Homework #5 Name: Flori Kusari

roblem 5.1: Compute the following limits:

a) $\lim_{x\to\infty} \sqrt{(x-a)(x-b)} - x$

$$\lim_{x\to\infty} \sqrt{x^2 - bx - ax + ab} - x$$

$$\lim_{x\to\infty} \left(\left(\sqrt{x^2 - bx - ax + ab} - x\right) \cdot \frac{\left(\sqrt{x^2 - bx - ax + ab} + x\right)}{\left(\sqrt{x^2 - bx - ax + ab} + x\right)} \right)$$

$$\lim_{x\to\infty} \left(\frac{x^2 - bx - ax + ab - x^2}{\sqrt{x^2 - bx - ax + ab} + x} \right)$$

$$\lim_{x\to\infty} \left(\frac{-bx - ax + ab}{\sqrt{x^2 - bx - ax + ab} + x} \right) \Rightarrow \lim_{x\to\infty} \left(\frac{\cancel{x} \cdot \left(-b - a + \frac{ab}{x}\right)}{\cancel{x} \cdot \left(\sqrt{1 - \frac{b}{x} - \frac{a}{x} + \frac{ab}{x^2}} + 1\right)} \right)$$

$$\frac{\lim_{x\to\infty} \left(-b - a + \frac{ab}{x}\right)}{\lim_{x\to\infty} \left(\sqrt{1 - \frac{b}{x} - \frac{a}{x} + \frac{ab}{x^2}} + 1\right)} \Rightarrow \frac{-b - a + ab \cdot \lim_{x\to\infty}\left(\frac{1}{x}\right)}{\sqrt{\lim_{x\to\infty}(1) - \lim_{x\to\infty}\left(\frac{b}{x}\right) - a \cdot \lim_{x\to\infty}\left(\frac{1}{x}\right) + ab \cdot \lim_{x\to\infty}\left(\frac{1}{x^2}\right)} + 1}$$

$$b \cdot \lim_{x\to\infty}\left(\frac{1}{x}\right)$$

$$\frac{-b - a + (ab \cdot 0)}{\sqrt{1 - (b \cdot 0) - (a \cdot 0) + (ab \cdot 0)} + 1}$$

$$\frac{-b - a}{\sqrt{1} + 1} = \boxed{\frac{-b - a}{2}}$$

b) $\lim\limits_{x\to 0} \frac{1-\cos(x)}{x\cdot\sin(2x)} \Rightarrow \frac{\frac{d}{dx}\left(1-\cos(x)\right)}{\frac{d}{dx}\left(x\cdot\sin(2x)\right)} \Rightarrow \frac{\sin(x)}{\sin(2x)+2x\cdot\cos(2x)}$

$$\lim_{x\to 0} \frac{\frac{d}{dx}\left(\sin(x)\right)}{\frac{d}{dx}\left(\sin(2x)+2x\cdot\cos(2x)\right)} \Rightarrow \frac{\cos(x)}{4\cdot\cos(2x)-4x\cdot(\sin(2x))}$$

Evaluate the limit

$$\frac{\cos(0)}{4\cdot\cos(2\cdot 0)-(4\cdot 0)\cdot\sin(2\cdot 0)} \Rightarrow \frac{1}{4\cdot 1-0\cdot 0} \Rightarrow \boxed{\frac{1}{4}}$$

$$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Problem 4.1:

c) $\lim\limits_{x\to\infty}\left(\frac{x^2+1}{x^2-1}\right)^{x^2} \Rightarrow \left(\frac{(x^2-1)+2}{(x^2-1)}\right)^{x^2}$

$$\lim_{x\to\infty}\left(1+\frac{2}{x^2-1}\right)^{x^2}$$

$$\lim_{x\to\infty}\left(1+\frac{2}{(x^2-1)}\cdot\frac{1/2}{1/2}\right)^{x^2} \Rightarrow \left(1+\frac{1}{1/2(x^2-1)}\right)^{x^2}$$

$\rightarrow$ to go further we need to transform this exponent to match the expression in the denominator

$$\lim_{x\to\infty}\left(1+\frac{1}{\frac{1}{2}(x^2-1)}\right)^{\frac{1}{2}\cdot(x^2-1)\cdot 2+1}$$

k is set to $\frac{1}{2}\cdot(x^2-1)$

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{k\cdot 2+1}$$

$$\lim_{k\to\infty}\left(\left(1+\frac{1}{k}\right)^{k\cdot 2}\cdot\left(1+\frac{1}{k}\right)^{1}\right)$$

$\downarrow$ Evaluating the limit using definition of "e"

$$e^2\cdot\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{1}$$

$$e^2\cdot 1$$

$$\boxed{e^2}$$

Problem 4.2: Find the derivatives of the following functions:

Googled expression

a) $y=\dfrac{x^2}{\sqrt{x^2+a^2}}$

Differentiation rule: $\dfrac{d}{dx}\left(\dfrac{f}{g}\right)=\dfrac{\frac{d}{dx}(f)\cdot g-f\cdot\frac{d}{dx}(g)}{g^2}$

$$y'=\frac{d}{dx}\left(\frac{x^2}{\sqrt{x^2+a^2}}\right)$$

$$y'=\frac{\frac{d}{dx}(x^2)\cdot\sqrt{x^2+a^2}-x^2\cdot\frac{d}{dx}\left(\sqrt{x^2+a^2}\right)}{\left(\sqrt{x^2+a^2}\right)^2}$$

$$y'=\frac{2x\sqrt{x^2+a^2}-x^2\cdot\frac{1}{2\sqrt{x^2+a^2}}\cdot 2x}{\left(\sqrt{x^2+a^2}\right)^2}$$

$$\boxed{y'=\frac{x^3+2a^2x}{\sqrt{x^2+a^2}\,(x^2+a^2)}}$$ $\rightarrow$ cannot simplify any further.

b)

Problem 4.2:

b) $y = (1 + \ln \sin(x))^n$

$$y' = \frac{d}{dx}\left(\left(1 + \ln(\sin(x))\right)^n\right)$$

$$y' = \frac{d}{dg}(g^n) \cdot \frac{d}{dx}\left(1 + \ln(\sin(x))\right)$$

$$y' = ng^{n-1} \cdot \frac{1}{\sin(x)} \cdot \cos(x)$$

$$y' = n \cdot \left(1 + \ln(\sin(x))\right)^{n-1} \cdot \frac{1}{\sin(x)} \cdot \cos(x)$$

$$\boxed{y' = \frac{n \cdot \left(1 + \ln(\sin(x))\right)^{n-1} \cdot \cos(x)}{\sin(x)}} \text{ or } \boxed{y' = n \cdot \left(1 + \ln(\sin(x))\right)^{n-1} \cdot \cot(x)}$$

$$\frac{\cos(x)}{\sin(x)}$$

Chain Rule Definition:

$$\frac{d}{dx}(f(g)) = \frac{d}{dg}(f(g)) \cdot \frac{d}{dx}(g)$$

where in this case $\boxed{g = 1 + \ln(\sin(x))}$

Problem 4.2:

c) $y = x^{\frac{1}{x}}$

$$\ln(y) = \ln\left(x^{\frac{1}{x}}\right)$$

$$\ln(y) = \frac{\ln(x)}{x}$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}\left(\frac{\ln(x)}{x}\right)$$

$$\frac{d}{dy}(\ln(y)) \cdot \frac{dy}{dx} = \frac{d}{dx}(\ln(x))$$

Chain Rule for these cases:

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dy}(\ln(y)) \cdot \frac{dy}{dx}$$

*Differentiation Rule $\boxed{\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{d}{dx}(f) \cdot g - f \cdot \frac{d}{dx}(g)}{g^2}}$

$$\frac{d}{dy}(\ln(y)) \cdot \frac{dy}{dx} = \frac{\frac{d}{dx}(\ln(x)) \cdot x - \ln(x) \cdot \frac{d}{dx}(x)}{x^2}$$

Previously: $\frac{d}{dx}(x)$

*We use $\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{\frac{1}{x} \cdot x - \ln(x) \cdot 1}{x}$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1 - \ln(x)}{x^2} \quad / \cdot y$$

*Initial Equation Substitution

$$\frac{dy}{dx} = y \cdot \frac{(1 - \ln(x))}{x^2} \qquad \boxed{y = x^{\frac{1}{x}}}$$

$$\frac{dy}{dx} = x^{\frac{1}{x}} \cdot \frac{(1 - \ln(x))}{x^2}$$

Solved for derivative using logarithmic differentiation

$$\frac{dy}{dx} = \boxed{x^{\frac{1-2x}{x}} \cdot (1 - \ln(x))}$$

Problem 4.3: Find the derivatives of implicitly defined functions $y(x)$:

a) $\sqrt{x} + \sqrt{y} = \sqrt{a}$ Differentiating both sides

$$\frac{d}{dx}\left(\sqrt{x} + \sqrt{y}\right) = \frac{d}{dx}\left(\sqrt{a}\right)$$

not dependent on any of the two therefore 0

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = \frac{-\sqrt{y}}{\sqrt{x}}}$$

$$\frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = \frac{-1}{2\sqrt{x}} \quad / \cdot 2\sqrt{y}$$

$$\frac{1}{1} \cdot \frac{dy}{dx} = \frac{-2\sqrt{y}}{2\sqrt{x}}$$

b) →

b) $2^x + 2^y = 2^{x+y}$

$2^x + 2^y - 2^{x+y} = 0$

To find $\frac{dy}{dx}$ we use the formula $\frac{dy}{dx} = -\frac{f_x}{f_y}$

$f(x,y) = 2^x + 2^y - 2^{x+y}$

$f_x = \ln(2) \cdot 2^x - \ln(2) \cdot 2^{x+y}$
$f_y = \ln(2) \cdot 2^y - \ln(2) \cdot 2^{x+y}$

$$\frac{dy}{dx} = -\frac{\ln(2) \cdot 2^x - \ln(2) \cdot 2^{x+y}}{\ln(2) \cdot 2^y - \ln(2) \cdot 2^{x+y}} \xrightarrow{\text{Factor out}} \frac{\cancel{\ln(2)} \cdot (2^x - 2^{x+y})}{\cancel{\ln(2)} \cdot (2^y - 2^{x+y})}$$

$$\frac{dy}{dx} = -\frac{2^x - 2^{x+y}}{2^y - 2^{x+y}} \Rightarrow \boxed{\frac{dy}{dx} = \frac{2^{x+y} - 2^x}{2^y - 2^{x+y}}}$$

or

$$\boxed{\frac{dy}{dx} = \frac{-2^x + 2^{x+y}}{2^y - 2^{x+y}}}$$

Same Answer just different order in writing.

THE END