Ordinary Differential Equation (ODE): let $n \in \mathbb{N}$ and let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ then $f\left(x, y(x), y'(x), \dots y^{(n)}(x)\right) = 0$.

Solution of an Ordinary Differential Equation: let $n \in \mathbb{N}$ let $\mathcal{U} \subseteq \mathbb{R}$ and let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ then a function $y \in C^n(\mathcal{U})$ such that $f\left(x, y(x), \dots, y^{(n)}(x)\right) = 0$ for all $x \in \mathcal{U}$.

Notation: let $n \in \mathbb{N}$ let $\mathcal{U} \subseteq \mathbb{R}$ and let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ then
$\text{sols}\left(f\left(x, y(x), y'(x), \dots y^{(n)}(x)\right) = 0\right) = \left\{y \in C^n(\mathcal{U}) \mid f\left(x, y(x), \dots, y^{(n)}(x)\right) = 0\right\}$.

First Order Ordinary Differential Equation: let $f : \mathbb{R}^3 \to \mathbb{R}$ then $f(x, y, y') = 0$.

Separable Ordinary Differential Equation: let $f, g : \mathbb{R} \to \mathbb{R}$ then $y' = f(y) g(x)$.

Claim Separation of Variables: let $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) \neq 0$ for all $x \in \mathbb{R}$ then
$\text{sols}(y' = f(y) g(x)) = \text{sols}\left(\int \frac{1}{f(y)} \mathrm{d}y = \int g(x) \mathrm{d}x\right)$.

Corollary: $\text{sols}(y = y') = \{Ce^x \mid C \in \mathbb{R}\}$.

Curves that foliate a domain: Let $\Omega \subseteq \mathbb{R}^2$ and let $I \subseteq \mathbb{R}$ then a set of curves $\{\gamma_\alpha\} \subseteq I \to \mathbb{R}$ such that $\biguplus \gamma_i = \Omega$.

Claim: $\text{sols}(y = y')$ foliate the plane.

Remark: let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and let $g \in \text{sols}\left(f\left(x, y, \dots, y^{(n)}\right)\right)$ such that g is piecewise continuous then each continuous branch of g is a solution of the ODE.

Initial Value Problem (IVP): let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and let $p \in \mathbb{R}^2$ then $\begin{cases} f\left(x, y', \dots y^{(n)}\right) = 0 \\ y(p_1) = p_2 \end{cases}$.

Theorem Cauchy-Peano: let $I, J \subseteq \mathbb{R}$ be closed intervals let $f \in C^1(I \times J)$ and let $p \in \text{int}(I \times J)$ then there exsits $\varepsilon > 0$ such that $\begin{cases} y' = f(x, y) \\ y(p_1) = p_2 \end{cases}$ has a solution on the interval $(p_1 - \varepsilon, p_1 + \varepsilon)$.

Theorem of Existence and Uniqueness: let $I, J \subseteq \mathbb{R}$ be closed intervals let $f \in C^1(I \times J)$ and let $p \in \text{int}(I \times J)$ then there exsits $\varepsilon > 0$ such that $\begin{cases} y' = f(x, y) \\ y(p_1) = p_2 \end{cases}$ has a unique solution on the interval $(p_1 - \varepsilon, p_1 + \varepsilon)$.

Corollary: let $I, J \subseteq \mathbb{R}$ be closed intervals and let $f \in C^1(I \times J)$ then $\text{sols}(y' = f(x, y))$ foliates $I \times J$.

Vector Field: let $\Omega \subseteq \mathbb{R}^2$ be a domain then $\nu \in C\left(\Omega, \mathbb{R}^2\right)$.

Notation: let X be a set and let $\gamma : \mathbb{R} \to X$ then $\dot{\gamma} = \frac{\mathrm{d}\gamma}{\mathrm{d}t}$.

Integral curve: let $\nu : \Omega \to \mathbb{R}^2$ be a vector field then a curve $\gamma : [a, b] \to \Omega$ such that $\dot{\gamma}(t) = \nu(g(t))$.

Claim: let $\nu : \Omega \to \mathbb{R}^2$ be a vector field such that $v(x) \neq 0$ for all $x \in \Omega$ and let $\gamma : [a, b] \to \Omega$ be a curve such that $v(\gamma(t)) \in T_{\gamma(t)}(\text{Im}(\gamma))$ for all $t \in [a, b]$ then there exists a curve $\eta : [\alpha, \beta] \to [a, b]$ such that $\gamma \circ \eta$ is an integral curve.

C^n **Vector Field**: let $\Omega \subseteq \mathbb{R}^2$ be a domain then $\nu \in C^n\left(\Omega, \mathbb{R}^2\right)$.

Claim: let $\nu : \Omega \to \mathbb{R}^2$ be a C^1 vector field then $\{\gamma : [a, b] \to \Omega \mid (a, b \in \mathbb{R}) \wedge (\gamma \text{ is an integral curve of } \nu)\}$ foliates Ω.

Theorem Peano: let $\nu : \Omega \to \mathbb{R}^2$ be a vector field then $\{\gamma : [a, b] \to \Omega \mid (a, b \in \mathbb{R}) \wedge (\gamma \text{ is an integral curve of } \nu)\}$ covers Ω.

Lemma: let $\Omega \subseteq \mathbb{R}^2$ be a domain and let $f \in C^1(\Omega)$ then $\left(\begin{smallmatrix} 1 \\ f \end{smallmatrix}\right)$ is a vector field.

Theorem of Existence and Uniqueness: let $\Omega \subseteq \mathbb{R}^2$ be a domain let $f \in C^1(\Omega)$ and let $g \in C^1(\mathbb{R})$ then
$(g' = f(x, g)) \iff \left(\left(\begin{smallmatrix} x \\ g \end{smallmatrix}\right) \text{ is an integral curve of } \left(\begin{smallmatrix} 1 \\ f \end{smallmatrix}\right)\right)$.

Autonomous Ordinary Differential Equations: let $f : \mathbb{R} \to \mathbb{R}$ then $y' = f(y)$.

Logistic Equation: let $L > 0$ then $\dot{P}(t) = P(t)(L - P(t))$.

Equilibrium Solution: let $f : \mathbb{R} \to \mathbb{R}$ then a function $g \in \text{sols}(y' = f(y))$ such that $f(g(x)) = 0$ for all $x \in \mathbb{R}$.

Corollary: let $f \in C(\mathbb{R})$ and let $g : \mathbb{R} \to \mathbb{R}$ be an equalibriem solution then g is constant.

Corollary: let $L > 0$ then $\{0, L\}$ are the equilibium solutions of the logistic equation.

Logistic Equation With Harvesting: let $L, k > 0$ then $\dot{P}(t) = P(t)(L - P(t)) - k$.

Stable Equilibrium Solution: let $f : \mathbb{R} \to \mathbb{R}$ then an equilibrium solution $g : \mathbb{R} \to \mathbb{R}$ such that for all $y \in \text{sols}(y' = f(y))$ and for all $\varepsilon > 0$ if there exists $\delta > 0$ and $p \in \mathbb{R}$ for which $|y(p) - g(p)| < \delta$ then $|y(t) - g(t)| < \varepsilon$ for all $t > p$.

Lotka–Volterra equations: let $\alpha > 0$ then $\begin{cases} \dot{x}(t) = x(t) \cdot (1 - y(t)) \\ \dot{y}(t) = \alpha \cdot y(t) \cdot (x(t) - 1) \end{cases}$.

Claim: let $\alpha > 0$ then $\left\{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)\right\}$ are the equilibium solutions of the lotka-volterra equation.

Claim: let $\alpha > 0$ and let $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in C^1\left(\mathbb{R}^2\right)$ be a solution to the lotka-volterra equation then
$\frac{\mathrm{d}}{\mathrm{d}t}((y(t) - 1 - \log(y(t))) + \alpha(x(t) - 1 - \log(x(t)))) = 0$.

Corollary: let $\alpha > 0$ let $p \in \mathbb{R}^2$ and let $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in C^1\left(\mathbb{R}^2\right)$ be a solution to $\begin{cases} \dot{x}(t) = x(t) \cdot (1 - y(t)) \\ \dot{y}(t) = \alpha \cdot y(t) \cdot (x(t) - 1) \\ \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)(0) = p \end{cases}$ then $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ is periodic.

Lemma: let $\alpha > 0$ let $p \in \mathbb{R}^2$ and let $\binom{x}{y} \in C^1\left(\mathbb{R}^2\right)$ be a solution to $\begin{cases} \dot{x}(t)=x(t)\cdot(1-y(t)) \\ \dot{y}(t)=\alpha\cdot y(t)\cdot(x(t)-1) \\ \binom{x}{y}(0)=p \end{cases}$ with priod T then $\frac{1}{T}\int_0^T x\mathrm{d}t = 1$ and $\frac{1}{T}\int_0^T y\mathrm{d}t = 1$.

Claim Linear Substitution: let $a, b, c \in \mathbb{R}$ let $f : \mathbb{R} \to \mathbb{R}$ and let $y \in \text{sols}\left(y' = f\left(ax + by + c\right)\right)$ then the function $z : \mathbb{R} \to \mathbb{R}$ which defined by $z\left(x\right) = ax + by + c$ satisfies $z' = a + bf\left(z\right)$.