Homework 9 for Bayesian Data Analysis

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Question 14.3

Assuming uniform prior distribution for $\beta|\sigma$, we have

$$p(\beta|\sigma, y) \propto p(y|\beta, \sigma)p(\beta|\sigma)$$
$$\propto p(y|\beta, \sigma) \propto \exp\left(-\frac{1}{2\sigma^2}(y - X\beta)^T(y - X\beta)\right).$$

Note the fact that

$$(y - X\beta)^T(y - X\beta) = (\beta - \hat{\beta})^T X^T X(\beta - \hat{\beta}) + \text{constant},$$

we have

$$p(\beta|\sigma, y) \propto \exp\left(-\frac{1}{2\sigma^2}(y - X\beta)^T(y - X\beta)\right)$$
$$\propto \exp\left(-\frac{1}{2\sigma^2}(\beta - \hat{\beta})^T X^T X(\beta - \hat{\beta})\right)$$
$$= \exp\left(-\frac{1}{2}(\beta - \hat{\beta})^T((X^T X)^{-1}\sigma^2)^{-1}(\beta - \hat{\beta})\right),$$

which implies that

$$\beta|\sigma, y \sim N(\hat{\beta}, (X^T X)^{-1}\sigma^2).$$

Question 14.4

Assuming the noninformative prior $p(\beta, \log\sigma) \propto 1$, or $p(\beta, \sigma^2) \propto \sigma^{-2}$, we obtain

$$p(\sigma|y) = p(\beta, \sigma^2|y)/p(\beta|\sigma^2, y) \propto p(\beta, \sigma^2)p(y|\beta, \sigma^2)/p(\beta|\sigma^2, y)$$
$$\propto \frac{\sigma^{-2} \cdot \sigma^{-n} \exp\left(-\frac{1}{2\sigma^2}(y - X\beta)^T(y - X\beta)\right)}{[\det((X^T X)^{-1}\sigma^2)]^{-1/2} \cdot \exp\left(-\frac{1}{2\sigma^2}(\beta - \hat{\beta})^T X^T X(\beta - \hat{\beta})\right)}$$

Fix $\beta = \hat{\beta}$ in the formula above, and it follows that

$$p(\sigma|y) \propto \frac{\sigma^{-2} \cdot \sigma^{-n} \exp\left(-\frac{1}{2\sigma^2}(y - X\hat{\beta})^T(y - X\hat{\beta})\right)}{[\det((X^T X)^{-1}\sigma^2)]^{-1/2}}$$
$$= \frac{\sigma^{-2} \cdot \sigma^{-n} \exp\left(-\frac{1}{2\sigma^2}(y - X\hat{\beta})^T(y - X\hat{\beta})\right)}{[\sigma^{2k} \cdot \det((X^T X)^{-1})]^{-1/2}}$$

$$\propto \sigma^{-2-n+k} \cdot \exp\left(-\frac{1}{2\sigma^2}(y - X\hat{\beta})^T(y - X\hat{\beta})\right).$$

Compare this expression to the Inverse-χ^2 distribution[1], we find that

$$s^2 = \frac{1}{n-k}(y - X\hat{\beta})^T(y - X\hat{\beta}).$$

Question 14.7

$\widetilde{y}$ conforms a normal distribution, as $\widetilde{y}$ is a linear combination of β, and $p(\beta|\sigma, y)$ is normal.

Longley data 1

The mean of Inverse-$\chi^2(n-k, s^2)$ is[2]

$$\frac{(n-k)s^2}{n-k-2}.$$

The calculated result is

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[1] 6.003243
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Longley data 2

The posterior mean of β under a conjugate prior $(\beta_0, \beta_1) \sim N(0, I)$ is

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(Intercept)           X
 -76.759663    1.519031
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Longley data 3

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> mod.full = BayesReg(dat$GNP.deflator, dat[, -1], g=nrow(dat))

PostMean  PostStError  Log10bf  EvidAgaH0
Intercept  101.6813      0.7431
x1          23.8697     25.1230  -0.3966
x2           3.1068      6.6053  -0.5603
x3           0.7078      2.5134  -0.5954
x4         -11.0111     10.9543  -0.3714
x5          -6.1556     32.7640  -0.6064
x6           0.7402     10.7025   -0.614


Posterior Mean of Sigma2: 8.8342
Posterior StError of Sigma2: 13.0037
```

[1] https://en.wikipedia.org/wiki/Inverse-chi-squared_distribution
[2] https://en.wikipedia.org/wiki/Scaled_inverse_chi-squared_distribution

```
> mod = BayesReg(dat$GNP.deflator, dat[, c(-1, -3)], g=nrow(dat))

PostMean PostStError Log10bf EvidAgaH0
Intercept 101.6813      0.7494
x1          14.6884     15.9493 -0.4094
x2          -0.3300      1.2139 -0.5968
x3          -9.9863     10.8264 -0.4087
x4           8.6301      9.3120 -0.4068
x5          -3.5398      5.6814 -0.5194


Posterior Mean of Sigma2: 8.9846
Posterior StError of Sigma2: 13.2249

> bf.full / bf
Bayes factor analysis
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[1] GNP + Unemployed + Armed.Forces + Population + Year + Employed : 0.1443632 ±0%

Against denominator:
GNP.deflator ~ GNP + Armed.Forces + Population + Year + Employed
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Bayes factor type: BFlinearModel, JZS
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We find that the reduced model has similar coefficients to the full model. And the ratio of Bayes factor is much less than 1, which means the reduced model is more likely.