Homework 8 for Bayesian Data Analysis

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Question 11.3

Separate model and Pooled model:

With the noninformative prior distribution, uniform for $(\theta, \log \sigma)$,

$$y_j | \theta, \sigma^2 \sim \mathrm{N}(\theta, \sigma^2),$$
$$p(\theta, \sigma^2) \propto \sigma^{-2}.$$

The conditional distributions are

$$p(\theta | \sigma^2) \propto 1,$$
$$p(\sigma^2 | \theta) \propto \sigma^{-2}.$$

It follows that

$$p(\theta | \sigma^2, y) \propto 1 \cdot \mathrm{N}(\bar{y}, \sigma^2/n) \propto \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\theta)^2\right),$$
$$p(\sigma^2 | \theta, y) \propto \sigma^{-2-n} \cdot \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\theta)^2\right]\right).$$

- For the separate model, plug in y for the sixth machine only. We can calculate the posterior distribution $\theta | y$ by simulation, using Metropolis-within-Gibbs on the two conditional distributions above.
  But we have no idea about the seventh machine, for each machine has its separate parameters. Therefore, we cannot obtain the predictive distributions or the posterior mean for the seventh machine.

- For the pooled model, plug in y for all the six machines. We can calculate the posterior distribution $\theta | y$ by simulation, using Metropolis-within-Gibbs on the two conditional distributions above.
  And we use this posterior distribution to predict the seventh machine, for the machines have pooled parameters.

Hierarchical model:

The four conditional distributions are given on p.289 of textbook:

- $$\theta_j | \mu, \sigma, \tau, y \sim \mathrm{N}(\hat{\theta}_j, V_{\theta_j}),$$
  where
  $$\hat{\theta}_j = \frac{\frac{1}{\tau^2}\mu + \frac{n_j}{\sigma^2}\bar{y}_{\cdot j}}{\frac{1}{\tau^2} + \frac{n_j}{\sigma^2}},$$
  $$V_{\theta_j} = \frac{1}{\frac{1}{\tau^2} + \frac{n_j}{\sigma^2}}.$$

- $$\mu|\theta, \sigma, \tau, y \sim \mathrm{N}(\hat{\mu}, \tau^2/J),$$

  where

  $$\hat{\mu} = \frac{1}{J}\sum_{j=1}^{J}\theta_j.$$

- $$\sigma^2|\theta, \mu, \tau, y \sim \mathrm{Inv}{-}\chi^2(n, \hat{\sigma}^2),$$

  where

  $$n = \sum_{j=1}^{J} n_j,$$

  $$\hat{\sigma}^2 = \frac{1}{n}\sum_{j=1}^{J}\sum_{i=1}^{n_j}(y_{ij} - \theta_j)^2.$$

- $$\tau^2|\theta, \mu, \sigma, y \sim \mathrm{Inv}{-}\chi^2(J-1, \hat{\tau}^2),$$

  where

  $$\hat{\tau}^2 = \frac{1}{J-1}\sum_{j=1}^{J}(\theta_j - \mu)^2.$$

Use Metropolis-within-Gibbs on the four conditional distributions above, we can draw θ_j, μ, σ^2, τ^2 by simulation. The posterior distribution $\theta_j|y$ can be obtained immediately. Then, we obtain $\mu|y$ and $\tau^2|y$, which then give $\theta_7|y$. Finally, the posterior distribution $y_{i7}|y$ can be estimated, given $\sigma^2|y$ obtained.

Reports

Source code: "src1.R"

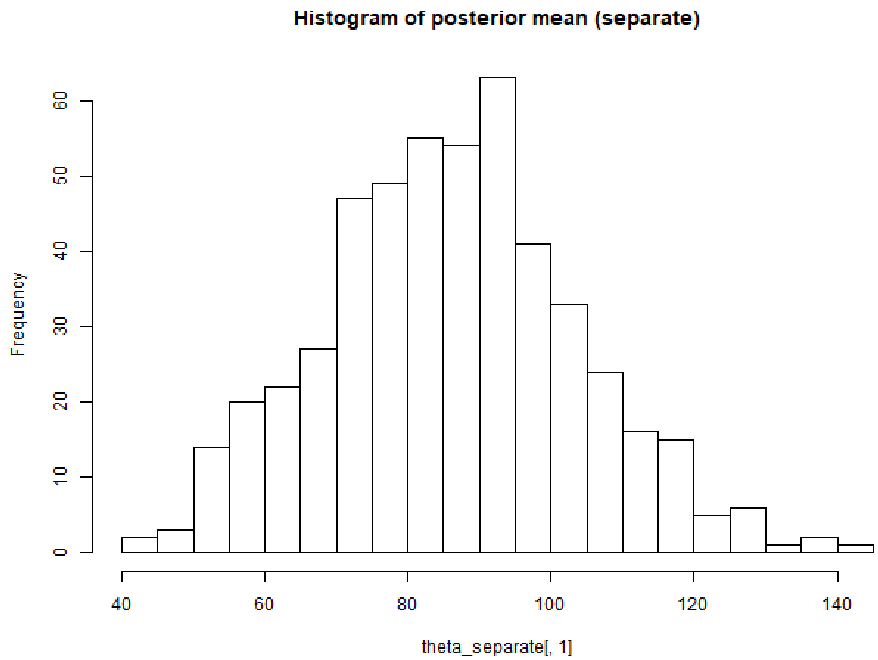

Figure 1:

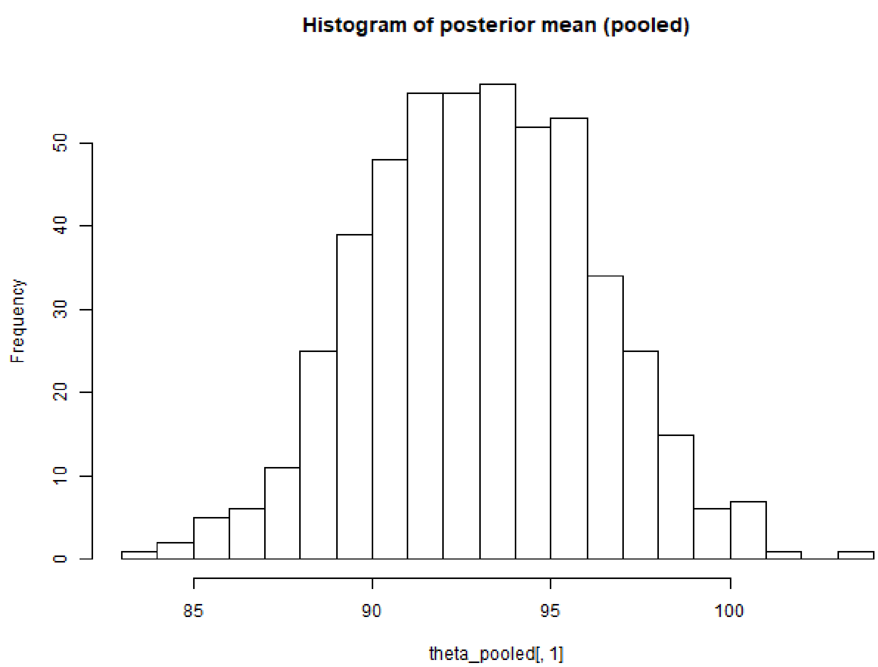

Figure 2:

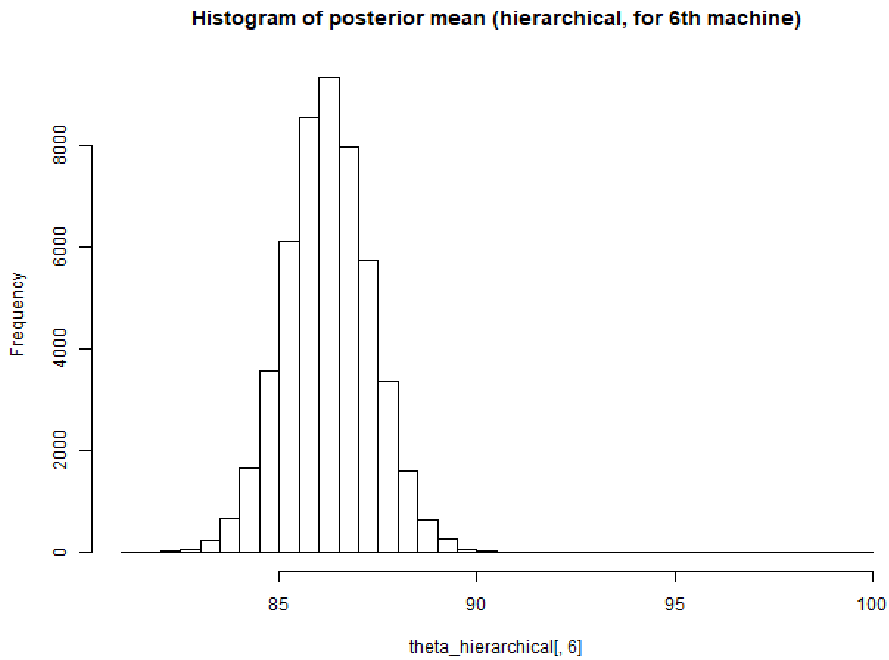

Figure 3:

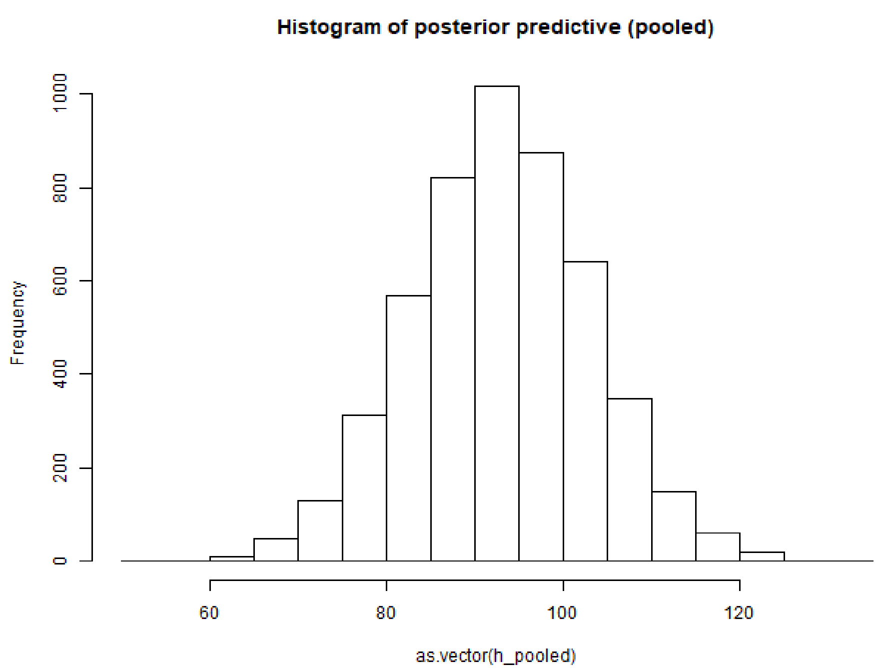

Figure 4:

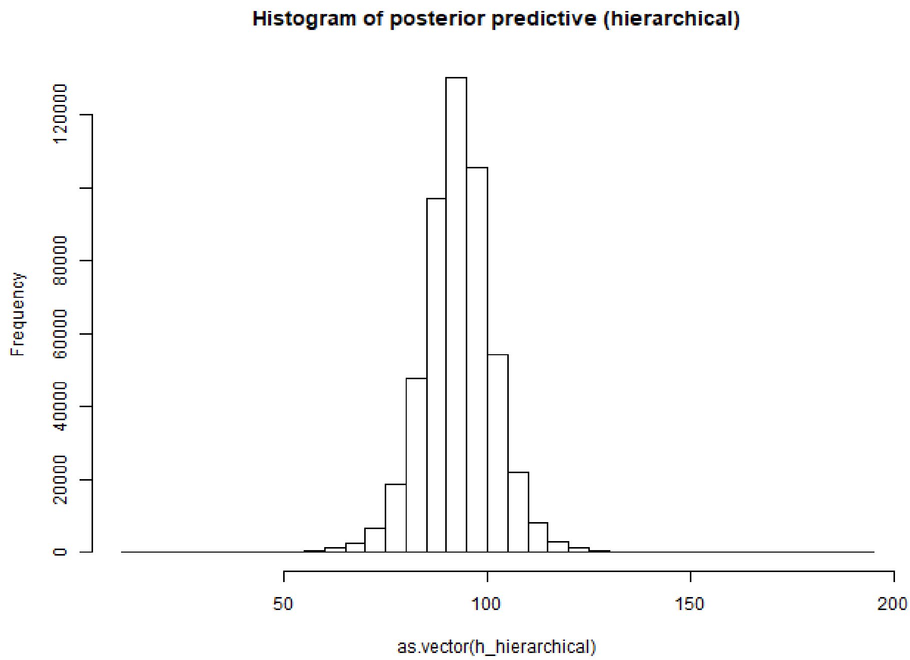

Figure 5:

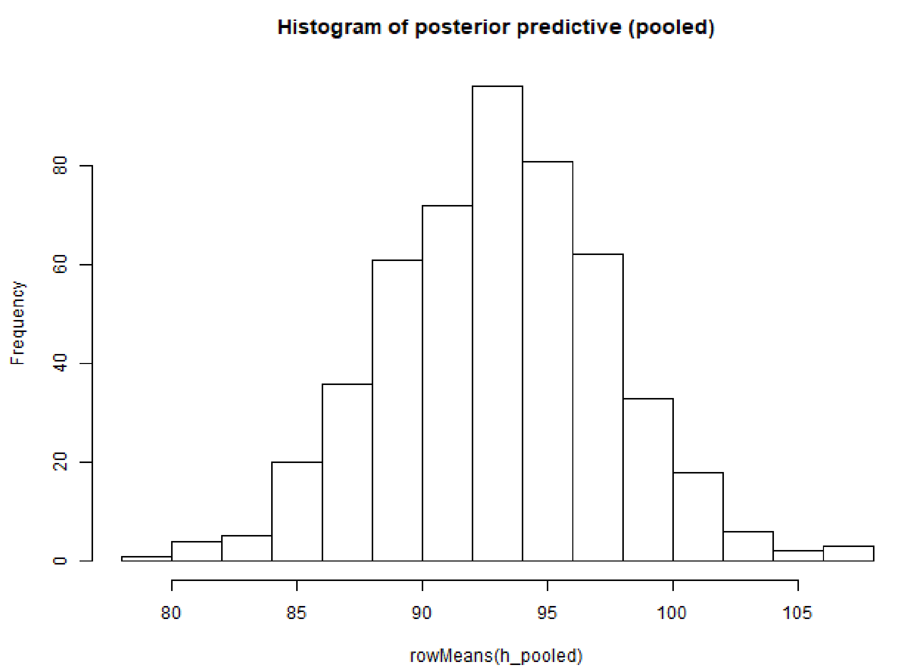

Figure 6:

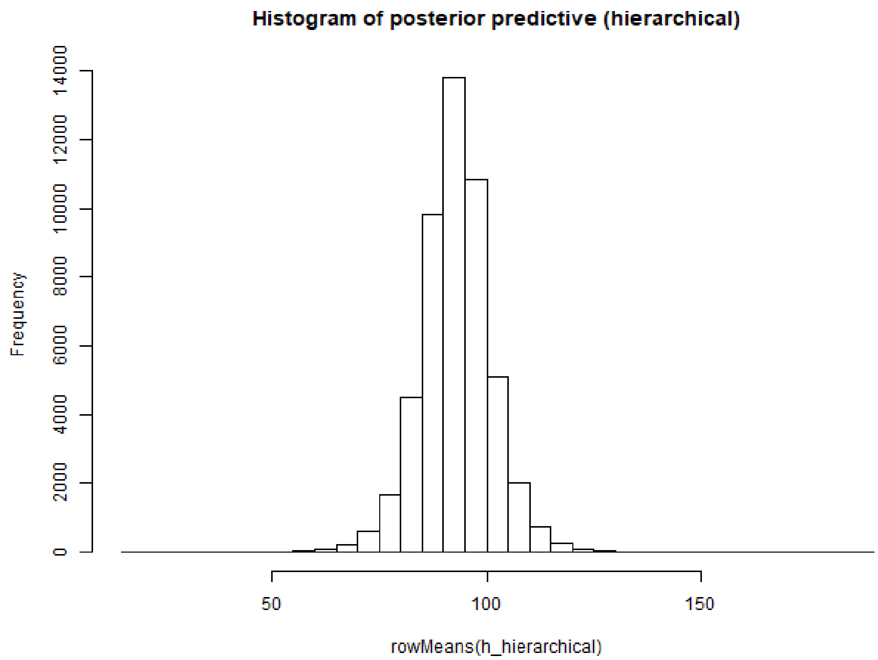

Figure 7:

Random-walk Metroplis

Source code: "src2.R"

To make the adaptive tune of α and β at each iteration, according to the empirical formula. For the first 20 iterations, we do not change the hyperparamters.

```
# Adapt
if (i > 20) {
  tune$alpha = 2.4^2 * var(alpha_keep[1:i]) / 2
  tune$beta = 2.4^2 * var(beta_keep[1:i]) / 2
}
```

The adaptived hyperparamters converge to $\alpha = 0.04629154$ and $\beta = 1.906087$.

```
$alpha
[1] 0.04629154

$beta
[1] 1.906087
```