$X = \{x_1, x_2, \cdots, x_{N_x}\}$

$Y = \{y_1, y_2, \cdots, y_{N_y}\}$

目标: $\min E(R,t) = \min \frac{1}{N_y}\sum_{i=1}^{N_y} \|x_i - Ry_i - t\|^2$ $\Downarrow$ $E(R,t) = \frac{1}{N_y}\sum_{i=1}^{N_y}\|x_i - Ry_i - t - u_x + Ru_y + u_x - Ru_y\|^2$

$= \frac{1}{N_y}\sum_{i=1}^{N_y}\|x_i - u_x - R(y_i-u_y) + (u_x - Ru_y - t)\|^2$

$= \frac{1}{N_y}\sum_{i=1}^{N_y}\left(\|x_i-u_x-R(y_i-u_y)\|^2 + \|u_x - Ru_y - t\|^2 + 2(x_i-u_x-R(y_i-u_y))^T(u_x-Ru_y-t)\right)$

$\because u_x = \frac{1}{N_x}\sum_{i=1}^{N_x} x_i \quad u_y = \frac{1}{N_y}\sum_{i=1}^{N_y} y_i$ (质心),

$N_x = N_y$

$\Downarrow$ $u_x N_x = \sum_{i=1}^{N_x} x_i$ $\Downarrow$ $\frac{1}{N_y}\sum_{i=1}^{N_y}(x_i - u_x - R(y_i-u_y)) = \frac{1}{N_y}u_x N_x - \frac{1}{N_y}\sum_{i=1}^{N_y} u_x - R\frac{1}{N_y}u_y N_x + R\frac{1}{N_y}\sum_{i=1}^{N_y} u_y$

$= u_x - u_x - R(u_y - u_y) = 0$.

$\Downarrow$ 原式 $= \frac{1}{N_y}\sum_{i=1}^{N_y}\left(\|x_i-u_x-R(y_i-u_y)\|^2 + \|u_x - Ru_y - t\|^2\right)$

$\Downarrow \quad E(R,t) = \underbrace{E_1(R,t)}_{\text{only } R} + \underbrace{E_2(R,t)}_{\text{solve}}$

~~再~~t代入 E_2 求 t，令

即先用 E_1 求 R，$E_2(R,t)=0$

$$\Downarrow \quad E_1(R,t) = \frac{1}{N_y}\sum_{i=1}^{N_y} \|\underbrace{x_i - u_x}_{x_i'} - R(\underbrace{y_i - u_y}_{y_i'})\|^2$$

$$\Downarrow = \frac{1}{N_y}\sum_{i=1}^{N_y} \|x_i' - R y_i'\|^2 =$$

$$\frac{N_y}{N_y}\sum_{i=1}^{N_y} (\underbrace{x_i'^T x_i' + y_i'^T R^T R y_i'}_{与R有关} - 2 x_i'^T R y_i')$$

单位阵

令 $E_1'(R,t) = \sum_{i=1}^{N_y} x_i'^T R y_i'$，则

$$\arg\min E_1(R,t) = \arg\max E_1'(R,t) = \arg\max_R \sum_{i=1}^{N_y} x_i'^T R y_i' \Downarrow =$$

$$E_1'(R,t) = \sum^{N_y} \mathrm{Trace}(x_i'^T R y_i') \qquad tr(AB) = tr(BA)$$

$$\Downarrow \quad 上式 = \sum_{i=1}^{N_y} \mathrm{Trace}(R y_i' x_i'^T)$$

$$= \mathrm{Trace}\left(\sum_{i=1}^{N_y} R y_i' x_i'^T\right) = \mathrm{Trace}\ RH$$

$$\Downarrow \cdot \text{ find } R \Rightarrow \arg\max(\mathrm{Trace}\ RH)$$

$\because$ 正定 AA^T，任意 B，有 $\mathrm{Trace}(AA^T) \geq \mathrm{Trace}(BAA^T)$

目的，find R，使 $\mathrm{Trace}(RH) = \mathrm{Trace}(AA^T)$

$\Downarrow$ SVD分解， $H = U\Sigma V^T$ 取 $R = VU^T$

则 $RH = VU^TU\Sigma V^T = V\Sigma V^T = V\Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}}V^T$

对称阵 $= V\Sigma^{\frac{1}{2}}(V\Sigma^{\frac{1}{2}})^T$

R确定后，由 $E_2(R,t)=0 \Rightarrow t = \mu_x - R\mu_y$