Homework 2 for Bayesian Data Analysis

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March 20, 2018

Question 2.7a

$$p(\theta) \propto \theta^{-1}(1-\theta)^{-1}.$$

Denote $\phi = \text{logit}(\theta) = \log(\theta/(1-\theta))$, one has $\theta = 1/(1+\exp(-\phi))$, and therefore,

$$p(\phi) \propto (\frac{1}{1+\exp(-\phi)})^{-1}(1-\frac{1}{1+\exp(-\phi)})^{-1} \cdot \left|\frac{\exp(-\phi)}{(1+\exp(-\phi))^2}\right| = 1,$$

which gives a uniform prior distribution for $\text{logit}(\theta)$, or the natural parameter of the exponential family.

Question 2.7b

$$p(\theta|y) \propto \theta^{-1}(1-\theta)^{-1} \cdot \theta^y(1-\theta)^{n-y} = \theta^{y-1}(1-\theta)^{n-y-1}.$$

If $y = 0$, then we have

$$p(\theta|y) \propto \theta^{-1}(1-\theta)^{n-1},$$

whose integral in interval $[0, 1]$ is

$$\int_0^1 \theta^{-1}(1-\theta)^{n-1}\mathrm{d}\theta = +\infty,$$

since $p(\theta|y)$ is of the same order as $1/\theta$ when $\theta \to 0$.

Likewise, if $y = n$, the integral is infinite when $\theta \to 1$. Therefore, the posterior distribution is improper if $y = 0$ or $y = n$.

Question 2.12

$$\log(p(y|\theta)) = \theta + y\log(\theta) - \log(y!),$$

$$J(\theta) = -\mathrm{E}(\frac{\mathrm{d}^2\log(p(y|\theta))}{\mathrm{d}\theta^2}|\theta) = -\mathrm{E}(-\frac{y}{\theta^2}|\theta) = \frac{\mathrm{E}(y|\theta)}{\theta^2} = \frac{1}{\theta}.$$

Hence, the Jeffery's prior density for θ is

$$p(\theta) \propto [J(\theta)]^{1/2} = \theta^{-1/2}.$$

Compared with a $\text{Gamma}(\alpha, \beta)$ distribution with prior $p(\theta) \propto \theta^{\alpha-1}\exp(-\beta\theta)$, we have $\alpha = 1/2$ and $\beta = 0$.

Question 3.1a

$$p(\theta) \propto \prod_{1}^{J} {\theta_j}^{\alpha_j - 1},$$

$$p(y|\theta) \propto \prod_{1}^{J} {\theta_j}^{y_j},$$

$$p(\theta|y) \propto \prod_{1}^{J} {\theta_j}^{y_j + \alpha_j - 1}.$$

Thus, by integrating $y_3, \cdots, y_J$, the joint posterior distribution of θ_1 and θ_2 is

$$p(\theta_1, \theta_2|y) \propto \theta_1^{y_1+\alpha_1-1} \theta_2^{y_2+\alpha_2-1}.$$

Under the variable substitution $\alpha = \theta_1/(\theta_1 + \theta_2)$ and $\beta = \theta_1 + \theta_2$, we have

$$\theta_1 = \alpha\beta, \quad \theta_2 = (1-\alpha)\beta,$$

and

$$\begin{aligned} p(\alpha, \beta|y) = p(\theta_1, \theta_2) \cdot \left| \frac{\partial(\theta_1, \theta_2)}{\partial(\alpha, \beta)} \right| &\propto (\alpha\beta)^{y_1+\alpha_1-1} \left((1-\alpha)\beta\right)^{y_2+\alpha_2-1} \cdot |\beta| \\ &= \alpha^{y_1+\alpha_1-1}(1-\alpha)^{y_2+\alpha_2-1}\beta^{y_1+y_2+\alpha_1+\alpha_2-1}. \end{aligned}$$

Therefore, the marginal posterior distribution of α is

$$p(\alpha|y) = \int p(\alpha, \beta|y) \mathrm{d}\beta \propto \alpha^{y_1+\alpha_1-1}(1-\alpha)^{y_2+\alpha_2-1},$$

which is $\alpha|y \sim \text{Gamma}(y_1 + \alpha_1, y_2 + \alpha_2)$.

Question 3.1b

For a Gamma(α_1, α_2) prior distribution, and a Binomial sample with y_1 independent observations out of $y_1 + y_2$ tests, we have the posterior distribution for the probability α is a Gamma$(y_1 + \alpha_1, y_2 + \alpha_2)$. (See Homework 1) This posterior distribution is identical to the distribution obtained in (a).

Question 3.9

It is known that

$$\begin{aligned} p(y|\mu, \sigma^2) &\propto \sigma^{-n} \exp\left(-\frac{1}{2\sigma^2}\sum_{1}^{n}(y_i - \mu)^2\right) \\ &= \sigma^{-n} \exp\left(-\frac{1}{2\sigma^2}\left[\sum_{1}^{n}(y_i - \bar{y})^2 + n(\bar{y} - \mu)^2\right]\right) \\ &= \sigma^{-n} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right), \end{aligned}$$

and

$$p(\mu, \sigma^2) = p(\sigma^2)p(\mu|\sigma^2) \propto \sigma^{-1}(\sigma^2)^{-(\nu_0/2+1)} \exp\left(-\frac{1}{2\sigma^2}\left[\nu_0\sigma_0^2 + \kappa_0(\mu_0 - \mu)^2\right]\right).$$

Therefore, we have the joint posterior distribution

$$
\begin{aligned}
p(\mu,\sigma^2|y) &\propto p(y|\mu,\sigma^2)p(\mu,\sigma^2) \\
&\propto \sigma^{-1}(\sigma^2)^{-((\nu_0+n)/2+1)}\exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2+n(\bar{y}-\mu)^2+\nu_0\sigma_0^2+\kappa_0(\mu_0-\mu)^2\right]\right),
\end{aligned}
$$

which is identical to a N-Inv-$\chi^2(\mu_n,\sigma_n^2;\nu_n,\sigma_n^2)$ distribution, or

$$\sigma^{-1}(\sigma^2)^{-(\nu_n/2+1)}\exp\left(-\frac{1}{2\sigma^2}\left[\nu_n\sigma_n^2+\kappa_n(\mu_n-\mu)^2\right]\right).$$

Thus, by comparing the coefficients, we have

$$\nu_n=\nu_0+n$$

and

$$(n-1)s^2+n(\bar{y}-\mu)^2+\nu_0\sigma_0^2+\kappa_0(\mu_0-\mu)^2=\nu_n\sigma_n^2+\kappa_n(\mu_n-\mu)^2,$$

or

$$
\begin{aligned}
n+\kappa_0&=\kappa_n,\\
-2n\bar{y}-2\kappa_0\mu_0&=-2\kappa_n\mu_n,\\
(n-1)s^2+n\bar{y}^2+\nu_0\sigma_0^2+\kappa_0\mu_0^2&=\nu_n\sigma_n^2+\kappa_n\mu_n^2,
\end{aligned}
$$

the solution to which is

$$
\begin{aligned}
\nu_n&=\nu_0+n,\\
\kappa_n&=\kappa_0+n,\\
\mu_n&=\frac{n}{n+\kappa_0}\bar{y}+\frac{\kappa_0}{n+\kappa_0}\mu_0,\\
\nu_n\sigma_n^2&=(n-1)s^2+\nu_0\sigma_0^2+n\bar{y}^2+\kappa_0\mu_0^2-\kappa_n\mu_n^2=(n-1)s^2+\nu_0\sigma_0^2+\frac{\kappa_0 n}{\kappa_0+n}(\bar{y}-\mu_0)^2.
\end{aligned}
$$

Question 3.10

From the independency conditions, we have the joint posterior distribution

$$p(\mu_1,\sigma_1^2,\mu_2,\sigma_2^2\mid y)\propto\sigma_1^{-n_1-2}\sigma_2^{-n_2-2}\exp\left(-\frac{1}{2\sigma_1^2}\sum_{j=1}^{n_1}(y_{1j}-\mu_1)^2-\frac{1}{2\sigma_2^2}\sum_{j=1}^{n_2}(y_{2j}-\mu_2)^2\right)$$

$$=\sigma_1^{-n_1-2}\sigma_2^{-n_2-2}\exp\left(-\frac{1}{2\sigma_1^2}\left[(n_1-1)s_1^2+n_1(\bar{y_1}-\mu_1)^2\right]\right)\exp\left(-\frac{1}{2\sigma_2^2}\left[(n_2-1)s_2^2+n_2(\bar{y_2}-\mu_2)^2\right]\right).$$

Recall that

$$\int\exp\left(-\frac{n_1}{2\sigma_1^2}(\bar{y_1}^2-\mu_1)^2\right)\mathrm{d}\mu_1=(2\pi)^{1/2}\sigma_1/\sqrt{n_1},$$

we have the integral

$$
\begin{aligned}
p(\sigma_1^2,\sigma_2^2\mid y)&=\int p(\mu_1,\sigma_1^2,\mu_2,\sigma_2^2|y)\mathrm{d}\mu_1\mathrm{d}\mu_2\\
&\propto\sigma_1^{-n_1-2}\sigma_2^{-n_2-2}\exp\left(-\frac{1}{2\sigma_1^2}\left[(n_1-1)s_1^2\right]\right)\exp\left(-\frac{1}{2\sigma_2^2}\left[(n_2-1)s_2^2\right]\right)\cdot\sigma_1\sigma_2\\
&\propto\sigma_1^{-n_1-1}\sigma_2^{-n_2-1}\exp\left(-\frac{(n_1-1)s_1^2}{2\sigma_1^2}\right)\exp\left(-\frac{(n_2-1)s_2^2}{2\sigma_2^2}\right)
\end{aligned}
$$

$$\propto \left(\frac{s_1^2}{\sigma_1^2}\right)^{(n_1+1)/2} \left(\frac{s_2^2}{\sigma_2^2}\right)^{(n_2+1)/2} \exp\left(-\frac{(n_1-1)s_1^2}{2\sigma_1^2}\right) \exp\left(-\frac{(n_2-1)s_2^2}{2\sigma_2^2}\right).$$

Denote $u_1 = \frac{s_1^2}{\sigma_1^2}$, $u_2 = \frac{s_2^2}{\sigma_2^2}$, we have

$$p(u_1, u_2 \mid y) \propto (u_1)^{(n_1+1)/2} (u_2)^{(n_2+1)/2} \exp\left(-\frac{(n_1-1)}{2}u_1\right) \exp\left(-\frac{(n_2-1)}{2}u_2\right) \cdot \left|\frac{1}{u_1^2}\frac{1}{u_1^2}\right|$$

$$= (u_1)^{(n_1-3)/2} (u_2)^{(n_2-3)/2} \exp\left(-\frac{(n_1-1)}{2}u_1\right) \exp\left(-\frac{(n_2-1)}{2}u_2\right)$$

and that u_1 is independent of u_2.

By variable substitution $v_1 = u_1/u_2$ and $v_2 = u_2$, or $u_1 = v_1 v_2$ and $u_2 = v_2$, we have

$$p(v_1, v_2 \mid y) \propto (v_1 v_2)^{(n_1-3)/2} (v_2)^{(n_2-3)/2} \exp\left(-\frac{(n_1-1)}{2}v_1 v_2\right) \exp\left(-\frac{(n_2-1)}{2}v_2\right) \cdot |v_2|$$

$$= (v_1)^{(n_1-2)/2} \cdot (v_2)^{(n_1+n_2-2)/2} \cdot \exp\left(-\frac{(n_1-1)v_1 + (n_2-1)}{2}v_2\right).$$

Note that the gamma distribution[1], we have

$$p(v_1 \mid y) = \int_0^\infty p(v_1, v_2 \mid y)\mathrm{d}v_2$$

$$= \int_0^\infty (v_1)^{(n_1-3)/2} \cdot (v_2)^{(n_1+n_2-4)/2} \cdot \exp\left(-\frac{(n_1-1)v_1 + (n_2-1)}{2}v_2\right)\mathrm{d}v_2$$

$$= (v_1)^{(n_1-3)/2} \cdot \frac{\Gamma(\alpha_0)}{\beta_0^{\alpha_0}}$$

$$\propto (v_1)^{(n_1-3)/2} \cdot \left(\frac{(n_1-1)v_1 + (n_2-1)}{2}\right)^{-(n_1+n_2-2)/2},$$

where $\alpha_0 = (n_1 + n_2 - 2)/2$ and $\beta_0 = (n_1 - 1)v_1 + (n_2 - 1))/2$.

Compare the expression above with the pdf of F distributions[2], we have

$$v_1 \mid y \sim F(n_1 - 1, n_2 - 1)$$

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[1] https://en.wikipedia.org/wiki/Gamma_distribution
[2] https://en.wikipedia.org/wiki/F-distribution