$\boxed{T3}$ $p(x) = \begin{cases} \frac{e^{-x/\theta}}{\theta}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \theta > 0$

$n=3$

$\tilde{\theta}_1 = \bar{x}$, $\tilde{\theta}_3 = x_{(2)}$

a) $M[\tilde{\theta}_1] = \theta$ -? Несмещённость

1) $M[\tilde{\theta}_1] = M[\bar{x}] = \cancel{M[\frac{1}{n}\sum_{i=1}^{n} x_i] = 1}$

$M[\xi] = \int_{-\infty}^{+\infty} x\, p(x)dx = \int_0^{+\infty} x \frac{e^{-x/\theta}}{\theta} dx = \theta \int_0^{+\infty} \frac{x}{\theta} e^{-x/\theta} d\left(\frac{x}{\theta}\right) =$

$= \theta \int_0^{+\infty} t e^{-t} dt = \theta \left(-te^{-t}\Big|_0^{+\infty} - \int_0^{+\infty} -e^{-t}dt\right) =$

$= -\theta\left(0 + e^{-t}\Big|_0^{+\infty}\right) = +\theta \Rightarrow$ несмещённая

$M[\tilde{\theta}_3] = M[x_{(2)}] = \int_0^{+\infty} t\, n\, p(t)\, C_2^1 F(t)(1-F(t))dt =$

$= 6\int_0^{+\infty} \left(t\frac{1}{\theta}\right) e^{-t/\theta} (1 - e^{-t/\theta})(1 - 1 + e^{-t/\theta})dt$

т.к. $F(t) = \int_{-\infty}^{t} p(x)dx = -e^{-x/\theta}\Big|^{t} = 1 - e^{-t/\theta}$

$\Rightarrow M[\tilde{\theta}_3] = 6\int_0^{+\infty} \frac{t}{\theta} e^{-2t/\theta}\left(1 - e^{-t/\theta}\right)dt =$

$= 6\theta\left(\frac{1}{4}\int_0^{+\infty} \frac{2t}{\theta} e^{-2t/\theta} d\left(\frac{2t}{\theta}\right) - \frac{1}{9}\int_0^{+\infty} \frac{3t}{\theta} e^{-3t/\theta} d\left(\frac{3t}{\theta}\right)\right) =$

$= 6\theta\left(\frac{1}{4}\int_0^{+\infty} s e^{-s} ds - \frac{1}{9}\int_0^{+\infty} \varphi e^{-\varphi} d\varphi\right) =$

$= 6\theta \cdot \frac{5}{36}\left(-se^{-s}\Big|_0^{+\infty} - e^{-s}\Big|_0^{+\infty}\right) = \frac{-5\theta}{6}(0 - 1) =$

$= \frac{5\theta}{6}$ — смещённая $\Rightarrow \tilde{\theta}_3' = \frac{6}{5} x_{(2)}$

б) $D[\tilde{\theta}_1] = M[\tilde{\theta}_1^2] - M^2[\tilde{\theta}_1] = M[\bar{x}^2] - M^2[\bar{x}] =$

$= M\left[\left(\frac{1}{n}\sum x_i\right)^2\right] - M^2[\xi] =$

$= \frac{1}{9} M\left[x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_3x_1\right] - \theta^2 =$

$= M[\xi^2] - \theta^2$

$M[\xi^2] = \int_{-\infty}^{+\infty} x^2 p(x)\,dx = 2\theta^2$ $\boxed{D[\tilde{\theta}_1] = \theta^2}$?

$D[\tilde{\theta}_3'] = M[\tilde{\theta}_3'^2] - M^2[\tilde{\theta}_3'] =$

$M[\tilde{\theta}_3'^2] = \left(\frac{6}{5}\right)^2 6 \int_0^{+\infty} t^2 \frac{1}{\theta} e^{-t/\theta}\left(1 - e^{-t/\theta}\right) e^{-t/\theta} dt =$

$= \left(\frac{6}{5}\right)^2 6 \cdot \theta^2 \left(\frac{1}{4^2}\int_0^{+\infty}\left(\frac{2t}{\theta}\right)^2 e^{-\frac{2t}{\theta}} d\left(\frac{2t}{\theta}\right) - \frac{1}{9^2}\int_0^{+\infty}\left(\frac{3t}{\theta}\right)^2 e^{-\frac{3t}{\theta}} d\left(\frac{3t}{\theta}\right)\right) =$

$= \left(\frac{6}{5}\right)^2 6\,\theta^2 \frac{\cancel{81-16}^{\,65}}{\underset{27\cdot 8}{81\cdot 16}} \cdot \int_0^{+\infty} s^2 e^{-s} ds = \left(\frac{6}{5}\right)^2 \cdot \frac{65}{216} \cdot$

$\cdot \left(\cancel{-s^2 e^{-s}\Big|_0^{+\infty}} + \int_0^{+\infty} 2s e^{-s} ds\right) = \left(\frac{6}{5}\right)^2 \frac{65}{108}\theta^2 \cdot (+1) =$

$= \frac{6\cdot 65}{6\cdot 108}\theta^2 = \left(\frac{6}{5}\right)\frac{13}{18}\theta^2 = \frac{38}{25}\theta^2$

$D[\tilde{\theta}_3'] = \frac{38}{25}\theta^2 - \theta^2 = \frac{13}{25}\theta^2 \Rightarrow$ $\tilde{\theta}_3'$ – лучше

~~c) $\tilde\theta_1$: a) $\tilde\theta_1$ – несмещённая~~

~~б) $\frac{\partial}{\partial\theta}\int_{-\infty}^{+\infty}\dots\int \bar{x} L(\bar x,\theta) = \int_{-\infty}^{+\infty}\dots\int \bar x \frac{\partial}{\partial\theta} L dx$ – ?~~

~~$D[\tilde\theta_1]$ – огр. (т.к. $D[\tilde\theta_1]=\theta^2$~~

c) $\tilde\theta_1$

Для н-ва Крамер-Рао оценка $\tilde\theta_1$ регулярная – ?

регулярна $\tilde\theta_1$, если

1) модель регулярная ?

2) $D[\tilde\theta_1]$ – огр. на $[0,+\infty)$ – верно, (т.к. $D[\tilde\theta_1]=\theta^2$)

3) $\tilde\theta_1$ – несмещённая – верно, ранее

Модель регулярная, если

1) $p(x,\theta)$ непр. дифф. по θ на $(0,+\infty)$ – верно

2) $\underbrace{\frac{\partial}{\partial\theta}\int_{-\infty}^{+\infty} p(x,\theta)dx}$ $= \int_{-\infty}^{+\infty}\frac{\partial}{\partial\theta}p(x,\theta)dx = 0$ – ?

$\int_{-\infty}^{+\infty}\frac{\partial}{\partial\theta}\left(e^{-x/\theta}\frac{1}{\theta}\right)dx = \int_{-\infty}^{+\infty}\left(-\frac{1}{\theta^2}e^{-x/\theta} - \frac{1}{\theta^2}e^{-x/\theta}\right)dx =$

$= -\frac{2}{\theta}\int_{-\infty}^{+\infty} e^{-t}dt = \frac{+2}{\theta}\left(e^{+\infty}\Big|^{0} - e^{+\infty}\Big|^{0}\right) = 0$ – верно

3) $I(\theta)>0$, непр – ?

$I(\theta_2) = M\left[\left(\frac{\partial \ln p}{\partial\theta_2}\right)^2\right] = \int_0^{+\infty}\left(\frac{\partial\left(\ln\frac{1}{\theta}-\frac{x}{\theta}\right)}{\partial\theta}\right)^2\frac{e^{-x/\theta}}{\theta}dx =$

$=\frac{3}{\theta^2}$ – верно, 1) 2) 3) $\Longrightarrow$ модель регулярная

$\Rightarrow$ Оценка регулярная $\Rightarrow$ $D[\tilde{\theta}_1] \geq \frac{1}{nI(\theta_?)}$

$\theta^2 \geq \frac{\theta^2}{3\cdot 3} \Leftrightarrow 1 \geq \frac{1}{9}$ — неизвестно ничего про эффективность

$\tilde{\theta}_3$ Для н-ва Крамер-Рао — оценка $\tilde{\theta}_3'$ — регулярная?

Оценка $\tilde{\theta}_3'$ регулярная, если

1) модель регул. — ? 2) $D[\tilde{\theta}_3']$ — огр на $(0, +\infty)$ — верно

3) $\tilde{\theta}_3'$ — несмещённ. — верно

Модель регул., если 1) $p(x, \theta)$ — непр. дифф. — верно

2) $\frac{\partial}{\partial\theta}\underbrace{\int_{-\infty}^{+\infty} p(x,\theta)\,dx}_{=1} = \int_{-\infty}^{+\infty} \frac{\partial}{\partial\theta} p(x,\theta)\,dx = 0$ — верно

3) $I(\tilde{\theta}_2') > 0$, непр — ?

$I(\theta_2) = \frac{3}{\theta^2}$ — верно $\Rightarrow$ модель регул. $\Rightarrow$

$\Rightarrow$ оценка регулярн. $\Rightarrow$ $D[\tilde{\theta}_3'] \geq \frac{1}{nI(\theta)}$

$\frac{13}{25}\theta^2 \geq \frac{\theta^2}{3\cdot 3} \Leftrightarrow \frac{13}{25} \geq \frac{1}{9}$ — неизвестно ничего про эффективность