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{Problem Sheet #7}

Problem 7.1: Sketch the graph of the following function with the detailed investigation: $y = \frac{x^3}{2-x^2}$

*** Find domain:** $y = \frac{x^3}{-x^2+2} \Rightarrow x \in \mathbb{R} \setminus \{-\sqrt{2}, \sqrt{2}\}$

*** Function is symetric about the origin ***

*** Horizontal Asymptotes:** $\lim_{x \to +\infty} \left(\frac{x^3}{2-x^2} \right) \Rightarrow -\infty$

Since the limits are infinite, the function has no horizontal asymptotes.

$$\lim_{x \to -\infty} \left(\frac{x^3}{2-x^2} \right) \Rightarrow +\infty$$

*** Vertical Asymptotes:** $x = -\sqrt{2}$

$x = \sqrt{2}$

*** Find Inflection Point:**

$$f(x) = \frac{x^3}{2-x^2}$$

$$f'(x) = \frac{6x^2 - x^4}{(2-x^2)^2}$$

$$f''(x) = \frac{24x + 4x^3}{(2-x^2)^3}$$

$$\Downarrow$$

$$0 = \frac{24x + 4x^3}{(2-x^2)^3}$$ Solve for "x"

$x_1 = 0 \Rightarrow$ Intervals $\langle -\sqrt{2}, 0 \rangle, \langle 0, \sqrt{2} \rangle$ Candidates for inflection points must include the excluded points $x = -\sqrt{2}$ and $x = \sqrt{2}$

$x_2 = \sqrt{2} \Rightarrow$ No inflection point

$x_3 = -\sqrt{2} \Rightarrow$ No inflection point

Choose two points from the intervals: $x_1 = -1$

$x_2 = 1$

$f''(-1) = -28$

$f''(1) = 28$

So inflection point is at $x = 0$

$$f(0) = \frac{x^3}{2-x^2} = \frac{0^3}{2-0^3} = \frac{0}{2} = 0$$

Inflection point is at $(0,0)$

* Finding the extrema

$$f(x) = \frac{x^3}{2-x^2}$$

$$f'(x) = \frac{6x^2 - x^4}{(2-x^2)^2}$$

$\Downarrow$

$$0 = \frac{6x^2 - x^4}{(2-x^2)^2}$$ Solve for "x"

$x_1 = 0 \implies \langle -\sqrt{2}, 0 \rangle, \langle 0, \sqrt{2} \rangle$

$x_2 = -\sqrt{6} \implies \langle -\infty, -\sqrt{6} \rangle, \langle -\sqrt{6}, -\sqrt{2} \rangle$

$x_3 = \sqrt{6} \implies \langle 2, \sqrt{6} \rangle, \langle \sqrt{6}, +\infty \rangle$

Choose 1 point from each interval:

$x_1 = -1 \implies f'(-1) = 5$

$x_2 = 1 \implies f'(1) = 5$

$x_3 = -3 \implies f'(-3) = -\frac{27}{49}$

$x_4 = -2 \implies f'(-2) = 2$

$x_5 = 2 \implies f'(2) = 2$

$x_6 = 3 \implies f'(3) = -\frac{27}{49}$

$\implies$

$f(x)$ at 0 has no extreme point

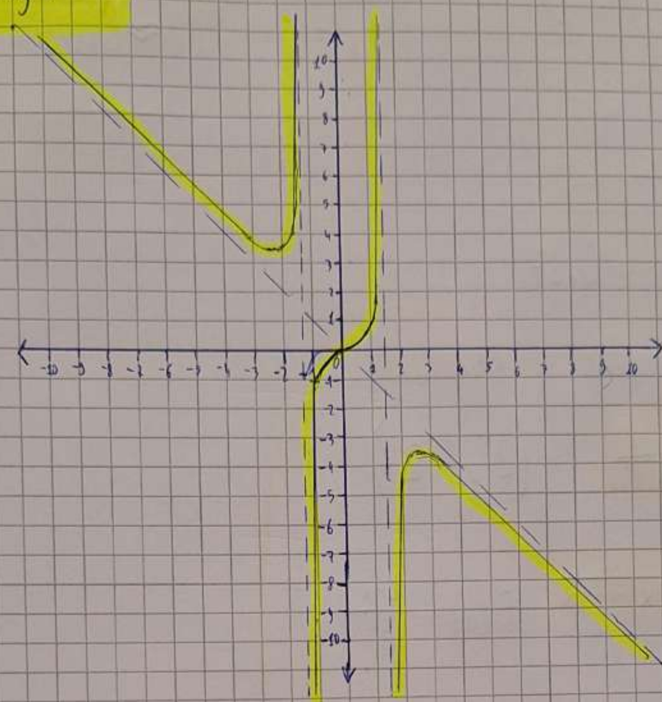
$f(x)$ at $\sqrt{6}$ has the maxima

$f(x)$ at $-\sqrt{6}$ has the minima

$$f(-\sqrt{6}) = \frac{3\sqrt{3}}{2}$$

$$f(\sqrt{6}) = -\frac{3\sqrt{3}}{2}$$

Graphing:



Problem 7.2: Sketch the graph of the following function with the detailed investigation: $y = \frac{x}{e^x}$

* Find domain: $y = \frac{x}{e^x} \Rightarrow x \in R$

* Not symmetric about the origin *

* Horizontal Asymptotes: $y = 0$

* Vertical Asymptote: N/A

* Find Inflection Point: $f(x) = \frac{x}{e^x}$

$$f'(x) = \frac{1-x}{e^x}$$

$$f''(x) = \frac{-2+x}{e^x}$$

$$0 = \frac{-2+x}{e^x}$$ * Solve for x

$x = 2 \Rightarrow \langle -\infty, 2 \rangle, \langle 2, +\infty \rangle$

* Determine the intervals *

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Choose points from each interval: $x_1 = 1 \Rightarrow f''(1) \approx -0.37$

$x_2 = 3 \Rightarrow f''(3) \approx 0.05$

$f(x) = \frac{x}{e^x}$; $x = 2$

$f(2) = \frac{2}{e^2} \approx 0.27$

So inflection point at $x = 2$

$\left(2, \frac{2}{e^2}\right)$ or $(2, 0.27)$

* Find the extrema

$f'(x) = \frac{1-x}{e^x}$

$0 = \frac{1-x}{e^x}$ Solve for "x"

$x = 1$

$\langle -\infty, 1 \rangle, \langle 1, +\infty \rangle$

$x_1 = 0$ $\quad$ $x_2 = 2$

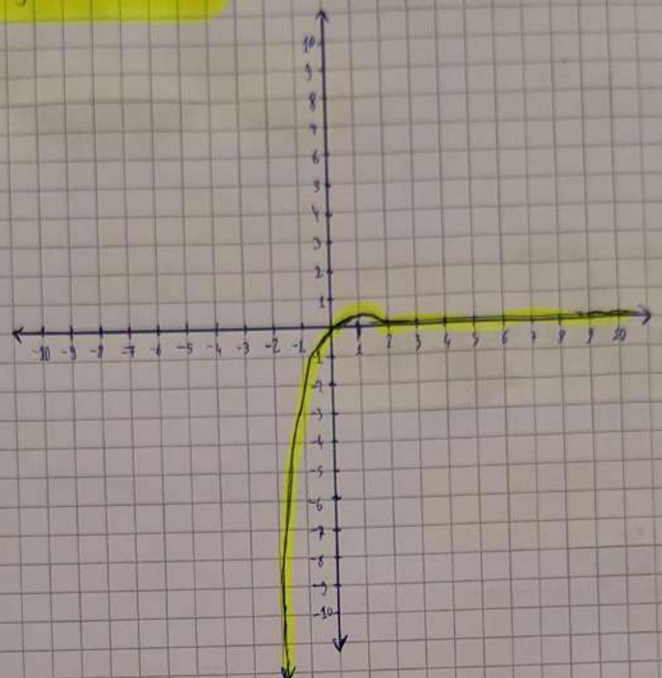
$f'(0) = 1$ $\quad$ $f'(2) \approx -0.136$

Since $f'(x)$ is positive for $x < 1$ and negative for $x > 1$ the function has a maxima at $x = 1$

$f(1) = \frac{1}{e^1}$

Maxima $\left(1, \frac{1}{e}\right)$

*Graphing the function:



Problem 7.3: Integrate the following functions:

a) $\int 2x\sqrt{x^2+1}\,dx$

$2\cdot\int x\sqrt{x^2+1}\,dx$ *Substitute "m" for x^2+1

$2\cdot\int \frac{1}{2}\cdot\sqrt{m}\,dm$

$2\cdot\frac{1}{2}\int\sqrt{m}\,dm$

$1\int\sqrt{m}\,dm$

$\int m^{\frac{1}{2}}\,dm$ $* \int x^n\,dx = \frac{x^{n+1}}{n+1},\ n\neq -1$

$\frac{2m\sqrt{m}}{3} \Rightarrow \boxed{\frac{2(x^2+1)\sqrt{x^2+1}}{3}+C}$

b)

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Problem 7.3: b) $\int \frac{dx}{(\arcsin(x))^3\sqrt{1-x^2}}$

* Substitute "m" with arcsin(x)

$\int \frac{1}{m^3}\,dm$

* $\int \frac{1}{x^n}\,dx = -\frac{1}{(n-1)\cdot x^{n-1}}$; $n \neq 1$

$-\frac{1}{2t^2}$

$$\boxed{-\frac{1}{2\cdot \arcsin(x)^2} + C,\ C \in \mathbb{R}}$$

c) $\int \frac{dx}{2x^2+9}$

$\int \frac{dx}{2(x^2+9/2)}$

$\frac{1}{2}\cdot\int \frac{dx}{x^2+\frac{9}{2}}$

* $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\cdot \arctan\left(\frac{x}{a}\right)$

$\frac{1}{2}\cdot\frac{1}{\sqrt{\frac{9}{2}}}\cdot\arctan\left(\frac{x}{\sqrt{\frac{9}{2}}}\right)$

$$\boxed{\frac{\sqrt{2}\arctan\left(\frac{\sqrt{2}x}{3}\right)}{6} + C}$$

Problem 7.4: Integrate using integration by parts:

ⓐ $\int \frac{\ln(x)}{x^3} dx$

$\int \ln(x) \cdot \frac{1}{x^3} dx$

$u = \ln(x)$ Find the differential using: $du = u' dx$ $\longrightarrow$ $du = \frac{1}{x} dx$

$dv = \frac{1}{x^3} dx$ $\longrightarrow$ $v = -\frac{1}{2x^2}$ ← Determine "v" by evaluating the integral.

Now we can:

* Property of Integrals: $\int a \cdot f(x) dx = a \cdot \int f(x) dx, a \in \mathbb{R}$

$\ln(x) \cdot \left(-\frac{1}{2x^2}\right) - \int -\frac{1}{2x^2} \cdot \frac{1}{x} dx$

$\ln(x) \cdot \left(\frac{-1}{2x^2}\right) - 1 \cdot \left(-\frac{1}{2}\right) \cdot \int \frac{1}{x^2} \cdot \frac{1}{x} dx$

$\ln(x) \cdot \left(\frac{-1}{2x^2}\right) + \frac{1}{2} \cdot \int \frac{1}{x^3} dx$

$\ln(x) \cdot \left(\frac{-1}{2x^2}\right) + \frac{1}{2} \cdot \left(\frac{-1}{2x^2}\right)$

$$\boxed{-\frac{\ln(x)}{2x^2} - \frac{1}{4x^2} + C}$$

ⓑ →

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Problem 7.4: b) $\int x^2 \cdot \sin(2x)\,dx$

$$u = x^2 \longrightarrow du = 2x\,dx$$
$$dv = \sin(2x)\,dx \longrightarrow v = \frac{-\cos(2x)}{2}$$

Solve with the integration by parts:

$$x^2 \cdot \left(\frac{-\cos(2x)}{2}\right) - \int -\frac{\cos(2x)}{\cancel{2}} \cdot \cancel{2}x\,dx$$

*Property of Integrals:

$$\int -f(x)\,dx = -\int f(x)\,dx$$

$$x^2 \cdot \left(\frac{-\cos(2x)}{2}\right) + \int x \cdot \cos(2x)\,dx$$

$$x^2 \cdot \left(-\frac{\cos(2x)}{2}\right) + x \cdot \frac{\sin(2x)}{2} - \int \frac{\sin(2x)}{2}\,dx$$

$$* \int a \cdot f(x)\,dx = a \cdot \int f(x)\,dx$$

$$x^2 \cdot \left(\frac{-\cos(2x)}{2}\right) + x \cdot \frac{\sin(2x)}{2} - \frac{1}{2} \cdot \int \sin(2x)\,dx$$

$$* \; m = 2x \; *$$

$$x^2 \cdot \left(\frac{-\cos(2x)}{2}\right) + x \cdot \frac{\sin(2x)}{2} - \frac{1}{2} \cdot \frac{1}{2} \cdot \int \sin(m)\,dm$$

$$x^2 \cdot \left(\frac{-\cos(2x)}{2}\right) + x \cdot \frac{\sin(2x)}{2} - \frac{1}{4} \cdot \left(-\cos(2x)\right)$$

$$\boxed{\frac{-x^2 \cdot \cos(2x) + x \cdot \sin(2x)}{2} + \frac{\cos(2x)}{4} + C}$$

C

Problem 7.4: (c) $\int \frac{x^3 dx}{\sqrt{1+x^2}}$

$* \; m = 1 + x^2 \; *$

$$\int \frac{m-1}{2\sqrt{m}} \, dm$$

$* \; \int a \cdot f(x) dx = a \cdot \int f(x) dx \; *$

$$\frac{1}{2} \cdot \int \frac{m-1}{\sqrt{m}} \, dm$$

$$\frac{1}{2} \cdot \int \frac{m}{m^{\frac{1}{2}}} - \frac{1}{m^{\frac{1}{2}}} \, dm$$

$*$ Property of Integrals: $\int f(x) \pm g(x) dx = \int f(x) dx \pm \int g(x) dx$

$$\frac{1}{2} \cdot \int m^{1/2} - \frac{1}{m^{\frac{1}{2}}}$$

$* \; \int x^n dx = \frac{x^{n+1}}{n+1} \; ; n \neq -1$

$* \; \int \frac{1}{x^n} dx = -\frac{1}{(n-1) \cdot x^{n-1}} \; ; n \neq 1$

$$\frac{1}{2} \cdot \left(\int m^{\frac{1}{2}} dm - \int \frac{1}{m^{\frac{1}{2}}} dm \right)$$

$$\frac{1}{2} \cdot \left(\frac{2 m \sqrt{m}}{3} - 2\sqrt{m} \right)$$

$$\boxed{\frac{(1+x^2)^{\frac{3}{2}}}{3} - \sqrt{1+x^2} + C}$$

THE END