Ordinary Differential Equation (ODE): let $n \in \mathbb{N}$ and let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ then $f\left(x, y(x), y'(x), \ldots y^{(n)}(x)\right) = 0$.

Solution of an Ordinary Differential Equation: let $n \in \mathbb{N}$ let $\mathcal{U} \subseteq \mathbb{R}$ and let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ then a function $y \in C^n(\mathcal{U})$ such that $f\left(x, y(x), \ldots, y^{(n)}(x)\right) = 0$ for all $x \in \mathcal{U}$.

Notation: let $n \in \mathbb{N}$ let $\mathcal{U} \subseteq \mathbb{R}$ and let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ then
$\text{sols}\left(f\left(x, y(x), y'(x), \ldots y^{(n)}(x)\right) = 0\right) = \left\{y \in C^n(\mathcal{U}) \mid f\left(x, y(x), \ldots, y^{(n)}(x)\right) = 0\right\}$.

First Order Ordinary Differential Equation: let $f : \mathbb{R}^3 \to \mathbb{R}$ then $f(x, y, y') = 0$.

Separable Ordinary Differential Equation: let $f, g : \mathbb{R} \to \mathbb{R}$ then $y' = f(y)\, g(x)$.

Claim Separation of Variables: let $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) \neq 0$ for all $x \in \mathbb{R}$ then
$\text{sols}(y' = f(y)\, g(x)) = \text{sols}\left(\int \frac{1}{f(y)} \mathrm{d}y = \int g(x)\, \mathrm{d}x\right)$.

Corollary: $\text{sols}(y = y') = \{Ce^x \mid C \in \mathbb{R}\}$.

Curves that foliate a domain: Let $\Omega \subseteq \mathbb{R}^2$ and let $I \subseteq \mathbb{R}$ then a set of curves $\{\gamma_\alpha\} \subseteq I \to \mathbb{R}$ such that $\biguplus \gamma_i = \Omega$.

Claim: $\text{sols}(y = y')$ foliate the plane.

Remark: let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and let $g \in \text{sols}\left(f\left(x, y, \ldots, y^{(n)}\right)\right)$ such that g is piecewise continuous then each continuous branch of g is a solution of the ODE.

Initial Value Problem (IVP): let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and let $p \in \mathbb{R}^2$ then $\begin{cases} f\left(x, y', \ldots y^{(n)}\right) = 0 \\ y(p_1) = p_2 \end{cases}$.

Theorem Cauchy-Peano: let $I, J \subseteq \mathbb{R}$ be closed intervals let $f \in C^1(I \times J)$ and let $p \in \text{int}(I \times J)$ then there exists $\varepsilon > 0$ such that $\begin{cases} y' = f(x, y) \\ y(p_1) = p_2 \end{cases}$ has a solution on the interval $(p_1 - \varepsilon, p_1 + \varepsilon)$.

Theorem of Existence and Uniqueness: let $I, J \subseteq \mathbb{R}$ be closed intervals let $f \in C^1(I \times J)$ and let $p \in \text{int}(I \times J)$ then there exists $\varepsilon > 0$ such that $\begin{cases} y' = f(x, y) \\ y(p_1) = p_2 \end{cases}$ has a unique solution on the interval $(p_1 - \varepsilon, p_1 + \varepsilon)$.

Corollary: let $I, J \subseteq \mathbb{R}$ be closed intervals and let $f \in C^1(I \times J)$ then $\text{sols}(y' = f(x, y))$ foliates $I \times J$.

Vector Field: let $\Omega \subseteq \mathbb{R}^2$ be a domain then $\nu \in C\left(\Omega, \mathbb{R}^2\right)$.

Integral curve: let $\nu : \Omega \to \mathbb{R}^2$ be a vector field then a curve $\gamma : [a, b] \to \Omega$ such that $\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = \nu(g(t))$.

Claim: let $\nu : \Omega \to \mathbb{R}^2$ be a vector field such that $v(x) \neq 0$ for all $x \in \Omega$ and let $\gamma : [a, b] \to \Omega$ be a curve such that $v(\gamma(t)) \in T_{\gamma(t)}(\text{Im}(\gamma))$ for all $t \in [a, b]$ then there exists a curve $\eta : [\alpha, \beta] \to [a, b]$ such that $\gamma \circ \eta$ is an integral curve.

C^n **Vector Field:** let $\Omega \subseteq \mathbb{R}^2$ be a domain then $\nu \in C^n\left(\Omega, \mathbb{R}^2\right)$.

Claim: let $\nu : \Omega \to \mathbb{R}^2$ be a C^1 vector field then $\{\gamma : [a, b] \to \Omega \mid (a, b \in \mathbb{R}) \wedge (\gamma \text{ is an integral curve of } \nu)\}$ foliates Ω.

Theorem Peano: let $\nu : \Omega \to \mathbb{R}^2$ be a vector field then $\{\gamma : [a, b] \to \Omega \mid (a, b \in \mathbb{R}) \wedge (\gamma \text{ is an integral curve of } \nu)\}$ covers Ω.

Lemma: let $\Omega \subseteq \mathbb{R}^2$ be a domain and let $f \in C^1(\Omega)$ then $\left(\begin{smallmatrix} 1 \\ f \end{smallmatrix}\right)$ is a vector field.

Theorem of Existence and Uniqueness: let $\Omega \subseteq \mathbb{R}^2$ be a domain let $f \in C^1(\Omega)$ and let $g \in C^1(\mathbb{R})$ then
$(g' = f(x, g)) \iff \left(\left(\begin{smallmatrix} x \\ g \end{smallmatrix}\right) \text{ is an integral curve of } \left(\begin{smallmatrix} 1 \\ f \end{smallmatrix}\right)\right)$.

Autonomous Ordinary Differential Equations: let $f : \mathbb{R} \to \mathbb{R}$ then $y' = f(y)$.

Logistic Equation: let $L > 0$ then $\dot{P}(t) = P(t)(L - P(t))$.

Equilibrium Solution: let $f : \mathbb{R} \to \mathbb{R}$ then a function $g \in \text{sols}(y' = f(y))$ such that $f(g(x)) = 0$ for all $x \in \mathbb{R}$.

Corollary: let $f \in C(\mathbb{R})$ and let $g : \mathbb{R} \to \mathbb{R}$ be an equilibrium solution then g is constant.

Corollary: let $L > 0$ then $\{0, L\}$ are the equilibrium solutions of the logistic equation.

Logistic Equation With Harvesting: let $L, k > 0$ then $\dot{P}(t) = P(t)(L - P(t)) - k$.

Stable Equilibrium Solution: let $f : \mathbb{R} \to \mathbb{R}$ then an equilibrium solution $g : \mathbb{R} \to \mathbb{R}$ such that for all $y \in \text{sols}(y' = f(y))$ and for all $\varepsilon > 0$ if there exists $\delta > 0$ and $p \in \mathbb{R}$ for which $|y(p) - g(p)| < \delta$ then $|y(t) - g(t)| < \varepsilon$ for all $t > p$.

Notation: let X be a set and let $\gamma : \mathbb{R} \to X$ then $\dot{\gamma} = \frac{\mathrm{d}\gamma}{\mathrm{d}t}$.

Notation: let X be a set and let $\gamma : \mathbb{R} \to X$ then $\ddot{\gamma} = \frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2}$.

Remark: the notation $\dot{\gamma}, \ddot{\gamma}$ is often used when γ is a function of time.

System of Ordinary Differential Equation (ODE): let $n \in \mathbb{N}$ let $m \in \mathbb{N}_+$ and let $f : \mathbb{R} \times (\mathbb{R}^m)^n \to \mathbb{R}^m$ then
$f\left(t, y(t), y'(t), \ldots y^{(n)}(t)\right) = 0$.

Remark: all the theory of first order ODEs also applies to vectors of first order ODEs.

Lotka–Volterra equations: let $\alpha > 0$ then $\begin{cases} \dot{x}(t) = x(t) \cdot (1 - y(t)) \\ \dot{y}(t) = \alpha \cdot y(t) \cdot (x(t) - 1) \end{cases}$.

Claim: let $\alpha>0$ then $\left\{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\right\}$ are the equilibrium solutions of the lotka-volterra equation.

Conserved Quantity: let $\mathcal{U}\subseteq\mathbb{R}^{n}$ and let $f:\mathcal{U}\to\mathbb{R}$ then a function $g\in C^{1}\left(\mathcal{U}\right)$ such that $g\circ y$ is constant for all $y\in\text{sols}\left(y'=f\left(y\right)\right)$.

Claim: let $\alpha>0$ and let $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\in C^{1}\left(\mathbb{R}^{2}\right)$ be a solution to the lotka-volterra equation then $\frac{\text{d}}{\text{d}t}\left(\left(y\left(t\right)-1-\log\left(y\left(t\right)\right)\right)+\alpha\left(x\left(t\right)-1-\log\left(x\left(t\right)\right)\right)\right)=0$.

Corollary: let $\alpha>0$ let $p\in\mathbb{R}^{2}$ and let $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\in C^{1}\left(\mathbb{R}^{2}\right)$ be a solution to $\begin{cases}\dot{x}(t)=x(t)\cdot(1-y(t))\\\dot{y}(t)=\alpha\cdot y(t)\cdot(x(t)-1)\\\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)(0)=p\end{cases}$ then $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)$ is periodic.

Lemma: let $\alpha>0$ let $p\in\mathbb{R}^{2}$ and let $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\in C^{1}\left(\mathbb{R}^{2}\right)$ be a solution to $\begin{cases}\dot{x}(t)=x(t)\cdot(1-y(t))\\\dot{y}(t)=\alpha\cdot y(t)\cdot(x(t)-1)\\\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)(0)=p\end{cases}$ with period T then $\frac{1}{T}\int_{0}^{T}x\text{d}t=1$ and $\frac{1}{T}\int_{0}^{T}y\text{d}t=1$.

Claim Linear Substitution: let $a,b,c\in\mathbb{R}$ let $f:\mathbb{R}\to\mathbb{R}$ and let $y\in\text{sols}\left(y'=f\left(ax+by+c\right)\right)$ then the function $z:\mathbb{R}\to\mathbb{R}$ which defined by $z\left(x\right)=ax+by+c$ satisfies $z'=a+bf\left(z\right)$.

Notation: let $\Omega\subseteq\mathbb{R}^{2}$ be a domain let $f:\Omega\to\mathbb{R}$ and let $p\in\mathbb{R}^{2}$ then $\left(\alpha_{-},\alpha_{+}\right)$ is the maximal interval where a solution to $\begin{cases}y'=f(x,y)\\y(p_{1})=p_{2}\end{cases}$ is defined.

Corollary: let $\Omega\subseteq\mathbb{R}^{2}$ be a domain let $f\in C^{1}\left(\Omega\right)$ and let $p\in\mathbb{R}^{2}$ then $\alpha_{-}<p_{1}<\alpha_{+}$.

Theorem: let $f\in C^{1}\left(\mathbb{R}^{2}\right)$ and let $p\in\mathbb{R}^{2}$ then

- $\left(\alpha_{+}<\infty\right)\iff\left(\lim_{x\to\alpha_{+}^{-}}\left|y\left(x\right)\right|=\infty\right)$.
- $\left(-\infty<\alpha_{-}\right)\iff\left(\lim_{x\to\alpha_{-}^{+}}\left|y\left(x\right)\right|=\infty\right)$.

Theorem Extensibility of Solutions: let $D\subseteq\mathbb{R}^{n+1}$ be a closed set let $f\in C^{1}\left(D,\mathbb{R}^{n}\right)$ and let $y\in\text{sols}\left(y'=f\left(x,y\right)\right)$ such that $\Gamma_{y}\cap D\neq\varnothing$ then there exists $\psi:\left[a,b\right]\to\mathbb{R}^{n}$ such that $\psi\in\text{sols}\left(y'=f\left(x,y\right)\right)$ and $\psi\left(x\right)=y\left(x\right)$ for all $x\in\text{Dom}\left(y\right)$ and $\left(\begin{smallmatrix}a\\\psi(a)\end{smallmatrix}\right),\left(\begin{smallmatrix}b\\\psi(b)\end{smallmatrix}\right)\in\partial D$.

Corollary: let $D\subseteq\mathbb{R}^{n+1}$ such that for all $\alpha,\beta\in\mathbb{R}$ the set $D\cap\left\{\alpha\le x_{1}\le\beta\right\}$ is bounded let $f\in C^{1}\left(D,\mathbb{R}^{n}\right)$ and let $y\in\text{sols}\left(y'=f\left(x,y\right)\right)$ such that $\Gamma_{y}\cap D\neq\varnothing$ then one of the next statements is true

- there exists $\psi:\left[a,b\right]\to\mathbb{R}^{n}$ such that $\psi\in\text{sols}\left(y'=f\left(x,y\right)\right)$ and $\psi\left(x\right)=y\left(x\right)$ for all $x\in\text{Dom}\left(y\right)$ and $\left(\begin{smallmatrix}a\\\psi(a)\end{smallmatrix}\right),\left(\begin{smallmatrix}b\\\psi(b)\end{smallmatrix}\right)\in\partial D$.
- there exists $\psi:\mathbb{R}\to\mathbb{R}^{n}$ such that $\psi\in\text{sols}\left(y'=f\left(x,y\right)\right)$ and $\psi\left(x\right)=y\left(x\right)$ for all $x\in\text{Dom}\left(y\right)$.

Corollary: let $I\subseteq\mathbb{R}$ be an open interval let $D\subseteq\mathbb{R}^{n}$ let $f\in C^{1}\left(I\times D,\mathbb{R}^{n}\right)$ and let $p,q:\mathbb{R}\to\mathbb{R}$ such that $\left\Vert f\left(x,y\right)\right\Vert\le p\left(x\right)\left\Vert y\right\Vert+q\left(x\right)$ then for all $y\in\text{sols}\left(y'=f\left(x,y\right)\right)$ such that $\Gamma_{y}\cap\left(I\times D\right)\neq\varnothing$ there exists $\psi:I\to\mathbb{R}^{n}$ such that $\psi\in\text{sols}\left(y'=f\left(x,y\right)\right)$ and $\psi\left(x\right)=y\left(x\right)$ for all $x\in\text{Dom}\left(y\right)$.

Theorem Continuous Dependence on Initial Conditions: let $\Omega\subseteq\mathbb{R}^{2}$ let $f\in C^{1}\left(\Omega\right)$ let $a,b,p\in\mathbb{R}$ and let y_{p} be a solution to $\begin{cases}y'=f(x,y)\\y(a)=p\end{cases}$ such that y_{p} is defined on $\left[a,b\right]$ then for all $\varepsilon>0$ there exists $\delta>0$ such that for all $q\in\mathbb{R}$ for which $\left|p-q\right|<\delta$ if y_{q} is a solution to $\begin{cases}y'=f(x,y)\\y(a)=q\end{cases}$ then y_{q} is defined on $\left[a,b\right]$ and $\left|y_{p}-y_{q}\right|<\varepsilon$.

Theorem: let $\Omega\subseteq\mathbb{R}^{2}$ let $f\in C^{1}\left(\Omega\right)$ let $a,b\in\mathbb{R}$ let $g\in C^{1}\left(\left[a,b\right]\right)$ and let $\left\{g_{n}\right\}_{n=0}^{\infty}\subseteq\text{sols}\left(y'=f\left(x,y\right)\right)$ such that g_{n} is defined on $\left[a,b\right]$ for all $n\in\mathbb{N}$ and $g_{n}\xrightarrow{p.w.}g$ then $g\in\text{sols}\left(y'=f\left(x,y\right)\right)$.

n-th Order Ordinary Differential Equation: let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ then $f\left(x,y\ldots y^{(n)}\right)=0$.

Claim: let $n\in\mathbb{N}_{+}$ let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ and let $g:\mathbb{R}\to\mathbb{R}$ then

$$\left(g\in\text{sols}\left(f\left(t,x\left(t\right)\ldots x^{(n)}\left(t\right)\right)=0\right)\right)\iff\left(\left(\begin{smallmatrix}g\\\vdots\\g^{(n)}\end{smallmatrix}\right)\in\text{sols}\left(\begin{cases}y_{1}=\dot{x}\\\vdots\\y_{n-1}=\dot{y_{n-2}}\\\dot{y_{n-1}}=f(t,x,y_{1}\ldots,y_{n-1})\end{cases}\right)\right)$$

Harmonic Oscillator/Spring Position Equation: $\ddot{x}=-x$.

Claim: the function $E\left(x,y\right)=\frac{1}{2}x^{2}+\frac{1}{2}y^{2}$ is a conserved quantity of $\begin{cases}\dot{x}=y\\\dot{y}=-x\end{cases}$.

Spring with Friction Position Equation: $\ddot{x}=-x-\dot{x}$.

Constant Tension Spring Position Equation: $\ddot{x}=-1$.

Claim: the function $E\left(x,y\right)=x+\frac{1}{2}y^{2}$ is a conserved quantity of $\begin{cases}\dot{x}=y\\\dot{y}=-1\end{cases}$.

Gravity in One Dimension Equation: $\ddot{x}=-\frac{1}{x^{2}}$.

Claim: the function $E(x,y)=\frac{1}{x}+\frac{1}{2}y^2$ is a conserved quantity of $\begin{cases}\dot{x}=y\\ \dot{y}=-\frac{1}{x^2}\end{cases}$.

Pendulum Equation: $\ddot{x}=-\sin(x)$.

Claim: the function $E(x,y)=\frac{1}{2}y^2-\cos(x)$ is a conserved quantity of $\begin{cases}\dot{x}=y\\ \dot{y}=-\sin(x)\end{cases}$.

Differential 1-form: let $P,Q:\mathbb{R}^2\to\mathbb{R}$ then $P(x,y)\,\mathrm{d}x+Q(x,y)\,\mathrm{d}y$.

Remark: in this course we won't explain the formal definition of a differential 1-form so we will use it as an object with certain properties.

Integral: let $\gamma:[a,b]\to\mathbb{R}^2$ be a curve and let $P,Q:\mathbb{R}^2\to\mathbb{R}$ then
$\int_\gamma(P(x,y)\,\mathrm{d}x+Q(x,y)\,\mathrm{d}y)=\lim_{\Delta\to0}\sum_{i=0}^{n-1}(P(x_i,y_i)(x_{i+1}-x_i)+Q(x_i,y_i)(y_{i+1}-y_i))$.

Remark: in the definition above $\lim_{\Delta\to0}$ is the limit of all partitions of γ to horizontal and vertical segments.

Claim: let $\gamma:[a,b]\to\mathbb{R}^n$ be a curve let $\nu:[\alpha,\beta]\to[a,b]$ such that $\gamma\circ\nu$ is a reparameterization of γ and let ω be a differential 1-form then $\int_\gamma\omega=\int_{\gamma\circ\nu}\omega$.

Claim: let $\gamma:[a,b]\to\mathbb{R}^n$ be a curve and let ω be a differential 1-form then $\int_\gamma\omega=-\int_{-\gamma}\omega$.

Integral: let $\gamma:[a,b]\to\mathbb{R}^2$ be a curve and let $f,g:\mathbb{R}^2\to\mathbb{R}$ then $\int_\gamma f\mathrm{d}g=\lim_{\Delta\to0}\sum_{i=0}^{n-1}(f(x_i,y_i)(g(x_{i+1},y_{i+1})-g(x_i,y_i)))$.

Claim: let $g\in C^1(\mathbb{R}^2)$ then $\mathrm{d}g=\frac{\partial g}{\partial x}\mathrm{d}x+\frac{\partial g}{\partial y}\mathrm{d}y$.

Corollary: let $\gamma:[a,b]\to\mathbb{R}^2$ be a curve let $f:\mathbb{R}^2\to\mathbb{R}$ and let $g\in C^1(\mathbb{R}^2)$ then $\int_\gamma f\mathrm{d}g=\int_\gamma\left(f\cdot\frac{\partial g}{\partial x}\mathrm{d}x+f\cdot\frac{\partial g}{\partial y}\mathrm{d}y\right)$.

Exact Differential 1-form: a differential 1-form ω such that there exists $g\in C^1(\mathbb{R}^2)$ for which $\omega=\mathrm{d}g$.

Theorem: let ω be a differential 1-form then TFAE

- ω is exact.
- there exists $f:\mathbb{R}\to\mathbb{R}$ such that for all curves $\gamma:[a,b]\to\mathbb{R}^2$ we have $\int_\gamma\omega=f(\gamma(b))-f(\gamma(a))$.
- for all closed curves $\gamma:[a,b]\to\mathbb{R}^2$ we have $\int_\gamma\omega=0$.

Primitive/Potential: let ω be an exact differential 1-form then $g\in C^1(\mathbb{R}^2)$ such that $\omega=\mathrm{d}g$.

Claim: let $X:\mathbb{R}\to\mathbb{R}^3$ and let $f:\mathbb{R}^3\to\mathbb{R}$ then $\frac{\mathrm{d}}{\mathrm{d}t}(f\circ X)=(\nabla f)\cdot\dot{X}$.