Notation: let X be a set and let $y:\mathbb{R}\to\mathbb{R}$ then $y^{(0)}=y$.
Notation: let X,Y be sets let $n\in\mathbb{N}_+$ and let $y\in C^n(X,Y)$ then $y^{(n)}=\left(y^{(n-1)}\right)'$.
Ordinary Differential Equation (ODE): let $n\in\mathbb{N}$ and let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ then $f\left(x,y(x),y'(x),\ldots y^{(n)}(x)\right)=0$.
Solution of an Ordinary Differential Equation: let $n\in\mathbb{N}$ let $\mathcal{U}\subseteq\mathbb{R}$ and let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ then a function $y\in C^n(\mathcal{U})$ such that $f\left(x,y(x),\ldots,y^{(n)}(x)\right)=0$ for all $x\in\mathcal{U}$.
Notation: let $n\in\mathbb{N}$ let $\mathcal{U}\subseteq\mathbb{R}$ and let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ then
$\text{sols}\left(f\left(x,y(x),y'(x),\ldots y^{(n)}(x)\right)=0\right)=\left\{y\in C^n(\mathcal{U})\mid f\left(x,y(x),\ldots,y^{(n)}(x)\right)=0\right\}$.
First Order Ordinary Differential Equation: let $f:\mathbb{R}^3\to\mathbb{R}$ then $f(x,y,y')=0$.
Separable Ordinary Differential Equation: let $f,g:\mathbb{R}\to\mathbb{R}$ then $y'=f(y)g(x)$.
Claim Separation of Variables: let $f,g:\mathbb{R}\to\mathbb{R}$ such that $f(x)\neq0$ for all $x\in\mathbb{R}$ then
$\text{sols}(y'=f(y)g(x))=\text{sols}\left(\int\frac{1}{f(y)}\mathrm{d}y=\int g(x)\,\mathrm{d}x\right)$.
Corollary: $\text{sols}(y=y')=\{Ce^x\mid C\in\mathbb{R}\}$.
Curves that foliate a domain: Let $\Omega\subseteq\mathbb{R}^2$ and let $I\subseteq\mathbb{R}$ then a set of curves $\{\gamma_\alpha\}\subseteq I\to\mathbb{R}$ such that $\biguplus\gamma_i=\Omega$.
Claim: $\text{sols}(y=y')$ foliate the plane.
Remark: let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ and let $g\in\text{sols}\left(f\left(x,y,\ldots,y^{(n)}\right)\right)$ such that g is piecewise continuous then each continuous branch of g is a solution of the ODE.
Initial Value Problem (IVP): let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ and let $p\in\mathbb{R}^2$ then $\begin{cases}f\left(x,y',\ldots y^{(n)}\right)=0\\ y(p_1)=p_2\end{cases}$.
Theorem Cauchy-Peano: let $I,J\subseteq\mathbb{R}$ be closed intervals let $f\in C^1(I\times J)$ and let $p\in\text{int}(I\times J)$ then there exists $\varepsilon>0$ such that $\begin{cases}y'=f(x,y)\\ y(p_1)=p_2\end{cases}$ has a solution on the interval $(p_1-\varepsilon,p_1+\varepsilon)$.
Theorem of Existence and Uniqueness: let $I,J\subseteq\mathbb{R}$ be closed intervals let $f\in C^1(I\times J)$ and let $p\in\text{int}(I\times J)$ then there exists $\varepsilon>0$ such that $\begin{cases}y'=f(x,y)\\ y(p_1)=p_2\end{cases}$ has a unique solution on the interval $(p_1-\varepsilon,p_1+\varepsilon)$.
Corollary: let $I,J\subseteq\mathbb{R}$ be closed intervals and let $f\in C^1(I\times J)$ then $\text{sols}(y'=f(x,y))$ foliates $I\times J$.
Vector Field: let $\Omega\subseteq\mathbb{R}^2$ be a domain then $\nu\in C\left(\Omega,\mathbb{R}^2\right)$.
Integral curve: let $\nu:\Omega\to\mathbb{R}^2$ be a vector field then a curve $\gamma:[a,b]\to\Omega$ such that $\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t)=\nu(g(t))$.
Claim: let $\nu:\Omega\to\mathbb{R}^2$ be a vector field such that $v(x)\neq0$ for all $x\in\Omega$ and let $\gamma:[a,b]\to\Omega$ be a curve such that $v(\gamma(t))\in T_{\gamma(t)}(\text{Im}(\gamma))$ for all $t\in[a,b]$ then there exists a curve $\eta:[\alpha,\beta]\to[a,b]$ such that $\gamma\circ\eta$ is an integral curve.
C^n **Vector Field:** let $\Omega\subseteq\mathbb{R}^2$ be a domain then $\nu\in C^n\left(\Omega,\mathbb{R}^2\right)$.
Claim: let $\nu:\Omega\to\mathbb{R}^2$ be a C^1 vector field then $\{\gamma:[a,b]\to\Omega\mid(a,b\in\mathbb{R})\wedge(\gamma\text{ is an integral curve of }\nu)\}$ foliates Ω.
Theorem Peano: let $\nu:\Omega\to\mathbb{R}^2$ be a vector field then $\{\gamma:[a,b]\to\Omega\mid(a,b\in\mathbb{R})\wedge(\gamma\text{ is an integral curve of }\nu)\}$ covers Ω.
Lemma: let $\Omega\subseteq\mathbb{R}^2$ be a domain and let $f\in C^1(\Omega)$ then $\left(\begin{smallmatrix}1\\ f\end{smallmatrix}\right)$ is a vector field.
Theorem of Existence and Uniqueness: let $\Omega\subseteq\mathbb{R}^2$ be a domain let $f\in C^1(\Omega)$ and let $g\in C^1(\mathbb{R})$ then
$(g'=f(x,g))\iff\left(\left(\begin{smallmatrix}x\\ g\end{smallmatrix}\right)\text{ is an integral curve of }\left(\begin{smallmatrix}1\\ f\end{smallmatrix}\right)\right)$.
Autonomous Ordinary Differential Equations: let $f:\mathbb{R}\to\mathbb{R}$ then $y'=f(y)$.
Logistic Equation: let $L>0$ then $\dot{P}(t)=P(t)(L-P(t))$.
Equilibrium Solution: let $f:\mathbb{R}\to\mathbb{R}$ then a function $g\in\text{sols}(y'=f(y))$ such that $f(g(x))=0$ for all $x\in\mathbb{R}$.
Corollary: let $f\in C(\mathbb{R})$ and let $g:\mathbb{R}\to\mathbb{R}$ be an equilibrium solution then g is constant.
Corollary: let $L>0$ then $\{0,L\}$ are the equilibrium solutions of the logistic equation.
Logistic Equation With Harvesting: let $L,k>0$ then $\dot{P}(t)=P(t)(L-P(t))-k$.
Stable Equilibrium Solution: let $f:\mathbb{R}\to\mathbb{R}$ then an equilibrium solution $g:\mathbb{R}\to\mathbb{R}$ such that for all $y\in\text{sols}(y'=f(y))$ and for all $\varepsilon>0$ if there exists $\delta>0$ and $p\in\mathbb{R}$ for which $|y(p)-g(p)|<\delta$ then $|y(t)-g(t)|<\varepsilon$ for all $t>p$.
Notation: let X be a set and let $\gamma:\mathbb{R}\to X$ then $\dot{\gamma}=\frac{\mathrm{d}\gamma}{\mathrm{d}t}$.
Notation: let X be a set and let $\gamma:\mathbb{R}\to X$ then $\ddot{\gamma}=\frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2}$.
Remark: the notation $\dot{\gamma},\ddot{\gamma}$ is often used when γ is a function of time.
System of Ordinary Differential Equation: let $n\in\mathbb{N}$ let $m\in\mathbb{N}_+$ and let $f:\mathbb{R}\times(\mathbb{R}^m)^n\to\mathbb{R}^m$ then
$f\left(t,y(t),y'(t),\ldots y^{(n)}(t)\right)=0$.

Remark: all the theory of first order ODEs also applies to vectors of first order ODEs.

Lotka–Volterra equations: let $\alpha>0$ then $\begin{cases}\dot{x}(t)=x(t)\cdot(1-y(t))\\ \dot{y}(t)=\alpha\cdot y(t)\cdot(x(t)-1)\end{cases}$.

Claim: let $\alpha>0$ then $\left\{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\right\}$ are the equilibrium solutions of the lotka-volterra equation.

Conserved Quantity: let $\mathcal{U}\subseteq\mathbb{R}^n$ and let $f:\mathcal{U}\to\mathbb{R}$ then a function $g\in C^1(\mathcal{U})$ such that $g\circ y$ is constant for all $y\in\text{sols}(y'=f(y))$.

Claim: let $\alpha>0$ and let $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\in C^1(\mathbb{R}^2)$ be a solution to the lotka-volterra equation then $\frac{\mathrm{d}}{\mathrm{d}t}\left((y(t)-1-\log(y(t)))+\alpha(x(t)-1-\log(x(t)))\right)=0$.

Corollary: let $\alpha>0$ let $p\in\mathbb{R}^2$ and let $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\in C^1(\mathbb{R}^2)$ be a solution to $\begin{cases}\dot{x}(t)=x(t)\cdot(1-y(t))\\ \dot{y}(t)=\alpha\cdot y(t)\cdot(x(t)-1)\\ \left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)(0)=p\end{cases}$ then $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)$ is periodic.

Lemma: let $\alpha>0$ let $p\in\mathbb{R}^2$ and let $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\in C^1(\mathbb{R}^2)$ be a solution to $\begin{cases}\dot{x}(t)=x(t)\cdot(1-y(t))\\ \dot{y}(t)=\alpha\cdot y(t)\cdot(x(t)-1)\\ \left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)(0)=p\end{cases}$ with period T then $\frac{1}{T}\int_0^T x\mathrm{d}t=1$ and $\frac{1}{T}\int_0^T y\mathrm{d}t=1$.

Claim Linear Substitution: let $a,b,c\in\mathbb{R}$ let $f:\mathbb{R}\to\mathbb{R}$ and let $y\in\text{sols}(y'=f(ax+by+c))$ then the function $z:\mathbb{R}\to\mathbb{R}$ which defined by $z(x)=ax+by+c$ satisfies $z'=a+bf(z)$.

Notation: let $\Omega\subseteq\mathbb{R}^2$ be a domain let $f:\Omega\to\mathbb{R}$ and let $p\in\mathbb{R}^2$ then (α_-,α_+) is the maximal interval where a solution to $\begin{cases}y'=f(x,y)\\ y(p_1)=p_2\end{cases}$ is defined.

Corollary: let $\Omega\subseteq\mathbb{R}^2$ be a domain let $f\in C^1(\Omega)$ and let $p\in\mathbb{R}^2$ then $\alpha_-<p_1<\alpha_+$.

Theorem: let $f\in C^1(\mathbb{R}^2)$ and let $p\in\mathbb{R}^2$ then

- $(\alpha_+<\infty)\iff\left(\lim_{x\to\alpha_+^-}|y(x)|=\infty\right)$.
- $(-\infty<\alpha_-)\iff\left(\lim_{x\to\alpha_-^+}|y(x)|=\infty\right)$.

Theorem Extensibility of Solutions: let $D\subseteq\mathbb{R}^{n+1}$ be a closed set let $f\in C^1(D,\mathbb{R}^n)$ and let $y\in\text{sols}(y'=f(x,y))$ such that $\Gamma_y\cap D\neq\varnothing$ then there exists $\psi:[a,b]\to\mathbb{R}^n$ such that $\psi\in\text{sols}(y'=f(x,y))$ and $\psi(x)=y(x)$ for all $x\in\text{Dom}(y)$ and $\left(\begin{smallmatrix}a\\ \psi(a)\end{smallmatrix}\right),\left(\begin{smallmatrix}b\\ \psi(b)\end{smallmatrix}\right)\in\partial D$.

Corollary: let $D\subseteq\mathbb{R}^{n+1}$ such that for all $\alpha,\beta\in\mathbb{R}$ the set $D\cap\{\alpha\leq x_1\leq\beta\}$ is bounded let $f\in C^1(D,\mathbb{R}^n)$ and let $y\in\text{sols}(y'=f(x,y))$ such that $\Gamma_y\cap D\neq\varnothing$ then one of the next statements is true

- there exists $\psi:[a,b]\to\mathbb{R}^n$ such that $\psi\in\text{sols}(y'=f(x,y))$ and $\psi(x)=y(x)$ for all $x\in\text{Dom}(y)$ and $\left(\begin{smallmatrix}a\\ \psi(a)\end{smallmatrix}\right),\left(\begin{smallmatrix}b\\ \psi(b)\end{smallmatrix}\right)\in\partial D$.
- there exists $\psi:\mathbb{R}\to\mathbb{R}^n$ such that $\psi\in\text{sols}(y'=f(x,y))$ and $\psi(x)=y(x)$ for all $x\in\text{Dom}(y)$.

Corollary: let $I\subseteq\mathbb{R}$ be an open interval let $D\subseteq\mathbb{R}^n$ let $f\in C^1(I\times D,\mathbb{R}^n)$ and let $p,q:\mathbb{R}\to\mathbb{R}$ such that $\|f(x,y)\|\leq p(x)\|y\|+q(x)$ then for all $y\in\text{sols}(y'=f(x,y))$ such that $\Gamma_y\cap(I\times D)\neq\varnothing$ there exists $\psi:I\to\mathbb{R}^n$ such that $\psi\in\text{sols}(y'=f(x,y))$ and $\psi(x)=y(x)$ for all $x\in\text{Dom}(y)$.

Theorem Continuous Dependence on Initial Conditions: let $\Omega\subseteq\mathbb{R}^2$ let $f\in C^1(\Omega)$ let $a,b,p\in\mathbb{R}$ and let y_p be a solution to $\begin{cases}y'=f(x,y)\\ y(a)=p\end{cases}$ such that y_p is defined on $[a,b]$ then for all $\varepsilon>0$ there exists $\delta>0$ such that for all $q\in\mathbb{R}$ for which $|p-q|<\delta$ if y_q is a solution to $\begin{cases}y'=f(x,y)\\ y(a)=q\end{cases}$ then y_q is defined on $[a,b]$ and $|y_p-y_q|<\varepsilon$.

Theorem: let $\Omega\subseteq\mathbb{R}^2$ let $f\in C^1(\Omega)$ let $a,b\in\mathbb{R}$ let $g\in C^1([a,b])$ and let $\{g_n\}_{n=0}^{\infty}\subseteq\text{sols}(y'=f(x,y))$ such that g_n is defined on $[a,b]$ for all $n\in\mathbb{N}$ and $g_n\xrightarrow{p.w.}g$ then $g\in\text{sols}(y'=f(x,y))$.

n-th Order Ordinary Differential Equation: let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ then $f\left(x,y\ldots y^{(n)}\right)=0$.

Claim: let $n\in\mathbb{N}_+$ let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ and let $g:\mathbb{R}\to\mathbb{R}$ then

$$\left(g\in\text{sols}\left(f\left(t,x(t)\ldots x^{(n)}(t)\right)=0\right)\right)\iff\left(\begin{pmatrix}g\\ \vdots\\ g^{(n)}\end{pmatrix}\in\text{sols}\left(\begin{cases}y_1=\dot{x}\\ \vdots\\ y_{n-1}=\dot{y_{n-2}}\\ \dot{y_{n-1}}=f(t,x,y_1\ldots,y_{n-1})\end{cases}\right)\right)$$

Harmonic Oscillator/Spring Position Equation: $\ddot{x}=-x$.

Claim: the function $E(x,y)=\frac{1}{2}x^2+\frac{1}{2}y^2$ is a conserved quantity of $\begin{cases}\dot{x}=y\\ \dot{y}=-x\end{cases}$.

Spring with Friction Position Equation: $\ddot{x}=-x-\dot{x}$.

Constant Tension Spring Position Equation: $\ddot{x}=-1$.

Claim: the function $E(x,y)=x+\frac{1}{2}y^2$ is a conserved quantity of $\begin{cases}\dot{x}=y\\ \dot{y}=-1\end{cases}$.
Gravity in One Dimension Equation: $\ddot{x}=-\frac{1}{x^2}$.
Claim: the function $E(x,y)=\frac{1}{x}+\frac{1}{2}y^2$ is a conserved quantity of $\begin{cases}\dot{x}=y\\ \dot{y}=-\frac{1}{x^2}\end{cases}$.
Pendulum Equation: $\ddot{x}=-\sin(x)$.
Claim: the function $E(x,y)=\frac{1}{2}y^2-\cos(x)$ is a conserved quantity of $\begin{cases}\dot{x}=y\\ \dot{y}=-\sin(x)\end{cases}$.
Differential 1-form: let $P,Q:\mathbb{R}^2\to\mathbb{R}$ then $P(x,y)\,\mathrm{d}x+Q(x,y)\,\mathrm{d}y$.
Remark: in this course we won't explain the formal definition of a differential 1-form so we will use it as an object with certain properties.
Integral: let $\gamma:[a,b]\to\mathbb{R}^2$ be a curve and let $P,Q:\mathbb{R}^2\to\mathbb{R}$ then $\int_\gamma(P(x,y)\,\mathrm{d}x+Q(x,y)\,\mathrm{d}y)=\lim_{\Delta\to0}\sum_{i=0}^{n-1}(P(x_i,y_i)(x_{i+1}-x_i)+Q(x_i,y_i)(y_{i+1}-y_i))$.
Remark: in the definition above $\lim_{\Delta\to0}$ is the limit of all partitions of γ to horizontal and vertical segments.
Claim: let $\gamma:[a,b]\to\mathbb{R}^n$ be a curve let $\nu:[\alpha,\beta]\to[a,b]$ such that $\gamma\circ\nu$ is a reparameterization of γ and let ω be a differential 1-form then $\int_\gamma\omega=\int_{\gamma\circ\nu}\omega$.
Claim: let $\gamma:[a,b]\to\mathbb{R}^n$ be a curve and let ω be a differential 1-form then $\int_\gamma\omega=-\int_{-\gamma}\omega$.
Integral: let $\gamma:[a,b]\to\mathbb{R}^2$ be a curve and let $f,g:\mathbb{R}^2\to\mathbb{R}$ then $\int_\gamma f\mathrm{d}g=\lim_{\Delta\to0}\sum_{i=0}^{n-1}(f(x_i,y_i)(g(x_{i+1},y_{i+1})-g(x_i,y_i)))$.
Claim: let $g\in C^1(\mathbb{R}^2)$ then $\mathrm{d}g=\frac{\partial g}{\partial x}\mathrm{d}x+\frac{\partial g}{\partial y}\mathrm{d}y$.
Corollary: let $\gamma:[a,b]\to\mathbb{R}^2$ be a curve let $f:\mathbb{R}^2\to\mathbb{R}$ and let $g\in C^1(\mathbb{R}^2)$ then $\int_\gamma f\mathrm{d}g=\int_\gamma\left(f\cdot\frac{\partial g}{\partial x}\mathrm{d}x+f\cdot\frac{\partial g}{\partial y}\mathrm{d}y\right)$.
Exact Differential 1-form: a differential 1-form ω such that there exists $g\in C^1(\mathbb{R}^2)$ for which $\omega=\mathrm{d}g$.
Theorem: let ω be a differential 1-form then TFAE

- ω is exact.
- there exists $f:\mathbb{R}\to\mathbb{R}$ such that for all curves $\gamma:[a,b]\to\mathbb{R}^2$ we have $\int_\gamma\omega=f(\gamma(b))-f(\gamma(a))$.
- for all closed curves $\gamma:[a,b]\to\mathbb{R}^2$ we have $\int_\gamma\omega=0$.

Primitive/Potential: let ω be an exact differential 1-form then $g\in C^1(\mathbb{R}^2)$ such that $\omega=\mathrm{d}g$.
Claim: let $X:\mathbb{R}\to\mathbb{R}^3$ and let $f:\mathbb{R}^3\to\mathbb{R}$ then $\frac{\mathrm{d}}{\mathrm{d}t}(f\circ X)=(\nabla f)\cdot\dot{X}$.
Conservative Vector Field: a vector field $F:\mathbb{R}^n\to\mathbb{R}^n$ such that there exists $U\in C^1(\mathbb{R}^n)$ for which $F=-\nabla U$.
Kinetic Energy: let $X\in C^1(\mathbb{R},\mathbb{R}^n)$ then $K:\mathbb{R}^n\to\mathbb{R}$ such that $K(X(t))=\frac{\|\dot{X}(t)\|^2}{2}$.
Total Energy: let $U\in C^1(\mathbb{R}^n)$ let $F:\mathbb{R}^n\to\mathbb{R}^n$ be a conservative vector field such that $F=-\nabla U$ let $X\in\mathrm{sols}\left(\ddot{X}=F(X)\right)$ and let $K:\mathbb{R}^n\to\mathbb{R}$ be the kinetic energy of X then $E:\mathbb{R}^n\to\mathbb{R}$ such that $E=K+U$.
Lemma: let $U\in C^1(\mathbb{R}^n)$ let $F:\mathbb{R}^n\to\mathbb{R}^n$ be a conservative vector field such that $F=-\nabla U$ let $X\in\mathrm{sols}\left(\ddot{X}=F(X)\right)$ and let $E:\mathbb{R}^n\to\mathbb{R}$ be the total energy of X then $\dot{E}(X(t))=0$.
Weighted Average/Center of Mass: let $p_1\dots p_n\in\mathbb{R}^n$ and let $w:\{p_1\dots p_n\}\to\mathbb{R}_+$ then $\frac{\sum_{i=1}^n w(p_i)\cdot p_i}{\sum_{i=1}^n w(p_i)}$.
Center of Mass: let $K\subseteq\mathbb{R}^2$ be compact and let $\rho:K\to\mathbb{R}$ then $\left(\frac{\int_K x\cdot\rho(x,y)\mathrm{d}x\mathrm{d}y}{\int_K\rho(x,y)\mathrm{d}x\mathrm{d}y},\frac{\int_K y\cdot\rho(x,y)\mathrm{d}x\mathrm{d}y}{\int_K\rho(x,y)\mathrm{d}x\mathrm{d}y}\right)$.
Line: let $A,B\in\mathbb{R}^2$ then $L_{A,B}=\{\lambda A+(1-\lambda)B\mid\lambda\in[0,1]\}$.
Triangle: let $A,B,C\in\mathbb{R}^2$ such that $A\notin L_{B,C}$ and $B\notin L_{A,C}$ and $C\notin L_{A,B}$ then $\{A,B,C\}$.
Theorem Ceva: let $A,B,C\in\mathbb{R}^2$ such that $\{A,B,C\}$ is a triangle let $A'\in L_{B,C}$ let $B'\in L_{A,C}$ and let $C'\in L_{A,C}$ then $(L_{A,A'}\cap L_{B,B'}\cap L_{C,C'}\neq\varnothing)\iff\left(\frac{d(A,B')}{d(B',C)}\cdot\frac{d(C,A')}{d(A',B)}\cdot\frac{d(B,C')}{d(C',A)}=1\right)$.
Special Orthogonal Group: let $n\in\mathbb{N}$ then $\mathrm{SO}(n)=\left\{A\in M_n(\mathbb{R})\mid(\det(A)=1)\wedge\left(A^T=A^{-1}\right)\right\}$.
Cross Product: let $x,y\in\mathbb{R}^3$ then $x\times y=\begin{pmatrix}x_2y_3-x_3y_2\\x_3y_1-x_1y_3\\x_1y_2-x_2y_1\end{pmatrix}$.
Claim Antisymmetry: let $x,y\in\mathbb{R}^3$ then $x\times y=-y\times x$.
Claim Orthogonality: let $x,y\in\mathbb{R}^3$ be linearly independent then $x\times y\perp x$ and $x\times y\perp y$.
Claim: let $x,y\in\mathbb{R}^3$ and let θ be the angle between x,y then $\|x\times y\|=\|x\|\cdot\|y\|\cdot\cos(\theta)$.
Corollary: let $x,y\in\mathbb{R}^3$ then $(x\times y=0)\iff(x,y$ are linearly dependent$)$.
Claim: let $x,y,z\in\mathbb{R}^2$ and let $\alpha,\beta\in\mathbb{R}$ then $(\alpha x+\beta y)\times z=\alpha(x\times z)+\beta(y\times z)$.
Claim: let $X,Y\in C^1(\mathbb{R}^3)$ then $\frac{\mathrm{d}}{\mathrm{d}t}(X\times Y)=\dot{X}\times Y+X\times\dot{Y}$.

Angular Momentum: let $X \in C^1\left(\mathbb{R}^3\right)$ then $X \times \dot{X}$.

Lemma: angular momentum is a conserved quantity of $\ddot{X} = -\frac{X}{\|X\|^2}$.

Exact Ordinary Differential Equation: let $\Omega \subseteq \mathbb{R}^2$ and let $F \in C^1(\Omega)$ then $\frac{\partial F}{\partial x}(x,y) + y' \cdot \frac{\partial F}{\partial y}(x,y) = 0$.

Remark: let $\Omega \subseteq \mathbb{R}^2$ and let $F \in C^1(\Omega)$ then $\frac{\partial F}{\partial x} + y' \cdot \frac{\partial F}{\partial y} = \mathrm{d}F$.

Claim: let $\Omega \subseteq \mathbb{R}^2$ let $F \in C^1(\Omega)$ and let $y \in C^1(\mathbb{R})$ then $\frac{\mathrm{d}}{\mathrm{d}t}(F(x, y(x))) = 0$.

Remark: let $\Omega \subseteq \mathbb{R}^2$ let $F \in C^1(\Omega)$ and let $c \in \mathbb{R}$ then $\{F = c\}$ are solutions to $\frac{\partial F}{\partial x}(x,y) + y' \cdot \frac{\partial F}{\partial y}(x,y) = 0$.

Claim: let $\Omega \subseteq \mathbb{R}^2$ be a simply connected domain and let $P, Q \in C^1(\Omega)$ then ($P(x,y) + y' \cdot Q(x,y) = 0$ is an exact ODE)$\Longleftrightarrow$($\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$).

Fibonacci Sequence: a sequence $F : \mathbb{N} \to \mathbb{N}$ such that $F_0 = 0$ and $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \in \mathbb{N}$.

Lemma: let $n, m \in \mathbb{N}$ then $F_{m+n+1} = F_m F_n + F_{m+1} F_{n+1}$.

Lemma Cassini Identity: let $n \in \mathbb{N}_+$ then $F_n^2 = F_{n+1} F_{n-1} + (-1)^{n-1}$.

Left Shift Operator: a function $\mathrm{LS} : (\mathbb{N} \to \mathbb{C}) \to (\mathbb{N} \to \mathbb{C})$ such that $\mathrm{LS}(x) = \lambda n \in \mathbb{N}.x_{n+1}$.

Remark; let $x \in \mathbb{N} \to \mathbb{C}$ then $(\forall n \in \mathbb{N}.x_{n+2} = x_{n+1} + x_n) \Longleftrightarrow \left(x \in \ker\left(\mathrm{LS}^2 - \mathrm{LS} - 1\right)\right)$.

Lemma: let $p, q \in \mathbb{C}[x]$ then $(p \cdot q)(\mathrm{LS}) = p(\mathrm{LS}) \cdot q(\mathrm{LS})$.

Corollary: $\mathrm{LS}^2 - \mathrm{LS} - 1 = \left(\mathrm{LS} - \frac{-1+\sqrt{5}}{2}\right)\left(\mathrm{LS} - \frac{-1-\sqrt{5}}{2}\right)$.

Homogeneous Linear Ordinary Differential Equation: let $n \in \mathbb{N}$ and let $a_0 \ldots a_{n-1}` : \mathbb{R} \to \mathbb{R}$ then $y^{(n)} + \sum_{i=0}^{n-1} a_i(x) \cdot y^{(i)} = 0$.

Remark: In this course we will discuss only homogeneous linear ODEs where the coefficients are constant.

Characteristic Polynomial of a Homogeneous Linear Ordinary Differential Equation: let $n \in \mathbb{N}$ and let $a_0 \ldots a_{n-1}` \in \mathbb{R}$ then $p(x) = x^n + \sum_{i=0}^{n-1} a_i \cdot x^i$.

Claim: let $n \in \mathbb{N}$ let $a_0 \ldots a_{n-1}` \in \mathbb{R}$ and let $f, g \in \mathrm{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i \cdot y^{(i)} = 0\right)$ then $f + g \in \mathrm{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i \cdot y^{(i)} = 0\right)$.

Theorem: let $n \in \mathbb{N}$ let $a_0 \ldots a_{n-1}` \in \mathbb{R}$ and let $\alpha \in \mathbb{R}$ be a solution to $p(x)$ with multiplicity ρ then $\left\{x^0 e^{\alpha x}, \ldots, x^{\rho-1} e^{\alpha x}\right\} \subseteq \mathrm{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i \cdot y^{(i)} = 0\right)$.

Claim: let $n \in \mathbb{N}$ let $a_0 \ldots a_{n-1}` \in \mathbb{R}$ and let $f \in \mathrm{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i \cdot y^{(i)} = 0\right)$ then $\mathrm{Re}(f), \mathrm{Im}(f) \in \mathrm{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i \cdot y^{(i)} = 0\right)$.

Linear System of Ordinary Differential Equation: let $n \in \mathbb{N}$ and let $A \in M_n(\mathbb{R})$ then $y'(x) = A \cdot y(x)$.

Definition: let $A \in M_n(\mathbb{R})$ then $\max(|A|) = \max\left\{\left|(A)_{i,j}\right| \mid i, j \in [n]\right\}$.

Lemma: let $n, k \in \mathbb{N}$ let $A \in M_n(\mathbb{R})$ and let $i, j \in [n]$ then $(A)_{i,j}^k \le n^{k-1} \cdot \max(|A|)^k$.

Corollary: let $n, k \in \mathbb{N}$ let $A \in M_n(\mathbb{R})$ and let $i, j \in [n]$ then $(A)_{i,j}^k \le (n \cdot \max(|A|))^k$.

Theorem: let $n \in \mathbb{N}$ and let $A \in M_n(\mathbb{R})$ then $\sum_{k=0}^{\infty} \frac{A^k}{k!} \in M_n(\mathbb{R})$.

Definition: let $n \in \mathbb{N}$ and let $A \in M_n(\mathbb{R})$ then $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$.

Lemma: let $A \in M_n(\mathbb{R})$ and let $s, t \in \mathbb{R}$ then $e^{sA} \cdot e^{tA} = e^{(s+t)A}$.

Matrix Differentiation: let $A : \mathbb{R} \to M_n(\mathbb{R})$ such that $(A)_{i,j} \in C^1(\mathbb{R})$ for all $i, j \in [n]$ then $(A'(t))_{i,j} = (A(t))'_{i,j}$.

Claim: let $A, B \in C^1(\mathbb{R}, M_n(\mathbb{R}))$ then $\frac{\mathrm{d}}{\mathrm{d}t}(AB) = \dot{A}B + A\dot{B}$.

Corollary: let $A \in C^1(\mathbb{R}, M_n(\mathbb{R}))$ and let $v \in C^1(\mathbb{R}, \mathbb{R}^n)$ then $\frac{\mathrm{d}}{\mathrm{d}t}(Av) = \dot{A}v + A\dot{v}$.

Lemma: let $A \in M_n(\mathbb{R})$ then $\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA}\right)\right)(0) = A$.

Corollary: let $A \in M_n(\mathbb{R})$ and let $\mu \in \mathbb{R}$ then $\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA}\right)\right)(\mu) = e^{\mu A} A$.

Theorem: let $n \in \mathbb{N}$ and let $A \in M_n(\mathbb{R})$ then $e^{tA} v$ is a solution to $\begin{cases} y' = A \cdot y \\ y(0) = v \end{cases}$.

Zeckendorff Representation: let $n \in \mathbb{N}$ let $k \in \mathbb{N}$ and let $c_0 \ldots c_k \in \mathbb{N}$ such that $c_0 \ge 2$ and $c_{i+1} > c_i + 1$ for all $i \in \{0 \ldots k-1\}$ and $n = \sum_{i=0}^{k} F_{c_i}$ then $(F_{c_0}, \ldots, F_{c_k})$.

Theorem Zeckendorff: let $n \in \mathbb{N}$ then

- Existence: there exists a zeckendorff representation for n.
- Uniqueness: let $(F_{c_0}, \ldots, F_{c_k}), (F_{d_0}, \ldots, F_{d_m})$ be zeckendorff representations for n then $k = m$ and $c_i = d_i$ for all $i \in \{0 \ldots k-1\}$.