Задание №1

T_1, T_2, T_3

T_1 $\xi \sim R[0, \theta]$; $\vec{X}_n$ — выборка, θ — пар. модели

1) $\tilde{\theta}_1 = 2\bar{X}$

a) Несмещённость : $M[\tilde{\theta}_1] = \theta$

$M[2\bar{X}] = M\left[2 \cdot \frac{1}{n}\sum_{j=1}^{n} x_i\right] = 2M[x_i] = 2M[\xi]$

$R[0,\theta] \sim p(x) = \frac{1}{\theta}\{(0,\theta)\} \Rightarrow \ldots = 2\int_0^\theta x\frac{1}{\theta}dx = \theta$ +

Состоятельность : $D[\tilde{\theta}_1] \to 0,\ n \to \infty$

$D[\tilde{\theta}_1] = D\left[\frac{2}{n}\sum_{i=1}^{n} x_i\right] = \frac{4}{n^2}\sum_{i=1}^{n} D[x_i] = \frac{4}{n}D[\xi] =$

$= \frac{4}{n}\left(M[\xi^2] - M^2[\xi]\right) = \frac{\theta^2}{3n} \to 0,\ n \to \infty$ +

2) $\tilde{\theta}_2 = X_{min}$

a) Несмещённость : $M[\tilde{\theta}_2] = \theta$

$M[x_{min}] = M[x_{(1)}] = \int_0^\theta y\,p(y)\,dy$ плотн. $x_{(1)}$

$p(y) = \Phi'(y) = \left((1-(1-F(y)))^n\right)' = n(1-F(y))^{n-1}F'(y) =$

$= n\left(1-\frac{y}{\theta}\right)^{n-1} \cdot \frac{1}{\theta} \Rightarrow M[\tilde{\theta}_2] = \int_0^\theta n\left(1-\frac{y}{\theta}\right)^{n-1} \cdot y \cdot \frac{1}{\theta}dy =$

$= \theta \cdot \frac{1}{n+1}$ — смещённая —

Состоятельность : $D[\tilde{\theta}_2] \to 0$

$D[x_{(1)}] = M[x_{(1)}^2] - M^2[x_{(1)}]$

$M[\tilde{\theta}_2^2] = \int_0^\theta y^2 n\left(1-\frac{y}{\theta}\right)^{n-1} \cdot \frac{1}{\theta}dy = \frac{2\theta^2}{(n+1)(n+2)}$

$D[x_{(1)}] = \frac{2\theta^2}{(n+1)(n+2)} - \theta^2 \cdot \frac{1}{(n+1)^2} = \frac{\theta^2 n}{(n+1)^2(n+2)} \xrightarrow[n\to\infty]{} 0$

! смещённые $\Rightarrow$ достаточн. усл-вие не выполнено ?

По опр: Состоятельность

$\tilde{\theta}_2 \xrightarrow{P} \theta \quad \forall \theta > 0 \Leftrightarrow \forall \varepsilon > 0 \quad P\left(\left|\tilde{\theta}_2 - \theta\right| \geq \varepsilon\right) \underset{n\to+\infty}{\to} 0$

$\left|\tilde{\theta}_2 - \theta\right| \geq \varepsilon \Leftrightarrow \left[\begin{array}{l}\tilde{\theta}_2 \geq \theta + \varepsilon \\ \tilde{\theta}_2 \leq \theta - \varepsilon\end{array}\right., \quad \tilde{\theta}_2 = x_{min} \Rightarrow$

$\Rightarrow P(x_{min} \geq \theta + \varepsilon) = 0 \Rightarrow (\Leftrightarrow P(\tilde{\theta}_2 \leq \theta - \varepsilon) \to 0, n \to \infty$

$P(x_{min} \leq \theta - \varepsilon) = P(x_{min} < \theta - \varepsilon) = \Phi(\theta - \varepsilon)$

$\Phi(y) = 1 - (1 - F(y))^n \Rightarrow P(x_{min} < \theta - \varepsilon) = 1 - \left(1 - \frac{\theta - \varepsilon}{\theta}\right)^n$

т.к. $|\Phi(y)| \leq 1 \Rightarrow 0 < \varepsilon < \theta \Rightarrow \Phi(\theta - \varepsilon) = 1 - \left(\frac{\varepsilon}{\theta}\right)^n \underset{n\to\infty}{\to} 1$

$\Rightarrow$ несостоят. $-$

Возьмём несмещённое $\tilde{\theta}_2' = (n+1) x_{min}$ $\left(\Rightarrow M\left[\tilde{\theta}_2'\right] = \theta\right)$

$D\left[\tilde{\theta}_2'\right] = \frac{\theta^2 n}{n+2} \underset{n\to\infty}{\not\to} 0 \Rightarrow$ дост. усл. не выполнено ?

Состоятельность по опр: $\tilde{\theta}_2' \xrightarrow{P} \theta \quad \forall \theta > 0$

$\Leftrightarrow P\left(\left|\tilde{\theta}_2' - \theta\right| \geq \varepsilon\right) \to 0, \; n \to \infty$

$|(n+1) x_{min} - \theta| \geq \varepsilon \Leftrightarrow \left[\begin{array}{l}(n+1) x_{min} \geq \varepsilon + \theta \\ (n+1) x_{min} \leq -\varepsilon + \theta\end{array}\right. \Leftrightarrow$

$\Leftrightarrow \left[\begin{array}{l}x_{min} \geq \frac{\varepsilon + \theta}{n+1} \\ x_{min} \leq \frac{\theta - \varepsilon}{n+1}\end{array}\right.$ т.к. $n \to \infty$, то $P\left(x_{min} \leq \frac{\theta - \varepsilon}{n+1}\right) = 0 \Rightarrow$

$\Rightarrow \Leftrightarrow P\left(x_{min} \geq \frac{\theta + \varepsilon}{n+1}\right) = 1 - P\left(x_{min} < \frac{\theta + \varepsilon}{n+1}\right) =$

$= 1 - \Phi\left(\frac{\theta + \varepsilon}{n+1}\right) = 1 - 1 + \left(1 - F\left(\frac{\theta + \varepsilon}{n+1}\right)\right)^n =$

$$\underset{n \geq n_0}{=} \left(1 - \frac{\theta+\varepsilon}{\theta(n+1)}\right)^n \underset{n\to\infty}{\to} e^{-\frac{\theta+\varepsilon}{\varepsilon}} > 0 \Rightarrow$$ не яв. состоят $-$

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3) $\tilde{\theta}_3 = X_{max}$

а) Несмещённость

$$M[\tilde{\theta}_3] = \int_0^\theta y\, q(y)\, dy \qquad \Psi(y) = (F(y))^n$$

$$q(y) = \Psi'(y) = n\left(\frac{y}{\theta}\right)^{n-1} \frac{1}{\theta} \{(0,\theta)\}$$

$$M[\tilde{\theta}_3] = \int_0^\theta y\, n\left(\frac{y}{\theta}\right)^{n-1} \frac{1}{\theta}\, dy = \frac{\theta n}{n+1}$$ — смещённ. $-$

Состоятельность

$$D[\tilde{\theta}_3] = M[\tilde{\theta}_3^2] - M^2[\tilde{\theta}_3]$$

$$M[\tilde{\theta}_3^2] = \int_0^\theta y^2 q(y)\, dy = \frac{n}{\theta^n} \cdot \frac{\theta^{n+2}}{n+2}$$

$$D[\tilde{\theta}_3] = \theta^2 \frac{n}{n+2} - \frac{\theta^2 n^2}{(n+1)^2} = \theta^2 \frac{n}{(n+2)(n+3)} \to 0$$

! смещённое $\Rightarrow$ достаточное усл. не выполнено ?

По опред. Состоятельность:

$$\tilde{\theta}_3 \xrightarrow{P} \theta \quad \forall \theta > 0 \Leftrightarrow \forall \varepsilon > 0 \ P(|\tilde{\theta}_3 - \theta| \geq \varepsilon) \underset{n\to\infty}{\to} 0$$

$$|\tilde{\theta}_3 - \theta| \geq \varepsilon \Leftrightarrow \left[\begin{array}{l}\tilde{\theta}_3 \geq \theta+\varepsilon \\ \tilde{\theta}_3 \leq \theta-\varepsilon\end{array}\right. \Leftrightarrow \left[\begin{array}{l}X_{max} \geq \theta+\varepsilon \\ X_{max} \leq \theta-\varepsilon\end{array}\right.$$

$$P(\tilde{\theta}_3 \geq \theta+\varepsilon) = 0 \Rightarrow \ldots \Leftrightarrow P(\tilde{\theta}_3 \leq \theta-\varepsilon) =$$

$$= (F(\theta-\varepsilon))^n = \left(\frac{\theta-\varepsilon}{\theta}\right)^n \to 0,\ n\to\infty \Rightarrow$$ состоят. $+$

3) $\tilde{\theta}_3^{!} = x_{max}\frac{n+1}{n}$ несмещённая

Состоят по опр: $\tilde{\theta}_3^{!} \xrightarrow{P} \theta \quad \forall \theta > 0 \Leftrightarrow$

$\forall \varepsilon > 0 \quad P\left(|\tilde{\theta}_3^{!} - \theta| \geq \varepsilon\right) \to 0, \; n \to \infty$

$$\left|\tilde{\theta}_3\left(\frac{n+1}{n}\right) - \theta\right| = \left|x_{max}\frac{n+1}{n} - \theta\right| \geq \varepsilon \Leftrightarrow \left[\begin{array}{l} x_{max}\frac{n+1}{n} \geq \varepsilon + \theta \\ x_{max}\frac{n+1}{n} \leq \theta - \varepsilon \end{array}\right.$$

$$\left[\begin{array}{l} x_{max} \geq \frac{n}{n+1}(\varepsilon + \theta) \\ x_{max} \leq \frac{n}{n+1}(\theta - \varepsilon) \end{array}\right. \Rightarrow \frac{n}{n+1} \to 1, \; n \to \infty \Rightarrow \ldots$$

$$\ldots \Leftrightarrow P\left(x_{max} \leq \frac{n}{n+1}(\varepsilon - \theta)\right) = P\left(x_{max} < \frac{n}{(n+1)}(\varepsilon - \theta)\right)$$

$$= F^n\left(\frac{n}{n+1}(\varepsilon - \theta)\right) = \left(\frac{n}{n+1}\,\frac{\varepsilon - \theta}{\theta}\right)^n = \left(\frac{1}{1+\frac{1}{n}}\right)^n \left(-1 + \frac{\varepsilon}{\theta}\right)^n \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow$ состоятельна $+$ $\quad \to 1 \quad \to 0$

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4) $\tilde{\theta}_5 = \left(x_1 + \frac{\sum_{k=2}^{n} x_k}{(n-1)}\right)$

а) Несмещённость

$$M[\tilde{\theta}_5] = M\left[x_1 + \frac{1}{n-1}\sum x_i\right] = M[x_1] + \frac{1}{n-1}\sum n x_i$$

$= M\xi + M\xi = 2M\xi = \theta$ — несмещ. $+$

Состоятельность

$$D[\tilde{\theta}_4] = D[x_1] + \frac{1}{(n+1)^2}\sum_{i=2}^{n} D[x_i] = \frac{\theta^2}{12} + \frac{1\cdot\theta^2}{(n-1)\cdot 12} \to$$

$\to \frac{\theta^2}{12} \; (\not\to 0), \; n \to \infty$ Дост. условия не выполн. ?

По опр.: (1) $\tilde{\theta}_5 \xrightarrow{P} \theta$ $\quad \left(\begin{array}{l} \xi_n \xrightarrow{P} \xi \quad \eta_n \xrightarrow{P} \eta \\ \xi_n + \eta_n \xrightarrow{P} \xi + \eta \end{array}\right)$

② ЗБЧ Хинчина: ξ_i - незав. и одинаков. распред., имеют $M\xi_i < \infty$, тогда $\frac{1}{n}\sum_{i=1}^{n} \xi_i \xrightarrow{P} M\xi_i$

$$\tilde{\theta}_5 = \underbrace{X_1}_{\xi_n} + \frac{1}{(n-1)}\sum_{i=2}^{n} x_i$$

$\xi_n \xrightarrow{P} X_i$

$\eta_n \xrightarrow{P} \underset{(\xi)}{M x_i} = \frac{\theta}{2}$

$\tilde{\theta}_5 \xrightarrow{P} \xi + \frac{\theta}{2}$: $\Rightarrow$ оценка не явл. состоятельной

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б) Сравнение эффективности:

$D[\tilde{\theta}_1] = \frac{\theta^2}{3n}$

$D[\tilde{\theta}_3'] = \frac{\theta^2}{(n+2)n}$

$\frac{1}{3n} > \frac{1}{(n+2)n}$ $n+2 > 3$

$n > 1$

$D[\tilde{\theta}_1] > D[\tilde{\theta}_3']$

$\tilde{\theta}_3'$ - эффективней $\tilde{\theta}_1$