

**Notation:** let  $X$  be a set and let  $y : \mathbb{R} \rightarrow \mathbb{R}$  then  $y^{(0)} = y$ .

**Notation:** let  $X, Y$  be sets let  $n \in \mathbb{N}_+$  and let  $y \in C^n(X, Y)$  then  $y^{(n)} = (y^{(n-1)})'$ .

**Ordinary Differential Equation (ODE):** let  $n \in \mathbb{N}$  and let  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  then  $f(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0$ .

**Solution of an Ordinary Differential Equation:** let  $n \in \mathbb{N}$  let  $\mathcal{U} \subseteq \mathbb{R}$  and let  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  then a function  $y \in C^n(\mathcal{U})$  such that  $f(x, y(x), \dots, y^{(n)}(x)) = 0$  for all  $x \in \mathcal{U}$ .

**Notation:** let  $n \in \mathbb{N}$  let  $\mathcal{U} \subseteq \mathbb{R}$  and let  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  then

$\text{sols}(f(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0) = \{y \in C^n(\mathcal{U}) \mid f(x, y(x), \dots, y^{(n)}(x)) = 0\}$ .

**First Order Ordinary Differential Equation:** let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  then  $f(x, y, y') = 0$ .

**Separable Ordinary Differential Equation:** let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  then  $y' = f(y)g(x)$ .

**Claim Separation of Variables:** let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(x) \neq 0$  for all  $x \in \mathbb{R}$  then

$\text{sols}(y' = f(y)g(x)) = \text{sols}\left(\int \frac{1}{f(y)} dy = \int g(x) dx\right)$ .

**Corollary:**  $\text{sols}(y = y') = \{Ce^x \mid C \in \mathbb{R}\}$ .

**Curves that foliate a domain:** Let  $\Omega \subseteq \mathbb{R}^2$  and let  $I \subseteq \mathbb{R}$  then a set of curves  $\{\gamma_\alpha\} \subseteq I \rightarrow \mathbb{R}$  such that  $\biguplus \gamma_i = \Omega$ .

**Claim:**  $\text{sols}(y = y')$  foliate the plane.

**Remark:** let  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  and let  $g \in \text{sols}(f(x, y, \dots, y^{(n)}))$  such that  $g$  is piecewise continuous then each continuous branch of  $g$  is a solution of the ODE.

**Initial Value Problem (IVP):** let  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  and let  $p \in \mathbb{R}^2$  then  $\begin{cases} f(x, y', \dots, y^{(n)})=0 \\ y(p_1)=p_2 \end{cases}$ .

**Theorem Cauchy-Peano:** let  $I, J \subseteq \mathbb{R}$  be closed intervals let  $f \in C^1(I \times J)$  and let  $p \in \text{int}(I \times J)$  then there exists  $\varepsilon > 0$  such that  $\begin{cases} y'=f(x,y) \\ y(p_1)=p_2 \end{cases}$  has a solution on the interval  $(p_1 - \varepsilon, p_1 + \varepsilon)$ .

**Theorem of Existence and Uniqueness:** let  $I, J \subseteq \mathbb{R}$  be closed intervals let  $f \in C^1(I \times J)$  and let  $p \in \text{int}(I \times J)$  then there exists  $\varepsilon > 0$  such that  $\begin{cases} y'=f(x,y) \\ y(p_1)=p_2 \end{cases}$  has a unique solution on the interval  $(p_1 - \varepsilon, p_1 + \varepsilon)$ .

**Corollary:** let  $I, J \subseteq \mathbb{R}$  be closed intervals and let  $f \in C^1(I \times J)$  then  $\text{sols}(y' = f(x, y))$  foliates  $I \times J$ .

**Vector Field:** let  $\Omega \subseteq \mathbb{R}^2$  be a domain then  $\nu \in C(\Omega, \mathbb{R}^2)$ .

**Integral curve:** let  $\nu : \Omega \rightarrow \mathbb{R}^2$  be a vector field then a curve  $\gamma : [a, b] \rightarrow \Omega$  such that  $\frac{d\gamma}{dt}(t) = \nu(g(t))$ .

**Claim:** let  $\nu : \Omega \rightarrow \mathbb{R}^2$  be a vector field such that  $\nu(x) \neq 0$  for all  $x \in \Omega$  and let  $\gamma : [a, b] \rightarrow \Omega$  be a curve such that  $\nu(\gamma(t)) \in T_{\gamma(t)}(\text{Im}(\gamma))$  for all  $t \in [a, b]$  then there exists a curve  $\eta : [\alpha, \beta] \rightarrow [a, b]$  such that  $\gamma \circ \eta$  is an integral curve.

**$C^n$  Vector Field:** let  $\Omega \subseteq \mathbb{R}^2$  be a domain then  $\nu \in C^n(\Omega, \mathbb{R}^2)$ .

**Claim:** let  $\nu : \Omega \rightarrow \mathbb{R}^2$  be a  $C^1$  vector field then  $\{\gamma : [a, b] \rightarrow \Omega \mid (a, b \in \mathbb{R}) \wedge (\gamma \text{ is an integral curve of } \nu)\}$  foliates  $\Omega$ .

**Theorem Peano:** let  $\nu : \Omega \rightarrow \mathbb{R}^2$  be a vector field then  $\{\gamma : [a, b] \rightarrow \Omega \mid (a, b \in \mathbb{R}) \wedge (\gamma \text{ is an integral curve of } \nu)\}$  covers  $\Omega$ .

**Lemma:** let  $\Omega \subseteq \mathbb{R}^2$  be a domain and let  $f \in C^1(\Omega)$  then  $(\frac{1}{f})$  is a vector field.

**Theorem of Existence and Uniqueness:** let  $\Omega \subseteq \mathbb{R}^2$  be a domain let  $f \in C^1(\Omega)$  and let  $g \in C^1(\mathbb{R})$  then  $(g' = f(x, g)) \iff ((\frac{x}{g}) \text{ is an integral curve of } (\frac{1}{f}))$ .

**Autonomous Ordinary Differential Equations:** let  $f : \mathbb{R} \rightarrow \mathbb{R}$  then  $y' = f(y)$ .

**Logistic Equation:** let  $L > 0$  then  $\dot{P}(t) = P(t)(L - P(t))$ .

**Equilibrium Solution:** let  $f : \mathbb{R} \rightarrow \mathbb{R}$  then a function  $g \in \text{sols}(y' = f(y))$  such that  $f(g(x)) = 0$  for all  $x \in \mathbb{R}$ .

**Corollary:** let  $f \in C(\mathbb{R})$  and let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be an equilibrium solution then  $g$  is constant.

**Corollary:** let  $L > 0$  then  $\{0, L\}$  are the equilibrium solutions of the logistic equation.

**Logistic Equation With Harvesting:** let  $L, k > 0$  then  $\dot{P}(t) = P(t)(L - P(t)) - k$ .

**Stable Equilibrium Solution:** let  $f : \mathbb{R} \rightarrow \mathbb{R}$  then an equilibrium solution  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that for all  $y \in \text{sols}(y' = f(y))$  and for all  $\varepsilon > 0$  if there exists  $\delta > 0$  and  $p \in \mathbb{R}$  for which  $|y(p) - g(p)| < \delta$  then  $|y(t) - g(t)| < \varepsilon$  for all  $t > p$ .

**Notation:** let  $X$  be a set and let  $\gamma : \mathbb{R} \rightarrow X$  then  $\dot{\gamma} = \frac{d\gamma}{dt}$ .

**Notation:** let  $X$  be a set and let  $\gamma : \mathbb{R} \rightarrow X$  then  $\ddot{\gamma} = \frac{d^2\gamma}{dt^2}$ .

**Remark:** the notation  $\dot{\gamma}, \ddot{\gamma}$  is often used when  $\gamma$  is a function of time.

**System of Ordinary Differential Equation:** let  $n \in \mathbb{N}$  let  $m \in \mathbb{N}_+$  and let  $f : \mathbb{R} \times (\mathbb{R}^m)^n \rightarrow \mathbb{R}^m$  then

$f(t, y(t), y'(t), \dots, y^{(n)}(t)) = 0$ .

**Remark:** all the theory of first order ODEs also applies to vectors of first order ODEs.

**Lotka-Volterra equations:** let  $\alpha > 0$  then  $\begin{cases} \dot{x}(t)=x(t) \cdot (1-y(t)) \\ \dot{y}(t)=\alpha \cdot y(t) \cdot (x(t)-1) \end{cases}$ .

**Claim:** let  $\alpha > 0$  then  $\{(\frac{0}{0}), (\frac{1}{1})\}$  are the equilibrium solutions of the lotka-volterra equation.

**Conserved Quantity:** let  $\mathcal{U} \subseteq \mathbb{R}^n$  and let  $f : \mathcal{U} \rightarrow \mathbb{R}$  then a function  $g \in C^1(\mathcal{U})$  such that  $g \circ y$  is constant for all  $y \in \text{sols}(y' = f(y))$ .

**Claim:** let  $\alpha > 0$  and let  $(\frac{x}{y}) \in C^1(\mathbb{R}^2)$  be a solution to the lotka-volterra equation then

$$\frac{d}{dt}((y(t) - 1 - \log(y(t))) + \alpha(x(t) - 1 - \log(x(t)))) = 0.$$

**Corollary:** let  $\alpha > 0$  let  $p \in \mathbb{R}^2$  and let  $(\frac{x}{y}) \in C^1(\mathbb{R}^2)$  be a solution to  $\begin{cases} \dot{x}(t)=x(t) \cdot (1-y(t)) \\ \dot{y}(t)=\alpha \cdot y(t) \cdot (x(t)-1) \\ (\frac{x}{y})(0)=p \end{cases}$  then  $(\frac{x}{y})$  is periodic.

**Lemma:** let  $\alpha > 0$  let  $p \in \mathbb{R}^2$  and let  $(\frac{x}{y}) \in C^1(\mathbb{R}^2)$  be a solution to  $\begin{cases} \dot{x}(t)=x(t) \cdot (1-y(t)) \\ \dot{y}(t)=\alpha \cdot y(t) \cdot (x(t)-1) \\ (\frac{x}{y})(0)=p \end{cases}$  with period  $T$  then  $\frac{1}{T} \int_0^T x dt = 1$

$$\text{and } \frac{1}{T} \int_0^T y dt = 1.$$

**Claim Linear Substitution:** let  $a, b, c \in \mathbb{R}$  let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and let  $y \in \text{sols}(y' = f(ax + by + c))$  then the function  $z : \mathbb{R} \rightarrow \mathbb{R}$  which defined by  $z(x) = ax + by + c$  satisfies  $z' = a + bf(z)$ .

**Notation:** let  $\Omega \subseteq \mathbb{R}^2$  be a domain let  $f : \Omega \rightarrow \mathbb{R}$  and let  $p \in \mathbb{R}^2$  then  $(\alpha_-, \alpha_+)$  is the maximal interval where a solution to  $\begin{cases} y' = f(x, y) \\ y(p_1) = p_2 \end{cases}$  is defined.

**Corollary:** let  $\Omega \subseteq \mathbb{R}^2$  be a domain let  $f \in C^1(\Omega)$  and let  $p \in \mathbb{R}^2$  then  $\alpha_- < p_1 < \alpha_+$ .

**Theorem:** let  $f \in C^1(\mathbb{R}^2)$  and let  $p \in \mathbb{R}^2$  then

- $(\alpha_+ < \infty) \iff \left(\lim_{x \rightarrow \alpha_+^-} |y(x)| = \infty\right)$ .
- $(-\infty < \alpha_-) \iff \left(\lim_{x \rightarrow \alpha_-^+} |y(x)| = \infty\right)$ .

**Theorem Extensibility of Solutions:** let  $D \subseteq \mathbb{R}^{n+1}$  be a closed set let  $f \in C^1(D, \mathbb{R}^n)$  and let  $y \in \text{sols}(y' = f(x, y))$  such that  $\Gamma_y \cap D \neq \emptyset$  then there exists  $\psi : [a, b] \rightarrow \mathbb{R}^n$  such that  $\psi \in \text{sols}(y' = f(x, y))$  and  $\psi(x) = y(x)$  for all  $x \in \text{Dom}(y)$  and  $(\begin{smallmatrix} a \\ \psi(a) \end{smallmatrix}), (\begin{smallmatrix} b \\ \psi(b) \end{smallmatrix}) \in \partial D$ .

**Corollary:** let  $D \subseteq \mathbb{R}^{n+1}$  such that for all  $\alpha, \beta \in \mathbb{R}$  the set  $D \cap \{\alpha \leq x_1 \leq \beta\}$  is bounded let  $f \in C^1(D, \mathbb{R}^n)$  and let  $y \in \text{sols}(y' = f(x, y))$  such that  $\Gamma_y \cap D \neq \emptyset$  then one of the next statements is true

- there exists  $\psi : [a, b] \rightarrow \mathbb{R}^n$  such that  $\psi \in \text{sols}(y' = f(x, y))$  and  $\psi(x) = y(x)$  for all  $x \in \text{Dom}(y)$  and  $(\begin{smallmatrix} a \\ \psi(a) \end{smallmatrix}), (\begin{smallmatrix} b \\ \psi(b) \end{smallmatrix}) \in \partial D$ .
- there exists  $\psi : \mathbb{R} \rightarrow \mathbb{R}^n$  such that  $\psi \in \text{sols}(y' = f(x, y))$  and  $\psi(x) = y(x)$  for all  $x \in \text{Dom}(y)$ .

**Corollary:** let  $I \subseteq \mathbb{R}$  be an open interval let  $D \subseteq \mathbb{R}^n$  let  $f \in C^1(I \times D, \mathbb{R}^n)$  and let  $p, q : \mathbb{R} \rightarrow \mathbb{R}$  such that

$\|f(x, y)\| \leq p(x)\|y\| + q(x)$  then for all  $y \in \text{sols}(y' = f(x, y))$  such that  $\Gamma_y \cap (I \times D) \neq \emptyset$  there exists  $\psi : I \rightarrow \mathbb{R}^n$  such that  $\psi \in \text{sols}(y' = f(x, y))$  and  $\psi(x) = y(x)$  for all  $x \in \text{Dom}(y)$ .

**Theorem Continuous Dependence on Initial Conditions:** let  $\Omega \subseteq \mathbb{R}^2$  let  $f \in C^1(\Omega)$  let  $a, b, p \in \mathbb{R}$  and let  $y_p$  be a solution to  $\begin{cases} y' = f(x, y) \\ y(a) = p \end{cases}$  such that  $y_p$  is defined on  $[a, b]$  then for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that for all  $q \in \mathbb{R}$  for which  $|p - q| < \delta$  if  $y_q$  is a solution to  $\begin{cases} y' = f(x, y) \\ y(a) = q \end{cases}$  then  $y_q$  is defined on  $[a, b]$  and  $|y_p - y_q| < \varepsilon$ .

**Theorem Continuous Dependence on Initial Conditions:** let  $\Omega \subseteq \mathbb{R}^2$  let  $f \in C^1(\Omega)$  let  $a, b \in \mathbb{R}$  let  $p \in \mathbb{R}$  let  $\{p_n\}_{n=0}^\infty \subseteq \mathbb{R}$  such that  $p_n \rightarrow p$  and let  $\{g_{p_n}\}_{n=0}^\infty \subseteq \mathbb{R} \rightarrow \mathbb{R}$  such that  $g_{p_n} \in \text{sols}\left(\begin{cases} y' = f(x, y) \\ y(a) = p_n \end{cases}\right)$  and  $g_{p_n}$  is defined on  $[a, b]$  for all  $n \in \mathbb{N}$  then there exists  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that  $g_{p_n} \xrightarrow{p.w.} g$  and  $g \in \text{sols}\left(\begin{cases} y' = f(x, y) \\ y(a) = p \end{cases}\right)$ .

**n-th Order Ordinary Differential Equation:** let  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  then  $f(x, y \dots y^{(n)}) = 0$ .

**Claim:** let  $n \in \mathbb{N}_+$  let  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  and let  $g : \mathbb{R} \rightarrow \mathbb{R}$  then

$$(g \in \text{sols}(f(t, x(t) \dots x^{(n)}(t)) = 0)) \iff \left(\begin{pmatrix} g \\ \vdots \\ g^{(n)} \end{pmatrix} \in \text{sols}\left(\begin{pmatrix} y_1 = \dot{x} \\ \vdots \\ y_{n-1} = \dot{y}_{n-2} \\ y_{n-1} = f(t, x, y_1, \dots, y_{n-1}) \end{pmatrix}\right)\right)$$

**Harmonic Oscillator/Spring Position Equation:**  $\ddot{x} = -x$ .

**Claim:** the function  $E(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$  is a conserved quantity of  $\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$ .

**Spring with Friction Position Equation:**  $\ddot{x} = -x - \dot{x}$ .

**Constant Tension Spring Position Equation:**  $\ddot{x} = -1$ .

**Claim:** the function  $E(x, y) = x + \frac{1}{2}y^2$  is a conserved quantity of  $\begin{cases} \dot{x}=y \\ \dot{y}=-1 \end{cases}$ .

**Gravity in One Dimension Equation:**  $\ddot{x} = -\frac{1}{x^2}$ .

**Claim:** the function  $E(x, y) = \frac{1}{x} + \frac{1}{2}y^2$  is a conserved quantity of  $\begin{cases} \dot{x}=y \\ \dot{y}=-\frac{1}{x^2} \end{cases}$ .

**Pendulum Equation:**  $\ddot{x} = -\sin(x)$ .

**Claim:** the function  $E(x, y) = \frac{1}{2}y^2 - \cos(x)$  is a conserved quantity of  $\begin{cases} \dot{x}=y \\ \dot{y}=-\sin(x) \end{cases}$ .

**Differential 1-form:** let  $P, Q : \mathbb{R}^2 \rightarrow \mathbb{R}$  then  $P(x, y) dx + Q(x, y) dy$ .

**Remark:** in this course we won't explain the formal definition of a differential 1-form so we will use it as an object with certain properties.

**Integral:** let  $\gamma : [a, b] \rightarrow \mathbb{R}^2$  be a curve and let  $P, Q : \mathbb{R}^2 \rightarrow \mathbb{R}$  then

$$\int_{\gamma} (P(x, y) dx + Q(x, y) dy) = \lim_{\Delta \rightarrow 0} \sum_{i=0}^{n-1} (P(x_i, y_i)(x_{i+1} - x_i) + Q(x_i, y_i)(y_{i+1} - y_i)).$$

**Remark:** in the definition above  $\lim_{\Delta \rightarrow 0}$  is the limit of all partitions of  $\gamma$  to horizontal and vertical segments.

**Claim:** let  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  be a curve let  $\nu : [\alpha, \beta] \rightarrow [a, b]$  such that  $\gamma \circ \nu$  is a reparameterization of  $\gamma$  and let  $\omega$  be a differential 1-form then  $\int_{\gamma} \omega = \int_{\gamma \circ \nu} \omega$ .

**Claim:** let  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  be a curve and let  $\omega$  be a differential 1-form then  $\int_{\gamma} \omega = -\int_{-\gamma} \omega$ .

**Integral:** let  $\gamma : [a, b] \rightarrow \mathbb{R}^2$  be a curve and let  $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$  then  $\int_{\gamma} f dg = \lim_{\Delta \rightarrow 0} \sum_{i=0}^{n-1} (f(x_i, y_i)(g(x_{i+1}, y_{i+1}) - g(x_i, y_i)))$ .

**Claim:** let  $g \in C^1(\mathbb{R}^2)$  then  $dg = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy$ .

**Corollary:** let  $\gamma : [a, b] \rightarrow \mathbb{R}^2$  be a curve let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  and let  $g \in C^1(\mathbb{R}^2)$  then  $\int_{\gamma} f dg = \int_{\gamma} \left( f \cdot \frac{\partial g}{\partial x} dx + f \cdot \frac{\partial g}{\partial y} dy \right)$ .

**Exact Differential 1-form:** a differential 1-form  $\omega$  such that there exists  $g \in C^1(\mathbb{R}^2)$  for which  $\omega = dg$ .

**Theorem:** let  $\omega$  be a differential 1-form then TFAE

- $\omega$  is exact.
- there exists  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that for all curves  $\gamma : [a, b] \rightarrow \mathbb{R}^2$  we have  $\int_{\gamma} \omega = f(\gamma(b)) - f(\gamma(a))$ .
- for all closed curves  $\gamma : [a, b] \rightarrow \mathbb{R}^2$  we have  $\int_{\gamma} \omega = 0$ .

**Primitive/Potential:** let  $\omega$  be an exact differential 1-form then  $g \in C^1(\mathbb{R}^2)$  such that  $\omega = dg$ .

**Claim:** let  $X : \mathbb{R} \rightarrow \mathbb{R}^3$  and let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  then  $\frac{d}{dt}(f \circ X) = (\nabla f) \cdot \dot{X}$ .

**Conservative Vector Field:** a vector field  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that there exists  $U \in C^1(\mathbb{R}^n)$  for which  $F = -\nabla U$ .

**Kinetic Energy:** let  $X \in C^1(\mathbb{R}, \mathbb{R}^n)$  then  $K : \mathbb{R} \rightarrow \mathbb{R}$  such that  $K(X(t)) = \frac{\|\dot{X}(t)\|^2}{2}$ .

**Total Energy:** let  $U \in C^1(\mathbb{R}^n)$  let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a conservative vector field such that  $F = -\nabla U$  let  $X \in \text{sols}(\ddot{X} = F(X))$  and let  $K : \mathbb{R}^n \rightarrow \mathbb{R}$  be the kinetic energy of  $X$  then  $E : \mathbb{R} \rightarrow \mathbb{R}$  such that  $E = K + U$ .

**Lemma:** let  $U \in C^1(\mathbb{R}^n)$  let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a conservative vector field such that  $F = -\nabla U$  let  $X \in \text{sols}(\ddot{X} = F(X))$  and let  $E : \mathbb{R} \rightarrow \mathbb{R}$  be the total energy of  $X$  then  $\dot{E}(X(t)) = 0$ .

**Weighted Average/Center of Mass:** let  $p_1 \dots p_n \in \mathbb{R}^n$  and let  $w : \{p_1 \dots p_n\} \rightarrow \mathbb{R}_+$  then  $\frac{\sum_{i=1}^n w(p_i) \cdot p_i}{\sum_{i=1}^n w(p_i)}$ .

**Center of Mass:** let  $K \subseteq \mathbb{R}^2$  be compact and let  $\rho : K \rightarrow \mathbb{R}$  then  $\left( \frac{\int_K x \cdot \rho(x, y) dx dy}{\int_K \rho(x, y) dx dy}, \frac{\int_K y \cdot \rho(x, y) dx dy}{\int_K \rho(x, y) dx dy} \right)$ .

**Line:** let  $A, B \in \mathbb{R}^2$  then  $L_{A,B} = \{\lambda A + (1 - \lambda) B \mid \lambda \in [0, 1]\}$ .

**Triangle:** let  $A, B, C \in \mathbb{R}^2$  such that  $A \notin L_{B,C}$  and  $B \notin L_{A,C}$  and  $C \notin L_{A,B}$  then  $\{A, B, C\}$ .

**Theorem Ceva:** let  $A, B, C \in \mathbb{R}^2$  such that  $\{A, B, C\}$  is a triangle let  $A' \in L_{B,C}$  let  $B' \in L_{A,C}$  and let  $C' \in L_{A,B}$  then  $(L_{A,A'} \cap L_{B,B'} \cap L_{C,C'} \neq \emptyset) \iff \left( \frac{d(A,B')}{d(B',C)} \cdot \frac{d(C,A')}{d(A',B)} \cdot \frac{d(B,C')}{d(C',A)} = 1 \right)$ .

**Special Orthogonal Group:** let  $n \in \mathbb{N}$  then  $\text{SO}(n) = \{A \in M_n(\mathbb{R}) \mid (\det(A) = 1) \wedge (A^T = A^{-1})\}$ .

**Cross Product:** let  $x, y \in \mathbb{R}^3$  then  $x \times y = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$ .

**Claim Antisymmetry:** let  $x, y \in \mathbb{R}^3$  then  $x \times y = -y \times x$ .

**Claim Orthogonality:** let  $x, y \in \mathbb{R}^3$  be linearly independent then  $x \times y \perp x$  and  $x \times y \perp y$ .

**Claim:** let  $x, y \in \mathbb{R}^3$  and let  $\theta$  be the angle between  $x, y$  then  $\|x \times y\| = \|x\| \cdot \|y\| \cdot \cos(\theta)$ .

**Corollary:** let  $x, y \in \mathbb{R}^3$  then  $(x \times y = 0) \iff (x, y \text{ are linearly dependent})$ .

**Claim:** let  $x, y, z \in \mathbb{R}^2$  and let  $\alpha, \beta \in \mathbb{R}$  then  $(\alpha x + \beta y) \times z = \alpha(x \times z) + \beta(y \times z)$ .

**Claim:** let  $X, Y \in C^1(\mathbb{R}^3)$  then  $\frac{d}{dt}(X \times Y) = \dot{X} \times Y + X \times \dot{Y}$ .

**Angular Momentum:** let  $X \in C^1(\mathbb{R}^3)$  then  $X \times \dot{X}$ .

**Lemma:** angular momentum is a conserved quantity of  $\ddot{X} = -\frac{X}{\|X\|^2}$ .

**Exact Ordinary Differential Equation:** let  $\Omega \subseteq \mathbb{R}^2$  and let  $F \in C^1(\Omega)$  then  $\frac{\partial F}{\partial x}(x, y) + y' \cdot \frac{\partial F}{\partial y}(x, y) = 0$ .

**Remark:** let  $\Omega \subseteq \mathbb{R}^2$  and let  $F \in C^1(\Omega)$  then  $\frac{\partial F}{\partial x} + y' \cdot \frac{\partial F}{\partial y} = dF$ .

**Claim:** let  $\Omega \subseteq \mathbb{R}^2$  let  $F \in C^1(\Omega)$  and let  $y \in \text{sols}\left(\frac{\partial F}{\partial x}(x, y) + y' \cdot \frac{\partial F}{\partial y}(x, y) = 0\right)$  then  $\frac{d}{dx}(F(x, y(x))) = 0$ .

**Remark:** let  $\Omega \subseteq \mathbb{R}^2$  let  $F \in C^1(\Omega)$  and let  $c \in \mathbb{R}$  then  $\{F = c\}$  are solutions to  $\frac{\partial F}{\partial x}(x, y) + y' \cdot \frac{\partial F}{\partial y}(x, y) = 0$ .

**Claim:** let  $\Omega \subseteq \mathbb{R}^2$  be a simply connected domain and let  $P, Q \in C^1(\Omega)$  then  $(P(x, y) + y' \cdot Q(x, y) = 0)$  is an exact ODE  $\iff (\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x})$ .

**Linear Ordinary Differential Equation:** let  $f : \mathbb{R} \rightarrow \mathbb{R}$  let  $n \in \mathbb{N}$  and let  $a_0 \dots a_{n-1} : \mathbb{R} \rightarrow \mathbb{R}$  then  $y^{(n)} + \sum_{i=0}^{n-1} a_i(x) \cdot y^{(i)} = f$ .

**Claim:** let  $I \subseteq \mathbb{R}$  be an interval let  $p, q : I \rightarrow \mathbb{R}$  and let  $\psi \in \text{sols}(y' + py = q)$  then  $\text{sols}(y' + py = q) = \text{sols}(y' + py = 0) + \psi$ .

**Bernoulli's Equation:** let  $\alpha \in \mathbb{R} \setminus \{0, 1\}$  and let  $p, q : \mathbb{R} \rightarrow \mathbb{R}$  then  $y' = py + qy^\alpha$ .

**Claim:** let  $\alpha \in \mathbb{R} \setminus \{0, 1\}$  and let  $p, q : \mathbb{R} \rightarrow \mathbb{R}$  then

$\text{sols}(y' = py + qy^\alpha) = \text{sols}\left(\frac{1}{1-\alpha} \cdot (y^{1-\alpha})' = p \cdot y^{1-\alpha} + q\right) \cup \{0 \mid \alpha > 0\}$ .

**Riccati's Equation:** let  $p, q, r : \mathbb{R} \rightarrow \mathbb{R}$  then  $y' + p + qy + ry^2 = 0$ .

**Claim:** let  $p, q, r : \mathbb{R} \rightarrow \mathbb{R}$  and let  $\psi \in \text{sols}(y' + p + qy + ry^2 = 0)$  then

$\text{sols}(y' + p + qy + ry^2 = 0) = \text{sols}\left((y + \psi)' + (q + 2r\psi)(y + \psi) + r(y + \psi)^2 = 0\right)$ .

**Fibonacci Sequence:** a sequence  $F : \mathbb{N} \rightarrow \mathbb{N}$  such that  $F_0 = 0$  and  $F_1 = 1$  and  $F_{n+2} = F_{n+1} + F_n$  for  $n \in \mathbb{N}$ .

**Lemma:** let  $n, m \in \mathbb{N}$  then  $F_{m+n+1} = F_m F_n + F_{m+1} F_{n+1}$ .

**Lemma Cassini Identity:** let  $n \in \mathbb{N}_+$  then  $F_n^2 = F_{n+1} F_{n-1} + (-1)^{n-1}$ .

**Left Shift Operator:** a function  $LS : (\mathbb{N} \rightarrow \mathbb{C}) \rightarrow (\mathbb{N} \rightarrow \mathbb{C})$  such that  $LS(x) = \lambda n \in \mathbb{N}.x_{n+1}$ .

**Remark:** let  $x \in \mathbb{N} \rightarrow \mathbb{C}$  then  $(\forall n \in \mathbb{N}.x_{n+2} = x_{n+1} + x_n) \iff (x \in \ker(LS^2 - LS - 1))$ .

**Lemma:** let  $p, q \in \mathbb{C}[x]$  then  $(p \cdot q)(LS) = p(LS) \cdot q(LS)$ .

**Corollary:**  $LS^2 - LS - 1 = \left(LS - \frac{-1+\sqrt{5}}{2}\right)\left(LS - \frac{-1-\sqrt{5}}{2}\right)$ .

**Homogeneous Linear Ordinary Differential Equation:** let  $n \in \mathbb{N}$  and let  $a_0 \dots a_{n-1} : \mathbb{R} \rightarrow \mathbb{R}$  then  $y^{(n)} + \sum_{i=0}^{n-1} a_i(x) \cdot y^{(i)} = 0$ .

**Remark:** In this course we will discuss only homogeneous linear ODEs where the coefficients are constant.

**Characteristic Polynomial of a Homogeneous Linear Ordinary Differential Equation:** let  $n \in \mathbb{N}$  and let  $a_0 \dots a_{n-1} \in \mathbb{R}$  then

$p(x) = x^n + \sum_{i=0}^{n-1} a_i \cdot x^i$ .

**Claim:** let  $n \in \mathbb{N}$  let  $a_0 \dots a_{n-1} \in \mathbb{R}$  and let  $f, g \in \text{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i \cdot y^{(i)} = 0\right)$  then

$f + g \in \text{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i \cdot y^{(i)} = 0\right)$ .

**Theorem:** let  $n \in \mathbb{N}$  let  $a_0 \dots a_{n-1} \in \mathbb{R}$  and let  $\alpha \in \mathbb{R}$  be a solution to  $p(x)$  with multiplicity  $\rho$  then

$\{x^0 e^{\alpha x}, \dots, x^{\rho-1} e^{\alpha x}\} \subseteq \text{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i \cdot y^{(i)} = 0\right)$ .

**Claim:** let  $n \in \mathbb{N}$  let  $a_0 \dots a_{n-1} \in \mathbb{R}$  and let  $f \in \text{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i \cdot y^{(i)} = 0\right)$  then

$\text{Re}(f), \text{Im}(f) \in \text{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i \cdot y^{(i)} = 0\right)$ .

**Complex Quasi Polynomial:** let  $\gamma \in \mathbb{C}$  and let  $p \in \mathbb{C}[x]$  then  $f : \mathbb{C} \rightarrow \mathbb{C}$  such that  $f(z) = e^{\gamma z} \cdot p(z)$ .

**Degree of a Complex Quasi Polynomial:** let  $\gamma \in \mathbb{C}$  and let  $p \in \mathbb{C}[x]$  then  $\deg(e^{\gamma z} \cdot p(z)) = \deg(p)$ .

**Exponent of a Complex Quasi Polynomial:** let  $\gamma \in \mathbb{C}$  and let  $p \in \mathbb{C}[x]$  then  $\gamma$ .

**Real Quasi Polynomial:** let  $\alpha, \beta \in \mathbb{R}$  and let  $p, q \in \mathbb{R}[x]$  then  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(x) = e^{\alpha x}(p(x) \cos(\beta x) + q(x) \sin(\beta x))$ .

**Degree of a Real Quasi Polynomial:** let  $\alpha, \beta \in \mathbb{R}$  and let  $p, q \in \mathbb{R}[x]$  then  $\deg(e^{\alpha x}(p(x) \cos(\beta x) + q(x) \sin(\beta x))) = \max\{\deg(p), \deg(q)\}$ .

**Claim:** let  $P : \mathbb{C} \rightarrow \mathbb{C}$  be a complex quasi polynomial then  $\text{Re}(P)$  is a real quasi polynomial.

**Claim:** let  $n \in \mathbb{N}$  let  $a_0 \dots a_{n-1} \in \mathbb{R}$  let  $\gamma \in \mathbb{C}$  let  $p \in \mathbb{C}[x]$  and let  $k \in \mathbb{N}$  the multiplicity of the root  $\gamma$  in  $x^n + \sum_{i=0}^{n-1} a_i \cdot x^i$  then there exists  $q \in \mathbb{C}[x]$  such that  $\deg(q) = \deg(p)$  and  $t^k q(t) e^{\gamma t} \in \text{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i \cdot y^{(i)} = p \cdot e^{\gamma t}\right)$ .

**Linear System of Ordinary Differential Equation:** let  $n \in \mathbb{N}$  and let  $A \in M_n(\mathbb{R})$  then  $y'(x) = A \cdot y(x)$ .

**Definition:** let  $A \in M_n(\mathbb{R})$  then  $\max(|A|) = \max \left\{ |(A)_{i,j}| \mid i, j \in [n] \right\}$ .

**Lemma:** let  $n, k \in \mathbb{N}$  let  $A \in M_n(\mathbb{R})$  and let  $i, j \in [n]$  then  $(A)_{i,j}^k \leq n^{k-1} \cdot \max(|A|)^k$ .

**Corollary:** let  $n, k \in \mathbb{N}$  let  $A \in M_n(\mathbb{R})$  and let  $i, j \in [n]$  then  $(A)_{i,j}^k \leq (n \cdot \max(|A|))^k$ .

**Theorem:** let  $n \in \mathbb{N}$  and let  $A \in M_n(\mathbb{R})$  then  $\sum_{k=0}^{\infty} \frac{A^k}{k!} \in M_n(\mathbb{R})$ .

**Definition:** let  $n \in \mathbb{N}$  and let  $A \in M_n(\mathbb{R})$  then  $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$ .

**Lemma:** let  $A \in M_n(\mathbb{R})$  and let  $s, t \in \mathbb{R}$  then  $e^{sA} \cdot e^{tA} = e^{(s+t)A}$ .

**Matrix Differentiation:** let  $A : \mathbb{R} \rightarrow M_n(\mathbb{R})$  such that  $(A)_{i,j} \in C^1(\mathbb{R})$  for all  $i, j \in [n]$  then  $(A'(t))_{i,j} = (A(t))'_{i,j}$ .

**Claim:** let  $A, B \in C^1(\mathbb{R}, M_n(\mathbb{R}))$  then  $\frac{d}{dt}(AB) = \dot{A}B + A\dot{B}$ .

**Corollary:** let  $A \in C^1(\mathbb{R}, M_n(\mathbb{R}))$  and let  $v \in C^1(\mathbb{R}, \mathbb{R}^n)$  then  $\frac{d}{dt}(Av) = \dot{A}v + A\dot{v}$ .

**Lemma:** let  $A \in M_n(\mathbb{R})$  then  $\left(\frac{d}{dt}(e^{tA})\right)(0) = A$ .

**Corollary:** let  $A \in M_n(\mathbb{R})$  and let  $\mu \in \mathbb{R}$  then  $\left(\frac{d}{dt}(e^{tA})\right)(\mu) = e^{\mu A}A$ .

**Theorem:** let  $n \in \mathbb{N}$  and let  $A \in M_n(\mathbb{R})$  then  $e^{tA}v$  is a solution to  $\begin{cases} y' = A \cdot y \\ y(0) = v \end{cases}$ .

**Zeckendorff Representation:** let  $n \in \mathbb{N}$  let  $k \in \mathbb{N}$  and let  $c_0 \dots c_k \in \mathbb{N}$  such that  $c_0 \geq 2$  and  $c_{i+1} > c_i + 1$  for all  $i \in \{0 \dots k-1\}$  and  $n = \sum_{i=0}^k F_{c_i}$  then  $(F_{c_0}, \dots, F_{c_k})$ .

**Theorem Zeckendorff:** let  $n \in \mathbb{N}$  then

- Existence: there exists a zeckendorff representation for  $n$ .
- Uniqueness: let  $(F_{c_0}, \dots, F_{c_k}), (F_{d_0}, \dots, F_{d_m})$  be zeckendorff representations for  $n$  then  $k = m$  and  $c_i = d_i$  for all  $i \in \{0 \dots k-1\}$ .

**Theorem Grönwall's Inequality:** let  $\Omega \subseteq \mathbb{R}^2$  let  $f_1, f_2 \in C(\Omega)$  such that  $f_2(x, y) > f_1(x, y)$  for all  $x > 0$  and  $y \in \mathbb{R}$  let  $a_1, a_2 \in \mathbb{R}$  such that  $a_2 > a_1$  let  $y_1 \in \text{sols} \left( \begin{cases} y' = f_1(x, y) \\ y(0) = a_1 \end{cases} \right)$  and let  $y_2 \in \text{sols} \left( \begin{cases} y' = f_2(x, y) \\ y(0) = a_2 \end{cases} \right)$  then  $y_2(x) > y_1(x)$  for all  $x > 0$ .

**Theorem:** let  $\Omega \subseteq \mathbb{R}^2$  let  $f_1, f_2 \in C^1(\Omega)$  such that  $f_2(x, y) \geq f_1(x, y)$  for all  $x > 0$  and  $y \in \mathbb{R}$  let  $a_1, a_2 \in \mathbb{R}$  such that  $a_2 \geq a_1$  let  $y_1 \in \text{sols} \left( \begin{cases} y' = f_1(x, y) \\ y(0) = a_1 \end{cases} \right)$  and let  $y_2 \in \text{sols} \left( \begin{cases} y' = f_2(x, y) \\ y(0) = a_2 \end{cases} \right)$  then  $y_2(x) \geq y_1(x)$  for all  $x > 0$ .

**Wronskian:** let  $\mathcal{U} \subseteq \mathbb{R}$  and let  $u, v \in C^1(\mathcal{U})$  then  $W_{u,v} : \mathcal{U} \rightarrow \mathbb{R}$  such that  $W_{u,v}(t) = \det \begin{pmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{pmatrix}$ .

**Corollary:** let  $\mathcal{U} \subseteq \mathbb{R}$  let  $u \in C^1(\mathcal{U})$  and let  $c \in \mathbb{R}$  then  $W_{u, c \cdot u} = 0$ .

**Remark:** let  $\mathcal{U} \subseteq \mathbb{R}$  and let  $u, v \in C^1(\mathcal{U})$  then  $W = W_{u,v}$ .

**Theorem Abel's Identity:** let  $I \subseteq \mathbb{R}$  be an open interval let  $p, q : I \rightarrow \mathbb{R}$  and let  $u, v \in \text{sols}(y'' + py' + qy = 0)$  then  $W_{u,v} \in \text{sols}(y' + py = 0)$ .

**Corollary:** let  $I \subseteq \mathbb{R}$  be an open interval let  $p, q : I \rightarrow \mathbb{R}$  and let  $u, v \in \text{sols}(y'' + py' + qy = 0)$  then there exists  $p \in I$  and  $c \in \mathbb{R}$  such that  $W_{u,v}(t) = \exp(c) \cdot W_{u,v}(p) \cdot \exp\left(-\int_p^t p(s) ds\right)$  for all  $t \in I$ .

**Corollary:** let  $I \subseteq \mathbb{R}$  be an open interval let  $p, q : I \rightarrow \mathbb{R}$  and let  $u, v \in \text{sols}(y'' + py' + qy = 0)$  then  $\text{sign}(W(t_1)) = \text{sign}(W(t_2))$  for all  $t_1, t_2 \in I$ .

**Corollary:** let  $I \subseteq \mathbb{R}$  be an open interval let  $p, q : I \rightarrow \mathbb{R}$  and let  $u, v \in \text{sols}(y'' + py' + qy = 0)$  then  $(u, v \text{ are linearly dependent}) \iff (\text{sign}(W) = 0)$ .

**Corollary Complementary Solution:** let  $I \subseteq \mathbb{R}$  be an open interval let  $p, q : I \rightarrow \mathbb{R}$  and let  $u \in \text{sols}(y'' + py' + qy = 0) \setminus \{0\}$  then  $u \int \frac{W}{u^2}$  is a solution of  $y'' + py' + qy = 0$  and  $u, u \int \frac{W}{u^2}$  are linearly independent.

**Claim Variation of Parameters:** let  $I \subseteq \mathbb{R}$  be an open interval let  $p, q, f : I \rightarrow \mathbb{R}$  let  $u, v \in \text{sols}(y'' + py' + qy = 0)$  such that  $u, v$  are linearly independent and let  $C_1, C_2 \in C^1(I)$  such that  $C_1u + C_2v$  is a solution of  $y'' + py' + qy = f$  and  $C_1'u + C_2'v = 0$  then  $\begin{cases} C_1'u + C_2'v = 0 \\ C_1'u' + C_2'v' = f \end{cases}$ .

**Corollary Lagrange's Formula:** let  $I \subseteq \mathbb{R}$  be an open interval let  $p, q, f : I \rightarrow \mathbb{R}$  and let  $u, v \in \text{sols}(y'' + py' + qy = 0)$  such that  $u, v$  are linearly independent then  $-u \int \frac{fv}{W_{u,v}} + v \int \frac{fu}{W_{u,v}}$  is a solution of  $y'' + py' + qy = f$ .

**Green Function:** let  $\mathcal{U} \subseteq \mathbb{R}$  and let  $u, v \in C^1(\mathcal{U})$  then  $G_{u,v} : \mathcal{U}^2 \rightarrow \mathbb{R}$  such that  $G_{u,v}(s, t) = \frac{v(t)u(s) - u(t)v(s)}{u(s)v'(s) - u'(s)v(s)}$ .

**Claim:** let  $\mathcal{U} \subseteq \mathbb{R}$  and let  $u, v \in C^1(\mathcal{U})$  then

- $G(t, t) = 0$  for all  $t \in \mathcal{U}$ .
- $\frac{\partial G}{\partial y}(t, t) = 1$  for all  $t \in \mathcal{U}$ .

**Corollary:** let  $I \subseteq \mathbb{R}$  be an open interval let  $p, q, f : I \rightarrow \mathbb{R}$  and let  $u, v \in \text{sols}(y'' + py' + qy = 0)$  such that  $u, v$  are linearly independent then there exists  $p \in I$  such that  $\int_p^t G_{u,v}(s, t) f(s) ds$  is a solution of  $y'' + py' + qy = f$ .

**Lemma:** let  $I \subseteq \mathbb{R}$  be an open interval let  $p, q, f : I \rightarrow \mathbb{R}$  let  $u, v \in \text{sols}(y'' + py' + qy = 0)$  such that  $u, v$  are linearly independent let  $p \in I$  such that  $\int_p^t G_{u,v}(s, t) f(s) ds$  is a solution of  $y'' + py' + qy = f$  and let  $g \in \text{sols} \left( \begin{array}{l} y'' + py' + qy = f \\ y(p) = 0 \\ y'(p) = 0 \end{array} \right)$  then  $g(t) = \int_p^t G_{u,v}(s, t) f(s) ds$  for all  $t \in I$ .

**Simple Zero:** let  $f \in C^1(\mathbb{R})$  then  $p \in \ker(f)$  such that  $f'(p) \neq 0$ .

**Isolated Zero:** let  $f : \mathbb{R} \rightarrow \mathbb{R}$  then  $p \in \ker(f)$  such that there exists a neighbourhood  $\mathcal{U}$  of  $p$  for which  $0 \notin f(\mathcal{U} \setminus \{p\})$ .

**Lemma:** let  $I \subseteq \mathbb{R}$  be an open interval let  $p, q : I \rightarrow \mathbb{R}$  let  $u \in \text{sols}(y'' + py' + qy = 0) \setminus \{0\}$  and let  $p \in \ker(u)$  then  $p$  is a simple and isolated zero of  $u$ .

**Lemma:** let  $I \subseteq \mathbb{R}$  be an open interval let  $p, q : I \rightarrow \mathbb{R}$  and let  $u, v \in \text{sols}(y'' + py' + qy = 0)$  such that  $u, v$  are linearly independent then  $\ker(u) \cap \ker(v) = \emptyset$ .

**Theorem Sturm's Separation Theorem:** let  $I \subseteq \mathbb{R}$  be an open interval let  $p, q : I \rightarrow \mathbb{R}$  and let  $u, v \in \text{sols}(y'' + py' + qy = 0)$  such that  $u, v$  are linearly independent then for all  $z, w \in \ker(u)$  if  $(z, w) \cap \ker(u) = \emptyset$  then  $|(z, w) \cap \ker(v)| = 1$ .

**Theorem Sturm's Comparison Theorem:** let  $I \subseteq \mathbb{R}$  be an open interval let  $p, q : I \rightarrow \mathbb{R}$  such that  $p \geq q$  and  $p \neq q$  let  $u \in \text{sols}(y'' + py = 0)$  let  $v \in \text{sols}(y'' + qy = 0)$  such that  $u, v$  are linearly independent and let  $z, w \in \ker(v)$  then  $(z, w) \cap \ker(u) \neq \emptyset$ .

**Claim:** let  $k < 0$  then  $e^{\sqrt{-k} \cdot t}$  is a solution of  $y'' + ky = 0$ .

**Claim:** the function 1 is a solution of  $y'' = 0$ .

**Corollary:** let  $I \subseteq \mathbb{R}$  be an open interval let  $q : I \rightarrow \mathbb{R}$  such that  $0 \geq q$  and let  $v \in \text{sols}(y'' + qy = 0)$  such that  $v \neq 0$  then  $|\ker(v)| \leq 1$ .

**Claim:** let  $k > 0$  then  $\sin(\sqrt{k} \cdot t)$  is a solution of  $y'' + ky = 0$ .

**Supremum Norm/ $L^\infty$  Norm:** let  $a, b \in \mathbb{R}$  such that  $a \leq b$  and let  $f \in C([a, b])$  then  $\|f\|_\infty = \max(\text{Im}(|f|))$ .

**Claim:** let  $a, b \in \mathbb{R}$  such that  $a \leq b$  then  $\|\cdot\|_\infty$  is a norm on  $C([a, b])$ .

**Supremum Metric:** let  $a, b \in \mathbb{R}$  such that  $a \leq b$  and let  $f, g \in C([a, b])$  then  $d_\infty(f, g) = \|f - g\|_\infty$ .

**Claim:** let  $a, b \in \mathbb{R}$  such that  $a \leq b$  then  $d_\infty$  is a metric on  $C([a, b])$ .

**Claim:** let  $a, b \in \mathbb{R}$  such that  $a \leq b$  let  $f \in C([a, b])$  and let  $\{f_n\}_{n=0}^\infty \subseteq C([a, b])$  then  $(f_n \xrightarrow{L^\infty} f) \iff (f_n \xrightarrow{u} f)$ .

**Theorem:** let  $a, b \in \mathbb{R}$  such that  $a \leq b$  then  $(C([a, b]), d_\infty)$  is complete.

**Closed Set:** let  $(X, d)$  be a metric space then  $A \subseteq X$  such that for all  $\{a_n\}_{n=0}^\infty \subseteq A$  and for all  $L \in X$  if  $a_n \rightarrow L$  then  $L \in A$ .

**Fixed Point:** let  $X$  be a set and let  $f : X \rightarrow X$  then  $p \in X$  such that  $f(p) = p$ .

**Contraction:** let  $(X, d)$  be a metric space then  $f : X \rightarrow X$  such that there exists  $\rho \in [0, 1)$  for which  $d(f(x), f(y)) \leq \rho \cdot d(x, y)$  for all  $x, y \in X$ .

**Theorem Contraction Mapping Theorem:** let  $(X, d)$  be a complete metric space such that  $X \neq \emptyset$  and let  $f : X \rightarrow X$  be a contraction then there exists a unique  $p \in X$  for which  $f(p) = p$ .

**Lipschitz Function:** let  $(X, d)$  be a metric space then  $f : X \rightarrow \mathbb{R}$  such that there exists  $L \in \mathbb{R}$  for which

$|f(x) - f(y)| \leq L \cdot d(x, y)$  for all  $x, y \in X$ .

**Lipschitz Constant:** let  $(X, d)$  be a metric space and let  $f : X \rightarrow \mathbb{R}$  be lipschitz then minimal  $L \in \mathbb{R}$  such that

$|f(x) - f(y)| \leq L \cdot d(x, y)$  for all  $x, y \in X$ .

**Claim:** let  $(X, d)$  be a metric space and let  $f : X \rightarrow \mathbb{R}$  be lipschitz then  $f \in C(X)$ .

**Locally Lipschitz Function:** let  $(X, d)$  be a metric space then  $f : X \rightarrow \mathbb{R}$  such that for all  $x \in X$  there exists a neighbourhood  $\mathcal{U}$  of  $x$  for which  $f|_{\mathcal{U}}$  is lipschitz.

**Theorem Picard-Lindelöf:** let  $I, J \subseteq \mathbb{R}$  be closed intervals let  $f \in C(I \times J)$  such that  $f$  is locally lipschitz in the second coordinate and let  $p \in \text{int}(I \times J)$  then there exists  $\varepsilon > 0$  such that  $\begin{cases} y' = f(x, y) \\ y(p_1) = p_2 \end{cases}$  has a unique solution on the interval  $(p_1 - \varepsilon, p_1 + \varepsilon)$ .

**Compact Set:** let  $(X, d)$  be a metric space then  $K \subseteq X$  such that for all  $\{x_n\}_{n=0}^\infty \subseteq K$  there exists a convergent subsequence.

**Uniformly Bounded Sequence of Functions:** let  $a, b \in \mathbb{R}$  such that  $a \leq b$  then  $\{f_n\}_{n=0}^\infty \subseteq C([a, b])$  such that  $\exists M \in \mathbb{R} . \forall x \in [a, b] . \forall n \in \mathbb{N} . |f_n(x)| \leq M$ .



**Uniformly Equicontinuous Sequence of Functions:** let  $a, b \in \mathbb{R}$  such that  $a \leq b$  then  $\{f_n\}_{n=0}^{\infty} \subseteq C([a, b])$  such that  $\forall \varepsilon > 0. \exists \delta > 0. \forall x, y \in [a, b]. |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$ .

**Theorem Arzela-Ascoli:** let  $a, b \in \mathbb{R}$  such that  $a \leq b$  and let  $\{f_n\}_{n=0}^{\infty} \subseteq C([a, b])$  then  $\{\{f_n\}_{n=0}^{\infty} \text{ has a convergent subsequence}\} \iff \{\{f_n\}_{n=0}^{\infty} \text{ is uniformly bounded and uniformly equicontinuous}\}$ .

**Notation:** let  $p \in \mathbb{R}^2$  and let  $f : \mathbb{R} \rightarrow \mathbb{R}$  then  $\eta_f(p) : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\eta_f(p)(x) = f(p) \cdot (x - p_1) + p_2$ .

**Algorithm Euler's Method:** let  $n \in \mathbb{N}$  let  $f \in C(\mathbb{R})$  let  $\varepsilon \in \mathbb{R}$  and let  $p \in \mathbb{R}^2$  then

**function** EulerMethod( $n, f, \varepsilon, p$ ):

```

    ( $x_0, y_0$ )  $\leftarrow p$ 
    for  $i \leftarrow \{1 \dots n\}$  do
         $x_i \leftarrow x_0 + \varepsilon \cdot \frac{i}{n}$ 
         $g_i \leftarrow (\lambda x \in [x_{i-1}, x_i]. (f(x_{i-1}, y_{i-1}) \cdot (x - x_{i-1}) + y_{i-1}))$ 
         $y_i \leftarrow g_i(x_i)$ 
    end
    return  $\bigcup_{i=1}^n g_i$ 

```

**Claim:** let  $n \in \mathbb{N}$  let  $f \in C(\mathbb{R})$  let  $\varepsilon \in \mathbb{R}$  and let  $p \in \mathbb{R}^2$  then EulerMethod( $n, f, \varepsilon, p$ ) :  $[p_1, p_1 + \varepsilon] \rightarrow \mathbb{R}$ .

**Notation:** let  $n \in \mathbb{N}$  let  $f \in C(\mathbb{R})$  let  $\varepsilon \in \mathbb{R}$  and let  $p \in \mathbb{R}^2$  then EulerMethod $_n = \text{EulerMethod}(n, f, \varepsilon, p)$ .

**Claim:** let  $f \in C(\mathbb{R})$  let  $\varepsilon \in \mathbb{R}$  let  $p \in \mathbb{R}^2$  then there exists a convergent subsequence  $\{\text{EulerMethod}_{n_k}\}$  such that  $\lim_{k \rightarrow \infty} \text{EulerMethod}_{n_k}$  is a solution of  $\begin{cases} y' = f(x, y) \\ y(p_1) = p_2 \end{cases}$  defined on  $[p_1, p_1 + \varepsilon]$ .

**Boundary Value Problem (BVP):** let  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  and let  $a, b \in \mathbb{R}$  then  $\begin{cases} f(x, y', \dots, y^{(n)}) = 0 \\ y(0) = a \\ y(1) = b \end{cases}$ .

**Theorem:** let  $k \in C([0, 1])$  such that  $k > 0$  and let  $a, b \in \mathbb{R}$  then  $\left| \text{sols} \left( \begin{cases} y'' = ky \\ y(0) = a \\ y(1) = b \end{cases} \right) \right| = 1$ .