Ordinary Differential Equation (ODE): let $n\in\mathbb{N}$ and let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ then $f\left(x,y\left(x\right),y'\left(x\right),\dots y^{(n)}\left(x\right)\right)=0$.

Solution of an Ordinary Differential Equation: let $n\in\mathbb{N}$ let $\mathcal{U}\subseteq\mathbb{R}$ and let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ then a function $y\in C^{n}\left(\mathcal{U}\right)$ such that $f\left(x,y\left(x\right),\dots,y^{(n)}\left(x\right)\right)=0$ for all $x\in\mathcal{U}$.

Notation: let $n\in\mathbb{N}$ let $\mathcal{U}\subseteq\mathbb{R}$ and let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ then
$\text{sols}\left(f\left(x,y\left(x\right),y'\left(x\right),\dots y^{(n)}\left(x\right)\right)=0\right)=\left\{y\in C^{n}\left(\mathcal{U}\right)\mid f\left(x,y\left(x\right),\dots,y^{(n)}\left(x\right)\right)=0\right\}$.

First Order Ordinary Differential Equation: let $f:\mathbb{R}^{3}\to\mathbb{R}$ then $f\left(x,y,y'\right)=0$.

Separable Ordinary Differential Equation: let $f,g:\mathbb{R}\to\mathbb{R}$ then $y'=f\left(y\right)g\left(x\right)$.

Claim Separation of Variables: let $f,g:\mathbb{R}\to\mathbb{R}$ such that $f\left(x\right)\neq0$ for all $x\in\mathbb{R}$ then
$\text{sols}\left(y'=f\left(y\right)g\left(x\right)\right)=\text{sols}\left(\int\frac{1}{f\left(y\right)}\text{d}y=\int g\left(x\right)\text{d}x\right)$.

Corollary: $\text{sols}\left(y=y'\right)=\left\{Ce^{x}\mid C\in\mathbb{R}\right\}$.

Curves that foliate a domain: Let $\Omega\subseteq\mathbb{R}^{2}$ and let $I\subseteq\mathbb{R}$ then a set of curves $\left\{\gamma_{\alpha}\right\}\subseteq I\to\mathbb{R}$ such that $\biguplus\gamma_{i}=\Omega$.

Claim: $\text{sols}\left(y=y'\right)$ foliate the plane.

Remark: let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ and let $g\in\text{sols}\left(f\left(x,y,\dots,y^{(n)}\right)\right)$ such that g is piecewise continuous then each continuous branch of g is a solution of the ODE.

Initial Value Problem (IVP): let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ and let $p\in\mathbb{R}^{2}$ then $\begin{cases}f\left(x,y',\dots y^{(n)}\right)=0\\ y\left(p_{1}\right)=p_{2}\end{cases}$.

Theorem Cauchy-Peano: let $I,J\subseteq\mathbb{R}$ be closed intervals let $f\in C^{1}\left(I\times J\right)$ and let $p\in\text{int}\left(I\times J\right)$ then there exsits $\varepsilon>0$ such that $\begin{cases}y'=f\left(x,y\right)\\ y\left(p_{1}\right)=p_{2}\end{cases}$ has a solution on the interval $\left(p_{1}-\varepsilon,p_{1}+\varepsilon\right)$.

Theorem of Existence and Uniqueness: let $I,J\subseteq\mathbb{R}$ be closed intervals let $f\in C^{1}\left(I\times J\right)$ and let $p\in\text{int}\left(I\times J\right)$ then there exsits $\varepsilon>0$ such that $\begin{cases}y'=f\left(x,y\right)\\ y\left(p_{1}\right)=p_{2}\end{cases}$ has a unique solution on the interval $\left(p_{1}-\varepsilon,p_{1}+\varepsilon\right)$.

Corollary: let $I,J\subseteq\mathbb{R}$ be closed intervals and let $f\in C^{1}\left(I\times J\right)$ then $\text{sols}\left(y'=f\left(x,y\right)\right)$ foliates $I\times J$.

Vector Field: let $\Omega\subseteq\mathbb{R}^{2}$ be a domain then $\nu\in C\left(\Omega,\mathbb{R}^{2}\right)$.

Notation: let X be a set and let $\gamma:\mathbb{R}\to X$ then $\dot{\gamma}=\frac{\text{d}\gamma}{\text{d}t}$.

Integral curve: let $\nu:\Omega\to\mathbb{R}^{2}$ be a vector field then a curve $\gamma:\left[a,b\right]\to\Omega$ such that $\dot{\gamma}\left(t\right)=\nu\left(g\left(t\right)\right)$.

Claim: let $\nu:\Omega\to\mathbb{R}^{2}$ be a vector field such that $v\left(x\right)\neq0$ for all $x\in\Omega$ and let $\gamma:\left[a,b\right]\to\Omega$ be a curve such that $v\left(\gamma\left(t\right)\right)\in T_{\gamma\left(t\right)}\left(\text{Im}\left(\gamma\right)\right)$ for all $t\in\left[a,b\right]$ then there exists a curve $\eta:\left[\alpha,\beta\right]\to\left[a,b\right]$ such that $\gamma\circ\eta$ is an integral curve.

C^{n} Vector Field: let $\Omega\subseteq\mathbb{R}^{2}$ be a domain then $\nu\in C^{n}\left(\Omega,\mathbb{R}^{2}\right)$.

Claim: let $\nu:\Omega\to\mathbb{R}^{2}$ be a C^{1} vector field then $\left\{\gamma:\left[a,b\right]\to\Omega\mid\left(a,b\in\mathbb{R}\right)\wedge\left(\gamma\text{ is an integral curve of }\nu\right)\right\}$ foliates Ω.

Theorem Peano: let $\nu:\Omega\to\mathbb{R}^{2}$ be a vector field then $\left\{\gamma:\left[a,b\right]\to\Omega\mid\left(a,b\in\mathbb{R}\right)\wedge\left(\gamma\text{ is an integral curve of }\nu\right)\right\}$ covers Ω.

Lemma: let $\Omega\subseteq\mathbb{R}^{2}$ be a domain and let $f\in C^{1}\left(\Omega\right)$ then $\left(\begin{smallmatrix}1\\ f\end{smallmatrix}\right)$ is a vector field.

Theorem of Existence and Uniqueness: let $\Omega\subseteq\mathbb{R}^{2}$ be a domain let $f\in C^{1}\left(\Omega\right)$ and let $g\in C^{1}\left(\mathbb{R}\right)$ then
$\left(g'=f\left(x,g\right)\right)\Longleftrightarrow\left(\left(\begin{smallmatrix}x\\ g\end{smallmatrix}\right)\text{ is an integral curve of }\left(\begin{smallmatrix}1\\ f\end{smallmatrix}\right)\right)$.

Autonomous Ordinary Differential Equations: let $f:\mathbb{R}\to\mathbb{R}$ then $y'=f\left(y\right)$.

Logistic Equation: let $L>0$ then $\dot{P}\left(t\right)=P\left(t\right)\left(L-P\left(t\right)\right)$.

Equilibrium Solution: let $f:\mathbb{R}\to\mathbb{R}$ then a function $g\in\text{sols}\left(y'=f\left(y\right)\right)$ such that $f\left(g\left(x\right)\right)=0$ for all $x\in\mathbb{R}$.

Corollary: let $f\in C\left(\mathbb{R}\right)$ and let $g:\mathbb{R}\to\mathbb{R}$ be an equalibriem solution then g is constant.

Corollary: let $L>0$ then $\left\{0,L\right\}$ are the equilibium solutions of the logistic equation.

Logistic Equation With Harvesting: let $L,k>0$ then $\dot{P}\left(t\right)=P\left(t\right)\left(L-P\left(t\right)\right)-k$.

Stable Equilibrium Solution: let $f:\mathbb{R}\to\mathbb{R}$ then an equilibrium solution $g:\mathbb{R}\to\mathbb{R}$ such that for all $y\in\text{sols}\left(y'=f\left(y\right)\right)$ and for all $\varepsilon>0$ if there exists $\delta>0$ and $p\in\mathbb{R}$ for which $\left|y\left(p\right)-g\left(p\right)\right|<\delta$ then $\left|y\left(t\right)-g\left(t\right)\right|<\varepsilon$ for all $t>p$.

Lotka–Volterra equations: let $\alpha>0$ then $\begin{cases}\dot{x}\left(t\right)=x\left(t\right)\cdot\left(1-y\left(t\right)\right)\\ \dot{y}\left(t\right)=\alpha\cdot y\left(t\right)\cdot\left(x\left(t\right)-1\right)\end{cases}$.

Claim: let $\alpha>0$ then $\left\{\left(\begin{smallmatrix}0\\ 0\end{smallmatrix}\right),\left(\begin{smallmatrix}1\\ 1\end{smallmatrix}\right)\right\}$ are the equilibium solutions of the lotka-volterra equation.

Claim: let $\alpha>0$ and let $\left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right)\in C^{1}\left(\mathbb{R}^{2}\right)$ be a solution to the lotka-volterra equation then
$\frac{\text{d}}{\text{d}t}\left(\left(y\left(t\right)-1-\log\left(y\left(t\right)\right)\right)+\alpha\left(x\left(t\right)-1-\log\left(x\left(t\right)\right)\right)\right)=0$.

Corollary: let $\alpha>0$ let $p\in\mathbb{R}^{2}$ and let $\left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right)\in C^{1}\left(\mathbb{R}^{2}\right)$ be a solution to $\begin{cases}\dot{x}\left(t\right)=x\left(t\right)\cdot\left(1-y\left(t\right)\right)\\ \dot{y}\left(t\right)=\alpha\cdot y\left(t\right)\cdot\left(x\left(t\right)-1\right)\\ \left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right)\left(0\right)=p\end{cases}$ then $\left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right)$ is periodic.

Lemma: let $\alpha > 0$ let $p \in \mathbb{R}^2$ and let $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in C^1\left(\mathbb{R}^2\right)$ be a solution to $\begin{cases} \dot{x}(t)=x(t)\cdot(1-y(t)) \\ \dot{y}(t)=\alpha\cdot y(t)\cdot(x(t)-1) \\ \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)(0)=p \end{cases}$ with priod T then $\frac{1}{T}\int_0^T x\mathrm{d}t = 1$ and $\frac{1}{T}\int_0^T y\mathrm{d}t = 1$.

Claim Linear Substitution: let $a, b, c \in \mathbb{R}$ let $f : \mathbb{R} \to \mathbb{R}$ and let $y \in \text{sols}\left(y' = f\left(ax + by + c\right)\right)$ then the function $z : \mathbb{R} \to \mathbb{R}$ which defined by $z\left(x\right) = ax + by + c$ satisfies $z' = a + bf\left(z\right)$.