Probability and Random Processes

(6.1) Consider a discrete random variable X with the probability mass function given by

$$p_X(x) = \begin{cases} k \cdot 2^x & \text{if } x = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the value of k.
(b) Find $\mathbb{P}[X \text{ is even}]$.
(c) Determine $\mathbb{E}[X]$ and $\text{Var}[X]$.

(6.2) Suppose X is a discrete random variable with $\mathbb{E}[X] = 10$ and $\text{Var}[X] = 15$. Find the values of $\mathbb{E}\left[X^2\right]$, $\mathbb{E}[2 - X]$, $\text{Var}[2X + 1]$.

(6.3) A random 4-digit integer N (that is, a random integer in the range 1000 to 9999) is chosen. Let X denote the number of distinct digits used in N. For instance, if $N = 1201$ then $X = 3$ and if $N = 8712$ then $X = 4$. Evaluate $\mathbb{E}[X]$.

(6.4) A fair coin is rolled N times until the sequence HH appears for the first time. Compute $\mathbb{E}[N]$.

(6.5) In order to determine which individuals in a group of 100 people are affected by a particular virus, the following strategy is adopted: The group is divided into blocks of m people. Sample from all people in a block are pooled. If the result from the pooled sample from a block is negative, then all the m people in that block are negative. Otherwise every person in that block is tested separately. Suppose that each individual has probability $p = 0.01$ of being infected. For simplicity assume that the events corresponding to different people being infected are independent. Let N be the total number of tests needed.

(a) Show that $\mathbb{E}(N) = \frac{100}{m} + 100(1 - (0.99)^m)$.
(b) Use the the Calculator Suite in GeoGebra to find the m that minimizes the expected value of the number of necessary tests. In order to do this, you need to give the function by writing $f(x) = ...$, click on the three dots and choose the Table of Values option.