Homework #4

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Problem 4.1: Compute the following limits:

a) The derivative of $f(x) = 2 + x - x^2$ is $f'(x) = \frac{d}{dx}(2) + \frac{d}{dx}(x) - \frac{d}{dx}(x^2)$

which is equal to $f'(x) = 0 + 1 - 2x$. Now we find the derivatives for the following points:

$f'(0) = 1 - 2(0) = \boxed{1}$

$f'\left(\frac{1}{2}\right) = 1 - 2\left(\frac{1}{2}\right) = \boxed{0}$

$f'(-10) = 1 - 2(-10) = \boxed{21}$

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b) $y(x) = \frac{x^3}{3} + \frac{x^2}{2} - 2x$

$y'(x) = \frac{d}{dx}\left(\frac{x^3}{3} + \frac{x^2}{2} - 2x\right)$

$y'(x) = x^2 + x - 2$

for $y'(x) = 10$

$x^2 + x - 2 = 10$

$x^2 + x - 12 = 0$

$(x+4)(x-3) = 0$

$x + 4 = 0$

$\boxed{x_1 = -4}$

$x - 3 = 0$

$\boxed{x_2 = 3}$

for $y'(x) = 0$

$x^2 + x - 2 = 0$

$(x-1)(x+2) = 0$

$x - 1 = 0$ $\quad$ $x + 2 = 0$

$\boxed{x_1 = 1}$ $\quad$ $\boxed{x_2 = -2}$

for $y'(x) = -2$

$x^2 + x - 2 = -2$

$x^2 + x = 0$

$x(x+1) = 0$

$\boxed{x_1 = 0}$ $\quad$ $x + 1 = 0$

$\boxed{x_2 = -1}$

Problem 4.2: Find the derivatives of the following functions:

a) $y = (x-a)(x-b)$ where a & b are constants.

*Product Rule: First Function $f^1(x) = x - a$

Second Function $f^2(x) = x - b$

The derivatives with respect to x: $f^{1'}(x) = 1$

$f^{2'}(x) = 1$

[redacted] $y'(x) = f^1(x) \cdot f^{2'}(x) + f^2(x) \cdot f^{1'}(x)$

$y'(x) = (x-a) \cdot 1 + 1 \cdot (x-b)$

$y'(x) = x - a + x - b$

$\boxed{y'(x) = 2x - (a+b)}$

Problem 4.2:

b) $y(x) = (x+1)(x+2)^2(x+3)^3$

* Product Rule: First Function: $f^1(x) = (x+1)$

Second Function: $f^2(x) = (x+2)^2(x+3)^3$

Derivatives:

✓ $f^{1'}(x) = 1$

$f^2(x) = (x+2)^2(x+3)^3$ → First Function $F_1^2(x) = (x+2)^2$

Second Function $f_2^2(x) = (x+3)^3$

→

$f_1^{2'}(x) = 2(x+2)$

$f_2^{2'}(x) = 3(x+3)^2$

Derivatives

Now apply the rule for $f^2(x)$:

$$f^2(x) = f_1^2(x)\cdot f_2^{2'}(x) + f_1^{2'}(x)\cdot f_2^2(x)$$

$$f^2(x) = (x+2)^2\cdot 3(x+3)^2 + 2(x+2)\cdot(x+3)^3$$

$$f'(x) = 1\cdot(x+2)^2(x+3)^3 + \left(2(x+2)(x+3)^3 + 3(x+3)^2(x+2)^2\right)(x+1)$$

$$f'(x) = 3x^2(x+3)^3 + 10x(x+3)^3 + 8(x+3)^3 + 3x(x+3)^2(x+2)^2 + 3(x+3)^2(x+2)^2$$

$$\underline{\underline{f'(x) = 6x^5 + 70x^4 + 320x^3 + 714x^2 + 774x + 324}}$$

Either can be taken as the derivative.

Problem 4.2:

c) $f(x) = (x-1)(x-2)^2(x-3)^3$

* Product Rule: First Function: $f^1(x) = (x-1)$

Second Function: $f^2(x) = (x-2)^2(x-3)^3$

Derivatives

✓ $f^{1'}(x) = 1$

$f^2(x) = (x-2)^2(x-3)^3$

First Function: $f_1^2(x) = (x-2)^2$

Second Function: $f_2^2(x) = (x-3)^3$

Derivatives: $f_1^{2'}(x) = 2(x-2)$

$f_2^{2'}(x) = 3(x-3)^2$

Now apply the rule for $f^2(x)$:

$$f^{2'}(x) = f_1^{2'}(x) \cdot f_2^{2}(x) + f_1^{2}(x) \cdot f_2^{2'}(x)$$

$$f^{2'}(x) = (x-2)^2 \cdot 3(x-3)^2 + 2(x-2) \cdot (x-3)^3$$

$$f'(x) = 1 \cdot (x-2)^2 (x-3)^3 + \left(2(x-2)(x-3)^3 + 3(x-3)^2(x-2)^2\right)(x-1)$$

$$f'(x) = 3x^2(x-3)^3 - 10x(x-3)^3 + 8(x-3)^3 + 3x(x-3)^2(x-2)^2 - 3(x-3)^2(x-2)^2$$

$$f'(x) = 6x^5 - 70x^4 + 320x^3 - 714x^2 + 774x - 324$$

Either can be taken as the derivative.

Problem 4.3: Find the derivatives of the following functions:

a) $f(x) = \frac{1}{x} + \frac{2}{x^2} + \frac{3}{x^3}$

Sum/Difference Rule: $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

$$\frac{d}{dx}\left(\frac{2}{x^2}\right) = -\frac{4}{x^3}$$

$$\frac{d}{dx}\left(\frac{3}{x^3}\right) = -\frac{9}{x^4}$$

$$f'(x) = -\frac{1}{x^2} - \frac{4}{x^3} - \frac{9}{x^4}$$

Problem 4.3: $f(x) = \dfrac{1+x-x^2}{1-x+x^2}$

b) * Quotient Rule : Original expression : $f(x) = \dfrac{g(x)}{h(x)}$

Derivative : $f'(x) = \dfrac{g'(x) \cdot h(x) - g(x) \cdot h'(x)}{(h(x))^2}$

$g(x) = 1 + x - x^2$

$h(x) = 1 - x + x^2$

$g'(x) = \dfrac{d}{dx}(1+x-x^2) = 1 - 2x$

$h'(x) = \dfrac{d}{dx}(1-x+x^2) = -1 + 2x$

$$f'(x) = \frac{(1-2x)(1-x+x^2) - (1+x-x^2)(-1+2x)}{(1-x+x^2)^2}$$

$$\boxed{f'(x) = \frac{2-4x}{(x^2-x+1)^2}}$$

Problem 4.3:

c) * Quotient Rule $f(x) = \dfrac{x^p(1-x)^q}{1+x}$ where p and q are constants.

$g(x) = x^p(1-x)^q$

$h(x) = 1 + x$

$g'(x) = \dfrac{d}{dx}(x^p(1-x)^q)$

$\dfrac{d}{dx}(x^p) \cdot (1-x)^q + x^p \cdot \dfrac{d}{dx}((1-x)^q)$

We use the product rule : $\dfrac{d}{dx}(x^p) = px^{p-1}$

$\dfrac{d}{dx}((1+x)^q) = q \cdot (x-1)^{q-1}(-1)$

$g'(x) = px^{p-1} \cdot (1-x)^q + x^p q \cdot (1-x)^{q-1} \cdot (-1))$

$$h'(x) = \frac{d}{dx}(1+x) = 1$$

$$f'(x) = \frac{\left(px^{p-1} \cdot (1-x)^q + x^p q \cdot (1-x)^{q-1} \cdot (-1)\right) \cdot (1+x) - x^p \cdot (1-x)^q \cdot 1}{(1+x)^2}$$

$$\boxed{f'(x) = \frac{\left(px^{p-1} \cdot (1-x)^q - qx^p \cdot (1-x)^{q-1}\right) \cdot (1+x) - x^p \cdot (1-x)^q}{(1+x)^2}}$$

4.4: Find the derivatives of the following functions:

a) $f(x) = \sqrt[3]{x^2} - \frac{2}{\sqrt{x}}$

$$f'(x) = \frac{d}{dx}\left(x^{\frac{2}{3}} - \frac{2}{\sqrt{x}}\right)$$

$$f'(x) = \frac{d}{dx}\left(x^{\frac{2}{3}}\right) - \frac{d}{dx}\left(\frac{2}{\sqrt{x}}\right)$$

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} - \left(-2 \cdot \frac{\frac{1}{2\sqrt{x}}}{\sqrt{x}^2}\right)$$

$$f'(x) \boxed{= \frac{2}{3\sqrt[3]{x}} + \frac{1}{x\sqrt{x}}} \text{ or } \boxed{= \frac{2}{3x^{\frac{1}{3}}} + \frac{1}{x^{\frac{3}{2}}}}$$

Problem 4.4:

*Product Rule

ⓑ $f(x) = x\sqrt{1+x^2}$

$f^1_{(x)} = x$

$f^2_{(x)} = \sqrt{1+x^2}$

$f^{1'}_{(x)} = 1$

$f^{2'}_{(x)} \Rightarrow$ Let $1+x^2$ be equal to t and $\sqrt{1+x^2} = \sqrt{t}$.

$$f^{2'}(x) = \frac{d}{dt}(\sqrt{t}) \cdot \frac{d}{dx}(t) = \frac{1}{2\sqrt{t}} \cdot (2x)$$

$$f'(x) = 1\sqrt{1+x^2} + x \cdot \frac{1}{2\sqrt{1+x^2}} \cdot 2x$$

$$\boxed{f'(x) = \frac{1+2x^2}{\sqrt{1+x^2}}}$$

ⓒ $f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}$

*Chain Rule

$$\frac{d}{dx}(f(g)) = \frac{d}{dg}(f(g)) \cdot \frac{d}{dx}(g),$$ where $g = x + \sqrt{x + \sqrt{x}}$

$$\frac{d}{dg}(\sqrt{g}) \cdot \frac{d}{dx}\left(x + \sqrt{x + \sqrt{x}}\right)$$

$$\frac{1}{2\sqrt{g}} \cdot \left(1 + \frac{1}{2\sqrt{x+\sqrt{x}}} \cdot \left(1 + \frac{1}{2\sqrt{x}}\right)\right)$$