Notation: let X be a set and let $y : \mathbb{R} \to \mathbb{R}$ then $y^{(0)} = y$.

Notation: let X, Y be sets let $n \in \mathbb{N}_+$ and let $y \in C^n(X, Y)$ then $y^{(n)} = \left(y^{(n-1)}\right)'$.

Ordinary Differential Equation (ODE): let $n \in \mathbb{N}$ and let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ then $f\left(x, y(x), y'(x), \dots y^{(n)}(x)\right) = 0$.

Solution of an Ordinary Differential Equation: let $n \in \mathbb{N}$ let $\mathcal{U} \subseteq \mathbb{R}$ and let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ then a function $y \in C^n(\mathcal{U})$ such that $f\left(x, y(x), \dots, y^{(n)}(x)\right) = 0$ for all $x \in \mathcal{U}$.

Notation: let $n \in \mathbb{N}$ let $\mathcal{U} \subseteq \mathbb{R}$ and let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ then

$\text{sols}\left(f\left(x, y(x), y'(x), \dots y^{(n)}(x)\right) = 0\right) = \left\{y \in C^n(\mathcal{U}) \mid f\left(x, y(x), \dots, y^{(n)}(x)\right) = 0\right\}$.

First Order Ordinary Differential Equation: let $f : \mathbb{R}^3 \to \mathbb{R}$ then $f(x, y, y') = 0$.

Separable Ordinary Differential Equation: let $f, g : \mathbb{R} \to \mathbb{R}$ then $y' = f(y)g(x)$.

Claim Separation of Variables: let $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) \neq 0$ for all $x \in \mathbb{R}$ then

$\text{sols}\left(y' = f(y)g(x)\right) = \text{sols}\left(\int \frac{1}{f(y)}\mathrm{d}y = \int g(x)\,\mathrm{d}x\right)$.

Corollary: $\text{sols}(y = y') = \{Ce^x \mid C \in \mathbb{R}\}$.

Curves that foliate a domain: Let $\Omega \subseteq \mathbb{R}^2$ and let $I \subseteq \mathbb{R}$ then a set of curves $\{\gamma_\alpha\} \subseteq I \to \mathbb{R}$ such that $\biguplus \gamma_i = \Omega$.

Claim: $\text{sols}(y = y')$ foliate the plane.

Remark: let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and let $g \in \text{sols}\left(f\left(x, y, \dots, y^{(n)}\right)\right)$ such that g is piecewise continuous then each continuous branch of g is a solution of the ODE.

Initial Value Problem (IVP): let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and let $p \in \mathbb{R}^2$ then $\begin{cases} f\left(x, y', \dots y^{(n)}\right) = 0 \\ y(p_1) = p_2 \end{cases}$.

Theorem Cauchy-Peano: let $I, J \subseteq \mathbb{R}$ be closed intervals let $f \in C^1(I \times J)$ and let $p \in \text{int}(I \times J)$ then there exists $\varepsilon > 0$ such that $\begin{cases} y' = f(x, y) \\ y(p_1) = p_2 \end{cases}$ has a solution on the interval $(p_1 - \varepsilon, p_1 + \varepsilon)$.

Theorem of Existence and Uniqueness: let $I, J \subseteq \mathbb{R}$ be closed intervals let $f \in C^1(I \times J)$ and let $p \in \text{int}(I \times J)$ then there exists $\varepsilon > 0$ such that $\begin{cases} y' = f(x, y) \\ y(p_1) = p_2 \end{cases}$ has a unique solution on the interval $(p_1 - \varepsilon, p_1 + \varepsilon)$.

Corollary: let $I, J \subseteq \mathbb{R}$ be closed intervals and let $f \in C^1(I \times J)$ then $\text{sols}(y' = f(x, y))$ foliates $I \times J$.

Vector Field: let $\Omega \subseteq \mathbb{R}^2$ be a domain then $\nu \in C(\Omega, \mathbb{R}^2)$.

Integral curve: let $\nu : \Omega \to \mathbb{R}^2$ be a vector field then a curve $\gamma : [a, b] \to \Omega$ such that $\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = \nu(g(t))$.

Claim: let $\nu : \Omega \to \mathbb{R}^2$ be a vector field such that $v(x) \neq 0$ for all $x \in \Omega$ and let $\gamma : [a, b] \to \Omega$ be a curve such that $v(\gamma(t)) \in T_{\gamma(t)}(\text{Im}(\gamma))$ for all $t \in [a, b]$ then there exists a curve $\eta : [\alpha, \beta] \to [a, b]$ such that $\gamma \circ \eta$ is an integral curve.

C^n Vector Field: let $\Omega \subseteq \mathbb{R}^2$ be a domain then $\nu \in C^n(\Omega, \mathbb{R}^2)$.

Claim: let $\nu : \Omega \to \mathbb{R}^2$ be a C^1 vector field then $\{\gamma : [a, b] \to \Omega \mid (a, b \in \mathbb{R}) \wedge (\gamma \text{ is an integral curve of } \nu)\}$ foliates Ω.

Theorem Peano: let $\nu : \Omega \to \mathbb{R}^2$ be a vector field then $\{\gamma : [a, b] \to \Omega \mid (a, b \in \mathbb{R}) \wedge (\gamma \text{ is an integral curve of } \nu)\}$ covers Ω.

Lemma: let $\Omega \subseteq \mathbb{R}^2$ be a domain and let $f \in C^1(\Omega)$ then $\left(\begin{smallmatrix} 1 \\ f \end{smallmatrix}\right)$ is a vector field.

Theorem of Existence and Uniqueness: let $\Omega \subseteq \mathbb{R}^2$ be a domain let $f \in C^1(\Omega)$ and let $g \in C^1(\mathbb{R})$ then $(g' = f(x, g)) \iff \left(\left(\begin{smallmatrix} x \\ g \end{smallmatrix}\right) \text{ is an integral curve of } \left(\begin{smallmatrix} 1 \\ f \end{smallmatrix}\right)\right)$.

Autonomous Ordinary Differential Equations: let $f : \mathbb{R} \to \mathbb{R}$ then $y' = f(y)$.

Logistic Equation: let $L > 0$ then $\dot{P}(t) = P(t)(L - P(t))$.

Equilibrium Solution: let $f : \mathbb{R} \to \mathbb{R}$ then a function $g \in \text{sols}(y' = f(y))$ such that $f(g(x)) = 0$ for all $x \in \mathbb{R}$.

Corollary: let $f \in C(\mathbb{R})$ and let $g : \mathbb{R} \to \mathbb{R}$ be an equilibrium solution then g is constant.

Corollary: let $L > 0$ then $\{0, L\}$ are the equilibrium solutions of the logistic equation.

Logistic Equation With Harvesting: let $L, k > 0$ then $\dot{P}(t) = P(t)(L - P(t)) - k$.

Stable Equilibrium Solution: let $f : \mathbb{R} \to \mathbb{R}$ then an equilibrium solution $g : \mathbb{R} \to \mathbb{R}$ such that for all $y \in \text{sols}(y' = f(y))$ and for all $\varepsilon > 0$ if there exists $\delta > 0$ and $p \in \mathbb{R}$ for which $|y(p) - g(p)| < \delta$ then $|y(t) - g(t)| < \varepsilon$ for all $t > p$.

Notation: let X be a set and let $\gamma : \mathbb{R} \to X$ then $\dot{\gamma} = \frac{\mathrm{d}\gamma}{\mathrm{d}t}$.

Notation: let X be a set and let $\gamma : \mathbb{R} \to X$ then $\ddot{\gamma} = \frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2}$.

Remark: the notation $\dot{\gamma}, \ddot{\gamma}$ is often used when γ is a function of time.

System of Ordinary Differential Equation: let $n \in \mathbb{N}$ let $m \in \mathbb{N}_+$ and let $f : \mathbb{R} \times (\mathbb{R}^m)^n \to \mathbb{R}^m$ then $f\left(t, y(t), y'(t), \dots y^{(n)}(t)\right) = 0$.

Remark: all the theory of first order ODEs also applies to vectors of first order ODEs.

Lotka–Volterra equations: let $\alpha > 0$ then $\begin{cases} \dot{x}(t)=x(t)\cdot(1-y(t)) \\ \dot{y}(t)=\alpha\cdot y(t)\cdot(x(t)-1) \end{cases}$.

Claim: let $\alpha > 0$ then $\left\{ \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \right\}$ are the equilibrium solutions of the lotka-volterra equation.

Conserved Quantity: let $\mathcal{U} \subseteq \mathbb{R}^n$ and let $f : \mathcal{U} \to \mathbb{R}$ then a function $g \in C^1(\mathcal{U})$ such that $g \circ y$ is constant for all $y \in \text{sols}\,(y' = f(y))$.

Claim: let $\alpha > 0$ and let $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in C^1\left(\mathbb{R}^2\right)$ be a solution to the lotka-volterra equation then

$\frac{\mathrm{d}}{\mathrm{d}t}\left((y(t) - 1 - \log(y(t))) + \alpha(x(t) - 1 - \log(x(t)))\right) = 0.$

Corollary: let $\alpha > 0$ let $p \in \mathbb{R}^2$ and let $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in C^1\left(\mathbb{R}^2\right)$ be a solution to $\begin{cases} \dot{x}(t)=x(t)\cdot(1-y(t)) \\ \dot{y}(t)=\alpha\cdot y(t)\cdot(x(t)-1) \\ \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)(0)=p \end{cases}$ then $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ is periodic.

Lemma: let $\alpha > 0$ let $p \in \mathbb{R}^2$ and let $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in C^1\left(\mathbb{R}^2\right)$ be a solution to $\begin{cases} \dot{x}(t)=x(t)\cdot(1-y(t)) \\ \dot{y}(t)=\alpha\cdot y(t)\cdot(x(t)-1) \\ \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)(0)=p \end{cases}$ with period T then $\frac{1}{T}\int_0^T x\mathrm{d}t = 1$

and $\frac{1}{T}\int_0^T y\mathrm{d}t = 1$.

Claim Linear Substitution: let $a, b, c \in \mathbb{R}$ let $f : \mathbb{R} \to \mathbb{R}$ and let $y \in \text{sols}\,(y' = f(ax + by + c))$ then the function $z : \mathbb{R} \to \mathbb{R}$ which defined by $z(x) = ax + by + c$ satisfies $z' = a + bf(z)$.

Notation: let $\Omega \subseteq \mathbb{R}^2$ be a domain let $f : \Omega \to \mathbb{R}$ and let $p \in \mathbb{R}^2$ then (α_-, α_+) is the maximal interval where a solution to $\begin{cases} y'=f(x,y) \\ y(p_1)=p_2 \end{cases}$ is defined.

Corollary: let $\Omega \subseteq \mathbb{R}^2$ be a domain let $f \in C^1(\Omega)$ and let $p \in \mathbb{R}^2$ then $\alpha_- < p_1 < \alpha_+$.

Theorem: let $f \in C^1\left(\mathbb{R}^2\right)$ and let $p \in \mathbb{R}^2$ then

- $(\alpha_+ < \infty) \Longleftrightarrow \left(\lim_{x \to \alpha_+^-} |y(x)| = \infty\right)$.
- $(-\infty < \alpha_-) \Longleftrightarrow \left(\lim_{x \to \alpha_-^+} |y(x)| = \infty\right)$.

Theorem Extensibility of Solutions: let $D \subseteq \mathbb{R}^{n+1}$ be a closed set let $f \in C^1(D, \mathbb{R}^n)$ and let $y \in \text{sols}\,(y' = f(x,y))$ such that $\Gamma_y \cap D \neq \varnothing$ then there exists $\psi : [a,b] \to \mathbb{R}^n$ such that $\psi \in \text{sols}\,(y' = f(x,y))$ and $\psi(x) = y(x)$ for all $x \in \text{Dom}\,(y)$ and $\left(\begin{smallmatrix} a \\ \psi(a) \end{smallmatrix}\right), \left(\begin{smallmatrix} b \\ \psi(b) \end{smallmatrix}\right) \in \partial D$.

Corollary: let $D \subseteq \mathbb{R}^{n+1}$ such that for all $\alpha, \beta \in \mathbb{R}$ the set $D \cap \{\alpha \leq x_1 \leq \beta\}$ is bounded let $f \in C^1(D, \mathbb{R}^n)$ and let $y \in \text{sols}\,(y' = f(x,y))$ such that $\Gamma_y \cap D \neq \varnothing$ then one of the next statements is true

- there exists $\psi : [a,b] \to \mathbb{R}^n$ such that $\psi \in \text{sols}\,(y' = f(x,y))$ and $\psi(x) = y(x)$ for all $x \in \text{Dom}\,(y)$ and $\left(\begin{smallmatrix} a \\ \psi(a) \end{smallmatrix}\right), \left(\begin{smallmatrix} b \\ \psi(b) \end{smallmatrix}\right) \in \partial D$.
- there exists $\psi : \mathbb{R} \to \mathbb{R}^n$ such that $\psi \in \text{sols}\,(y' = f(x,y))$ and $\psi(x) = y(x)$ for all $x \in \text{Dom}\,(y)$.

Corollary: let $I \subseteq \mathbb{R}$ be an open interval let $D \subseteq \mathbb{R}^n$ let $f \in C^1(I \times D, \mathbb{R}^n)$ and let $p, q : \mathbb{R} \to \mathbb{R}$ such that $\|f(x,y)\| \leq p(x)\|y\| + q(x)$ then for all $y \in \text{sols}\,(y' = f(x,y))$ such that $\Gamma_y \cap (I \times D) \neq \varnothing$ there exists $\psi : I \to \mathbb{R}^n$ such that $\psi \in \text{sols}\,(y' = f(x,y))$ and $\psi(x) = y(x)$ for all $x \in \text{Dom}\,(y)$.

Theorem Continuous Dependence on Initial Conditions: let $\Omega \subseteq \mathbb{R}^2$ let $f \in C^1(\Omega)$ let $a, b, p \in \mathbb{R}$ and let y_p be a solution to $\begin{cases} y'=f(x,y) \\ y(a)=p \end{cases}$ such that y_p is defined on $[a,b]$ then for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $q \in \mathbb{R}$ for which $|p - q| < \delta$ if y_q is a solution to $\begin{cases} y'=f(x,y) \\ y(a)=q \end{cases}$ then y_q is defined on $[a,b]$ and $|y_p - y_q| < \varepsilon$.

Theorem Continuous Dependence on Initial Conditions: let $\Omega \subseteq \mathbb{R}^2$ let $f \in C^1(\Omega)$ let $a, b \in \mathbb{R}$ let $p \in \mathbb{R}$ let $\{p_n\}_{n=0}^\infty \subseteq \mathbb{R}$ such that $p_n \to p$ and let $\{g_{p_n}\}_{n=0}^\infty \subseteq \mathbb{R} \to \mathbb{R}$ such that $g_{p_n} \in \text{sols}\left(\begin{cases} y'=f(x,y) \\ y(a)=p_n \end{cases}\right)$ and g_{p_n} is defined on $[a,b]$ for all $n \in \mathbb{N}$ then there exists $g : \mathbb{R} \to \mathbb{R}$ such that $g_{p_n} \xrightarrow{p.w.} g$ and $g \in \text{sols}\left(\begin{cases} y'=f(x,y) \\ y(a)=p \end{cases}\right)$.

n-th Order Ordinary Differential Equation: let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ then $f\left(x, y \ldots y^{(n)}\right) = 0$.

Claim: let $n \in \mathbb{N}_+$ let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}$ then

$\left(g \in \text{sols}\left(f\left(t, x(t) \ldots x^{(n)}(t)\right) = 0\right)\right) \Longleftrightarrow \left(\left(\begin{smallmatrix} g \\ \vdots \\ g^{(n)} \end{smallmatrix}\right) \in \text{sols}\left(\begin{cases} \dot{y_1}=\dot{x} \\ \vdots \\ \dot{y}_{n-1}=y_{n-2} \\ \dot{y}_{n-1}=f(t,x,y_1\ldots,y_{n-1}) \end{cases}\right)\right)$

Harmonic Oscillator/Spring Position Equation: $\ddot{x} = -x$.

Claim: the function $E(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$ is a conserved quantity of $\begin{cases} \dot{x}=y \\ \dot{y}=-x \end{cases}$.

Spring with Friction Position Equation: $\ddot{x} = -x - \dot{x}$.

Constant Tension Spring Position Equation: $\ddot{x} = -1$.

Claim: the function $E(x,y) = x + \frac{1}{2}y^2$ is a conserved quantity of $\begin{cases} \dot{x}=y \\ \dot{y}=-1 \end{cases}$.

Gravity in One Dimension Equation: $\ddot{x} = -\frac{1}{x^2}$.

Claim: the function $E(x,y) = \frac{1}{x} + \frac{1}{2}y^2$ is a conserved quantity of $\begin{cases} \dot{x}=y \\ \dot{y}=-\frac{1}{x^2} \end{cases}$.

Pendulum Equation: $\ddot{x} = -\sin(x)$.

Claim: the function $E(x,y) = \frac{1}{2}y^2 - \cos(x)$ is a conserved quantity of $\begin{cases} \dot{x}=y \\ \dot{y}=-\sin(x) \end{cases}$.

Differential 1-form: let $P, Q : \mathbb{R}^2 \to \mathbb{R}$ then $P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$.

Remark: in this course we won't explain the formal definition of a differential 1-form so we will use it as an object with certain properties.

Integral: let $\gamma : [a,b] \to \mathbb{R}^2$ be a curve and let $P, Q : \mathbb{R}^2 \to \mathbb{R}$ then

$\int_\gamma (P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y) = \lim_{\Delta \to 0} \sum_{i=0}^{n-1} (P(x_i,y_i)(x_{i+1}-x_i) + Q(x_i,y_i)(y_{i+1}-y_i))$.

Remark: in the definition above $\lim_{\Delta \to 0}$ is the limit of all partitions of γ to horizontal and vertical segments.

Claim: let $\gamma : [a,b] \to \mathbb{R}^n$ be a curve let $\nu : [\alpha,\beta] \to [a,b]$ such that $\gamma \circ \nu$ is a reparameterization of γ and let ω be a differential 1-form then $\int_\gamma \omega = \int_{\gamma \circ \nu} \omega$.

Claim: let $\gamma : [a,b] \to \mathbb{R}^n$ be a curve and let ω be a differential 1-form then $\int_\gamma \omega = -\int_{-\gamma} \omega$.

Integral: let $\gamma : [a,b] \to \mathbb{R}^2$ be a curve and let $f, g : \mathbb{R}^2 \to \mathbb{R}$ then $\int_\gamma f\mathrm{d}g = \lim_{\Delta \to 0} \sum_{i=0}^{n-1} (f(x_i,y_i)(g(x_{i+1},y_{i+1}) - g(x_i,y_i)))$.

Claim: let $g \in C^1(\mathbb{R}^2)$ then $\mathrm{d}g = \frac{\partial g}{\partial x}\mathrm{d}x + \frac{\partial g}{\partial y}\mathrm{d}y$.

Corollary: let $\gamma : [a,b] \to \mathbb{R}^2$ be a curve let $f : \mathbb{R}^2 \to \mathbb{R}$ and let $g \in C^1(\mathbb{R}^2)$ then $\int_\gamma f\mathrm{d}g = \int_\gamma \left(f \cdot \frac{\partial g}{\partial x}\mathrm{d}x + f \cdot \frac{\partial g}{\partial y}\mathrm{d}y \right)$.

Exact Differential 1-form: a differential 1-form ω such that there exists $g \in C^1(\mathbb{R}^2)$ for which $\omega = \mathrm{d}g$.

Theorem: let ω be a differential 1-form then TFAE

- ω is exact.
- there exists $f : \mathbb{R} \to \mathbb{R}$ such that for all curves $\gamma : [a,b] \to \mathbb{R}^2$ we have $\int_\gamma \omega = f(\gamma(b)) - f(\gamma(a))$.
- for all closed curves $\gamma : [a,b] \to \mathbb{R}^2$ we have $\int_\gamma \omega = 0$.

Primitive/Potential: let ω be an exact differential 1-form then $g \in C^1(\mathbb{R}^2)$ such that $\omega = \mathrm{d}g$.

Claim: let $X : \mathbb{R} \to \mathbb{R}^3$ and let $f : \mathbb{R}^3 \to \mathbb{R}$ then $\frac{\mathrm{d}}{\mathrm{d}t}(f \circ X) = (\nabla f) \cdot \dot{X}$.

Conservative Vector Field: a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ such that there exists $U \in C^1(\mathbb{R}^n)$ for which $F = -\nabla U$.

Kinetic Energy: let $X \in C^1(\mathbb{R}, \mathbb{R}^n)$ then $K : \mathbb{R}^n \to \mathbb{R}$ such that $K(X(t)) = \frac{\left\| \dot{X}(t) \right\|^2}{2}$.

Total Energy: let $U \in C^1(\mathbb{R}^n)$ let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a conservative vector field such that $F = -\nabla U$ let $X \in \mathrm{sols}\left(\ddot{X} = F(X) \right)$ and let $K : \mathbb{R}^n \to \mathbb{R}$ be the kinetic energy of X then $E : \mathbb{R}^n \to \mathbb{R}$ such that $E = K + U$.

Lemma: let $U \in C^1(\mathbb{R}^n)$ let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a conservative vector field such that $F = -\nabla U$ let $X \in \mathrm{sols}\left(\ddot{X} = F(X) \right)$ and let $E : \mathbb{R}^n \to \mathbb{R}$ be the total energy of X then $\dot{E}(X(t)) = 0$.

Weighted Average/Center of Mass: let $p_1 \ldots p_n \in \mathbb{R}^n$ and let $w : \{p_1 \ldots p_n\} \to \mathbb{R}_+$ then $\frac{\sum_{i=1}^n w(p_i) \cdot p_i}{\sum_{i=1}^n w(p_i)}$.

Center of Mass: let $K \subseteq \mathbb{R}^2$ be compact and let $\rho : K \to \mathbb{R}$ then $\left(\frac{\int_K x \cdot \rho(x,y)\mathrm{d}x\mathrm{d}y}{\int_K \rho(x,y)\mathrm{d}x\mathrm{d}y}, \frac{\int_K y \cdot \rho(x,y)\mathrm{d}x\mathrm{d}y}{\int_K \rho(x,y)\mathrm{d}x\mathrm{d}y} \right)$.

Line: let $A, B \in \mathbb{R}^2$ then $L_{A,B} = \{\lambda A + (1-\lambda)B \mid \lambda \in [0,1]\}$.

Triangle: let $A, B, C \in \mathbb{R}^2$ such that $A \notin L_{B,C}$ and $B \notin L_{A,C}$ and $C \notin L_{A,B}$ then $\{A, B, C\}$.

Theorem Ceva: let $A, B, C \in \mathbb{R}^2$ such that $\{A, B, C\}$ is a triangle let $A' \in L_{B,C}$ let $B' \in L_{A,C}$ and let $C' \in L_{A,C}$ then $(L_{A,A'} \cap L_{B,B'} \cap L_{C,C'} \neq \varnothing) \iff \left(\frac{d(A,B')}{d(B',C)} \cdot \frac{d(C,A')}{d(A',B)} \cdot \frac{d(B,C')}{d(C',A)} = 1 \right)$.

Special Orthogonal Group: let $n \in \mathbb{N}$ then $\mathrm{SO}(n) = \{A \in M_n(\mathbb{R}) \mid (\det(A) = 1) \wedge (A^T = A^{-1})\}$.

Cross Product: let $x, y \in \mathbb{R}^3$ then $x \times y = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$.

Claim Antisymmetry: let $x, y \in \mathbb{R}^3$ then $x \times y = -y \times x$.

Claim Orthogonality: let $x, y \in \mathbb{R}^3$ be linearly independent then $x \times y \perp x$ and $x \times y \perp y$.

Claim: let $x, y \in \mathbb{R}^3$ and let θ be the angle between x, y then $\|x \times y\| = \|x\| \cdot \|y\| \cdot \cos(\theta)$.

Corollary: let $x, y \in \mathbb{R}^3$ then $(x \times y = 0) \iff (x, y$ are linearly dependent$)$.

Claim: let $x, y, z \in \mathbb{R}^2$ and let $\alpha, \beta \in \mathbb{R}$ then $(\alpha x + \beta y) \times z = \alpha(x \times z) + \beta(y \times z)$.

Claim: let $X, Y \in C^1\left(\mathbb{R}^3\right)$ then $\frac{\mathrm{d}}{\mathrm{d}t}\left(X \times Y\right) = \dot{X} \times Y + X \times \dot{Y}$.

Angular Momentum: let $X \in C^1\left(\mathbb{R}^3\right)$ then $X \times \dot{X}$.

Lemma: angular momentum is a conserved quantity of $\ddot{X} = -\frac{X}{\|X\|^2}$.

Exact Ordinary Differential Equation: let $\Omega \subseteq \mathbb{R}^2$ and let $F \in C^1\left(\Omega\right)$ then $\frac{\partial F}{\partial x}\left(x, y\right) + y' \cdot \frac{\partial F}{\partial y}\left(x, y\right) = 0$.

Remark: let $\Omega \subseteq \mathbb{R}^2$ and let $F \in C^1\left(\Omega\right)$ then $\frac{\partial F}{\partial x} + y' \cdot \frac{\partial F}{\partial y} = \mathrm{d}F$.

Claim: let $\Omega \subseteq \mathbb{R}^2$ let $F \in C^1\left(\Omega\right)$ and let $y \in \mathrm{sols}\left(\frac{\partial F}{\partial x}\left(x, y\right) + y' \cdot \frac{\partial F}{\partial y}\left(x, y\right) = 0\right)$ then $\frac{\mathrm{d}}{\mathrm{d}x}\left(F\left(x, y\left(x\right)\right)\right) = 0$.

Remark: let $\Omega \subseteq \mathbb{R}^2$ let $F \in C^1\left(\Omega\right)$ and let $c \in \mathbb{R}$ then $\{F = c\}$ are solutions to $\frac{\partial F}{\partial x}\left(x, y\right) + y' \cdot \frac{\partial F}{\partial y}\left(x, y\right) = 0$.

Claim: let $\Omega \subseteq \mathbb{R}^2$ be a simply connected domain and let $P, Q \in C^1\left(\Omega\right)$ then $\left(P\left(x, y\right) + y' \cdot Q\left(x, y\right) = 0\right.$ is an exact ODE$\left.\right) \Longleftrightarrow \left(\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}\right)$.

Linear Ordinary Differential Equation: let $f : \mathbb{R} \to \mathbb{R}$ let $n \in \mathbb{N}$ and let $a_0 \dots a_{n-1}\cdot : \mathbb{R} \to \mathbb{R}$ then $y^{(n)} + \sum_{i=0}^{n-1} a_i\left(x\right) \cdot y^{(i)} = f$.

Claim: let $I \subseteq \mathbb{R}$ be an interval let $p, q : I \to \mathbb{R}$ and let $\psi \in \mathrm{sols}\left(y' + py = q\right)$ then $\mathrm{sols}\left(y' + py = q\right) = \mathrm{sols}\left(y' + py = 0\right) + \psi$.

Bernoulli's Equation: let $\alpha \in \mathbb{R}\backslash\{0, 1\}$ and let $p, q : \mathbb{R} \to \mathbb{R}$ then $y' = py + qy^\alpha$.

Claim: let $\alpha \in \mathbb{R}\backslash\{0, 1\}$ and let $p, q : \mathbb{R} \to \mathbb{R}$ then
$$\mathrm{sols}\left(y' = py + qy^\alpha\right) = \mathrm{sols}\left(\frac{1}{1-\alpha} \cdot \left(y^{1-\alpha}\right)' = p \cdot y^{1-\alpha} + q\right) \cup \{0 \mid \alpha > 0\}.$$

Riccati's Equation: let $p, q, r : \mathbb{R} \to \mathbb{R}$ then $y' + p + qy + ry^2 = 0$.

Claim: let $p, q, r : \mathbb{R} \to \mathbb{R}$ and let $\psi \in \mathrm{sols}\left(y' + p + qy + ry^2 = 0\right)$ then
$$\mathrm{sols}\left(y' + p + qy + ry^2 = 0\right) = \mathrm{sols}\left(\left(y + \psi\right)' + \left(q + 2r\psi\right)\left(y + \psi\right) + r\left(y + \psi\right)^2 = 0\right).$$

Fibonacci Sequence: a sequence $F : \mathbb{N} \to \mathbb{N}$ such that $F_0 = 0$ and $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \in \mathbb{N}$.

Lemma: let $n, m \in \mathbb{N}$ then $F_{m+n+1} = F_m F_n + F_{m+1} F_{n+1}$.

Lemma Cassini Identity: let $n \in \mathbb{N}_+$ then $F_n^2 = F_{n+1}F_{n-1} + \left(-1\right)^{n-1}$.

Left Shift Operator: a function $\mathrm{LS} : \left(\mathbb{N} \to \mathbb{C}\right) \to \left(\mathbb{N} \to \mathbb{C}\right)$ such that $\mathrm{LS}\left(x\right) = \lambda n \in \mathbb{N}.x_{n+1}$.

Remark; let $x \in \mathbb{N} \to \mathbb{C}$ then $\left(\forall n \in \mathbb{N}.x_{n+2} = x_{n+1} + x_n\right) \Longleftrightarrow \left(x \in \ker\left(\mathrm{LS}^2 - \mathrm{LS} - 1\right)\right)$.

Lemma: let $p, q \in \mathbb{C}\left[x\right]$ then $\left(p \cdot q\right)\left(\mathrm{LS}\right) = p\left(\mathrm{LS}\right) \cdot q\left(\mathrm{LS}\right)$.

Corollary: $\mathrm{LS}^2 - \mathrm{LS} - 1 = \left(\mathrm{LS} - \frac{-1+\sqrt{5}}{2}\right)\left(\mathrm{LS} - \frac{-1-\sqrt{5}}{2}\right)$.

Homogeneous Linear Ordinary Differential Equation: let $n \in \mathbb{N}$ and let $a_0 \dots a_{n-1}\cdot : \mathbb{R} \to \mathbb{R}$ then $y^{(n)} + \sum_{i=0}^{n-1} a_i\left(x\right) \cdot y^{(i)} = 0$.

Remark: In this course we will discuss only homogeneous linear ODEs where the coefficients are constant.

Characteristic Polynomial of a Homogeneous Linear Ordinary Differential Equation: let $n \in \mathbb{N}$ and let $a_0 \dots a_{n-1}\cdot \in \mathbb{R}$ then $p\left(x\right) = x^n + \sum_{i=0}^{n-1} a_i \cdot x^i$.

Claim: let $n \in \mathbb{N}$ let $a_0 \dots a_{n-1}\cdot \in \mathbb{R}$ and let $f, g \in \mathrm{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i \cdot y^{(i)} = 0\right)$ then
$$f + g \in \mathrm{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i \cdot y^{(i)} = 0\right).$$

Theorem: let $n \in \mathbb{N}$ let $a_0 \dots a_{n-1}\cdot \in \mathbb{R}$ and let $\alpha \in \mathbb{R}$ be a solution to $p\left(x\right)$ with multiplicity ρ then
$$\left\{x^0 e^{\alpha x}, \dots, x^{\rho-1} e^{\alpha x}\right\} \subseteq \mathrm{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i \cdot y^{(i)} = 0\right).$$

Claim: let $n \in \mathbb{N}$ let $a_0 \dots a_{n-1}\cdot \in \mathbb{R}$ and let $f \in \mathrm{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i \cdot y^{(i)} = 0\right)$ then
$$\mathrm{Re}\left(f\right), \mathrm{Im}\left(f\right) \in \mathrm{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i \cdot y^{(i)} = 0\right).$$

Claim: let $f : \mathbb{R} \to \mathbb{R}$ let $n \in \mathbb{N}$ let $a_0 \dots a_{n-1}\cdot : \mathbb{R} \to \mathbb{R}$ and let $\psi \in \mathrm{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i\left(x\right) \cdot y^{(i)} = f\right)$ then
$$\mathrm{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i\left(x\right) \cdot y^{(i)} = f\right) = \mathrm{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i\left(x\right) \cdot y^{(i)} = 0\right) + \psi.$$

Complex Quasi Polynomial: let $\gamma \in \mathbb{C}$ and let $p \in \mathbb{C}\left[x\right]$ then $f : \mathbb{C} \to \mathbb{C}$ such that $f\left(z\right) = e^{\gamma z} \cdot p\left(z\right)$.

Degree of a Complex Quasi Polynomial: let $\gamma \in \mathbb{C}$ and let $p \in \mathbb{C}\left[x\right]$ then $\deg\left(e^{\gamma z} \cdot p\left(z\right)\right) = \deg\left(p\right)$.

Exponent of a Complex Quasi Polynomial: let $\gamma \in \mathbb{C}$ and let $p \in \mathbb{C}\left[x\right]$ then γ.

Real Quasi Polynomial: let $\alpha, \beta \in \mathbb{R}$ and let $p, q \in \mathbb{R}\left[x\right]$ then $f : \mathbb{R} \to \mathbb{R}$ such that $f\left(x\right) = e^{\alpha x}\left(p\left(x\right)\cos\left(\beta x\right) + q\left(x\right)\sin\left(\beta x\right)\right)$.

Degree of a Real Quasi Polynomial: let $\alpha, \beta \in \mathbb{R}$ and let $p, q \in \mathbb{R}\left[x\right]$ then $\deg\left(e^{\alpha x}\left(p\left(x\right)\cos\left(\beta x\right) + q\left(x\right)\sin\left(\beta x\right)\right)\right) = \max\left\{\deg\left(p\right), \deg\left(q\right)\right\}$.

Claim: let $P : \mathbb{C} \to \mathbb{C}$ be a complex quasi polynomial then $\mathrm{Re}\left(P\right)$ is a real quasi polynomial.

Claim: let $n \in \mathbb{N}$ let $a_0 \ldots a_{n-1} \in \mathbb{R}$ let $\gamma \in \mathbb{C}$ let $p \in \mathbb{C}[x]$ and let $k \in \mathbb{N}$ the multiplicity of the root γ in $x^n + \sum_{i=0}^{n-1} a_i \cdot x^i$ then there exists $q \in \mathbb{C}[x]$ such that $\deg(q) = \deg(p)$ and $t^k q(t) e^{\gamma t} \in \text{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i \cdot y^{(i)} = p \cdot e^{\gamma t} \right)$.

Linear System of Ordinary Differential Equation: let $n \in \mathbb{N}$ and let $A \in M_n(\mathbb{R})$ then $y'(x) = A \cdot y(x)$.

Definition: let $A \in M_n(\mathbb{R})$ then $\max(|A|) = \max\left\{ \left| (A)_{i,j} \right| \mid i,j \in [n] \right\}$.

Lemma: let $n, k \in \mathbb{N}$ let $A \in M_n(\mathbb{R})$ and let $i, j \in [n]$ then $(A)_{i,j}^k \leq n^{k-1} \cdot \max(|A|)^k$.

Corollary: let $n, k \in \mathbb{N}$ let $A \in M_n(\mathbb{R})$ and let $i, j \in [n]$ then $(A)_{i,j}^k \leq (n \cdot \max(|A|))^k$.

Theorem: let $n \in \mathbb{N}$ and let $A \in M_n(\mathbb{R})$ then $\sum_{k=0}^{\infty} \frac{A^k}{k!} \in M_n(\mathbb{R})$.

Definition: let $n \in \mathbb{N}$ and let $A \in M_n(\mathbb{R})$ then $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$.

Lemma: let $A \in M_n(\mathbb{R})$ and let $s, t \in \mathbb{R}$ then $e^{sA} \cdot e^{tA} = e^{(s+t)A}$.

Matrix Differentiation: let $A : \mathbb{R} \to M_n(\mathbb{R})$ such that $(A)_{i,j} \in C^1(\mathbb{R})$ for all $i, j \in [n]$ then $(A'(t))_{i,j} = (A(t))'_{i,j}$.

Claim: let $A, B \in C^1(\mathbb{R}, M_n(\mathbb{R}))$ then $\frac{\mathrm{d}}{\mathrm{d}t}(AB) = \dot{A}B + A\dot{B}$.

Corollary: let $A \in C^1(\mathbb{R}, M_n(\mathbb{R}))$ and let $v \in C^1(\mathbb{R}, \mathbb{R}^n)$ then $\frac{\mathrm{d}}{\mathrm{d}t}(Av) = \dot{A}v + A\dot{v}$.

Lemma: let $A \in M_n(\mathbb{R})$ then $\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA} \right) \right)(0) = A$.

Corollary: let $A \in M_n(\mathbb{R})$ and let $\mu \in \mathbb{R}$ then $\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA} \right) \right)(\mu) = e^{\mu A} A$.

Theorem: let $n \in \mathbb{N}$ and let $A \in M_n(\mathbb{R})$ then $e^{tA}v$ is a solution to $\begin{cases} y' = A \cdot y \\ y(0) = v \end{cases}$.

Zeckendorff Representation: let $n \in \mathbb{N}$ let $k \in \mathbb{N}$ and let $c_0 \ldots c_k \in \mathbb{N}$ such that $c_0 \geq 2$ and $c_{i+1} > c_i + 1$ for all $i \in \{0 \ldots k-1\}$ and $n = \sum_{i=0}^{k} F_{c_i}$ then $(F_{c_0}, \ldots, F_{c_k})$.

Theorem Zeckendorff: let $n \in \mathbb{N}$ then

- Existence: there exists a zeckendorff representation for n.

- Uniqueness: let $(F_{c_0}, \ldots, F_{c_k}), (F_{d_0}, \ldots, F_{d_m})$ be zeckendorff representations for n then $k = m$ and $c_i = d_i$ for all $i \in \{0 \ldots k-1\}$.

Theorem Grönwall's Inequality: let $\Omega \subseteq \mathbb{R}^2$ let $f_1, f_2 \in C(\Omega)$ such that $f_2(x,y) > f_1(x,y)$ for all $x > 0$ and $y \in \mathbb{R}$ let $a_1, a_2 \in \mathbb{R}$ such that $a_2 > a_1$ let $y_1 \in \text{sols}\left(\begin{cases} y' = f_1(x,y) \\ y(0) = a_1 \end{cases} \right)$ and let $y_2 \in \text{sols}\left(\begin{cases} y' = f_2(x,y) \\ y(0) = a_2 \end{cases} \right)$ then $y_2(x) > y_1(x)$ for all $x > 0$.

Theorem: let $\Omega \subseteq \mathbb{R}^2$ let $f_1, f_2 \in C^1(\Omega)$ such that $f_2(x,y) \geq f_1(x,y)$ for all $x > 0$ and $y \in \mathbb{R}$ let $a_1, a_2 \in \mathbb{R}$ such that $a_2 \geq a_1$ let $y_1 \in \text{sols}\left(\begin{cases} y' = f_1(x,y) \\ y(0) = a_1 \end{cases} \right)$ and let $y_2 \in \text{sols}\left(\begin{cases} y' = f_2(x,y) \\ y(0) = a_2 \end{cases} \right)$ then $y_2(x) \geq y_1(x)$ for all $x > 0$.

Wronskian: let $\mathcal{U} \subseteq \mathbb{R}$ and let $u, v \in C^1(\mathcal{U})$ then $W_{u,v} : \mathcal{U} \to \mathbb{R}$ such that $W_{u,v}(t) = \det\begin{pmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{pmatrix}$.

Corollary: let $\mathcal{U} \subseteq \mathbb{R}$ let $u \in C^1(\mathcal{U})$ and let $c \in \mathbb{R}$ then $W_{u, c \cdot u} = 0$.

Remark: let $\mathcal{U} \subseteq \mathbb{R}$ and let $u, v \in C^1(\mathcal{U})$ then $W = W_{u,v}$.

Theorem Abel's Identity: let $I \subseteq \mathbb{R}$ be an open interval let $p, q : I \to \mathbb{R}$ and let $u, v \in \text{sols}(y'' + py' + qy = 0)$ then $W_{u,v} \in \text{sols}(y' + py = 0)$.

Corollary: let $I \subseteq \mathbb{R}$ be an open interval let $p, q : I \to \mathbb{R}$ and let $u, v \in \text{sols}(y'' + py' + qy = 0)$ then there exists $p \in I$ and $c \in \mathbb{R}$ such that $W_{u,v}(t) = \exp(c) \cdot W_{u,v}(p) \cdot \exp\left(-\int_p^t p(s)\,\mathrm{d}s \right)$ for all $t \in I$.

Corollary: let $I \subseteq \mathbb{R}$ be an open interval let $p, q : I \to \mathbb{R}$ and let $u, v \in \text{sols}(y'' + py' + qy = 0)$ then $\text{sign}(W(t_1)) = \text{sign}(W(t_2))$ for all $t_1, t_2 \in I$.

Corollary: let $I \subseteq \mathbb{R}$ be an open interval let $p, q : I \to \mathbb{R}$ and let $u, v \in \text{sols}(y'' + py' + qy = 0)$ then $(u, v$ are linearly dependent$) \Longleftrightarrow (\text{sign}(W) = 0)$.

Corollary Complementary Solution: let $I \subseteq \mathbb{R}$ be an open interval let $p, q : I \to \mathbb{R}$ and let $u \in \text{sols}(y'' + py' + qy = 0) \setminus \{0\}$ then $u \int \frac{W}{u^2}$ is a solution of $y'' + py' + qy = 0$ and $u, u \int \frac{W}{u^2}$ are linearly independent.

Claim Variation of Parameters: let $I \subseteq \mathbb{R}$ be an open interval let $p, q, f : I \to \mathbb{R}$ let $u, v \in \text{sols}(y'' + py' + qy = 0)$ such that u, v are linearly independent and let $C_1, C_2 \in C^1(I)$ such that $C_1 u + C_2 v$ is a solution of $y'' + py' + qy = f$ and $C_1' u + C_2' v = 0$ then $\begin{cases} C_1' u + C_2' v = 0 \\ C_1' u' + C_2' v' = f \end{cases}$.

Corollary Lagrange's Formula: let $I \subseteq \mathbb{R}$ be an open interval let $p, q, f : I \to \mathbb{R}$ and let $u, v \in \text{sols}(y'' + py' + qy = 0)$ such that u, v are linearly independent then $-u \int \frac{fv}{W_{u,v}} + v \int \frac{fu}{W_{u,v}}$ is a solution of $y'' + py' + qy = f$.

Green Function: let $\mathcal{U} \subseteq \mathbb{R}$ and let $u, v \in C^1(\mathcal{U})$ then $G_{u,v} : \mathcal{U}^2 \to \mathbb{R}$ such that $G_{u,v}(s,t) = \frac{v(t)u(s) - u(t)v(s)}{u(s)v'(s) - u'(s)v(s)}$.

Claim: let $\mathcal{U} \subseteq \mathbb{R}$ and let $u, v \in C^1(\mathcal{U})$ then

- $G(t,t) = 0$ for all $t \in \mathcal{U}$.
- $\frac{\partial G}{\partial y}(t,t) = 1$ for all $t \in \mathcal{U}$.

Corollary: let $I \subseteq \mathbb{R}$ be an open interval let $p, q, f : I \to \mathbb{R}$ and let $u, v \in \text{sols}(y'' + py' + qy = 0)$ such that u, v are linearly independent then there exists $p \in I$ such that $\int_p^t G_{u,v}(s,t) f(s) \, ds$ is a solution of $y'' + py' + qy = f$.

Lemma: let $I \subseteq \mathbb{R}$ be an open interval let $p, q, f : I \to \mathbb{R}$ let $u, v \in \text{sols}(y'' + py' + qy = 0)$ such that u, v are linearly independent let $p \in I$ such that $\int_p^t G_{u,v}(s,t) f(s) \, ds$ is a solution of $y'' + py' + qy = f$ and let $g \in \text{sols}\begin{pmatrix} y''+py'+qy=f \\ y(p)=0 \\ y'(p)=0 \end{pmatrix}$ then $g(t) = \int_p^t G_{u,v}(s,t) f(s) \, ds$ for all $t \in I$.

Simple Zero: let $f \in C^1(\mathbb{R})$ then $p \in \ker(f)$ such that $f'(p) \neq 0$.

Isolated Zero: let $f : \mathbb{R} \to \mathbb{R}$ then $p \in \ker(f)$ such that there exists a neighbourhood $\mathcal{U}$ of p for which $0 \notin f(\mathcal{U} \setminus \{p\})$.

Lemma: let $I \subseteq \mathbb{R}$ be an open interval let $p, q : I \to \mathbb{R}$ let $u \in \text{sols}(y'' + py' + qy = 0) \setminus \{0\}$ and let $p \in \ker(u)$ then p is a simple and isolated zero of u.

Lemma: let $I \subseteq \mathbb{R}$ be an open interval let $p, q : I \to \mathbb{R}$ and let $u, v \in \text{sols}(y'' + py' + qy = 0)$ such that u, v are linearly independent then $\ker(u) \cap \ker(v) = \varnothing$.

Theorem Sturm's Separation Theorem: let $I \subseteq \mathbb{R}$ be an open interval let $p, q : I \to \mathbb{R}$ and let $u, v \in \text{sols}(y'' + py' + qy = 0)$ such that u, v are linearly independent then for all $z, w \in \ker(u)$ if $(z,w) \cap \ker(u) = \varnothing$ then $|(z,w) \cap \ker(v)| = 1$.

Theorem Sturm's Comparison Theorem: let $I \subseteq \mathbb{R}$ be an open interval let $p, q : I \to \mathbb{R}$ such that $p \geq q$ and $p \neq q$ let $u \in \text{sols}(y'' + py = 0)$ let $v \in \text{sols}(y'' + qy = 0)$ such that u, v are linearly independent and let $z, w \in \ker(v)$ then $(z,w) \cap \ker(u) \neq \varnothing$.

Claim: let $k < 0$ then $e^{\sqrt{-k} \cdot t}$ is a solution of $y'' + ky = 0$.

Claim: the function 1 is a solution of $y'' = 0$.

Corollary: let $I \subseteq \mathbb{R}$ be an open interval let $q : I \to \mathbb{R}$ such that $0 \geq q$ and let $v \in \text{sols}(y'' + qy = 0)$ such that $v \neq 0$ then $|\ker(v)| \leq 1$.

Claim: let $k > 0$ then $\sin\left(\sqrt{k} \cdot t\right)$ is a solution of $y'' + ky = 0$.

Supremum Norm/L^∞ Norm: let $a, b \in \mathbb{R}$ such that $a \leq b$ and let $f \in C([a,b])$ then $\|f\|_\infty = \max(\text{Im}(|f|))$.

Claim: let $a, b \in \mathbb{R}$ such that $a \leq b$ then $\|\cdot\|_\infty$ is a norm on $C([a,b])$.

Supremum Metric: let $a, b \in \mathbb{R}$ such that $a \leq b$ and let $f, g \in C([a,b])$ then $d_\infty(f,g) = \|f - g\|_\infty$.

Claim: let $a, b \in \mathbb{R}$ such that $a \leq b$ then d_∞ is a metric on $C([a,b])$.

Claim: let $a, b \in \mathbb{R}$ such that $a \leq b$ let $f \in C([a,b])$ and let $\{f_n\}_{n=0}^\infty \subseteq C([a,b])$ then $\left(f_n \xrightarrow{L^\infty} f\right) \iff \left(f_n \xrightarrow{u} f\right)$.

Theorem: let $a, b \in \mathbb{R}$ such that $a \leq b$ then $(C([a,b]), d_\infty)$ is complete.

Closed Set: let (X, d) be a metric space then $A \subseteq X$ such that for all $\{a_n\}_{n=0}^\infty \subseteq A$ and for all $L \in X$ if $a_n \to L$ then $L \in A$.

Fixed Point: let X be a set and let $f : X \to X$ then $p \in X$ such that $f(p) = p$.

Contraction: let (X, d) be a metric space then $f : X \to X$ such that there exists $\rho \in [0, 1)$ for which $d(f(x), f(y)) \leq \rho \cdot d(x, y)$ for all $x, y \in X$.

Theorem Contraction Mapping Theorem: let (X, d) be a complete metric space such that $X \neq \varnothing$ and let $f : X \to X$ be a contraction then there exists a unique $p \in X$ for which $f(p) = p$.

Lipschitz Function: let (X, d) be a metric space then $f : X \to \mathbb{R}$ such that there exists $L \in \mathbb{R}$ for which $|f(x) - f(y)| \leq L \cdot d(x, y)$ for all $x, y \in X$.

Lipschitz Constant: let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be lipschitz then minimal $L \in \mathbb{R}$ such that $|f(x) - f(y)| \leq L \cdot d(x, y)$ for all $x, y \in X$.

Claim: let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be lipschitz then $f \in C(X)$.

Locally Lipschitz Function: let (X, d) be a metric space then $f : X \to \mathbb{R}$ such that for all $x \in X$ there exists a neighbourhood $\mathcal{U}$ of x for which $f_{\restriction \mathcal{U}}$ is lipschitz.

Theorem Picard-Lindelöf: let $I, J \subseteq \mathbb{R}$ be closed intervals let $f \in C(I \times J)$ such that f is locally lipschitz in the second coordinate and let $p \in \text{int}(I \times J)$ then there exists $\varepsilon > 0$ such that $\begin{cases} y' = f(x,y) \\ y(p_1) = p_2 \end{cases}$ has a unique solution on the interval $(p_1 - \varepsilon, p_1 + \varepsilon)$.

Compact Set: let (X, d) be a metric space then $K \subseteq X$ such that for all $\{x_n\}_{n=0}^\infty \subseteq K$ there exists a convergent subsequence.

Uniformly Bounded Sequence of Functions: let $a, b \in \mathbb{R}$ such that $a \leq b$ then $\{f_n\}_{n=0}^{\infty} \subseteq C\left([a, b]\right)$ such that $\exists M \in \mathbb{R}.\forall x \in [a, b].\forall n \in \mathbb{N}.\left|f_n\left(x\right)\right| \leq M.$

Uniformly Equicontinuous Sequence of Functions: let $a, b \in \mathbb{R}$ such that $a \leq b$ then $\{f_n\}_{n=0}^{\infty} \subseteq C\left([a, b]\right)$ such that $\forall \varepsilon > 0.\exists \delta > 0.\forall x, y \in [a, b].\left|x - y\right| < \delta \implies \left|f\left(x\right) - f\left(y\right)\right| < \varepsilon.$

Theorem Arzela-Ascoli: let $a, b \in \mathbb{R}$ such that $a \leq b$ and let $\{f_n\}_{n=0}^{\infty} \subseteq C\left([a, b]\right)$ then $(\{f_n\}_{n=0}^{\infty}$ has a convergent subsequence$)\Longleftrightarrow(\{f_n\}_{n=0}^{\infty}$ is uniformly bounded and uniformly equicontinuous$).$

Notation: let $p \in \mathbb{R}^2$ and let $f : \mathbb{R} \to \mathbb{R}$ then $\eta_f\left(p\right) : \mathbb{R} \to \mathbb{R}$ such that $\eta_f\left(p\right)\left(x\right) = f\left(p\right) \cdot \left(x - p_1\right) + p_2.$

Algorithm Euler's Method: let $n \in \mathbb{N}$ let $f \in C\left(\mathbb{R}\right)$ let $\varepsilon \in \mathbb{R}$ and let $p \in \mathbb{R}^2$ then

function EulerMethod(n, f, ε, p):

$\quad (x_0, y_0) \leftarrow p$

$\quad$ **for** $i \leftarrow \{1 \ldots n\}$ **do**

$\quad\quad x_i \leftarrow x_0 + \varepsilon \cdot \frac{i}{n}$

$\quad\quad g_i \leftarrow (\lambda x \in [x_{i-1}, x_i].(f\left(x_{i-1}, y_{i-1}\right) \cdot \left(x - x_{i-1}\right) + y_{i-1}))$

$\quad\quad y_i \leftarrow g_i(x_i)$

$\quad$ **end**

$\quad$ **return** $\bigcup_{i=1}^{n} g_i$

Claim: let $n \in \mathbb{N}$ let $f \in C\left(\mathbb{R}\right)$ let $\varepsilon \in \mathbb{R}$ and let $p \in \mathbb{R}^2$ then EulerMethod $(n, f, \varepsilon, p) : [p_1, p_1 + \varepsilon] \to \mathbb{R}.$

Notation: let $n \in \mathbb{N}$ let $f \in C\left(\mathbb{R}\right)$ let $\varepsilon \in \mathbb{R}$ and let $p \in \mathbb{R}^2$ then EulerMethod$_n =$ EulerMethod $(n, f, \varepsilon, p).$

Claim: let $f \in C\left(\mathbb{R}\right)$ let $\varepsilon \in \mathbb{R}$ let $p \in \mathbb{R}^2$ then there exists a convergent subsequence $\{$EulerMethod$_{n_k}\}$ such that $\lim_{k \to \infty}$ EulerMethod$_{n_k}$ is a solution of $\begin{cases} y' = f(x,y) \\ y(p_1) = p_2 \end{cases}$ defined on $[p_1, p_1 + \varepsilon].$

Boundary Value Problem (BVP): let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and let $a, b \in \mathbb{R}$ then $\begin{cases} f\left(x, y', \ldots y^{(n)}\right) = 0 \\ y(0) = a \\ y(1) = b \end{cases}$.

Theorem: let $k \in C\left([0, 1]\right)$ such that $k > 0$ and let $a, b \in \mathbb{R}$ then $\left| \text{sols} \left(\begin{cases} y'' = ky \\ y(0) = a \\ y(1) = b \end{cases} \right) \right| = 1.$