Name: Flori Kusari Problem Sheet #9

Problem 9.1: Expand the following functions in the Taylor series near b=0 and determine for which "x" it converges

a) $y = x^2 e^x$

Step 1. Find the derivatives

$f(x) = x^2 e^x$

$f'(x) = (2x + x^2) e^x$

$f''(x) = (2 + 4x + x^2) e^x$

$f'''(x) = (4 + 8x + 2x^2) e^x$

$f''''(x) = (8 + 16x + 4x^2) e^x$

Step 2: Evaluate:

$f(0) = 0$

$f'(0) = 0$

$f''(0) = 2$

$f'''(0) = 4$

$f''''(0) = 8$

Taylor Series Formula $y = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$y = \cancel{0} + \cancel{0x} + \frac{2}{2!} x^2 + \frac{4}{3!} x^3 + \frac{8}{4!} x^4 + \ldots$

$y = \frac{2}{2!} x^2 + \frac{4}{3!} x^3 + \frac{8}{4!} x^4 + \ldots$

$y = x^2 + \frac{2}{3} x^3 + \frac{1}{3} x^4 + \frac{1}{6} x^5 + \ldots$

Use ratio test: $\lim_{x \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$

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$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{2}{(n+1)(n+2)}}{\frac{2}{n(n+1)}}\, x\right| = \lim_{n\to\infty}\left|\frac{n(n+1)}{(n+1)(n+2)}\right| x$$

$$\Rightarrow \lim_{n\to\infty}|x| = |x|$$

Convergence for the Taylor Series for $y = x^2 e^x$: $-\infty < x < \infty$

(b) $y = \cos(x+a)$, where $a \in \mathbb{R}$ is a constant.

Taylor Series formula for the cosine function: $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (x+a)^{2n}$$

$y = \cos(x+a) \Rightarrow y(0) = \cos(a)$
$y' = -\sin(x+a) \Rightarrow y(0) = -\sin(a)$
$y'' = -\cos(x+a) \Rightarrow y(0) = -\cos(a)$
$y''' = \sin(x+a) \Rightarrow y(0) = \sin(a)$
$y'''' = \cos(x+a) \Rightarrow y(0) = \cos(a)$

$$y = \cos(a) - \sin(a)x - \frac{\cos(a)}{2!}x^2 + \frac{\sin(a)}{3!}x^3 - \frac{\cos(a)}{4!}x^4 + \dots$$

The Taylor Series for $y = \cos(x+a)$ converges for all real values of "x" because both "cos" and "sin" functions of this series converge for all real values due to their properties and the behaviour of Taylor series.

Converges for $x \in \mathbb{R}$ or $-\infty < x < \infty$

(c) $y = \ln(1+x)$

$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$. Considering the given hint to find the first derivative

$y'(x) = \frac{1}{1+x}$

Using the ratio test we see that the interval of convergence is $-1 < x \leq 1$

$y'(0) = \frac{1}{1+0} = 1$

Problem 9.2: Compute areas of the following domains:

a) between parabolas $y^2+8x=16$ and $y^2-24x=48$

$$y^2+8x=16 \Rightarrow 2-\frac{y^2}{8}=x$$

$$y^2-24x=48 \Rightarrow \frac{y^2-48}{24}=x$$

$$\frac{y^2-48}{24}=2-\frac{y^2}{8} \quad /\cdot 24$$

$$48-3y^2=y^2-48$$

$$4y^2=96 \quad /:4$$

$$y^2=24 \quad /\sqrt{\ }$$

$$y=\pm 2\sqrt{6}$$

$$A=\int_{-2\sqrt{6}}^{2\sqrt{6}} \Big[\Big(\frac{1}{8}(16-y^2)\Big)+\frac{1}{24}(48-y^2)\Big)\,dy\Big]$$

After calculating the result will be $A=32\sqrt{\frac{2}{3}}\approx 26.1\ldots$ square units

b) of both parts in which parabola $\frac{x^2}{2}$ divides the circle $x^2+y^2=8$.

$$x^2+\left(\frac{x^2}{2}\right)^2=8$$

$$x^2+\frac{x^4}{4}=8 \quad /\cdot 4$$

$$4x^2+x^4=32$$

$$x^4+4x^2-32=0$$

Subsitute $x^2=u$

$$u^2+4u-32=0$$

$$u=\frac{-4\pm\sqrt{4^2-4\cdot1\cdot(-32)}}{2\cdot1}$$

$$u=\frac{-4\pm\sqrt{144}}{2}$$

$$u=\frac{-4\pm12}{2}$$

$$u_1=4$$

$$u_2=\frac{-16}{2}=-8$$

since $u=x^2$ the negative value is disregarded

$$u=4$$

$$x^2=4$$

$$\boxed{x=\pm 2}$$

So the points of intersection are $(2,2)$ and $(-2,2)$.

$$A_1 = \int_{-2}^{2} \left(\sqrt{8-x^2} - \frac{x^2}{2}\right) dx = \boxed{2\pi + \frac{4}{3}}$$

$$A_2 = \int_{2}^{-2} \left(\sqrt{8-x^2} - \frac{x^2}{2}\right) dx = \boxed{-2\pi - \frac{4}{3}}$$

Total Area $= |A_1| + |A_2| = \frac{6\pi + 4}{3} - \frac{-6\pi - 4}{3} = \boxed{\frac{12\pi + 8}{3}}$ square units

or $\boxed{4\pi + \frac{8}{3}}$ square units

Problem 9.2 : © bounded by curve $\frac{1}{1+x^2}$ and parabola $\frac{x^2}{2}$

$$\frac{1}{1+x^2} \times \frac{x^2}{2}$$

$$2 = x^2(x^2+1)$$

$$2 = x^4 + x^2$$

$*u = x^2*$

$$x^4 + x^2 - 2 = 0$$

$$u^2 + u - 2 = 0$$

$$\frac{-1 \pm \sqrt{1^2 - 4(1)(-2)}}{2 \cdot (1)} = \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2}$$

$$u_1 = \frac{-1+3}{2} = 1$$

$$u_2 = \frac{-1-3}{2} = -2$$

Disregard the negative possibility

$$x^2 = 1$$

$$\boxed{x = \pm 1}$$

$$\int_{-1}^{1} \left[\frac{1}{1+x^2} - \frac{x^2}{2} dx\right]$$

$$\int \frac{1}{1+x^2} dx - \int \frac{x^2}{2} dx$$

$$\left(\arctan(x) - \frac{x^3}{6}\right)\Bigg|_{-1}^{1}$$

$$\arctan(1) - \frac{1^3}{6} - \left(\arctan(-1) - \frac{(-1)^3}{6}\right)$$

$$\frac{\pi}{2} - \frac{1}{3} = A$$

or

$A \approx 1.23746$ units squared

Problem 9.3: Compute the following improper integrals:

(a) $\int_1^{+\infty} \frac{1}{x^4} dx$

$$\lim_{a \to +\infty} \left(\int_1^a \frac{1}{x^4} dx \right)$$

$$\lim_{a \to +\infty} \left(-\frac{1}{3a^3} + \frac{1}{3} \right) \Rightarrow \frac{1}{3}$$

Problem 9.3: (b) $\int_{-\infty}^{+\infty} \frac{dx}{x^2+2x+2}$

$$\int_{-\infty}^{0} \frac{1}{x^2+2x+2} dx + \int_0^{+\infty} \frac{1}{x^2+2x+2} dx$$

$$\lim_{a \to -\infty} \left(\int_a^0 \frac{1}{x^2+2x+2} dx \right) + \lim_{a \to +\infty} \left(\int_0^a \frac{1}{x^2+2x+2} dx \right)$$

$$\lim_{a \to -\infty} \left(\frac{\pi}{4} - \arctan(a+1) \right) + \lim_{a \to +\infty} \left(\arctan(a+1) - \frac{\pi}{4} \right)$$

$$\frac{3\pi}{4} + \frac{\pi}{4}$$

$$\frac{\cancel{4}\pi}{\cancel{4}} = \boxed{\pi}$$

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Problem 9.3: (c) $\int_1^{\infty} \frac{\arctan(x)}{x^2}\, dx$

$$\lim_{a\to\infty}\left(\int_1^a \arctan(x)\cdot\frac{1}{x^2}\,dx\right)$$

$$\lim_{a\to\infty}\left(-\frac{\arctan(a)}{a}+\frac{1}{2}\ln|1+a^2|-\left(-\frac{\arctan(1)}{1}+\frac{1}{2}\ln|1+1^2|\right)\right)$$

$$\Rightarrow \frac{\pi}{4}+\infty-\left(\frac{\pi}{4}+0\right)=+\infty$$

Problem 9.4: Find the volume of the rotation body:

a) obtained by a rotation of the figure bounded by curve $y=xe^x$ and lines $x=1$ and $y=0$, around axis OX.

$$V=\int_a^b \pi\,|(f(x))^2-(g(x))^2|\,dx$$

$$\int_0^1 \pi\,|x|^2|e^{2x}|\,dx$$

$$\int_0^1 \pi\cdot|x|^2\cdot|e^{2x}|\,dx$$

$$\pi\cdot\int_0^1 |x|^2\cdot|e^{2x}|\,dx$$

$$\pi\cdot\int_0^1 x^2\cdot e^{2x}\,dx$$

$$V=\boxed{\frac{\pi e^2-\pi}{4}}$$

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ⓑ obtained by a rotation of a figure bounded by parabolas $y=x^2$ and $x=y^2$ around axis OX,

$$\int_0^1 \pi \left| x^4 - x \right| dx$$

$$\left| \int \pi x^4 dx - \int \pi x dx \right|$$

$$\left| \frac{\pi x^5}{5} - \frac{\pi x^2}{2} \right| \Bigg|_0^1$$

$$\left| \frac{\pi \cdot 1^5}{5} - \frac{\pi \cdot 1^2}{2} - \left(\frac{\pi \cdot 0^5}{5} - \frac{\pi \cdot 0^2}{2} \right) \right|$$

$$\left| -\frac{3\pi}{10} \right|$$

$$V = \frac{3\pi}{10}$$

THE END