Name: Flori Kusari — Problem Sheet #5 — Date: 13/10/2024

Problem 5.1:

$\Omega = 36$ possible outcomes

$X=1$ doesn't exist

$X=2 \Rightarrow \boxed{\frac{1}{36}}$

$X=3 \Rightarrow \frac{2}{36} = \boxed{\frac{1}{18}}$

$X=4 \Rightarrow \frac{3}{36} = \boxed{\frac{1}{12}}$

$X=5 \Rightarrow \frac{4}{36} = \boxed{\frac{1}{9}}$

$X=6 \Rightarrow \boxed{\frac{5}{36}}$

$X=7 \Rightarrow \frac{6}{36} = \boxed{\frac{1}{6}}$

$X=8 \Rightarrow \boxed{\frac{5}{36}}$

$X=9 \Rightarrow \frac{4}{36} = \boxed{\frac{1}{9}}$

$X=10 \Rightarrow \frac{3}{36} = \boxed{\frac{1}{12}}$

$X=11 \Rightarrow \frac{2}{36} = \boxed{\frac{1}{18}}$

$X=12 \Rightarrow \boxed{\frac{1}{36}}$

Y:

$Y=1 \Rightarrow (1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(3,1),(4,1)(5,1)(6,1)$

11 outcomes so $\boxed{\frac{11}{36}}$ for $Y=1$

$Y=2 \Rightarrow (2,2),(2,3)(2,4)(2,5)(2,6)(3,2),(4,2),(5,2)(6,2)$

9 outcomes for $Y=2$ so $\boxed{\frac{9}{36}}$ or $\boxed{\frac{1}{4}}$

$Y=3 \Rightarrow (3,3),(3,4),(3,5),(3,6),(4,3),(5,3),(6,3)$

7 outcomes for $Y=3$ so $\boxed{\frac{7}{36}}$

$Y=4 \Rightarrow (4,4),(4,5),(4,6),(5,4),(6,4)$

5 outcomes for $Y=4$ so $\boxed{5/36}$

$Y=5 \Rightarrow (5,5),(5,6),(6,5)$

3 outcomes for $Y=5$ so $\boxed{3/36}$ or $\boxed{1/12}$

$Y=6 \Rightarrow (6,6)$

1 outcome for $Y=6$ so $\boxed{1/36}$

Problem 5.2:

$x \in \{1,2,3,4\}$ $k=?$ $P[X=x] = kx^2$

(a) $P[X=1] + P[X=2] + P[X=3] + P[X=4] = 1$

$k + 4k + 9k + 16k = 1$

$30k = 1$

$\boxed{k = 1/30}$

(b) $P[X>2]=?$

$P[X=1]=k=\frac{1}{30}$

$P[X=2]=4k=\frac{4}{30}$

$P[(X=2)+(X=1)]=\frac{5}{30}=1/6$

$P[X>2]=1-P[X<2]$

$P[X>2]=1-1/6$

$\boxed{P[X>2]=5/6}$

Problem 5.3: (n,p) $n=4$. $P[X \text{ is even}]=?$

$\uparrow$ Binomial Distribution

$P[X \text{ is even}]=1-4p+12p^2-16p^3+8p^4$

Prove!

$P(x=k)=\binom{n}{k}p^k(1-p)^{n-k}$

Even:
$$\begin{cases} P(x=0)=\binom{4}{0}p^0(1-p)^4=(1-p)^4 \\ P(x=2)=\binom{4}{2}p^2(1-p)^2=6p^2(1-p)^2 \\ P(x=4)=\binom{4}{4}p^4(1-p)^0=p^4 \end{cases}$$

$P[\text{Even } X]=((1-p)^4)+(6p^2(1-p)^2)+p^4$

$\boxed{P[\text{Even } X]=1-4p+12p^2-16p^3+8p^4}$ Proven!

Problem 5.4:

(a) $P(X=k)=\frac{\lambda^k e^{-\lambda}}{k!}$ } Poisson Distribution

$\lambda=1$

$P[X\leq 2]=\sum_{k=0}^{2}\frac{\lambda^k e^{-\lambda}}{k!}=P(X=0)+P(X=1)+P(X=2)$

$=\frac{1^0 e^{-1}}{0!}+\frac{1^1 e^{-1}}{1!}+\frac{1^2 e^{-1}}{2!}$

$=\frac{e^{-1}}{1}+\frac{e^{-1}}{1}+\frac{e^{-1}}{2}$

$=\frac{2e^{-1}}{1}+\frac{e^{-1}}{2}=\boxed{\frac{5e^{-1}}{2}}\approx\boxed{\frac{5}{2e}}\approx\boxed{0.919699}$ $\rightarrow$

ⓑ $P[X \geq 2) = 1 - P(X<2) = 1 - (e^{-1} + e^{-1})$

$$= \boxed{1 - 2e^{-1}} \text{ or } \boxed{1 - \frac{2}{e}} \approx \boxed{0,264241}$$

Problem 5.5:

ⓐ Show $G = 5X - m$

Suppose a student answers 8 question and is right on 5 of them but wrong on 3 of them,

Total $= \underset{\uparrow \text{ Points per correct}}{4}(5) - \underset{\uparrow \text{ Penalty for wrong answers}}{1}(3) = 20 - 3 = \boxed{17}$

Now lets test the formula:

$G = 5(5) - 8 = 25 - 8 = \boxed{17}$

$17 == 17$ so the formula Works.

ⓑ Uncertain if this is the question but: $P[G>0] \Rightarrow X > \left\lceil \frac{m}{5} \right\rceil$

$P(X=k) = \binom{m}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{m-k} = \binom{m}{k}\left(\frac{1}{2}\right)^m$ // Probability of k correct out of m.

Case 1: Probability of getting them all wrong is $\left(\frac{1}{2}\right)^m$

We need only 1 question correctly for a positive grade.

$P[G>0] = 1 - P[X=0] = \boxed{1 - (3/4)^m}$ // proven

Case 2: We need $X > m/5$ rounded up for a positive grade

$P[G>0] = 1 - P[X=0] - P[X=1]$

$P[G>0] = 1 - (3/4)^m - m(1/4)(3/4)^{m-1}$ // proven

Case 3: We need $X > 2$ for a positive grade

$P[G>0] = 1 - P[X=0] - P[X=1] - P[X=2]$

$P[G>0] = 1 - (3/4)^{10} - (10/4)(3/4)^9 - (45/16)(3/4)^8$ // proven

ⓒ Find value of m for which $P[G>0]$ is maximized.

$m=1: P[G>0] = 1-(\frac{3}{4}) = 0.25$

$m=2: P[G>0] = 1-(3/4)^2 = 0.4375$

$m=3: P[G>0] = 1-(3/4)^3 = 0.5781$

$m=4: P[G>0] = 1-(3/4)^4 = 0.6838$

$m=5: P[G>0] = 1-(3/4)^5 - 5(1/4)(3/4)^4 = 0.7627$

$m=6: P[G>0] = 1-(3/4)^6 - 6(1/4)(3/4)^5 = 0.7724$

$m=7: P[G>0] = 1-(3/4)^7 - 7(1/4)(3/4)^6 = 0.7525$

$m=8: P[G>0] = 1-(3/4)^8 - 8(1/4)(3/4)^7 = 0.7101$

$m=9: P[G>0] = 1-(3/4)^9 - 9(1/4)(3/4)^8 = 0.6494$

$m=10: P[G>0] = 1-(3/4)^{10} - 10(1/4)^9 - (45/16)(3/4)^8 = 0.5744$

maximized at $\boxed{m=6}$ with value $\boxed{0.7724}$

THE END