Notation: let X be a set and let $y:\mathbb{R}\to\mathbb{R}$ then $y^{(0)}=y$.
Notation: let X,Y be sets let $n\in\mathbb{N}_{+}$ and let $y\in C^{n}\left(X,Y\right)$ then $y^{(n)}=\left(y^{(n-1)}\right)'$.
Ordinary Differential Equation (ODE): let $n\in\mathbb{N}$ and let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ then $f\left(x,y\left(x\right),y'\left(x\right),\ldots y^{(n)}\left(x\right)\right)=0$.
Solution of an Ordinary Differential Equation: let $n\in\mathbb{N}$ let $\mathcal{U}\subseteq\mathbb{R}$ and let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ then a function $y\in C^{n}\left(\mathcal{U}\right)$ such that $f\left(x,y\left(x\right),\ldots,y^{(n)}\left(x\right)\right)=0$ for all $x\in\mathcal{U}$.
Notation: let $n\in\mathbb{N}$ let $\mathcal{U}\subseteq\mathbb{R}$ and let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ then
$\text{sols}\left(f\left(x,y\left(x\right),y'\left(x\right),\ldots y^{(n)}\left(x\right)\right)=0\right)=\left\{y\in C^{n}\left(\mathcal{U}\right)\mid f\left(x,y\left(x\right),\ldots,y^{(n)}\left(x\right)\right)=0\right\}$.
First Order Ordinary Differential Equation: let $f:\mathbb{R}^{3}\to\mathbb{R}$ then $f\left(x,y,y'\right)=0$.
Separable Ordinary Differential Equation: let $f,g:\mathbb{R}\to\mathbb{R}$ then $y'=f\left(y\right)g\left(x\right)$.
Claim Separation of Variables: let $f,g:\mathbb{R}\to\mathbb{R}$ such that $f\left(x\right)\neq0$ for all $x\in\mathbb{R}$ then
$\text{sols}\left(y'=f\left(y\right)g\left(x\right)\right)=\text{sols}\left(\int\frac{1}{f\left(y\right)}\mathrm{d}y=\int g\left(x\right)\mathrm{d}x\right)$.
Corollary: $\text{sols}\left(y=y'\right)=\left\{Ce^{x}\mid C\in\mathbb{R}\right\}$.
Curves that foliate a domain: Let $\Omega\subseteq\mathbb{R}^{2}$ and let $I\subseteq\mathbb{R}$ then a set of curves $\left\{\gamma_{\alpha}\right\}\subseteq I\to\mathbb{R}$ such that $\biguplus\gamma_{i}=\Omega$.
Claim: $\text{sols}\left(y=y'\right)$ foliate the plane.
Remark: let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ and let $g\in\text{sols}\left(f\left(x,y,\ldots,y^{(n)}\right)\right)$ such that g is piecewise continuous then each continuous branch of g is a solution of the ODE.
Initial Value Problem (IVP): let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ and let $p\in\mathbb{R}^{2}$ then $\begin{cases}f\left(x,y',\ldots y^{(n)}\right)=0\\y\left(p_{1}\right)=p_{2}\end{cases}$.
Theorem Cauchy-Peano: let $I,J\subseteq\mathbb{R}$ be closed intervals let $f\in C^{1}\left(I\times J\right)$ and let $p\in\text{int}\left(I\times J\right)$ then there exists $\varepsilon>0$ such that $\begin{cases}y'=f\left(x,y\right)\\y\left(p_{1}\right)=p_{2}\end{cases}$ has a solution on the interval $\left(p_{1}-\varepsilon,p_{1}+\varepsilon\right)$.
Theorem of Existence and Uniqueness: let $I,J\subseteq\mathbb{R}$ be closed intervals let $f\in C^{1}\left(I\times J\right)$ and let $p\in\text{int}\left(I\times J\right)$ then there exists $\varepsilon>0$ such that $\begin{cases}y'=f\left(x,y\right)\\y\left(p_{1}\right)=p_{2}\end{cases}$ has a unique solution on the interval $\left(p_{1}-\varepsilon,p_{1}+\varepsilon\right)$.
Corollary: let $I,J\subseteq\mathbb{R}$ be closed intervals and let $f\in C^{1}\left(I\times J\right)$ then $\text{sols}\left(y'=f\left(x,y\right)\right)$ foliates $I\times J$.
Vector Field: let $\Omega\subseteq\mathbb{R}^{2}$ be a domain then $\nu\in C\left(\Omega,\mathbb{R}^{2}\right)$.
Integral curve: let $\nu:\Omega\to\mathbb{R}^{2}$ be a vector field then a curve $\gamma:\left[a,b\right]\to\Omega$ such that $\frac{\mathrm{d}\gamma}{\mathrm{d}t}\left(t\right)=\nu\left(g\left(t\right)\right)$.
Claim: let $\nu:\Omega\to\mathbb{R}^{2}$ be a vector field such that $v\left(x\right)\neq0$ for all $x\in\Omega$ and let $\gamma:\left[a,b\right]\to\Omega$ be a curve such that $v\left(\gamma\left(t\right)\right)\in T_{\gamma\left(t\right)}\left(\text{Im}\left(\gamma\right)\right)$ for all $t\in\left[a,b\right]$ then there exists a curve $\eta:\left[\alpha,\beta\right]\to\left[a,b\right]$ such that $\gamma\circ\eta$ is an integral curve.
C^{n} **Vector Field:** let $\Omega\subseteq\mathbb{R}^{2}$ be a domain then $\nu\in C^{n}\left(\Omega,\mathbb{R}^{2}\right)$.
Claim: let $\nu:\Omega\to\mathbb{R}^{2}$ be a C^{1} vector field then $\left\{\gamma:\left[a,b\right]\to\Omega\mid\left(a,b\in\mathbb{R}\right)\wedge\left(\gamma\text{ is an integral curve of }\nu\right)\right\}$ foliates Ω.
Theorem Peano: let $\nu:\Omega\to\mathbb{R}^{2}$ be a vector field then $\left\{\gamma:\left[a,b\right]\to\Omega\mid\left(a,b\in\mathbb{R}\right)\wedge\left(\gamma\text{ is an integral curve of }\nu\right)\right\}$ covers Ω.
Lemma: let $\Omega\subseteq\mathbb{R}^{2}$ be a domain and let $f\in C^{1}\left(\Omega\right)$ then $\begin{pmatrix}1\\f\end{pmatrix}$ is a vector field.
Theorem of Existence and Uniqueness: let $\Omega\subseteq\mathbb{R}^{2}$ be a domain let $f\in C^{1}\left(\Omega\right)$ and let $g\in C^{1}\left(\mathbb{R}\right)$ then
$\left(g'=f\left(x,g\right)\right)\iff\left(\begin{pmatrix}x\\g\end{pmatrix}\text{ is an integral curve of }\begin{pmatrix}1\\f\end{pmatrix}\right)$.
Autonomous Ordinary Differential Equations: let $f:\mathbb{R}\to\mathbb{R}$ then $y'=f\left(y\right)$.
Logistic Equation: let $L>0$ then $\dot{P}\left(t\right)=P\left(t\right)\left(L-P\left(t\right)\right)$.
Equilibrium Solution: let $f:\mathbb{R}\to\mathbb{R}$ then a function $g\in\text{sols}\left(y'=f\left(y\right)\right)$ such that $f\left(g\left(x\right)\right)=0$ for all $x\in\mathbb{R}$.
Corollary: let $f\in C\left(\mathbb{R}\right)$ and let $g:\mathbb{R}\to\mathbb{R}$ be an equilibrium solution then g is constant.
Corollary: let $L>0$ then $\left\{0,L\right\}$ are the equilibrium solutions of the logistic equation.
Logistic Equation With Harvesting: let $L,k>0$ then $\dot{P}\left(t\right)=P\left(t\right)\left(L-P\left(t\right)\right)-k$.
Stable Equilibrium Solution: let $f:\mathbb{R}\to\mathbb{R}$ then an equilibrium solution $g:\mathbb{R}\to\mathbb{R}$ such that for all $y\in\text{sols}\left(y'=f\left(y\right)\right)$ and for all $\varepsilon>0$ if there exists $\delta>0$ and $p\in\mathbb{R}$ for which $\left|y\left(p\right)-g\left(p\right)\right|<\delta$ then $\left|y\left(t\right)-g\left(t\right)\right|<\varepsilon$ for all $t>p$.
Notation: let X be a set and let $\gamma:\mathbb{R}\to X$ then $\dot{\gamma}=\frac{\mathrm{d}\gamma}{\mathrm{d}t}$.
Notation: let X be a set and let $\gamma:\mathbb{R}\to X$ then $\ddot{\gamma}=\frac{\mathrm{d}^{2}\gamma}{\mathrm{d}t^{2}}$.
Remark: the notation $\dot{\gamma},\ddot{\gamma}$ is often used when γ is a function of time.
System of Ordinary Differential Equation: let $n\in\mathbb{N}$ let $m\in\mathbb{N}_{+}$ and let $f:\mathbb{R}\times\left(\mathbb{R}^{m}\right)^{n}\to\mathbb{R}^{m}$ then
$f\left(t,y\left(t\right),y'\left(t\right),\ldots y^{(n)}\left(t\right)\right)=0$.

Remark: all the theory of first order ODEs also applies to vectors of first order ODEs.

Lotka–Volterra equations: let $\alpha > 0$ then $\begin{cases} \dot{x}(t)=x(t)\cdot(1-y(t)) \\ \dot{y}(t)=\alpha\cdot y(t)\cdot(x(t)-1) \end{cases}$.

Claim: let $\alpha > 0$ then $\left\{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\right\}$ are the equilibrium solutions of the lotka-volterra equation.

Conserved Quantity: let $\mathcal{U} \subseteq \mathbb{R}^n$ and let $f : \mathcal{U} \to \mathbb{R}$ then a function $g \in C^1(\mathcal{U})$ such that $g \circ y$ is constant for all $y \in \mathrm{sols}(y' = f(y))$.

Claim: let $\alpha > 0$ and let $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) \in C^1(\mathbb{R}^2)$ be a solution to the lotka-volterra equation then $\frac{\mathrm{d}}{\mathrm{d}t}\left((y(t) - 1 - \log(y(t))) + \alpha(x(t) - 1 - \log(x(t)))\right) = 0$.

Corollary: let $\alpha > 0$ let $p \in \mathbb{R}^2$ and let $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) \in C^1(\mathbb{R}^2)$ be a solution to $\begin{cases} \dot{x}(t)=x(t)\cdot(1-y(t)) \\ \dot{y}(t)=\alpha\cdot y(t)\cdot(x(t)-1) \\ \left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)(0)=p \end{cases}$ then $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)$ is periodic.

Lemma: let $\alpha > 0$ let $p \in \mathbb{R}^2$ and let $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) \in C^1(\mathbb{R}^2)$ be a solution to $\begin{cases} \dot{x}(t)=x(t)\cdot(1-y(t)) \\ \dot{y}(t)=\alpha\cdot y(t)\cdot(x(t)-1) \\ \left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)(0)=p \end{cases}$ with period T then $\frac{1}{T}\int_0^T x\mathrm{d}t = 1$ and $\frac{1}{T}\int_0^T y\mathrm{d}t = 1$.

Claim Linear Substitution: let $a, b, c \in \mathbb{R}$ let $f : \mathbb{R} \to \mathbb{R}$ and let $y \in \mathrm{sols}(y' = f(ax + by + c))$ then the function $z : \mathbb{R} \to \mathbb{R}$ which defined by $z(x) = ax + by + c$ satisfies $z' = a + bf(z)$.

Notation: let $\Omega \subseteq \mathbb{R}^2$ be a domain let $f : \Omega \to \mathbb{R}$ and let $p \in \mathbb{R}^2$ then (α_-, α_+) is the maximal interval where a solution to $\begin{cases} y'=f(x,y) \\ y(p_1)=p_2 \end{cases}$ is defined.

Corollary: let $\Omega \subseteq \mathbb{R}^2$ be a domain let $f \in C^1(\Omega)$ and let $p \in \mathbb{R}^2$ then $\alpha_- < p_1 < \alpha_+$.

Theorem: let $f \in C^1(\mathbb{R}^2)$ and let $p \in \mathbb{R}^2$ then

- $(\alpha_+ < \infty) \iff \left(\lim_{x \to \alpha_+^-} |y(x)| = \infty\right)$.
- $(-\infty < \alpha_-) \iff \left(\lim_{x \to \alpha_-^+} |y(x)| = \infty\right)$.

Theorem Extensibility of Solutions: let $D \subseteq \mathbb{R}^{n+1}$ be a closed set let $f \in C^1(D, \mathbb{R}^n)$ and let $y \in \mathrm{sols}(y' = f(x, y))$ such that $\Gamma_y \cap D \neq \varnothing$ then there exists $\psi : [a, b] \to \mathbb{R}^n$ such that $\psi \in \mathrm{sols}(y' = f(x, y))$ and $\psi(x) = y(x)$ for all $x \in \mathrm{Dom}(y)$ and $\left(\begin{smallmatrix}a\\\psi(a)\end{smallmatrix}\right), \left(\begin{smallmatrix}b\\\psi(b)\end{smallmatrix}\right) \in \partial D$.

Corollary: let $D \subseteq \mathbb{R}^{n+1}$ such that for all $\alpha, \beta \in \mathbb{R}$ the set $D \cap \{\alpha \le x_1 \le \beta\}$ is bounded let $f \in C^1(D, \mathbb{R}^n)$ and let $y \in \mathrm{sols}(y' = f(x, y))$ such that $\Gamma_y \cap D \neq \varnothing$ then one of the next statements is true

- there exists $\psi : [a, b] \to \mathbb{R}^n$ such that $\psi \in \mathrm{sols}(y' = f(x, y))$ and $\psi(x) = y(x)$ for all $x \in \mathrm{Dom}(y)$ and $\left(\begin{smallmatrix}a\\\psi(a)\end{smallmatrix}\right), \left(\begin{smallmatrix}b\\\psi(b)\end{smallmatrix}\right) \in \partial D$.
- there exists $\psi : \mathbb{R} \to \mathbb{R}^n$ such that $\psi \in \mathrm{sols}(y' = f(x, y))$ and $\psi(x) = y(x)$ for all $x \in \mathrm{Dom}(y)$.

Corollary: let $I \subseteq \mathbb{R}$ be an open interval let $D \subseteq \mathbb{R}^n$ let $f \in C^1(I \times D, \mathbb{R}^n)$ and let $p, q : \mathbb{R} \to \mathbb{R}$ such that $\|f(x, y)\| \le p(x)\|y\| + q(x)$ then for all $y \in \mathrm{sols}(y' = f(x, y))$ such that $\Gamma_y \cap (I \times D) \neq \varnothing$ there exists $\psi : I \to \mathbb{R}^n$ such that $\psi \in \mathrm{sols}(y' = f(x, y))$ and $\psi(x) = y(x)$ for all $x \in \mathrm{Dom}(y)$.

Theorem Continuous Dependence on Initial Conditions: let $\Omega \subseteq \mathbb{R}^2$ let $f \in C^1(\Omega)$ let $a, b, p \in \mathbb{R}$ and let y_p be a solution to $\begin{cases} y'=f(x,y) \\ y(a)=p \end{cases}$ such that y_p is defined on $[a, b]$ then for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $q \in \mathbb{R}$ for which $|p - q| < \delta$ if y_q is a solution to $\begin{cases} y'=f(x,y) \\ y(a)=q \end{cases}$ then y_q is defined on $[a, b]$ and $|y_p - y_q| < \varepsilon$.

Theorem Continuous Dependence on Initial Conditions: let $\Omega \subseteq \mathbb{R}^2$ let $f \in C^1(\Omega)$ let $a, b \in \mathbb{R}$ let $p \in \mathbb{R}$ let $\{p_n\}_{n=0}^{\infty} \subseteq \mathbb{R}$ such that $p_n \to p$ and let $\{g_{p_n}\}_{n=0}^{\infty} \subseteq \mathbb{R} \to \mathbb{R}$ such that $g_{p_n} \in \mathrm{sols}\left(\begin{cases} y'=f(x,y) \\ y(a)=p_n \end{cases}\right)$ and g_{p_n} is defined on $[a, b]$ for all $n \in \mathbb{N}$ then there exists $g : \mathbb{R} \to \mathbb{R}$ such that $g_{p_n} \xrightarrow{p.w.} g$ and $g \in \mathrm{sols}\left(\begin{cases} y'=f(x,y) \\ y(a)=p \end{cases}\right)$.

n-th Order Ordinary Differential Equation: let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ then $f\left(x, y \ldots y^{(n)}\right) = 0$.

Claim: let $n \in \mathbb{N}_+$ let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}$ then

$$\left(g \in \mathrm{sols}\left(f\left(t, x(t) \ldots x^{(n)}(t)\right) = 0\right)\right) \iff \left(\begin{pmatrix} g \\ \vdots \\ g^{(n)} \end{pmatrix} \in \mathrm{sols}\left(\begin{cases} y_1=\dot{x} \\ \vdots \\ y_{n-1}=\dot{y_{n-2}} \\ \dot{y_{n-1}}=f(t,x,y_1\ldots,y_{n-1}) \end{cases}\right)\right)$$

Harmonic Oscillator/Spring Position Equation: $\ddot{x} = -x$.

Claim: the function $E(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$ is a conserved quantity of $\begin{cases} \dot{x}=y \\ \dot{y}=-x \end{cases}$.

Spring with Friction Position Equation: $\ddot{x} = -x - \dot{x}$.

Constant Tension Spring Position Equation: $\ddot{x} = -1$.

Claim: the function $E(x,y) = x + \frac{1}{2}y^2$ is a conserved quantity of $\begin{cases} \dot{x}=y \\ \dot{y}=-1 \end{cases}$.

Gravity in One Dimension Equation: $\ddot{x} = -\frac{1}{x^2}$.

Claim: the function $E(x,y) = \frac{1}{x} + \frac{1}{2}y^2$ is a conserved quantity of $\begin{cases} \dot{x}=y \\ \dot{y}=-\frac{1}{x^2} \end{cases}$.

Pendulum Equation: $\ddot{x} = -\sin(x)$.

Claim: the function $E(x,y) = \frac{1}{2}y^2 - \cos(x)$ is a conserved quantity of $\begin{cases} \dot{x}=y \\ \dot{y}=-\sin(x) \end{cases}$.

Differential 1-form: let $P, Q : \mathbb{R}^2 \to \mathbb{R}$ then $P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$.

Remark: in this course we won't explain the formal definition of a differential 1-form so we will use it as an object with certain properties.

Integral: let $\gamma : [a,b] \to \mathbb{R}^2$ be a curve and let $P, Q : \mathbb{R}^2 \to \mathbb{R}$ then $\int_\gamma (P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y) = \lim_{\Delta\to 0} \sum_{i=0}^{n-1} (P(x_i,y_i)(x_{i+1}-x_i) + Q(x_i,y_i)(y_{i+1}-y_i))$.

Remark: in the definition above $\lim_{\Delta\to 0}$ is the limit of all partitions of γ to horizontal and vertical segments.

Claim: let $\gamma : [a,b] \to \mathbb{R}^n$ be a curve let $\nu : [\alpha,\beta] \to [a,b]$ such that $\gamma \circ \nu$ is a reparameterization of γ and let ω be a differential 1-form then $\int_\gamma \omega = \int_{\gamma\circ\nu} \omega$.

Claim: let $\gamma : [a,b] \to \mathbb{R}^n$ be a curve and let ω be a differential 1-form then $\int_\gamma \omega = -\int_{-\gamma} \omega$.

Integral: let $\gamma : [a,b] \to \mathbb{R}^2$ be a curve and let $f, g : \mathbb{R}^2 \to \mathbb{R}$ then $\int_\gamma f\mathrm{d}g = \lim_{\Delta\to 0} \sum_{i=0}^{n-1} (f(x_i,y_i)(g(x_{i+1},y_{i+1}) - g(x_i,y_i)))$.

Claim: let $g \in C^1(\mathbb{R}^2)$ then $\mathrm{d}g = \frac{\partial g}{\partial x}\mathrm{d}x + \frac{\partial g}{\partial y}\mathrm{d}y$.

Corollary: let $\gamma : [a,b] \to \mathbb{R}^2$ be a curve let $f : \mathbb{R}^2 \to \mathbb{R}$ and let $g \in C^1(\mathbb{R}^2)$ then $\int_\gamma f\mathrm{d}g = \int_\gamma \left(f \cdot \frac{\partial g}{\partial x}\mathrm{d}x + f \cdot \frac{\partial g}{\partial y}\mathrm{d}y\right)$.

Exact Differential 1-form: a differential 1-form ω such that there exists $g \in C^1(\mathbb{R}^2)$ for which $\omega = \mathrm{d}g$.

Theorem: let ω be a differential 1-form then TFAE

- ω is exact.
- there exists $f : \mathbb{R} \to \mathbb{R}$ such that for all curves $\gamma : [a,b] \to \mathbb{R}^2$ we have $\int_\gamma \omega = f(\gamma(b)) - f(\gamma(a))$.
- for all closed curves $\gamma : [a,b] \to \mathbb{R}^2$ we have $\int_\gamma \omega = 0$.

Primitive/Potential: let ω be an exact differential 1-form then $g \in C^1(\mathbb{R}^2)$ such that $\omega = \mathrm{d}g$.

Claim: let $X : \mathbb{R} \to \mathbb{R}^3$ and let $f : \mathbb{R}^3 \to \mathbb{R}$ then $\frac{\mathrm{d}}{\mathrm{d}t}(f \circ X) = (\nabla f) \cdot \dot{X}$.

Conservative Vector Field: a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ such that there exists $U \in C^1(\mathbb{R}^n)$ for which $F = -\nabla U$.

Kinetic Energy: let $X \in C^1(\mathbb{R}, \mathbb{R}^n)$ then $K : \mathbb{R}^n \to \mathbb{R}$ such that $K(X(t)) = \frac{\|\dot{X}(t)\|^2}{2}$.

Total Energy: let $U \in C^1(\mathbb{R}^n)$ let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a conservative vector field such that $F = -\nabla U$ let $X \in \text{sols}\left(\ddot{X} = F(X)\right)$ and let $K : \mathbb{R}^n \to \mathbb{R}$ be the kinetic energy of X then $E : \mathbb{R}^n \to \mathbb{R}$ such that $E = K + U$.

Lemma: let $U \in C^1(\mathbb{R}^n)$ let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a conservative vector field such that $F = -\nabla U$ let $X \in \text{sols}\left(\ddot{X} = F(X)\right)$ and let $E : \mathbb{R}^n \to \mathbb{R}$ be the total energy of X then $\dot{E}(X(t)) = 0$.

Weighted Average/Center of Mass: let $p_1 \dots p_n \in \mathbb{R}^n$ and let $w : \{p_1 \dots p_n\} \to \mathbb{R}_+$ then $\frac{\sum_{i=1}^n w(p_i)\cdot p_i}{\sum_{i=1}^n w(p_i)}$.

Center of Mass: let $K \subseteq \mathbb{R}^2$ be compact and let $\rho : K \to \mathbb{R}$ then $\left(\frac{\int_K x\cdot\rho(x,y)\mathrm{d}x\mathrm{d}y}{\int_K \rho(x,y)\mathrm{d}x\mathrm{d}y}, \frac{\int_K y\cdot\rho(x,y)\mathrm{d}x\mathrm{d}y}{\int_K \rho(x,y)\mathrm{d}x\mathrm{d}y}\right)$.

Line: let $A, B \in \mathbb{R}^2$ then $L_{A,B} = \{\lambda A + (1-\lambda)B \mid \lambda \in [0,1]\}$.

Triangle: let $A, B, C \in \mathbb{R}^2$ such that $A \notin L_{B,C}$ and $B \notin L_{A,C}$ and $C \notin L_{A,B}$ then $\{A, B, C\}$.

Theorem Ceva: let $A, B, C \in \mathbb{R}^2$ such that $\{A, B, C\}$ is a triangle let $A' \in L_{B,C}$ let $B' \in L_{A,C}$ and let $C' \in L_{A,C}$ then $(L_{A,A'} \cap L_{B,B'} \cap L_{C,C'} \neq \varnothing) \iff \left(\frac{d(A,B')}{d(B',C)} \cdot \frac{d(C,A')}{d(A',B)} \cdot \frac{d(B,C')}{d(C',A)} = 1\right)$.

Special Orthogonal Group: let $n \in \mathbb{N}$ then $\mathrm{SO}(n) = \left\{A \in M_n(\mathbb{R}) \mid (\det(A) = 1) \wedge \left(A^T = A^{-1}\right)\right\}$.

Cross Product: let $x, y \in \mathbb{R}^3$ then $x \times y = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}$.

Claim Antisymmetry: let $x, y \in \mathbb{R}^3$ then $x \times y = -y \times x$.

Claim Orthogonality: let $x, y \in \mathbb{R}^3$ be linearly independent then $x \times y \perp x$ and $x \times y \perp y$.

Claim: let $x, y \in \mathbb{R}^3$ and let θ be the angle between x, y then $\|x \times y\| = \|x\| \cdot \|y\| \cdot \cos(\theta)$.

Corollary: let $x, y \in \mathbb{R}^3$ then $(x \times y = 0) \iff (x, y$ are linearly dependent$)$.

Claim: let $x, y, z \in \mathbb{R}^2$ and let $\alpha, \beta \in \mathbb{R}$ then $(\alpha x + \beta y) \times z = \alpha(x \times z) + \beta(y \times z)$.

Claim: let $X, Y \in C^1(\mathbb{R}^3)$ then $\frac{d}{dt}(X \times Y) = \dot{X} \times Y + X \times \dot{Y}$.
Angular Momentum: let $X \in C^1(\mathbb{R}^3)$ then $X \times \dot{X}$.
Lemma: angular momentum is a conserved quantity of $\ddot{X} = -\frac{X}{\|X\|^2}$.
Exact Ordinary Differential Equation: let $\Omega \subseteq \mathbb{R}^2$ and let $F \in C^1(\Omega)$ then $\frac{\partial F}{\partial x}(x,y) + y' \cdot \frac{\partial F}{\partial y}(x,y) = 0$.
Remark: let $\Omega \subseteq \mathbb{R}^2$ and let $F \in C^1(\Omega)$ then $\frac{\partial F}{\partial x} + y' \cdot \frac{\partial F}{\partial y} = \mathrm{d}F$.
Claim: let $\Omega \subseteq \mathbb{R}^2$ let $F \in C^1(\Omega)$ and let $y \in \text{sols}\left(\frac{\partial F}{\partial x}(x,y) + y' \cdot \frac{\partial F}{\partial y}(x,y) = 0\right)$ then $\frac{\mathrm{d}}{\mathrm{d}x}(F(x, y(x))) = 0$.
Remark: let $\Omega \subseteq \mathbb{R}^2$ let $F \in C^1(\Omega)$ and let $c \in \mathbb{R}$ then $\{F = c\}$ are solutions to $\frac{\partial F}{\partial x}(x,y) + y' \cdot \frac{\partial F}{\partial y}(x,y) = 0$.
Claim: let $\Omega \subseteq \mathbb{R}^2$ be a simply connected domain and let $P, Q \in C^1(\Omega)$ then ($P(x,y) + y' \cdot Q(x,y) = 0$ is an exact ODE)$\Longleftrightarrow$($\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$).
Linear Ordinary Differential Equation: let $f : \mathbb{R} \to \mathbb{R}$ let $n \in \mathbb{N}$ and let $a_0 \dots a_{n-1}' : \mathbb{R} \to \mathbb{R}$ then $y^{(n)} + \sum_{i=0}^{n-1} a_i(x) \cdot y^{(i)} = f$.
Claim: let $I \subseteq \mathbb{R}$ be an interval let $p, q : I \to \mathbb{R}$ and let $\psi \in \text{sols}(y' + py = q)$ then $\text{sols}(y' + py = q) = \text{sols}(y' + py = 0) + \psi$.
Bernoulli's Equation: let $\alpha \in \mathbb{R} \backslash \{0, 1\}$ and let $p, q : \mathbb{R} \to \mathbb{R}$ then $y' = py + qy^\alpha$.
Claim: let $\alpha \in \mathbb{R} \backslash \{0, 1\}$ and let $p, q : \mathbb{R} \to \mathbb{R}$ then
$\text{sols}(y' = py + qy^\alpha) = \text{sols}\left(\frac{1}{1-\alpha} \cdot (y^{1-\alpha})' = p \cdot y^{1-\alpha} + q\right) \cup \{0 \mid \alpha > 0\}$.
Riccati's Equation: let $p, q, r : \mathbb{R} \to \mathbb{R}$ then $y' + p + qy + ry^2 = 0$.
Claim: let $p, q, r : \mathbb{R} \to \mathbb{R}$ and let $\psi \in \text{sols}(y' + p + qy + ry^2 = 0)$ then
$\text{sols}(y' + p + qy + ry^2 = 0) = \text{sols}\left((y + \psi)' + (q + 2r\psi)(y + \psi) + r(y + \psi)^2 = 0\right)$.
Fibonacci Sequence: a sequence $F : \mathbb{N} \to \mathbb{N}$ such that $F_0 = 0$ and $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \in \mathbb{N}$.
Lemma: let $n, m \in \mathbb{N}$ then $F_{m+n+1} = F_m F_n + F_{m+1} F_{n+1}$.
Lemma Cassini Identity: let $n \in \mathbb{N}_+$ then $F_n^2 = F_{n+1} F_{n-1} + (-1)^{n-1}$.
Left Shift Operator: a function $\text{LS} : (\mathbb{N} \to \mathbb{C}) \to (\mathbb{N} \to \mathbb{C})$ such that $\text{LS}(x) = \lambda n \in \mathbb{N}.x_{n+1}$.
Remark; let $x \in \mathbb{N} \to \mathbb{C}$ then $(\forall n \in \mathbb{N}.x_{n+2} = x_{n+1} + x_n) \Longleftrightarrow \left(x \in \ker\left(\text{LS}^2 - \text{LS} - 1\right)\right)$.
Lemma: let $p, q \in \mathbb{C}[x]$ then $(p \cdot q)(\text{LS}) = p(\text{LS}) \cdot q(\text{LS})$.
Corollary: $\text{LS}^2 - \text{LS} - 1 = \left(\text{LS} - \frac{-1+\sqrt{5}}{2}\right)\left(\text{LS} - \frac{-1-\sqrt{5}}{2}\right)$.
Homogeneous Linear Ordinary Differential Equation: let $n \in \mathbb{N}$ and let $a_0 \dots a_{n-1}' : \mathbb{R} \to \mathbb{R}$ then $y^{(n)} + \sum_{i=0}^{n-1} a_i(x) \cdot y^{(i)} = 0$.
Remark: In this course we will discuss only homogeneous linear ODEs where the coefficients are constant.
Characteristic Polynomial of a Homogeneous Linear Ordinary Differential Equation: let $n \in \mathbb{N}$ and let $a_0 \dots a_{n-1}' \in \mathbb{R}$ then $p(x) = x^n + \sum_{i=0}^{n-1} a_i \cdot x^i$.
Claim: let $n \in \mathbb{N}$ let $a_0 \dots a_{n-1}' \in \mathbb{R}$ and let $f, g \in \text{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i \cdot y^{(i)} = 0\right)$ then
$f + g \in \text{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i \cdot y^{(i)} = 0\right)$.
Theorem: let $n \in \mathbb{N}$ let $a_0 \dots a_{n-1}' \in \mathbb{R}$ and let $\alpha \in \mathbb{R}$ be a solution to $p(x)$ with multiplicity ρ then
$\left\{x^0 e^{\alpha x}, \dots, x^{\rho-1} e^{\alpha x}\right\} \subseteq \text{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i \cdot y^{(i)} = 0\right)$.
Claim: let $n \in \mathbb{N}$ let $a_0 \dots a_{n-1}' \in \mathbb{R}$ and let $f \in \text{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i \cdot y^{(i)} = 0\right)$ then
$\text{Re}(f), \text{Im}(f) \in \text{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i \cdot y^{(i)} = 0\right)$.
Claim: let $f : \mathbb{R} \to \mathbb{R}$ let $n \in \mathbb{N}$ let $a_0 \dots a_{n-1}' : \mathbb{R} \to \mathbb{R}$ and let $\psi \in \text{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i(x) \cdot y^{(i)} = f\right)$ then
$\text{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i(x) \cdot y^{(i)} = f\right) = \text{sols}\left(y^{(n)} + \sum_{i=0}^{n-1} a_i(x) \cdot y^{(i)} = 0\right) + \psi$.
Complex Quasi Polynomial: let $\gamma \in \mathbb{C}$ and let $p \in \mathbb{C}[x]$ then $f : \mathbb{C} \to \mathbb{C}$ such that $f(z) = e^{\gamma z} \cdot p(z)$.
Degree of a Complex Quasi Polynomial: let $\gamma \in \mathbb{C}$ and let $p \in \mathbb{C}[x]$ then $\deg(e^{\gamma z} \cdot p(z)) = \deg(p)$.
Exponent of a Complex Quasi Polynomial: let $\gamma \in \mathbb{C}$ and let $p \in \mathbb{C}[x]$ then γ.
Real Quasi Polynomial: let $\alpha, \beta \in \mathbb{R}$ and let $p, q \in \mathbb{R}[x]$ then $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = e^{\alpha x}(p(x)\cos(\beta x) + q(x)\sin(\beta x))$.
Degree of a Real Quasi Polynomial: let $\alpha, \beta \in \mathbb{R}$ and let $p, q \in \mathbb{R}[x]$ then $\deg(e^{\alpha x}(p(x)\cos(\beta x) + q(x)\sin(\beta x))) = \max\{\deg(p), \deg(q)\}$.
Claim: let $P : \mathbb{C} \to \mathbb{C}$ be a complex quasi polynomial then $\text{Re}(P)$ is a real quasi polynomial.

Claim: let $n\in\mathbb{N}$ let $a_0\ldots a_{n-1^{\cdot}}\in\mathbb{R}$ let $\gamma\in\mathbb{C}$ let $p\in\mathbb{C}[x]$ and let $k\in\mathbb{N}$ the multiplicity of the root γ in $x^n+\sum_{i=0}^{n-1}a_i\cdot x^i$ then there exists $q\in\mathbb{C}[x]$ such that $\deg(q)=\deg(p)$ and $t^kq(t)e^{\gamma t}\in\text{sols}\left(y^{(n)}+\sum_{i=0}^{n-1}a_i\cdot y^{(i)}=p\cdot e^{\gamma t}\right)$.

Linear System of Ordinary Differential Equation: let $n\in\mathbb{N}$ and let $A\in M_n(\mathbb{R})$ then $y'(x)=A\cdot y(x)$.

Definition: let $A\in M_n(\mathbb{R})$ then $\max(|A|)=\max\left\{\left|(A)_{i,j}\right|\mid i,j\in[n]\right\}$.

Lemma: let $n,k\in\mathbb{N}$ let $A\in M_n(\mathbb{R})$ and let $i,j\in[n]$ then $(A)_{i,j}^k\le n^{k-1}\cdot\max(|A|)^k$.

Corollary: let $n,k\in\mathbb{N}$ let $A\in M_n(\mathbb{R})$ and let $i,j\in[n]$ then $(A)_{i,j}^k\le(n\cdot\max(|A|))^k$.

Theorem: let $n\in\mathbb{N}$ and let $A\in M_n(\mathbb{R})$ then $\sum_{k=0}^{\infty}\frac{A^k}{k!}\in M_n(\mathbb{R})$.

Definition: let $n\in\mathbb{N}$ and let $A\in M_n(\mathbb{R})$ then $e^A=\sum_{k=0}^{\infty}\frac{A^k}{k!}$.

Lemma: let $A\in M_n(\mathbb{R})$ and let $s,t\in\mathbb{R}$ then $e^{sA}\cdot e^{tA}=e^{(s+t)A}$.

Matrix Differentiation: let $A:\mathbb{R}\to M_n(\mathbb{R})$ such that $(A)_{i,j}\in C^1(\mathbb{R})$ for all $i,j\in[n]$ then $(A'(t))_{i,j}=(A(t))_{i,j}'$.

Claim: let $A,B\in C^1(\mathbb{R},M_n(\mathbb{R}))$ then $\frac{\mathrm{d}}{\mathrm{d}t}(AB)=\dot{A}B+A\dot{B}$.

Corollary: let $A\in C^1(\mathbb{R},M_n(\mathbb{R}))$ and let $v\in C^1(\mathbb{R},\mathbb{R}^n)$ then $\frac{\mathrm{d}}{\mathrm{d}t}(Av)=\dot{A}v+A\dot{v}$.

Lemma: let $A\in M_n(\mathbb{R})$ then $\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA}\right)\right)(0)=A$.

Corollary: let $A\in M_n(\mathbb{R})$ and let $\mu\in\mathbb{R}$ then $\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA}\right)\right)(\mu)=e^{\mu A}A$.

Theorem: let $n\in\mathbb{N}$ and let $A\in M_n(\mathbb{R})$ then $e^{tA}v$ is a solution to $\left\{\begin{smallmatrix}y'=A\cdot y\\y(0)=v\end{smallmatrix}\right.$.

Zeckendorff Representation: let $n\in\mathbb{N}$ let $k\in\mathbb{N}$ and let $c_0\ldots c_k\in\mathbb{N}$ such that $c_0\ge2$ and $c_{i+1}>c_i+1$ for all $i\in\{0\ldots k-1\}$ and $n=\sum_{i=0}^{k}F_{c_i}$ then $(F_{c_0},\ldots,F_{c_k})$.

Theorem Zeckendorff: let $n\in\mathbb{N}$ then

- Existence: there exists a zeckendorff representation for n.
- Uniqueness: let $(F_{c_0},\ldots,F_{c_k}),(F_{d_0},\ldots,F_{d_m})$ be zeckendorff representations for n then $k=m$ and $c_i=d_i$ for all $i\in\{0\ldots k-1\}$.

Theorem Grönwall's Inequality: let $\Omega\subseteq\mathbb{R}^2$ let $f_1,f_2\in C(\Omega)$ such that $f_2(x,y)>f_1(x,y)$ for all $x>0$ and $y\in\mathbb{R}$ let $a_1,a_2\in\mathbb{R}$ such that $a_2>a_1$ let $y_1\in\text{sols}\left(\left\{\begin{smallmatrix}y'=f_1(x,y)\\y(0)=a_1\end{smallmatrix}\right.\right)$ and let $y_2\in\text{sols}\left(\left\{\begin{smallmatrix}y'=f_2(x,y)\\y(0)=a_2\end{smallmatrix}\right.\right)$ then $y_2(x)>y_1(x)$ for all $x>0$.

Theorem: let $\Omega\subseteq\mathbb{R}^2$ let $f_1,f_2\in C^1(\Omega)$ such that $f_2(x,y)\ge f_1(x,y)$ for all $x>0$ and $y\in\mathbb{R}$ let $a_1,a_2\in\mathbb{R}$ such that $a_2\ge a_1$ let $y_1\in\text{sols}\left(\left\{\begin{smallmatrix}y'=f_1(x,y)\\y(0)=a_1\end{smallmatrix}\right.\right)$ and let $y_2\in\text{sols}\left(\left\{\begin{smallmatrix}y'=f_2(x,y)\\y(0)=a_2\end{smallmatrix}\right.\right)$ then $y_2(x)\ge y_1(x)$ for all $x>0$.

Wronskian: let $\mathcal{U}\subseteq\mathbb{R}$ and let $u,v\in C^1(\mathcal{U})$ then $W_{u,v}:\mathcal{U}\to\mathbb{R}$ such that $W_{u,v}(t)=\det\left(\begin{smallmatrix}u(t)&v(t)\\u'(t)&v'(t)\end{smallmatrix}\right)$.

Corollary: let $\mathcal{U}\subseteq\mathbb{R}$ let $u\in C^1(\mathcal{U})$ and let $c\in\mathbb{R}$ then $W_{u,c\cdot u}=0$.

Remark: let $\mathcal{U}\subseteq\mathbb{R}$ and let $u,v\in C^1(\mathcal{U})$ then $W=W_{u,v}$.

Theorem Abel's Identity: let $I\subseteq\mathbb{R}$ be an open interval let $p,q:I\to\mathbb{R}$ and let $u,v\in\text{sols}(y''+py'+qy=0)$ then $W_{u,v}\in\text{sols}(y'+py=0)$.

Corollary: let $I\subseteq\mathbb{R}$ be an open interval let $p,q:I\to\mathbb{R}$ and let $u,v\in\text{sols}(y''+py'+qy=0)$ then there exists $p\in I$ and $c\in\mathbb{R}$ such that $W_{u,v}(t)=\exp(c)\cdot W_{u,v}(p)\cdot\exp\left(-\int_p^t p(s)\,\mathrm{d}s\right)$ for all $t\in I$.

Corollary: let $I\subseteq\mathbb{R}$ be an open interval let $p,q:I\to\mathbb{R}$ and let $u,v\in\text{sols}(y''+py'+qy=0)$ then
$\text{sign}(W(t_1))=\text{sign}(W(t_2))$ for all $t_1,t_2\in I$.

Corollary: let $I\subseteq\mathbb{R}$ be an open interval let $p,q:I\to\mathbb{R}$ and let $u,v\in\text{sols}(y''+py'+qy=0)$ then (u,v are linearly dependent)$\Longleftrightarrow$($\text{sign}(W)=0$).

Corollary Complementary Solution: let $I\subseteq\mathbb{R}$ be an open interval let $p,q:I\to\mathbb{R}$ and let $u\in\text{sols}(y''+py'+qy=0)\setminus\{0\}$ then $u\int\frac{W}{u^2}$ is a solution of $y''+py'+qy=0$ and $u,u\int\frac{W}{u^2}$ are linearly independent.

Claim Variation of Parameters: let $I\subseteq\mathbb{R}$ be an open interval let $p,q,f:I\to\mathbb{R}$ let $u,v\in\text{sols}(y''+py'+qy=0)$ such that u,v are linearly independent and let $C_1,C_2\in C^1(I)$ such that C_1u+C_2v is a solution of $y''+py'+qy=f$ and $C_1'u+C_2'v=0$ then $\left\{\begin{smallmatrix}C_1'u+C_2'v=0\\C_1'u'+C_2'v'=f\end{smallmatrix}\right.$.

Corollary Lagrange's Formula: let $I\subseteq\mathbb{R}$ be an open interval let $p,q,f:I\to\mathbb{R}$ and let $u,v\in\text{sols}(y''+py'+qy=0)$ such that u,v are linearly independent then $-u\int\frac{fv}{W_{u,v}}+v\int\frac{fu}{W_{u,v}}$ is a solution of $y''+py'+qy=f$.

Green Function: let $\mathcal{U}\subseteq\mathbb{R}$ and let $u,v\in C^1(\mathcal{U})$ then $G_{u,v}:\mathcal{U}^2\to\mathbb{R}$ such that $G_{u,v}(s,t)=\frac{v(t)u(s)-u(t)v(s)}{u(s)v'(s)-u'(s)v(s)}$.

Claim: let $\mathcal{U}\subseteq\mathbb{R}$ and let $u,v\in C^1(\mathcal{U})$ then

- $G(t,t) = 0$ for all $t \in \mathcal{U}$.
- $\frac{\partial G}{\partial y}(t,t) = 1$ for all $t \in \mathcal{U}$.

Corollary: let $I \subseteq \mathbb{R}$ be an open interval let $p, q, f : I \to \mathbb{R}$ and let $u, v \in \text{sols}(y'' + py' + qy = 0)$ such that u, v are linearly independent then there exists $p \in I$ such that $\int_p^t G_{u,v}(s,t) f(s)\, ds$ is a solution of $y'' + py' + qy = f$.

Lemma: let $I \subseteq \mathbb{R}$ be an open interval let $p, q, f : I \to \mathbb{R}$ let $u, v \in \text{sols}(y'' + py' + qy = 0)$ such that u, v are linearly independent let $p \in I$ such that $\int_p^t G_{u,v}(s,t) f(s)\, ds$ is a solution of $y'' + py' + qy = f$ and let $g \in \text{sols}\begin{pmatrix} y''+py'+qy=f \\ y(p)=0 \\ y'(p)=0 \end{pmatrix}$ then $g(t) = \int_p^t G_{u,v}(s,t) f(s)\, ds$ for all $t \in I$.

Simple Zero: let $f \in C^1(\mathbb{R})$ then $p \in \ker(f)$ such that $f'(p) \neq 0$.

Isolated Zero: let $f : \mathbb{R} \to \mathbb{R}$ then $p \in \ker(f)$ such that there exists a neighbourhood $\mathcal{U}$ of p for which $0 \notin f(\mathcal{U} \backslash \{p\})$.

Lemma: let $I \subseteq \mathbb{R}$ be an open interval let $p, q : I \to \mathbb{R}$ let $u \in \text{sols}(y'' + py' + qy = 0) \setminus \{0\}$ and let $p \in \ker(u)$ then p is a simple and isolated zero of u.

Lemma: let $I \subseteq \mathbb{R}$ be an open interval let $p, q : I \to \mathbb{R}$ and let $u, v \in \text{sols}(y'' + py' + qy = 0)$ such that u, v are linearly independent then $\ker(u) \cap \ker(v) = \varnothing$.

Theorem Sturm's Separation Theorem: let $I \subseteq \mathbb{R}$ be an open interval let $p, q : I \to \mathbb{R}$ and let $u, v \in \text{sols}(y'' + py' + qy = 0)$ such that u, v are linearly independent then for all $z, w \in \ker(u)$ if $(z,w) \cap \ker(u) = \varnothing$ then $|(z,w) \cap \ker(v)| = 1$.

Theorem Sturm's Comparison Theorem: let $I \subseteq \mathbb{R}$ be an open interval let $p, q : I \to \mathbb{R}$ such that $p \geq q$ and $p \neq q$ let $u \in \text{sols}(y'' + py = 0)$ let $v \in \text{sols}(y'' + qy = 0)$ such that u, v are linearly independent and let $z, w \in \ker(v)$ then $(z,w) \cap \ker(u) \neq \varnothing$.

Claim: let $k < 0$ then $e^{\sqrt{-k} \cdot t}$ is a solution of $y'' + ky = 0$.

Claim: the function 1 is a solution of $y'' = 0$.

Corollary: let $I \subseteq \mathbb{R}$ be an open interval let $q : I \to \mathbb{R}$ such that $0 \geq q$ and let $v \in \text{sols}(y'' + qy = 0)$ such that $v \neq 0$ then $|\ker(v)| \leq 1$.

Claim: let $k > 0$ then $\sin\left(\sqrt{k} \cdot t\right)$ is a solution of $y'' + ky = 0$.

Supremum Norm/L^∞ Norm: let $a, b \in \mathbb{R}$ such that $a \leq b$ and let $f \in C([a,b])$ then $\|f\|_\infty = \max(\text{Im}(|f|))$.

Claim: let $a, b \in \mathbb{R}$ such that $a \leq b$ then $\|\cdot\|_\infty$ is a norm on $C([a,b])$.

Supremum Metric: let $a, b \in \mathbb{R}$ such that $a \leq b$ and let $f, g \in C([a,b])$ then $d_\infty(f,g) = \|f - g\|_\infty$.

Claim: let $a, b \in \mathbb{R}$ such that $a \leq b$ then d_∞ is a metric on $C([a,b])$.

Claim: let $a, b \in \mathbb{R}$ such that $a \leq b$ let $f \in C([a,b])$ and let $\{f_n\}_{n=0}^\infty \subseteq C([a,b])$ then $\left(f_n \xrightarrow{L^\infty} f\right) \iff \left(f_n \xrightarrow{u} f\right)$.

Theorem: let $a, b \in \mathbb{R}$ such that $a \leq b$ then $(C([a,b]), d_\infty)$ is complete.

Closed Set: let (X,d) be a metric space then $A \subseteq X$ such that for all $\{a_n\}_{n=0}^\infty \subseteq A$ and for all $L \in X$ if $a_n \to L$ then $L \in A$.

Fixed Point: let X be a set and let $f : X \to X$ then $p \in X$ such that $f(p) = p$.

Contraction: let (X,d) be a metric space then $f : X \to X$ such that there exists $\rho \in [0,1)$ for which $d(f(x), f(y)) \leq \rho \cdot d(x,y)$ for all $x, y \in X$.

Theorem Contraction Mapping Theorem: let (X,d) be a complete metric space such that $X \neq \varnothing$ and let $f : X \to X$ be a contraction then there exists a unique $p \in X$ for which $f(p) = p$.

Lipschitz Function: let (X,d) be a metric space then $f : X \to \mathbb{R}$ such that there exists $L \in \mathbb{R}$ for which $|f(x) - f(y)| \leq L \cdot d(x,y)$ for all $x, y \in X$.

Lipschitz Constant: let (X,d) be a metric space and let $f : X \to \mathbb{R}$ be lipschitz then minimal $L \in \mathbb{R}$ such that $|f(x) - f(y)| \leq L \cdot d(x,y)$ for all $x, y \in X$.

Claim: let (X,d) be a metric space and let $f : X \to \mathbb{R}$ be lipschitz then $f \in C(X)$.

Locally Lipschitz Function: let (X,d) be a metric space then $f : X \to \mathbb{R}$ such that for all $x \in X$ there exists a neighbourhood $\mathcal{U}$ of x for which $f_{\restriction \mathcal{U}}$ is lipschitz.

Theorem Picard-Lindelöf: let $I, J \subseteq \mathbb{R}$ be closed intervals let $f \in C(I \times J)$ such that f is locally lipschitz in the second coordinate and let $p \in \text{int}(I \times J)$ then there exists $\varepsilon > 0$ such that $\begin{cases} y' = f(x,y) \\ y(p_1) = p_2 \end{cases}$ has a unique solution on the interval $(p_1 - \varepsilon, p_1 + \varepsilon)$.

Compact Set: let (X,d) be a metric space then $K \subseteq X$ such that for all $\{x_n\}_{n=0}^\infty \subseteq K$ there exists a convergent subsequence.

Uniformly Bounded Sequence of Functions: let $a,b\in\mathbb{R}$ such that $a\leq b$ then $\{f_n\}_{n=0}^{\infty}\subseteq C([a,b])$ such that $\exists M\in\mathbb{R}.\forall x\in[a,b].\forall n\in\mathbb{N}.|f_n(x)|\leq M$.

Uniformly Equicontinuous Sequence of Functions: let $a,b\in\mathbb{R}$ such that $a\leq b$ then $\{f_n\}_{n=0}^{\infty}\subseteq C([a,b])$ such that $\forall\varepsilon>0.\exists\delta>0.\forall x,y\in[a,b].|x-y|<\delta\Longrightarrow|f(x)-f(y)|<\varepsilon$.

Theorem Arzela-Ascoli: let $a,b\in\mathbb{R}$ such that $a\leq b$ and let $\{f_n\}_{n=0}^{\infty}\subseteq C([a,b])$ then ($\{f_n\}_{n=0}^{\infty}$ has a convergent subsequence)$\Longleftrightarrow$($\{f_n\}_{n=0}^{\infty}$ is uniformly bounded and uniformly equicontinuous).

Notation: let $p\in\mathbb{R}^2$ and let $f:\mathbb{R}\to\mathbb{R}$ then $\eta_f(p):\mathbb{R}\to\mathbb{R}$ such that $\eta_f(p)(x)=f(p)\cdot(x-p_1)+p_2$.

Algorithm Euler's Method: let $n\in\mathbb{N}$ let $f\in C(\mathbb{R})$ let $\varepsilon\in\mathbb{R}$ and let $p\in\mathbb{R}^2$ then

function `EulerMethod`(n,f,ε,p):
  $(x_0,y_0)\leftarrow p$
  for $i\leftarrow\{1\ldots n\}$ **do**
    $x_i\leftarrow x_0+\varepsilon\cdot\frac{i}{n}$
    $g_i\leftarrow(\lambda x\in[x_{i-1},x_i].(f(x_{i-1},y_{i-1})\cdot(x-x_{i-1})+y_{i-1}))$
    $y_i\leftarrow g_i(x_i)$
  end
  return $\bigcup_{i=1}^{n}g_i$

Claim: let $n\in\mathbb{N}$ let $f\in C(\mathbb{R})$ let $\varepsilon\in\mathbb{R}$ and let $p\in\mathbb{R}^2$ then $\text{EulerMethod}(n,f,\varepsilon,p):[p_1,p_1+\varepsilon]\to\mathbb{R}$.

Notation: let $n\in\mathbb{N}$ let $f\in C(\mathbb{R})$ let $\varepsilon\in\mathbb{R}$ and let $p\in\mathbb{R}^2$ then $\text{EulerMethod}_n=\text{EulerMethod}(n,f,\varepsilon,p)$.

Claim: let $f\in C(\mathbb{R})$ let $\varepsilon\in\mathbb{R}$ let $p\in\mathbb{R}^2$ then there exists a convergent subsequence $\{\text{EulerMethod}_{n_k}\}$ such that $\lim_{k\to\infty}\text{EulerMethod}_{n_k}$ is a solution of $\begin{cases}y'=f(x,y)\\y(p_1)=p_2\end{cases}$ defined on $[p_1,p_1+\varepsilon]$.

Boundary Value Problem (BVP): let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ and let $a,b\in\mathbb{R}$ then $\begin{cases}f\left(x,y',\ldots y^{(n)}\right)=0\\y(0)=a\\y(1)=b\end{cases}$.

Theorem: let $k\in C([0,1])$ such that $k>0$ and let $a,b\in\mathbb{R}$ then $\left|\text{sols}\left(\begin{cases}y''=ky\\y(0)=a\\y(1)=b\end{cases}\right)\right|=1$.