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Problem 6.1: Find the intervals of monotonicity of the following functions (where it increases and where it decreases).

(a) $f(x)=(x-2)^5(2x-1)^4$; Domain $x \in R$

Domain is not always necessary but it can help if found in the beginning of the solution

Substitute for $f'(x)=0$

$f'(x)=5(x-2)^4\cdot(2x-1)^4+8(x-2)^5\cdot(2x-1)^3$; Domain $x \in R$

$0=5(x-2)^4\cdot(2x-1)^4+8(x-2)^5\cdot(2x-1)^3$

Skipped the solving the equation part to save space since it is not really necessary as long as we are only using it to find the intervals.

$x_1=2$; $x=\frac{1}{2}$; $x=\frac{7}{6}$

Determine the intervals

$\left\langle\frac{7}{6},2\right\rangle$, $\langle 2,+\infty\rangle$

$\left\langle-\infty,\frac{1}{2}\right\rangle$, $\left\langle\frac{1}{2},\frac{7}{6}\right\rangle$

$\left\langle\frac{1}{2},\frac{7}{6}\right\rangle$, $\left\langle\frac{7}{6},2\right\rangle$

For the first and the last we can do the same work since they are the same. This is also the case for the fourth and fifth

Give the selected points from said intervals

$x_{1,6}=\frac{19}{10}$

$x_2=3$

$x_3=0$

$x_{4,5}=1$

Input the selected points into the derivative

for 1, 6 : $f'\left(\frac{19}{10}\right) = \frac{11319}{390625}$

for 2 : $f'(3) = 4125$

for 3 : $f'(0) = 336$

for 4, 5 : $f'(1) = -3$

*For negatives it decreases for positives it increases thus:

$$\boxed{\begin{array}{l} \text{Increasing: } -\infty < x < \frac{1}{2} \\ \text{Decreasing: } \frac{1}{2} < x \frac{7}{6} \\ \text{Increasing: } \frac{7}{6} < x < 2 \\ \text{Increasing: } 2 < x < \infty \end{array}}$$

Problem 6.1: b $f(x) = 2x^2 - \ln(x)$; $x \in \langle 0, +\infty \rangle$ Domain

Substitute for $f'(x) = 0$

$f'(x) = \frac{4x^2 - 1}{x}$; $x \in R \setminus \{0\}$

$0 = \frac{4x^2 - 1}{1}$

$x = -\frac{1}{2}$; $x = \frac{1}{2}$

* Intervals determined: $\langle -\infty, -\frac{1}{2} \rangle, \langle -\frac{1}{2}, 0 \rangle$
$\langle 0, \frac{1}{2} \rangle, \langle \frac{1}{2}, +\infty \rangle$

* Select Points in each interval:

$x_1 = -1$; $x_2 = -\frac{1}{10}$; $x_3 = \frac{1}{10}$; $x_4 = 1$

Input all selected points into the derivative

$f'(-1) = -3$

$f'\left(\frac{-1}{10}\right) = \frac{48}{5}$

$f'\left(\frac{1}{10}\right) = \frac{-48}{5}$

$f'(1) = 3$

* For negatives it decreases for positives it increases thus:

Decreasing for: $0 < x < 1/2$

Increasing for: $1/2 < x < \infty$

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Problem 6.2: Find the extrema, minima, and maxima, of the following functions:

(a) $y = \frac{3x^2+4x+4}{x^2+x+1} \rightarrow f(x) = \frac{3x^2+4x+4}{x^2+x+1}$; Domain $x \in \mathbb{R}$

$f'(x) = -\frac{x^2+2x}{(x^2+x+1)^2}$; Domain $x \in \mathbb{R}$

* Substitute "$f'(x)$" for "$=0$".

$0 = -\frac{x^2+2x}{(x^2+x+1)^2}$

$\boxed{x=0}$ $\boxed{x=-2}$

*Skipped the part of solving the equation to save space since it isn't that important to show it.

* Now we determine the interval.

$x=0 \rightarrow \langle -2, 0\rangle, \langle 0, +\infty\rangle$

$x=-2 \rightarrow \langle -\infty, -2\rangle, \langle -2, 0\rangle$

* We select one point from each interval *

$x_1 = -1$; $x_2 = 1$; $x_3 = -3$; $x_4 = -1$

* We find the value of the derivative for the selected point.

$f'(-1) = 1$; $f'(1) = \frac{-1}{3}$; $f'(-3) = \frac{-3}{49}$; $f'(-1) = 1$

Since the derivative is positive at $-2 < x < 0$ and negative for $x > 0$ the function has a maxima at $x=0$:

$x=0 \Rightarrow f(x) = \frac{3x^2+4x+4}{x^2+x+1} = \frac{4}{1} = 4$

$$f'(-3) = -\frac{3}{49} \; ; \; f'(-1) = 1$$

Since the derivative is negative at $x<-2$ and positive at $-2<x<0$ the function has a minima at $\boxed{x=-1}$

$$x=-2 \Rightarrow f(x) = \frac{3x^2+4x+4}{x^2+x+1} = \frac{3(-2)^2+4(-2)+4}{(-2)^2+(-2)+1} = \frac{12-8+4}{4-2+1} = \frac{8}{3}$$

$$\boxed{\text{Minima } \left(-2, \frac{8}{3}\right) \; ; \; \text{Maxima } (0,4)}$$

Problem 6.2: b

$$y = \frac{4\sqrt{3}}{9x\sqrt{1-x}} \Rightarrow f(x) = \frac{4\sqrt{3}}{9x\sqrt{1-x}} \; ; \; \overset{\text{Domain}}{x \in \langle -\infty, 0\rangle \cup \langle 0, 1\rangle}$$

$$f'(x) = \frac{6\sqrt{3}\,x - 4\sqrt{3}}{9x^2\sqrt{1-x}\,(1-x)} \; ; \; \overset{\text{Domain}}{x \in \langle -\infty, 0\rangle \cup \langle 0, 1\rangle}$$

*Substitute "f'(x)" for "=0":

$$0 = \frac{6\sqrt{3}\,x - 4\sqrt{3}}{9x^2\sqrt{1-x}\,(1-x)}$$

$$\Downarrow$$

$$x = \frac{2}{3}$$

* Determine the Intervals *

$$\left\langle 0, \frac{2}{3}\right\rangle, \left\langle \frac{2}{3}, 1\right\rangle$$

* We select one point from each interval *

$$x_1 = \frac{1}{10} \; ; \; x = \frac{9}{10}$$

* Find the value of the derivative for the selected points:

$f'\left(\frac{1}{10}\right) \approx -76.6......$

$f'\left(\frac{9}{10}\right) \approx 10.5.....$

Used calculator for estimate since exact measurement is not necessary

Since the derivative is negative at $0<x<\frac{2}{3}$ and positive at $\frac{2}{3}<x<1$ the function has a minima at $\boxed{x=\frac{2}{3}}$:

$$x=\frac{2}{3} \Rightarrow f(x)=\frac{4\sqrt{3}}{9x\sqrt{1-x}} = \frac{4\sqrt{3}}{9\cdot\frac{2}{3}\sqrt{1-\frac{2}{3}}} = \frac{4\sqrt{3}}{3\cdot 2\sqrt{\frac{1}{3}}} = \frac{\sqrt{3}}{3\cdot\sqrt{\frac{1}{3}}}$$

$$= \frac{2\sqrt{3}}{3\cdot\frac{1}{\sqrt{3}}} = \frac{2\sqrt{3}}{\frac{3}{\sqrt{3}}} = \frac{6}{3} = 2$$

$\boxed{\text{Minima } \left(\frac{2}{3}, 2\right)}$

Problem **6.3**: Find the minima and maxima of the following functions on given intervals:

ⓐ $y=x^5-5x^4+5x^3+1$, at $[-1,2] \Rightarrow f(x)=x^5-5x^4+5x^3+1$; $x \in R$ (Domain)

$f'(x)=5x^4-20x^3+15x^2$; $x \in R$ (Domain) $\rightarrow$ Not necessary in this case but force of habit.

Intervals are given $[-1,2]$

Select Inputs for the calculation

$x_1=-1$; $x_2=1$; $x_3=2$

*Find the value of the function from the selected points:

$f(-1)=x^5-5x^4+5x^3+1=(-1)^5-5(-1)^4+5(-1)^3+1=-10 \Rightarrow (-1,-10)$

$f(1)=x^5-5x^4+5x^3+1=(1)^5-5(1)^4+5(1)^3+1=2 \Rightarrow (1,2)$

$f(2)=x^5-5x^4+5x^3+1=(2)^5-5(2)^4+5(2)^3+1=-7 \Rightarrow (2,-7)$

$\boxed{\text{Minimums}: (-1,-10)}$ and $(2,-7)$ $\boxed{\text{Maximum}: (1,2)}$

Since lower

Problem 6.3: (b) $y=\frac{x-1}{x+1}$; for $[0 \leq x \leq 4]$

Not needed but force of habit

$$y=\frac{x-1}{x+1} \Rightarrow f(x)=\frac{x-1}{x+1} \; ; \; \underbrace{x \in R \setminus \{-1\}}_{\text{Domain}}$$

$$f'(x)=\frac{2}{(x+1)^2} \; ; \; \underbrace{x \in R \setminus \{-1\}}_{\text{Domain}}$$

* Select points using the given $[0 \leq x \leq 4]$

$x_1 = 0$; $x_2 = 4$

* Solve for the selected points:

$f(0)=\frac{x-1}{x+1}=-1$

$f(4)=\frac{x-1}{x+1}=\frac{4-1}{4+1}=\frac{3}{5}$

$\boxed{\text{Minimal}: (0,-1); \text{ Maximal}: (4,\tfrac{3}{5})}$

Problem 6.4: Find the domains of convexity and concavity of the following functions:

(a) $f(x)=x^4-12x^3+48x^2-50$; $\underbrace{x \in R}_{\text{Domain}}$

First Derivative: $f'(x)=4x^3-36x^2+96x$

Second Derivative: $f''(x)=12x^2-72x+96$; $\underbrace{x \in R}_{\text{Domain}}$

Set to "0": $0=12x^2-72x+96$

Solve Function: $x=2$; $x=4$

Determine Intervals: $\langle -\infty, 2\rangle, \langle 2,4\rangle ; \langle 4,+\infty\rangle$

* For 2 and 3 we can choose the same because they are the same interval *

* Choose points: $x_1 = 1$

$x_{2,3} = 3$

$x_4 = 5$

* Solve for the second derivative:

$f''(1) = 12(1)^2 - 72(1) + 96 = -60 + 96 = 36$

$f''(3) = 12(3)^2 - 72(3) + 96 = 108 - 216 + 96 = -12$

$f''(5) = 12(5)^2 - 72(5) + 96 = 300 - 360 + 96 = 36$

* From these calculations we can conclude that the function is:

Concave Upward: $\boxed{-\infty < x < 2}$

Concave Downward: $\boxed{2 < x < 4}$

Concave Upward: $\boxed{4 < x < \infty}$

Problem 6.4: (b) $f(x) = \dfrac{x^3}{x^2 + 3a^2}$, $a > 0$

Find First Derivative: $f'(x) = \dfrac{d}{dx}\left(\dfrac{x^3}{x^2+3a^2}\right)$

$$= \frac{d/dx(x^3) \cdot (x^2+3a^2) - x^3 \cdot d/dx(x^2+3a^2)}{(x^2+3a^2)^2}$$

$$= \frac{3x^2 \cdot (x^2+3a^2) - x^3 \cdot 2x}{(x^2+3a^2)^2} = \boxed{\frac{x^4 + 9a^2x^2}{(x^2+3a^2)^2}}$$

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Find second derivative:

$$\boxed{f''(x)} = \frac{\frac{d}{dx}\left(x^4+9a^2x^2\right)\cdot\left(x^2+3a^2\right)^2-\left(x^4+9a^2x^2\right)\cdot\frac{d}{dx}\left(\left(x^2+3a^2\right)^2\right)}{\left(\left(x^2+3a^2\right)^2\right)^2}$$

$$= \frac{\left(4x^3+9a^2\cdot 2x\right)\cdot\left(x^2+3a^2\right)^2-\left(x^4+9a^2x^2\right)\cdot \boxed{2\left(x^2+3a^2\right)}\cdot 2x}{\left(\left(x^2+3a^2\right)^2\right)^2}$$

$$f''(x) \boxed{=} \frac{-6a^2x^3+54a^4x}{\left(x^2+3a^2\right)^3}$$

Now we transform the equation into it's equivalent **for $a=2$ since** it fills the requirement for $a>0$.

$$f''(x)=\frac{-24x^3+864x}{\left(x^2+12\right)^3}\ ;\ x\in\mathbb{R}\ ;\ a=2$$

$$0=\frac{-24x^3+864x}{\left(x^2+12\right)^3}$$

Solve the equation:
$x_1=0 \rightarrow \langle -6,0\rangle, \langle 0,6\rangle$
$x_2=-6 \rightarrow \langle -\infty,-6\rangle, \langle -6,0\rangle$
$x_3=6 \rightarrow \langle 0,6\rangle, \langle 6,+\infty\rangle$

Select points for each interval:

$x_{1,4}=-5 \Longrightarrow f''(-5)=-\frac{1320}{50653}$

$\boxed{x_{2,5}}=1 \Longrightarrow f''(1)=\frac{840}{2197}$

$x_{3}=-7 \Longrightarrow f''(7)=\frac{2184}{226981}$

$x_6=7 \Longrightarrow f''(-7)=-\frac{2184}{226981}$

From these calculations we can conclude that the function is:

$\longrightarrow$

Concave Upward: $\boxed{-\infty < x < -6}$

Concave Downward: $\boxed{-6 < x < 0}$

Concave Upward: $\boxed{0 < x < 6}$

Concave Downward: $\boxed{6 < x < \infty}$

Note for TA: Please keep in mind that these are formulas that I know and may not be the generally accepted solving methods so please indicate any issue with my solution in the feedback in as much detail as needed since I do not want to repeat any mistakes, I may have, in the future.

THE END

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