A) = P(B) = 0.5$

- $\frac{P(C|A)}{P(C|B)} = \frac{\left(\frac{7}{10}\right)^4}{\left(\frac{3}{10}\right)^4} = \left(\frac{7}{3}\right)^4$

$$\left. \begin{array}{l} \frac{P(A|C)}{P(B|C)} = \left(\frac{7}{3}\right)^4 \\ P(A|C) + P(B|C) = 1 \end{array} \right\} P(A|C) = \frac{\left(\frac{7}{3}\right)^4}{\left(\frac{7}{3}\right)^4 + 1}$$

Dramatic Taxicab

- A cab was involved in a hit-and-run at night.
- Two cab companies operate in the city, with green and blue cabs, respectively.
- 85% of the cabs are green.
- A witness identified the cab as blue.
- The witness correctly identified the two colors 80% of the time under night-time testing.
- What is the probability that the witness was right?

Independent Events

- Two events are independent when the occurrence of one does not affect the probability of the other.
 - tossing coins
 - rolling dice
- Events A and B are independent iff:
$$P(A \cap B) = P(A) \times P(B)$$

$$P(A \cap \bar{B})$$

- If A and B are independent events, so are A and \bar{B} .
- From set theory:
 - $(A \cap B) \cup (A \cap \bar{B}) = A$
 - $(A \cap B) \cap (A \cap \bar{B}) = \emptyset$
- $P((A \cap B) \cup (A \cap \bar{B})) = P(A \cap B) + P(A \cap \bar{B}) = P(A)$
- $P(A \cap \bar{B}) = P(A) - P(A \cap B) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(\bar{B})$

Loaded Coin

- A coin is loaded so that the probability of heads is 0.6. After 10 tosses, what is the probability of obtaining 8 heads?
- Consider HHHHHHHHTT
- $P(\text{HHHHHHHHTT}) = 0.6^8 \times 0.4^2$
- How many ways can you get 8 heads with 10 tosses? – $C(10, 8)$
- $P(8 \text{ heads}) = C(10, 8) \times 0.6^8 \times 0.4^2 \approx 0.12$