Vector space: V is v.s. over $\mathbb{F}$ if $(\forall u, v \in V. u + v \in V) \wedge (\forall v \in V. \forall a \in \mathbb{F}. a \cdot v \in V)$.

Inner product: $\langle \cdot, \cdot \rangle : V^2 \to \mathbb{F}$ s.t. (multilinear by first argument)$\wedge$($\forall u \in V \setminus \{0\} . \langle u, u \rangle > 0$).

Norm: $\|\cdot\| : V \to \mathbb{F}$ s.t. $\|v\| = \sqrt{\langle v, v \rangle}$.

Orthogonal: $\langle v, u \rangle = 0$.

Orthogonal set: ($\forall v \neq u \in V. \langle v, u \rangle = 0$)$\wedge$($\forall v \in V. \|v\| \neq 0$).

Orthonormal set: (orthogonal set)$\wedge$($\forall v \in V. \|v\| = 1$).

Homomorphismic function: f s.t. $f(xy) = f(x) f(y)$.

Expected value: $\mathbb{E}_{x\sim A}[f(x)] = \frac{1}{|A|} \sum_{x \in A} f(x)$.

Boolean function: function f such that $|\text{Im}(f)| \leq 2$.

definition: $\mathcal{F}_n = \{\pm 1\}^n \to \{\pm 1\} \equiv \{0, 1\}^n \to \{0, 1\}$.

definition: for $f, g \in \mathcal{F}_n$ define $\langle f, g \rangle = \mathbb{E}_{x\sim\{0,1\}^n}[f(x) \cdot g(x)]$.

claim: F_n is a vector space over $\{0, 1\}$.

Characteristic vector: let $S \subseteq [n]$, define $\chi_S : \{\pm 1\}^n \to \{\pm 1\}$ as $\chi_S(x) = \prod\limits_{i \in S} x_i$.

claim: $\{\chi_S\}_{S\subseteq[n]}$ is orthonormal basis for F_n.

claim: every bool func $f(x) \in F_n$ can be represented as $f(x) = \sum_{S\subseteq[n]} \chi_S(x) \cdot \hat{f}(S)$ s.t. $\hat{f}(S) \in [-1, 1]$.

Norm 2: let $f \in F_n$ then $\|f\|_2 = \sqrt{\mathbb{E}_{x\in\{0,1\}^n}\left[f(x)^2\right]}$.

claim: $\forall S, T \subseteq [n] . \langle \chi_S(x), \chi_T(x) \rangle = \begin{cases} 1 & S = T \\ 0 & else \end{cases}$.

statement: $f \cdot \chi_S = \hat{f}(S)$.

definition: $\overline{J} = [n] \setminus J$.

Restricted function: let $f \in F_n$, $J \subseteq [n]$, $z \in \{\pm 1\}^{|\overline{J}|}$ define $f_{\overline{J}\to z} : \{\pm 1\}^{|J|} \to \{\pm 1\}$ as

$$f_{\overline{J}\to z}(y) = f\left(\lambda m \in [n] . \begin{cases} y_{|\{x\in J | x \leq m\}|} & m \in J \\ z_{\left|\left\{x\in \overline{J} | x\leq m\right\}\right|} & m \in \overline{J} \end{cases}\right).$$

claim: let $T \subseteq J \subseteq [n]$ then $\mathbb{E}_{z\in\{\pm1\}^{|\overline{J}|}}\left[\widehat{f_{\overline{J}\to z}}(T)^2\right] = \sum\limits_{S\in\{S\subseteq[n] | S\cap J = T\}} \hat{f}(S)^2$.

Heavy coefficients: let $\gamma \in \mathbb{R}$, $f \in F_n$ then $\left|\left\{S \subseteq [n] \mid \left|\hat{f}(S)\right| \geq \gamma\right\}\right| \leq \frac{1}{\gamma^2}$.

t-sparse function: $\exists A \subseteq [n] . \left(f(x) = \sum\limits_{S\in A} \hat{f}(S) \chi_S(x)\right) \wedge (|A| = t)$.

(t, ε)-sparse function: $\exists g \in F_n. \left(\exists A \subseteq [n] . \left(g(x) = \sum\limits_{S\in A} \hat{g}(S) \chi_S(x)\right) \wedge (|A| = t)\right) \wedge \left(\|f - g\|_2^2 \leq \varepsilon\right)$.

statement: let L be the output of the GL algorithm, we can conclude that $f(x) \approx \text{sign}\left(\sum\limits_{T\in L} \chi_T(x) \cdot \hat{f}(T)\right)$.

definition: let G be a finite group then $V_G = G \to \mathbb{C}$.

definition: for $f, g \in V_G$ define $\langle f, g \rangle = \mathbb{E}_{x\sim G}\left[f(x) \cdot \overline{g(x)}\right]$.

Cyclic group: a group G s.t. $\exists g \in G. \langle g \rangle = G$.

claim: $\langle \mathbb{Z}_n, + \rangle$ is a cyclic group.

Homomorphism: func $f: G \to H$ s.t. $f(xy) = f(x) f(y)$.

Isomorphism: homomorphismic bijection.

symbol: $G \cong H$ is there exists an homomorphismic bijection between them.

Theorem: let G be a finite group then there exists $\mathbb{Z}_{q_1} \ldots \mathbb{Z}_{q_n}$ cyclic groups s.t. $G \cong \mathbb{Z}_{q_1} \oplus \ldots \oplus \mathbb{Z}_{q_n}$.

claim: let G be cyclic abelian group then $G \cong \mathbb{Z}_{|G|}$.

General basis: for every $\mathbb{Z}_n$ define $\chi_y : \mathbb{Z}_n \to \mathbb{C}$ as $\chi_y(x) = e^{\frac{2\pi i x y}{n}}$.

claim: $\{\chi_g\}_{g \in G}$ is the only orthonormal homomorphismic basis of G.

definition: $x^{\otimes i} = \begin{cases} x_j & i \neq j \\ -x_i & i = j \end{cases}$.

Derivative: let f be a bool func and $i \in [n]$, $\partial_i f : \{\pm 1\}^n \to \{\pm 1\}$ as

$\partial_i f(y) = \frac{1}{2}\left(f(y_1, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_{n-1}) - f(y_1, \ldots, y_{i-1}, -1, y_{i+1}, \ldots, y_{n-1})\right)$.

Event probability: $\Pr_{x \sim A}[Q(x)] = \frac{1}{|A|}\sum_{x \in A} Q(x)$, where Q is 1 at the event and 0 else.

Influence: $I_i[f] = \Pr_{x \in \{\pm 1\}^n}[f(x) \neq f(x^{\otimes i})]$.

Total influence: $I[f] = \sum_{i \in [n]} I_i[f]$.

claim: $I_i[f] = \langle \partial_i f, \partial_i f \rangle$.

Variance: $\mathrm{var}(f) = \mathbb{E}_{x \in \{\pm 1\}^n}\left[\left(f(x) - \mathbb{E}_{y \in \{\pm 1\}^n}[f(y)]\right)^2\right]$.

Balanced boolean function: $\mathbb{E}_{x \in \{\pm 1\}^n}[f(x)] = 0$.