Constructor University
Assignment 3

29 September 2023

Probability and Random Processes

(**3.1**) Suppose x and y are randomly chosen in the interval $[0, 2]$. Find the probability of the event that $xy \leq 1$.

Hint: Plot the graph of $y = 1/x$ and identify the region in which $xy \leq 1$.

(**3.2**) A bias coin has the probability $1/3$ of turning up heads. The coin is thrown 4 times.
(a) What is the probability that the total number of heads shown is 2?
(b) Suppose that we know that outcome of the first throw is a head. Find the probability that the total number of heads shown is 2.
(c) If we know that the total number of heads shown is 2, find the probability that the outcome of the first throw was a head.

(**3.3**) A fair die is rolled and outcome k has appeared. The same die is subsequently rolled k times. We consider the rolls a success if the sum of all $k + 1$ outcomes (including the initial one) is 10. For instance, if the outcomes of the first die is $k = 3$ and the outcomes of the subsequent three rolls are $1, 2, 4$ then we have had a success. Find the probability of Success.

(**3.4**) A number x is randomly chosen from the set $\{1, 2, \ldots, 4\}$. Then a number y is chosen from the set $\{1, \ldots, x\}$ with equal probability. For instance, if $x = 3$, the y is randomly chosen from $\{1, 2, \ldots, 3\}$, each with probability $1/3$.
(a) Find the probability of $y = i$ for $i = 1, 2, 3, 4$.
(b) Find the probability of the event that $y = x$.
(c) After having chosen y suppose z is chosen from the set $\{1, \ldots, y\}$. What is the probability of the event that $z = 1$?

(**3.5**) The logical statement "P implies Q" is equivalent to its contrapositive, "not Q implies not P". Suppose A and B are events with probabilities different from 0 or 1.
(a) Show that if $\mathbb{P}[B|A] = 1$ then $\mathbb{P}[A^c|B^c] = 1$.
(b) Show that the approximate version of this is not true: give examples of events A and B such that $\mathbb{P}[B|A] > 0.99$ but $\mathbb{P}[A^c|B^c] < 0.01$.

Hint: Consider a coin is flipped 20 times. Let A denote the event that a Heads appear at least once the first 10 flips, and let B denote the event the event that a Heads appear at least once the second 10 flips.