№3 $p(x) = \begin{cases} \frac{e^{-x/\theta}}{\theta}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \theta > 0$

$n = 3$

$\tilde{\theta}_1 = \bar{x}$, $\tilde{\theta}_3 = x_{(2)}$

а) $M[\tilde{\theta}_1] = \theta$ – ? Несмещённость

1) $M[\tilde{\theta}_1] = M[\bar{x}] = M\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] = \frac{1}{n}\sum_{i=1}^{n} M[x_i] = \frac{1}{n} \cdot n\theta = \theta$

$M[x] = \int_{-\infty}^{+\infty} x\, p(x)\,dx = \int_0^{+\infty} x \frac{e^{-x/\theta}}{\theta}\,dx = \theta \int_0^{+\infty} \frac{x}{\theta} e^{-x/\theta}\, d\left(\frac{x}{\theta}\right) =$

$= \theta \int_0^{+\infty} t e^{-t}\,dt = \theta\left(-te^{-t}\Big|_0^{+\infty} - \int_0^{+\infty} -e^{-t}\,dt\right) =$

$= -\theta\left(0 + e^{-t}\Big|_0^{+\infty}\right) = +\theta \Rightarrow$ несмещённая

$M[\tilde{\theta}_2] = M[x_{(2)}] = \int_0^{+\infty} t\, u''_1\, p(t)\, C_2^1 F(t)(1-F(t))\,dt =$

$= 6\int_0^{+\infty} \left(t + \frac{1}{\theta}\right) e^{-t/\theta}\left(1 - e^{-t/\theta}\right)\left(1 - 1 + e^{-t/\theta}\right) dt$

т.к. $F(t) = \int_{-\infty}^{t} p(x)\,dx = -e^{-x/\theta}\Big|^t = 1 - e^{-t/\theta}$

$\Rightarrow M[\tilde{\theta}_2] = 6\int_0^{+\infty} \frac{t}{\theta} e^{-2t/\theta}\left(1 - e^{-t/\theta}\right) dt =$

$= 6\theta\left(\frac{1}{4}\int_0^{+\infty} \frac{2t}{\theta} e^{-2t/\theta}\, d\left(\frac{2t}{\theta}\right) - \frac{1}{9}\int_0^{+\infty} \frac{3t}{\theta} e^{-3t/\theta}\, d\left(\frac{3t}{\theta}\right)\right) =$

$= 6\theta\left(\frac{1}{4}\int_0^{+\infty} s e^{-s}\,ds - \frac{1}{9}\int_0^{+\infty} u e^{-u}\,du\right) =$

$= 6\theta \cdot \frac{5}{36}\left(-se^{-s}\Big|_0^{+\infty} - e^{-s}\Big|_0^{+\infty}\right) = -\frac{5\theta}{6}(0 - 1) =$

$= \frac{5\theta}{6}$ — смещённая $\Rightarrow \tilde{\theta}_3 = \frac{6}{5} x_{(2)}$

б) $D[\tilde{\theta}_1] = M[\tilde{\theta}_1^2] - M^2[\tilde{\theta}_1] = M[\bar{x}^2] - M^2[\bar{x}] =$

$= M\left[\left(\frac{1}{n}\sum x_i\right)^2\right] - M^2[\xi] =$

$= \frac{1}{9} M[x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_3x_1] - \theta^2 =$

$= M[\xi^2] - \theta^2$

$M[\xi^2] = \int_{-\infty}^{+\infty} x^2 p(x)\,dx = 2\theta^2$ $\quad \boxed{D[\tilde{\theta}_1] = \theta^2}$?

$D[\tilde{\theta}_3'] = M[\tilde{\theta}_3'^2] - M^2[\tilde{\theta}_3']$

$M[\tilde{\theta}_3'^2] = \left(\frac{6}{5}\right)^2 6 \int_0^{+\infty} t^2 \frac{1}{\theta} e^{-t/\theta}\left(1 - e^{-t/\theta}\right) e^{-t/\theta}\,dt =$

$= \left(\frac{6}{5}\right)^2 6\cdot\theta^2\left(\frac{1}{4}\int_0^{+\infty}\left(\frac{2t}{\theta}\right)^2 e^{-\frac{2t}{\theta}}\,d\left(\frac{2t}{\theta}\right) - \frac{1}{9}\int_0^{+\infty}\left(\frac{3t}{\theta}\right)^2 e^{-\frac{3t}{\theta}}\,d\left(\frac{3t}{\theta}\right)\right) =$

$= \left(\frac{6}{5}\right)^4 8\theta^2 \frac{81-16}{81\cdot 16}\cdot\int_0^{+\infty} s^2 e^{-s}\,ds = \left(\frac{6}{5}\right)^2\cdot\frac{65}{216}\cdot$

$\cdot\left(-s^2 e^{-s}\Big|_0^{+\infty} + \int_0^{+\infty} 2s e^{-s}\,ds\right) = \left(\frac{6}{5}\right)^2 \frac{65}{108}\theta^2\cdot(+1) =$

$= \frac{6\cdot 65}{6\cdot 108}\theta^2 = \left(\frac{6}{5}\right)\frac{13}{18}\theta^2 = \frac{38}{25}\theta^2$

$D[\tilde{\theta}_3'] = \frac{38}{25}\theta^2 - \theta^2 = \frac{13}{25}\theta^2 \Rightarrow$ $\tilde{\theta}_3'$ – лучше

c) ~~$\tilde{\theta}_1$: а) $\tilde{\theta}_1$ – несмещённая~~

~~б) $\frac{\partial}{\partial\theta}\int_{-\infty}^{+\infty}\dots\int x\,L(x,\theta) = \int_{-\infty}^{+\infty}\dots\int x\frac{\partial}{\partial\theta}L\,dx$ – ?~~

~~$D[\tilde{\theta}_1]$ – ст. р. (т.к. $D[\tilde{\theta}_1]=\theta^2$~~

c) $\tilde{\theta}_1$

Для н-ва Крамер-Рао оценка $\tilde{\theta}_1$ регулярная – ?

~~если~~ регулярна $\tilde{\theta}_1$, если

1) модель регулярная ?

2) $D[\tilde{\theta}_1]$ – огр. на $(0,+\infty)$ – верно, (т.к. $D[\tilde{\theta}_1]=\theta^2$)

3) $\tilde{\theta}_1$ – несмещённая – верно, далее.

Модель регулярная, если

1) $P(x,\theta)$ непр. диффр. по θ на $(0,+\infty)$ – верно

2) $\frac{\partial}{\partial\theta}\int_{-\infty}^{+\infty}P(x,\theta)\,dx = \int_{-\infty}^{+\infty}\frac{\partial}{\partial\theta}P(x,\theta)\,dx = 0$ – ?

$$\int_{-\infty}^{+\infty}\frac{\partial}{\partial\theta}\left(e^{-x/\theta}\frac{1}{\theta}\right)dx = \int_{-\infty}^{+\infty}\left(-\frac{1}{\theta^2}e^{-\frac{x}{\theta}} - \frac{1}{\theta^2}e^{-x/\theta}\right)dx =$$

$$= -\frac{2}{\theta}\int_{-\infty}^{+\infty}e^{-t}\,dt = \frac{+2}{\theta}\left(e^{+\infty}-e^{+\infty}\right) = 0$$ – верно

3) $I(\theta)>0$, непр – ?

$$I(\theta_2) = M\left[\left(\frac{\partial(\ln P)}{\partial\theta}\right)^2\right] = \int_0^{+\infty}\left(\frac{\partial(\ln\frac{1}{\theta}-\frac{x}{\theta})}{\partial\theta}\right)^2\frac{e^{-x/\theta}}{\theta}\,dx =$$

$$= \frac{3}{\theta^2}$$ – верно, $\overset{1)}{2)}\overset{}{3)}\Rightarrow$ модель регулярная

$\Rightarrow$ Оценка регулярная $\Rightarrow$ $D[\tilde{\theta}_1] \geq \frac{1}{n I(\theta)}$

$\theta^2 \geq \frac{\theta^2}{3 \cdot 3} \Leftrightarrow 1 \geq \frac{1}{9}$ — неизвестно ничего про эффективность

$\tilde{\theta}_3$ Для п-ва Крамер-Рао — оценка $\tilde{\theta}_3'$ регулярна?

Оценка $\tilde{\theta}_3'$ регулярная, если

1) модель регул. — ? 2) $D[\tilde{\theta}_3']$ — огр на $(0, +\infty)$ — верно,

3) $\tilde{\theta}_3'$ — несмещённ. — верно,

Модель регул., если 1) $P(x, \theta)$ — непр. дифф. — верно,

2) $\frac{\partial}{\partial \theta} \int_{-\infty}^{+\infty} P(x, \theta) dx = \int_{-\infty}^{+\infty} \frac{\partial}{\partial \theta} P(x, \theta) dx = 0$ — верно,

$\int_{-\infty}^{+\infty} P(x, \theta) dx = 1$

3) $I(\tilde{\theta}_2') > 0$, инф — ?

$I(\theta_2) = \frac{3}{\theta^2}$ — верно, $\Rightarrow$ модель регул. $\Rightarrow$

$\Rightarrow$ оценка регулярн. $\Rightarrow D[\tilde{\theta}_3'] \geq \frac{1}{n I(\theta)}$

$\frac{13}{25} \theta^2 \geq \frac{\theta^2}{3 \cdot 3} \Leftrightarrow \frac{13}{25} \geq \frac{1}{9}$ — неизвестно ничего про эффективность