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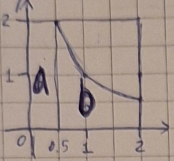
① $x\cancel{y} \le 1$ interval $[0,2]$ Find $P[A]$



$$P[A] = \frac{A[a] + A[b]}{4}$$

$$P[A] = \frac{(2 \cdot 0.5) + \int_{0.5}^{2} \frac{1}{x}\,dx}{4}$$

$A[\sigma] = 2 \times 2$

$$P[A] = \frac{1 + 2\ln(2)}{4}$$

$$P[A] = \frac{1 + \ln(4)}{4} \approx \boxed{0.596574}$$

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② ⓐ $\vec{r} = 4$ (rolls)

$\vec{h} = 2$ (heads needed)

$H = \left(\frac{1}{3}\right)$ (head probability)

$T = \left(\frac{2}{3}\right)$ (tail probability)

* Formula found online for rigged dice

This gives us the formula $P[a] = \binom{\vec{r}}{\vec{h}} (H)^{\vec{h}} (T)^{\vec{r}-\vec{h}}$

$$P[a] = \binom{4}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2$$

$$P[a] = 6 \cdot \frac{1}{9} \cdot \frac{4}{9}$$

$$P[a] = \frac{2}{3} \cdot \frac{4}{9}$$

$$P[a] = \frac{8}{27} \approx \boxed{0.2962962963}$$

ⓑ Now we can think of it as the first toss was a head now we need 1 more

so: $r = 3$

$\vec{h} = 1$

$H = 1/3$

$T = 2/3$

$$P[b] = \binom{3}{1} \cdot \left(\frac{1}{3}\right)^1 \cdot \left(\frac{2}{3}\right)^2$$

$$P[b] = 3 \cdot \frac{1}{3} \cdot \left(\frac{2}{3}\right)^2$$

$$P[b] = \frac{4}{9} \approx 0.4\overline{4}$$

ⓒ

$P[H] \to$ probability of first being heads

$$P[c] = \frac{P[H] \text{ and } P[b]}{P[a]} = \frac{\frac{1}{3} \cdot \frac{4}{9}}{8/27} = \frac{4}{\cancel{27}} \cdot \frac{\cancel{27}}{8} = \frac{4}{8} = \frac{1}{2}$$

$$P[c] = \frac{1}{2} = 0.5$$

③ Assuming a fair die has 6 sides we have to know:

for initial throw

$$P[\text{Success}] = \left[\sum_{k=1}^{6} \frac{P[\text{successful } k \text{ combination of throws}]}{\underbrace{6^{k}}_{\text{number of throws dictates probability}}} \right] \cdot \frac{1}{6}$$

Lets go step by step.

1 more throw

<u>For $k=1$:</u> $1 + k_1 = 10$

$k_1 = 9$ since it is impossible for a 6 sided die to give more than 6 we ignore ①.

$P[A1] = 0$

<u>For $k=2$:</u> $2 + k_1 + k_2 = 10$

$k_1 + k_2 = 8$

$8 \Rightarrow 5+3,\ 4+4,\ 2+6 \rightarrow 3(\text{outcomes}) \cdot 2(\text{for order}) - 1(\text{for } 4+4) = \underline{5 \text{ outcomes}}$

<u>For $k=3$:</u> $3 + k_1 + k_2 + k_3 = 10$
$k_1 + k_2 + k_3 = 7$

$(3,2,2), (5,1,1), (1,3,3), (1,4,2), (1,1,5) (1,2,4), (1,5,1), (2,1,4), (2,2,3), (2,4,1), (3,1,3), (3,3,1), (4,1,2), (4,2,1), (2,3,2)$

15 Possibilities

<u>For $k=4$:</u> $4 + k_1 + k_2 + k_3 + k_4 = 10$
$k_1 + k_2 + k_3 + k_4 = 6$

Combinations:
$(1,1,1,3), (1,1,3,1), (1,3,1,1), (3,1,1,1), (1,1,2,2), (1,2,1,2), (1,2,2,1), (2,1,2,1), (2,2,1,1), (2,1,1,2)$

10 Possibilities.

<u>For $k=5$:</u> $5 + k_1 + k_2 + k_3 + k_4 + k_5 = 10$
$k_1 + k_2 + k_3 + k_4 + k_5 = 5$

$(1,1,1,1,1)$

1 Possibility

<u>For $k=6$:</u> $6 + k_1 + k_2 + k_3 + k_4 + k_5 + k_6 = 10$
$k_1 + k_2 + k_3 + k_4 + k_5 + k_6 = 4$

Not possible so 0 possibilities

Final result

$$P[\text{Success}] = \left[0 + \frac{5}{6^2} + \frac{15}{6^3} + \frac{10}{6^4} + \frac{1}{6^5} + 0 \right] \cdot \frac{1}{6}$$

$$\left[\boxed{= \frac{1681}{7776} \approx 0.216178} \right] \cdot \underbrace{\frac{1}{6}}_{\text{initial throw}} \Rightarrow \boxed{\frac{1681}{46656} \approx 0.0360297}$$

④ ⓐ $X \to \{1, 2, 3, 4\}$ $\quad \boxed{y = i \text{ for each } i = 1, 2, 3, 4}$

$y \to$ [redacted] $\Rightarrow \{1, \ldots, x\}$

For $y = 1$:

If $y=1$ and $x=1 \Rightarrow P = 1$
If $y=1$ and $x=2 \Rightarrow P = 1/2$
If $y=1$ and $x=3 \Rightarrow P = 1/3$
If $y=1$ and $x=4 \Rightarrow P = 1/4$

chances of it being 1

$P[y=1] = \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) \cdot \boxed{\frac{1}{4}}$

$\boxed{P[y=1] = \frac{25}{48} \approx 0.52083}$

For $y = 2$:

If $y=2$ and $x=1 \Rightarrow P=0$
If $y=2$ and $x=2 \Rightarrow P = 1/2$
If $y=2$ and $x=3 \Rightarrow P = 1/3$
If $y=2$ and $x=4 \Rightarrow P = 1/4$

$P[y=2] = \left(0 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) \cdot \frac{1}{4}$

$\boxed{P[y=2] = \frac{13}{48} \approx 0.27083}$

For $y = 3$:

If $y=3$ and $x=1 \Rightarrow P=0$
If $y=3$ and $x=2 \Rightarrow P=0$
If $y=3$ and $x=3 \Rightarrow P=1/3$
If $y=3$ and $x=4 \Rightarrow P=1/4$

$P[y=3] = \left(0 + 0 + \frac{1}{3} + \frac{1}{4}\right) \cdot \frac{1}{4}$

$\boxed{P[y=3] = \frac{7}{48} \approx 0.14583}$

For $y = 4$:

If $y=4$ and $x=1 \Rightarrow P=0$
If $y=4$ and $x=2 \Rightarrow P=0$
If $y=4$ and $x=3 \Rightarrow P=0$
If $y=4$ and $x=4 \Rightarrow P=1/4$

$P[y=4] = \left(0+0+0+\frac{1}{4}\right) \cdot \frac{1}{4}$

$\boxed{P[y=4] = \frac{1}{16} \approx 0.0625}$

ⓑ $y = x \Rightarrow$ Event A

For $x=1$, $y=1 \Rightarrow 1$
For $x=2$, $y=\{1,2\} \Rightarrow 1/2$
For $x=3$, $y=\{1,2,3\} \Rightarrow 1/3$
For $x=4$, $y=\{1,2,3,4\} \Rightarrow 1/4$

$P[A] = \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = \boxed{\frac{25}{48} \approx 0.52083}$

ⓒ $z \Rightarrow \{1, \ldots, y\}$. $P[z=1]$

If $y=1$, $z \Rightarrow P = 1$
If $y=2$, $z \Rightarrow P = 1/2$
If $y=3$, $z \Rightarrow P = 1/3$
If $y=4$, $z \Rightarrow P = 1/4$

$$P[z=1] = \left[P[z=1 \mid y=1] \cdot P[y=1]\right] + \left[P[z=1 \mid y=2] \cdot P[y=2]\right] + \left[P[z=1 \mid y=3] \cdot P[y=3]\right] + \left[P[z=1 \mid y=4] \cdot P[y=4]\right]$$

$$= \left(\frac{25}{48} \cdot 1\right) + \left(\frac{13}{48} \cdot \frac{1}{2}\right) + \left(\frac{7}{48} \cdot \frac{1}{3}\right) + \left(\frac{1}{16} \cdot \frac{1}{4}\right) = \boxed{\frac{415}{576}} \approx \boxed{0.7204861}$$

5 a Show that if $P(B|A)=1$ $P(A^c|B^c)=1$

$P(B|A)=1$ states that "If event A occurs, then event B definatelly occurs."

$$P(B|A)=\frac{P(B\cap A)}{P(A)}=1$$

Which means $P(A)=P(B\cap A)$ understood as whenever A happens, B must happen aswell.

<u>Contrapositive:</u> $P(A^c|B^c)=1$ "If B^c does happen, then A^c also happens"

Following the logic of relationship $A\subseteq B$, $B^c\subseteq A^c$, <u>A^c also must occur</u>, since if B doesn't occurr, A also doesn't.

Therefore, $P(A^c|B^c)=1$, which completes <u>the statement.</u>

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b <u>Show that an approximate version is not true.</u>

$P(B|A)>0.99$

$P(A^c|B^c)<0.01$

$\boxed{\text{20 throws}}$:
- Event A → Heads appears at least once after 10 first throws
- Event B → Heads appears at least once in last 10 throws

$$P(B|A)=1-\frac{1}{1024}=\frac{1023}{1024}\approx 0.999$$

A^c → no heads in first 10
B^c → no heads in last 10

Now we find $P(A^c|B^c)$:

$$P(A^c|B^c)=\frac{P(A^c\cap B^c)}{P(B^c)}=\frac{1/1024}{1/1024}=1$$

So this approximate version is <u>wrong.</u>

THE END