Ordinary Differential Equation (ODE): let $n \in \mathbb{N}$ and let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ then $f\left(x, y(x), y'(x), \ldots y^{(n)}(x)\right) = 0$.

Solution of an Ordinary Differential Equation: let $n \in \mathbb{N}$ let $\mathcal{U} \subseteq \mathbb{R}$ and let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ then a function $y \in C^n(\mathcal{U})$ such that $f\left(x, y(x), \ldots, y^{(n)}(x)\right) = 0$ for all $x \in \mathcal{U}$.

Notation: let $n \in \mathbb{N}$ let $\mathcal{U} \subseteq \mathbb{R}$ and let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ then $\text{sols}\left(f\left(x, y(x), y'(x), \ldots y^{(n)}(x)\right) = 0\right) = \left\{y \in C^n(\mathcal{U}) \mid f\left(x, y(x), \ldots, y^{(n)}(x)\right) = 0\right\}$.

First Order Ordinary Differential Equation: let $f : \mathbb{R}^3 \to \mathbb{R}$ then $f(x, y, y') = 0$.

Separable Ordinary Differential Equation: let $f, g : \mathbb{R} \to \mathbb{R}$ then $y' = f(y)\,g(x)$.

Claim Separation of Variables: let $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) \neq 0$ for all $x \in \mathbb{R}$ then $\text{sols}\left(y' = f(y)\,g(x)\right) = \text{sols}\left(\int \frac{1}{f(y)} \mathrm{d}y = \int g(x)\,\mathrm{d}x\right)$.

Corollary: $\text{sols}(y = y') = \{Ce^x \mid C \in \mathbb{R}\}$.

Curves that foliate a domain: Let $\Omega \subseteq \mathbb{R}^2$ and let $I \subseteq \mathbb{R}$ then a set of curves $\{\gamma_\alpha\} \subseteq I \to \mathbb{R}$ such that $\biguplus \gamma_i = \Omega$.

Claim: $\text{sols}(y = y')$ foliate the plane.

Remark: let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and let $g \in \text{sols}\left(f\left(x, y, \ldots, y^{(n)}\right)\right)$ such that g is piecewise continuous then each continuous branch of g is a solution of the ODE.

Initial Value Problem (IVP): let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and let $p \in \mathbb{R}^2$ then $\begin{cases} f\left(x, y', \ldots y^{(n)}\right) = 0 \\ y(p_1) = p_2 \end{cases}$.

Theorem Cauchy-Peano: let $I, J \subseteq \mathbb{R}$ be closed intervals let $f \in C^1(I \times J)$ and let $p \in \text{int}(I \times J)$ then there exists $\varepsilon > 0$ such that $\begin{cases} y' = f(x, y) \\ y(p_1) = p_2 \end{cases}$ has a solution on the interval $(p_1 - \varepsilon, p_1 + \varepsilon)$.

Theorem of Existence and Uniqueness: let $I, J \subseteq \mathbb{R}$ be closed intervals let $f \in C^1(I \times J)$ and let $p \in \text{int}(I \times J)$ then there exists $\varepsilon > 0$ such that $\begin{cases} y' = f(x, y) \\ y(p_1) = p_2 \end{cases}$ has a unique solution on the interval $(p_1 - \varepsilon, p_1 + \varepsilon)$.

Corollary: let $I, J \subseteq \mathbb{R}$ be closed intervals and let $f \in C^1(I \times J)$ then $\text{sols}(y' = f(x, y))$ foliates $I \times J$.

Vector Field: let $\Omega \subseteq \mathbb{R}^2$ be a domain then $\nu \in C\left(\Omega, \mathbb{R}^2\right)$.

Integral curve: let $\nu : \Omega \to \mathbb{R}^2$ be a vector field then a curve $\gamma : [a, b] \to \Omega$ such that $\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = \nu(g(t))$.

Claim: let $\nu : \Omega \to \mathbb{R}^2$ be a vector field such that $v(x) \neq 0$ for all $x \in \Omega$ and let $\gamma : [a, b] \to \Omega$ be a curve such that $v(\gamma(t)) \in T_{\gamma(t)}(\text{Im}(\gamma))$ for all $t \in [a, b]$ then there exists a curve $\eta : [\alpha, \beta] \to [a, b]$ such that $\gamma \circ \eta$ is an integral curve.

C^n **Vector Field:** let $\Omega \subseteq \mathbb{R}^2$ be a domain then $\nu \in C^n\left(\Omega, \mathbb{R}^2\right)$.

Claim: let $\nu : \Omega \to \mathbb{R}^2$ be a C^1 vector field then $\{\gamma : [a, b] \to \Omega \mid (a, b \in \mathbb{R}) \wedge (\gamma \text{ is an integral curve of } \nu)\}$ foliates Ω.

Theorem Peano: let $\nu : \Omega \to \mathbb{R}^2$ be a vector field then $\{\gamma : [a, b] \to \Omega \mid (a, b \in \mathbb{R}) \wedge (\gamma \text{ is an integral curve of } \nu)\}$ covers Ω.

Lemma: let $\Omega \subseteq \mathbb{R}^2$ be a domain and let $f \in C^1(\Omega)$ then $\left(\begin{smallmatrix} 1 \\ f \end{smallmatrix}\right)$ is a vector field.

Theorem of Existence and Uniqueness: let $\Omega \subseteq \mathbb{R}^2$ be a domain let $f \in C^1(\Omega)$ and let $g \in C^1(\mathbb{R})$ then $(g' = f(x, g)) \iff \left(\left(\begin{smallmatrix} x \\ g \end{smallmatrix}\right) \text{ is an integral curve of } \left(\begin{smallmatrix} 1 \\ f \end{smallmatrix}\right)\right)$.

Autonomous Ordinary Differential Equations: let $f : \mathbb{R} \to \mathbb{R}$ then $y' = f(y)$.

Logistic Equation: let $L > 0$ then $\dot{P}(t) = P(t)(L - P(t))$.

Equilibrium Solution: let $f : \mathbb{R} \to \mathbb{R}$ then a function $g \in \text{sols}(y' = f(y))$ such that $f(g(x)) = 0$ for all $x \in \mathbb{R}$.

Corollary: let $f \in C(\mathbb{R})$ and let $g : \mathbb{R} \to \mathbb{R}$ be an equilibrium solution then g is constant.

Corollary: let $L > 0$ then $\{0, L\}$ are the equilibrium solutions of the logistic equation.

Logistic Equation With Harvesting: let $L, k > 0$ then $\dot{P}(t) = P(t)(L - P(t)) - k$.

Stable Equilibrium Solution: let $f : \mathbb{R} \to \mathbb{R}$ then an equilibrium solution $g : \mathbb{R} \to \mathbb{R}$ such that for all $y \in \text{sols}(y' = f(y))$ and for all $\varepsilon > 0$ if there exists $\delta > 0$ and $p \in \mathbb{R}$ for which $|y(p) - g(p)| < \delta$ then $|y(t) - g(t)| < \varepsilon$ for all $t > p$.

Notation: let X be a set and let $\gamma : \mathbb{R} \to X$ then $\dot{\gamma} = \frac{\mathrm{d}\gamma}{\mathrm{d}t}$.

Notation: let X be a set and let $\gamma : \mathbb{R} \to X$ then $\ddot{\gamma} = \frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2}$.

Remark: the notation $\dot{\gamma}, \ddot{\gamma}$ is often used when γ is a function of time.

System of Ordinary Differential Equation (ODE): let $n \in \mathbb{N}$ let $m \in \mathbb{N}_+$ and let $f : \mathbb{R} \times (\mathbb{R}^m)^n \to \mathbb{R}^m$ then $f\left(t, y(t), y'(t), \ldots y^{(n)}(t)\right) = 0$.

Remark: all the theory of first order ODEs also applies to vectors of first order ODEs.

Lotka–Volterra equations: let $\alpha > 0$ then $\begin{cases} \dot{x}(t) = x(t) \cdot (1 - y(t)) \\ \dot{y}(t) = \alpha \cdot y(t) \cdot (x(t) - 1) \end{cases}$.

Claim: let $\alpha > 0$ then $\left\{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\right\}$ are the equilibrium solutions of the lotka-volterra equation.

Conserved Quantity: let $\mathcal{U} \subseteq \mathbb{R}^n$ and let $f : \mathcal{U} \to \mathbb{R}$ then a function $g \in C^1(\mathcal{U})$ such that $g \circ y$ is constant for all $y \in \text{sols}(y' = f(y))$.

Claim: let $\alpha > 0$ and let $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) \in C^1(\mathbb{R}^2)$ be a solution to the lotka-volterra equation then $\frac{\mathrm{d}}{\mathrm{d}t}\left((y(t) - 1 - \log(y(t))) + \alpha(x(t) - 1 - \log(x(t)))\right) = 0$.

Corollary: let $\alpha > 0$ let $p \in \mathbb{R}^2$ and let $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) \in C^1(\mathbb{R}^2)$ be a solution to $\begin{cases} \dot{x}(t)=x(t)\cdot(1-y(t)) \\ \dot{y}(t)=\alpha\cdot y(t)\cdot(x(t)-1) \\ \left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)(0)=p \end{cases}$ then $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)$ is periodic.

Lemma: let $\alpha > 0$ let $p \in \mathbb{R}^2$ and let $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) \in C^1(\mathbb{R}^2)$ be a solution to $\begin{cases} \dot{x}(t)=x(t)\cdot(1-y(t)) \\ \dot{y}(t)=\alpha\cdot y(t)\cdot(x(t)-1) \\ \left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)(0)=p \end{cases}$ with period T then $\frac{1}{T}\int_0^T x\mathrm{d}t = 1$ and $\frac{1}{T}\int_0^T y\mathrm{d}t = 1$.

Claim Linear Substitution: let $a, b, c \in \mathbb{R}$ let $f : \mathbb{R} \to \mathbb{R}$ and let $y \in \text{sols}(y' = f(ax + by + c))$ then the function $z : \mathbb{R} \to \mathbb{R}$ which defined by $z(x) = ax + by + c$ satisfies $z' = a + bf(z)$.

Notation: let $\Omega \subseteq \mathbb{R}^2$ be a domain let $f : \Omega \to \mathbb{R}$ and let $p \in \mathbb{R}^2$ then (α_-, α_+) is the maximal interval where a solution to $\begin{cases} y'=f(x,y) \\ y(p_1)=p_2 \end{cases}$ is defined.

Corollary: let $\Omega \subseteq \mathbb{R}^2$ be a domain let $f \in C^1(\Omega)$ and let $p \in \mathbb{R}^2$ then $\alpha_- < p_1 < \alpha_+$.

Theorem: let $f \in C^1(\mathbb{R}^2)$ and let $p \in \mathbb{R}^2$ then

- $(\alpha_+ < \infty) \iff \left(\lim_{x \to \alpha_+^-} |y(x)| = \infty\right)$.
- $(-\infty < \alpha_-) \iff \left(\lim_{x \to \alpha_-^+} |y(x)| = \infty\right)$.

Theorem Extensibility of Solutions: let $D \subseteq \mathbb{R}^{n+1}$ be a closed set let $f \in C^1(D, \mathbb{R}^n)$ and let $y \in \text{sols}(y' = f(x,y))$ such that $\Gamma_y \cap D \neq \varnothing$ then there exists $\psi : [a,b] \to \mathbb{R}^n$ such that $\psi \in \text{sols}(y' = f(x,y))$ and $\psi(x) = y(x)$ for all $x \in \text{Dom}(y)$ and $\left(\begin{smallmatrix}a\\\psi(a)\end{smallmatrix}\right), \left(\begin{smallmatrix}b\\\psi(b)\end{smallmatrix}\right) \in \partial D$.

Corollary: let $D \subseteq \mathbb{R}^{n+1}$ such that for all $\alpha, \beta \in \mathbb{R}$ the set $D \cap \{\alpha \le x_1 \le \beta\}$ is bounded let $f \in C^1(D, \mathbb{R}^n)$ and let $y \in \text{sols}(y' = f(x,y))$ such that $\Gamma_y \cap D \neq \varnothing$ then one of the next statements is true

- there exists $\psi : [a,b] \to \mathbb{R}^n$ such that $\psi \in \text{sols}(y' = f(x,y))$ and $\psi(x) = y(x)$ for all $x \in \text{Dom}(y)$ and $\left(\begin{smallmatrix}a\\\psi(a)\end{smallmatrix}\right), \left(\begin{smallmatrix}b\\\psi(b)\end{smallmatrix}\right) \in \partial D$.
- there exists $\psi : \mathbb{R} \to \mathbb{R}^n$ such that $\psi \in \text{sols}(y' = f(x,y))$ and $\psi(x) = y(x)$ for all $x \in \text{Dom}(y)$.

Corollary: let $I \subseteq \mathbb{R}$ be an open interval let $D \subseteq \mathbb{R}^n$ let $f \in C^1(I \times D, \mathbb{R}^n)$ and let $p, q : \mathbb{R} \to \mathbb{R}$ such that $\|f(x,y)\| \le p(x)\|y\| + q(x)$ then for all $y \in \text{sols}(y' = f(x,y))$ such that $\Gamma_y \cap (I \times D) \neq \varnothing$ there exists $\psi : I \to \mathbb{R}^n$ such that $\psi \in \text{sols}(y' = f(x,y))$ and $\psi(x) = y(x)$ for all $x \in \text{Dom}(y)$.

Theorem Continuous Dependence on Initial Conditions: let $\Omega \subseteq \mathbb{R}^2$ let $f \in C^1(\Omega)$ let $a, b, p \in \mathbb{R}$ and let y_p be a solution to $\begin{cases} y'=f(x,y) \\ y(a)=p \end{cases}$ such that y_p is defined on $[a,b]$ then for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $q \in \mathbb{R}$ for which $|p - q| < \delta$ if y_q is a solution to $\begin{cases} y'=f(x,y) \\ y(a)=q \end{cases}$ then y_q is defined on $[a,b]$ and $|y_p - y_q| < \varepsilon$.

Theorem: let $\Omega \subseteq \mathbb{R}^2$ let $f \in C^1(\Omega)$ let $a, b \in \mathbb{R}$ let $g \in C^1([a,b])$ and let $\{g_n\}_{n=0}^{\infty} \subseteq \text{sols}(y' = f(x,y))$ such that g_n is defined on $[a,b]$ for all $n \in \mathbb{N}$ and $g_n \xrightarrow{p.w.} g$ then $g \in \text{sols}(y' = f(x,y))$.

n-th Order Ordinary Differential Equation: let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ then $f\left(x, y \ldots y^{(n)}\right) = 0$.

Claim: let $n \in \mathbb{N}_+$ let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}$ then

$$\left(g \in \text{sols}\left(f\left(t, x(t) \ldots x^{(n)}(t)\right) = 0\right)\right) \iff \left(\begin{pmatrix} g \\ \vdots \\ g^{(n)} \end{pmatrix} \in \text{sols}\left(\begin{cases} y_1=\dot{x} \\ \vdots \\ y_{n-1}=\dot{y_{n-2}} \\ \dot{y_{n-1}}=f(t,x,y_1\ldots,y_{n-1}) \end{cases}\right)\right)$$

Harmonic Oscillator/Spring Position Equation: $\ddot{x} = -x$.

Claim: the function $E(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$ is a conserved quantity of $\begin{cases} \dot{x}=y \\ \dot{y}=-x \end{cases}$.

Spring with Friction Position Equation: $\ddot{x} = -x - \dot{x}$.

Constant Tension Spring Position Equation: $\ddot{x} = -1$.

Claim: the function $E(x,y) = x + \frac{1}{2}y^2$ is a conserved quantity of $\begin{cases} \dot{x}=y \\ \dot{y}=-1 \end{cases}$.

Gravity in One Dimension Equation: $\ddot{x} = -\frac{1}{x^2}$.

Claim: the function $E(x,y)=\frac{1}{x}+\frac{1}{2}y^2$ is a conserved quantity of $\begin{cases}\dot{x}=y\\ \dot{y}=-\frac{1}{x^2}\end{cases}$.

Pendulum Equation: $\ddot{x}=-\sin(x)$.

Claim: the function $E(x,y)=\frac{1}{2}y^2-\cos(x)$ is a conserved quantity of $\begin{cases}\dot{x}=y\\ \dot{y}=-\sin(x)\end{cases}$.

Differential 1-form: let $P,Q:\mathbb{R}^2\to\mathbb{R}$ then $P(x,y)\,\mathrm{d}x+Q(x,y)\,\mathrm{d}y$.

Remark: in this course we won't explain the formal definition of a differential 1-form so we will use it as an object with certain properties.

Integral: let $\gamma:[a,b]\to\mathbb{R}^2$ be a curve and let $P,Q:\mathbb{R}^2\to\mathbb{R}$ then
$\int_\gamma(P(x,y)\,\mathrm{d}x+Q(x,y)\,\mathrm{d}y)=\lim_{\Delta\to0}\sum_{i=0}^{n-1}(P(x_i,y_i)(x_{i+1}-x_i)+Q(x_i,y_i)(y_{i+1}-y_i))$.

Remark: in the definition above $\lim_{\Delta\to0}$ is the limit of all partitions of γ to horizontal and vertical segments.

Claim: let $\gamma:[a,b]\to\mathbb{R}^n$ be a curve let $\nu:[\alpha,\beta]\to[a,b]$ such that $\gamma\circ\nu$ is a reparameterization of γ and let ω be a differential 1-form then $\int_\gamma\omega=\int_{\gamma\circ\nu}\omega$.

Claim: let $\gamma:[a,b]\to\mathbb{R}^n$ be a curve and let ω be a differential 1-form then $\int_\gamma\omega=-\int_{-\gamma}\omega$.

Integral: let $\gamma:[a,b]\to\mathbb{R}^2$ be a curve and let $f,g:\mathbb{R}^2\to\mathbb{R}$ then $\int_\gamma f\mathrm{d}g=\lim_{\Delta\to0}\sum_{i=0}^{n-1}(f(x_i,y_i)(g(x_{i+1},y_{i+1})-g(x_i,y_i)))$.

Claim: let $g\in C^1(\mathbb{R}^2)$ then $\mathrm{d}g=\frac{\partial g}{\partial x}\mathrm{d}x+\frac{\partial g}{\partial y}\mathrm{d}y$.

Corollary: let $\gamma:[a,b]\to\mathbb{R}^2$ be a curve let $f:\mathbb{R}^2\to\mathbb{R}$ and let $g\in C^1(\mathbb{R}^2)$ then $\int_\gamma f\mathrm{d}g=\int_\gamma\left(f\cdot\frac{\partial g}{\partial x}\mathrm{d}x+f\cdot\frac{\partial g}{\partial y}\mathrm{d}y\right)$.

Exact Differential 1-form: a differential 1-form ω such that there exists $g\in C^1(\mathbb{R}^2)$ for which $\omega=\mathrm{d}g$.

Theorem: let ω be a differential 1-form then TFAE

- ω is exact.
- there exists $f:\mathbb{R}\to\mathbb{R}$ such that for all curves $\gamma:[a,b]\to\mathbb{R}^2$ we have $\int_\gamma\omega=f(\gamma(b))-f(\gamma(a))$.
- for all closed curves $\gamma:[a,b]\to\mathbb{R}^2$ we have $\int_\gamma\omega=0$.

Primitive/Potential: let ω be an exact differential 1-form then $g\in C^1(\mathbb{R}^2)$ such that $\omega=\mathrm{d}g$.

Claim: let $X:\mathbb{R}\to\mathbb{R}^3$ and let $f:\mathbb{R}^3\to\mathbb{R}$ then $\frac{\mathrm{d}}{\mathrm{d}t}(f\circ X)=(\nabla f)\cdot\dot{X}$.

Conservative Vector Field: a vector field $F:\mathbb{R}^n\to\mathbb{R}^n$ such that there exists $U\in C^1(\mathbb{R}^n)$ for which $F=-\nabla U$.

Kinetic Energy: let $X\in C^1(\mathbb{R},\mathbb{R}^n)$ then $K:\mathbb{R}^n\to\mathbb{R}$ such that $K(X(t))=\frac{\|\dot{X}(t)\|^2}{2}$.

Total Energy: let $U\in C^1(\mathbb{R}^n)$ let $F:\mathbb{R}^n\to\mathbb{R}^n$ be a conservative vector field such that $F=-\nabla U$ let $X\in\text{sols}\left(\ddot{X}=F(X)\right)$ and let $K:\mathbb{R}^n\to\mathbb{R}$ be the kinetic energy of X then $E:\mathbb{R}^n\to\mathbb{R}$ such that $E=K+U$.

Lemma: let $U\in C^1(\mathbb{R}^n)$ let $F:\mathbb{R}^n\to\mathbb{R}^n$ be a conservative vector field such that $F=-\nabla U$ let $X\in\text{sols}\left(\ddot{X}=F(X)\right)$ and let $E:\mathbb{R}^n\to\mathbb{R}$ be the total energy of X then $\dot{E}(X(t))=0$.

Weighted Average/Center of Mass: let $p_1\dots p_n\in\mathbb{R}^n$ and let $w:\{p_1\dots p_n\}\to\mathbb{R}_+$ then $\frac{\sum_{i=1}^n w(p_i)\cdot p_i}{\sum_{i=1}^n w(p_i)}$.

Center of Mass: let $K\subseteq\mathbb{R}^2$ be compact and let $\rho:K\to\mathbb{R}$ then $\left(\frac{\int_K x\cdot\rho(x,y)\mathrm{d}x\mathrm{d}y}{\int_K\rho(x,y)\mathrm{d}x\mathrm{d}y},\frac{\int_K y\cdot\rho(x,y)\mathrm{d}x\mathrm{d}y}{\int_K\rho(x,y)\mathrm{d}x\mathrm{d}y}\right)$.

Line: let $A,B\in\mathbb{R}^2$ then $L_{A,B}=\{\lambda A+(1-\lambda)B\mid\lambda\in[0,1]\}$.

Triangle: let $A,B,C\in\mathbb{R}^2$ such that $A\notin L_{B,C}$ and $B\notin L_{A,C}$ and $C\notin L_{A,B}$ then $\{A,B,C\}$.

Theorem Ceva: let $A,B,C\in\mathbb{R}^2$ such that $\{A,B,C\}$ is a triangle let $A'\in L_{B,C}$ let $B'\in L_{A,C}$ and let $C'\in L_{A,C}$ then $(L_{A,A'}\cap L_{B,B'}\cap L_{C,C'}\neq\varnothing)\iff\left(\frac{d(A,B')}{d(B',C)}\cdot\frac{d(C,A')}{d(A',B)}\cdot\frac{d(B,C')}{d(C',A)}=1\right)$.

Special Orthogonal Group: let $n\in\mathbb{N}$ then $\mathrm{SO}(n)=\left\{A\in M_n(\mathbb{R})\mid(\det(A)=1)\wedge\left(A^T=A^{-1}\right)\right\}$.

Cross Product: let $x,y\in\mathbb{R}^3$ then $x\times y=\begin{pmatrix}x_2y_3-x_3y_2\\x_3y_1-x_1y_3\\x_1y_2-x_2y_1\end{pmatrix}$.

Claim Antisymmetry: let $x,y\in\mathbb{R}^3$ then $x\times y=-y\times x$.

Claim Orthogonality: let $x,y\in\mathbb{R}^3$ be linearly independent then $x\times y\perp x$ and $x\times y\perp y$.

Claim: let $x,y\in\mathbb{R}^3$ and let θ be the angle between x,y then $\|x\times y\|=\|x\|\cdot\|y\|\cdot\cos(\theta)$.

Corollary: let $x,y\in\mathbb{R}^3$ then $(x\times y=0)\Longleftrightarrow(x,y$ are linearly dependent).

Claim: let $x,y,z\in\mathbb{R}^2$ and let $\alpha,\beta\in\mathbb{R}$ then $(\alpha x+\beta y)\times z=\alpha(x\times z)+\beta(y\times z)$.

Claim: let $X,Y\in C^1(\mathbb{R}^3)$ then $\frac{\mathrm{d}}{\mathrm{d}t}(X\times Y)=\dot{X}\times Y+X\times\dot{Y}$.

Angular Momentum: let $X\in C^1(\mathbb{R}^3)$ then $\mathcal{L}=X\times\dot{X}$.

Lemma: angular momentum is a conserved quantity of $\ddot{X} = -\frac{X}{\|X\|^2}$.

Exact Ordinary Differential Equation: let $\Omega \subseteq \mathbb{R}^2$ and let $F \in C^1(\Omega)$ then $\frac{\partial F}{\partial x}(x,y) + y' \cdot \frac{\partial F}{\partial y}(x,y) = 0$.

Claim: let $\Omega \subseteq \mathbb{R}^2$ let $F \in C^1(\Omega)$ and let $y \in C^1(\mathbb{R})$ then $\frac{\mathrm{d}}{\mathrm{d}t}(F(x,y(x))) = 0$.

Remark: let $\Omega \subseteq \mathbb{R}^2$ let $F \in C^1(\Omega)$ and let $c \in \mathbb{R}$ then $\{F = c\}$ are solutions to $\frac{\partial F}{\partial x}(x,y) + y' \cdot \frac{\partial F}{\partial y}(x,y) = 0$.

Claim: let $\Omega \subseteq \mathbb{R}^2$ be a simply connected domain and let $P, Q \in C^1(\Omega)$ then ($P(x,y) + y' \cdot Q(x,y) = 0$ is an exact ODE)$\Longleftrightarrow$($\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$).