MTH 9879 Market Microstructure Models, Spring 2016

Lecture 4: Understanding the bid-ask spread

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Outline of Lecture 4

- The Roll model
- TAQ data cleaning and trade signing
- A generalization of the Roll model
- The Glosten and Milgrom model
- The Foucault (1999) model
- The Glosten and Harris model
- The Madhavan-Richardson-Roomans model
- The Huang and Stoll model
- PIN - the probability of informed trading

The Roll model

- We think of the efficient price $m_t = \mathbb{E}[\tilde{S}|\mathcal{F}_t]$ as being the expected value of the cashflows of a security, conditioning on all public information.

- We model trade prices p_t as

  (1)

  $$p_t = m_t + c\,\epsilon_t$$

  where (under competition), c is the market maker($\mathcal{M}$)'s cost to trade and $\epsilon_t = \pm 1$ is a trade sign indicator.

- We further assume that trade signs are serially independent with equal probabilities of buys and sells.

The Roll model

The Roll model

Under these (very unrealistic) assumptions, we have

$$\begin{aligned}\gamma_0 := \mathrm{Var}[\Delta p_t] &= \mathbb{E}[\Delta p_t^2]\\ &= \mathbb{E}[\Delta m_t^2] + c^2\ \mathrm{Var}[\epsilon_t - \epsilon_{t-1}]\\ &= \sigma^2 + 2\,c^2\end{aligned}$$

where σ is the volatility of the stock per trade.

Also,

$$\begin{aligned}\gamma_1 :\ &= \mathrm{Cov}[\Delta p_{t-1}, \Delta p_t]\\ &= \mathbb{E}[\Delta p_{t-1}\ \Delta p_t]\\ &= \mathbb{E}[\{\Delta m_{t-1} + c\,(\epsilon_{t-1} - \epsilon_{t-2})\}\ \{\Delta m_t + c\,(\epsilon_t - \epsilon_{t-1})\}]\\ &= -\,c^2\end{aligned}$$

Higher order autocovariances are all zero.

The Roll model

We conclude that the effective half-spread is given by

$$c = \sqrt{-\gamma_1}$$

and

$$\sigma^2 = \gamma_0 + 2\,\gamma_1$$

The first order autocovariance is negative because of bid-ask bounce.

Autocorrelation of BAC price changes

In [17]:

```
download.file(url="http://mfe.baruch.cuny.edu/wp-content/uploads/2015/02/tq.zip", de
unzip(zipfile="tq.zip")
download.file(url="http://mfe.baruch.cuny.edu/wp-content/uploads/2015/02/tqBAC_2013(
unzip(zipfile="tqBAC_20130731.zip")
```

In [18]:

```
library(xts)
library(highfrequency)
library(quantmod)

load("tq.rData")
tqdata <- tqBAC
```

In [11]:

```
data.frame(head(tqdata))
```

	SYMBOL	EX	PRICE	SIZE	COND	BID	BIDSIZ	OFR	OFRSIZ
2012-05-04 09:30:00	BAC	T	7.89	38538	F	7.89	523	7.9	952
2012-05-04 09:30:01	BAC	Z	7.885	288	@	7.88	61033	7.9	92866
2012-05-04 09:30:03	BAC	X	7.89	1000	@	7.88	1974	7.89	333
2012-05-04 09:30:07	BAC	T	7.89	19052	F	7.88	1058	7.89	218
2012-05-04 09:30:08	BAC	Y	7.89	85053	F	7.88	108101	7.9	31104
2012-05-04 09:30:09	BAC	D	7.8901	10219	@	7.89	268	7.9	291

In [19]:

```
#-----------------------------------------------------------------------------
# Figure 1: Autocorrelation of BAC price changes
p <- as.numeric(tqdata$PRICE)
dp <- diff(p)
ac <- acf(dp,lag=100,plot=FALSE)
plot(ac,col="red",main=NA)
```

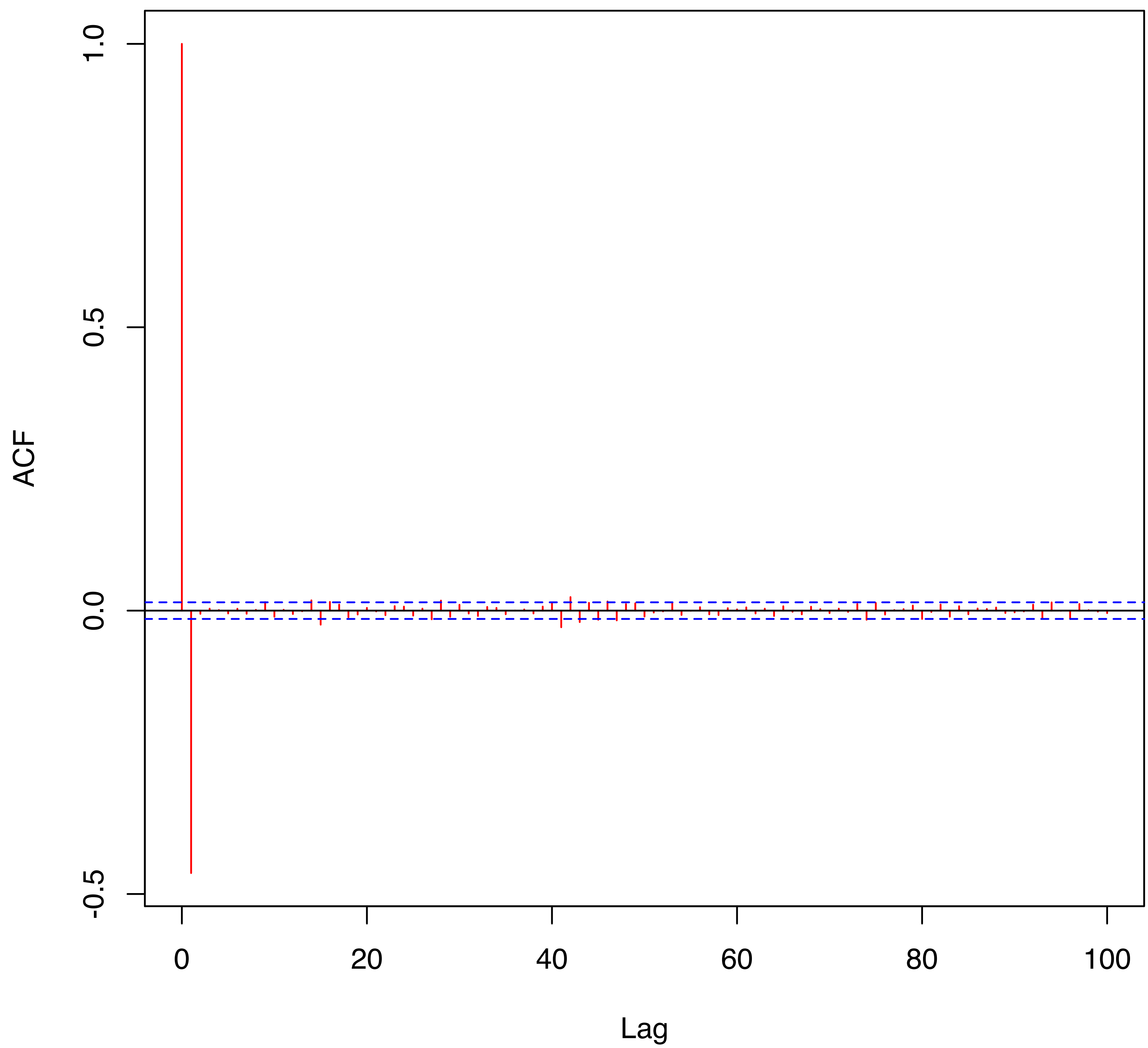


Figure 1: Autocorrelation of BAC price changes (04-May-2012)

Roll estimate of bid-ask spread

In [14]:

```
(c <- sqrt(-ac$acf[2])*sd(dp))
```

0.00295272617733834

5 minutes of BAC trades

In [15]:

```
# 5 minutes of BAC trades
min5 <- '2012-05-04 11:50:00::2012-05-04 11:55:00'
plot(tqdata$BID[min5],ylim=c(7.7,7.8),main=NA)
lines(tqdata$OFR[min5],col="red")
points(tqdata$PRICE[min5][tqdata$EX[min5]=="D"],col="red",pch=20)
points(tqdata$PRICE[min5][tqdata$EX[min5]=="B"],col="blue",pch=20)
points(tqdata$PRICE[min5][tqdata$EX[min5]=="Z"],col="orange",pch=20)
points(tqdata$PRICE[min5][tqdata$EX[min5]=="Y"],col="purple",pch=20)
points(tqdata$PRICE[min5][tqdata$EX[min5]=="P"],col="dark green",pch=20)
legend("topright", c("D - FINRA           ",
                     "B - NASDAQ OMX BX",
                     "Z - BATS Exchange",
                     "Y - BATS Y-Exchange",
                     "P - NYSE Arca SM"), text.width = strwidth("Y - BATS Y-Exchange
       col=c("red","blue","orange","purple", "dark green"), pch=20, inset=0.05)
```

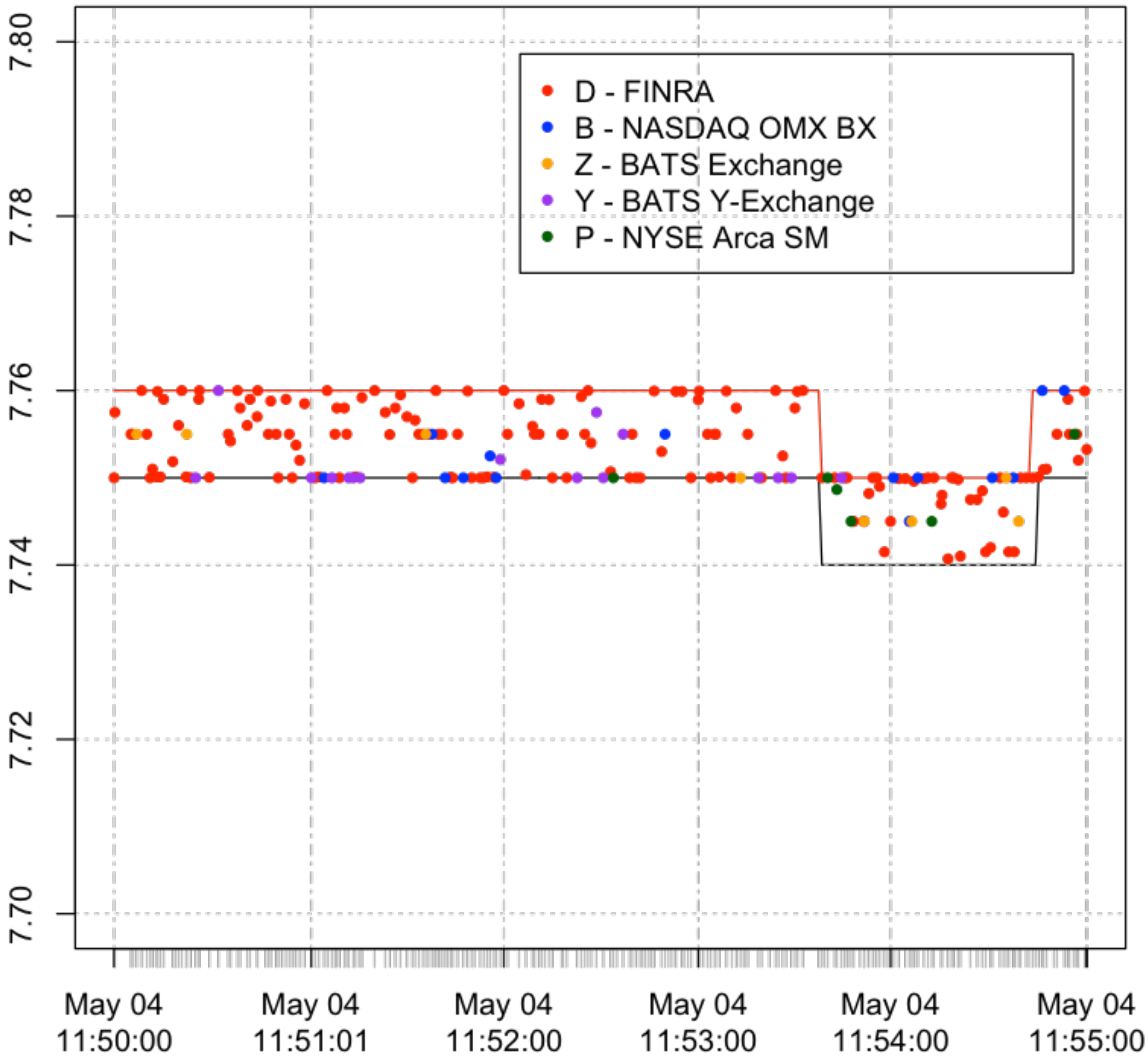


Figure 2: BAC trades (04-May-2012)

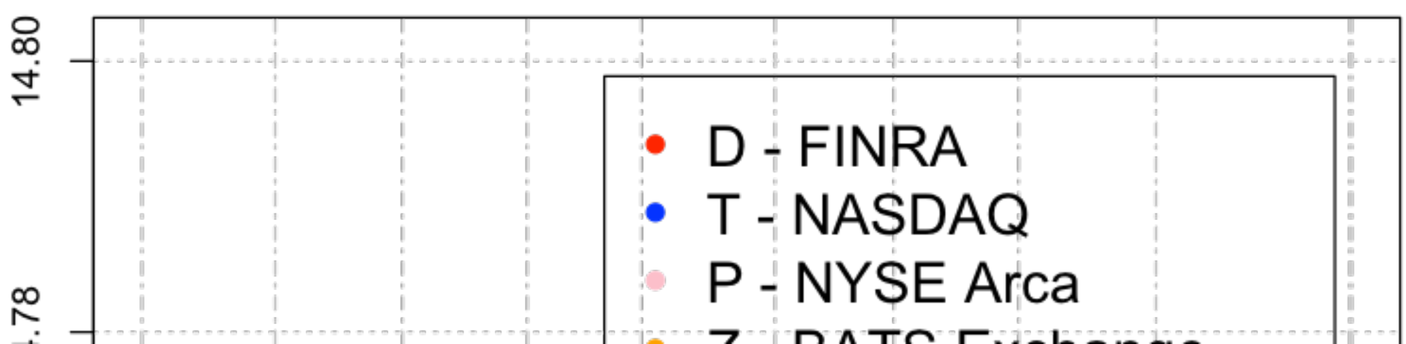
10 minutes of BAC trades from a later date

In [16]:

```
load("tqBAC_20130731.rData")

# The data is a list of xts objects by exchange.  Consolidate into one object
tqdata <- NULL
for(i in 1:8){tqdata <- xts(rbind(tqdata,tqBAC[[i]]))}
```

In [17]:

```
#---------------------------------------------------------------------------
# Figure 3: 10 minutes of BAC trades
min10 <- '2013-07-31 11:15:00::2013-07-31 11:25:00'

plot(tqdata$BID[min10],ylim=c(14.72,14.80),main=NA)
lines(tqdata$OFR[min10],col="red")
points(as.numeric(tqdata$PRICE[min10][tqdata$EX[min10]=="D"]),col="red",cex=1.5,pch=
points(tqdata$PRICE[min10][tqdata$EX[min10]=="T"],col="blue",cex=1.5,pch=20)
points(tqdata$PRICE[min10][tqdata$EX[min10]=="P"],col="pink",cex=1.5,pch=20)
points(tqdata$PRICE[min10][tqdata$EX[min10]=="Z"],col="orange",cex=1.5,pch=20)
points(tqdata$PRICE[min10][tqdata$EX[min10]=="Y"],col="purple",cex=1.5,pch=20)
points(tqdata$PRICE[min10][tqdata$EX[min10]=="K"],col="dark green",cex=1.5,pch=20)
points(tqdata$PRICE[min10][tqdata$EX[min10]=="B"],col="green",cex=1.5,pch=20)
points(tqdata$PRICE[min10][tqdata$EX[min10]=="N"],col="brown",cex=1.5,pch=20)

legend("topright", c("D - FINRA           ",
                     "T - NASDAQ",
                     "P - NYSE Arca",
                     "Z - BATS Exchange",
                     "Y - BATS Y-Exchange",
                     "K - DirectEdge X",
                     "B - NASDAQ BX",
                     "N - NYSE"
                     ), text.width = strwidth("Y - BATS Y-Exchange Y 1,000,000"),
       col=c("red","blue","pink","orange","purple", "dark green","green","brown"), c
```



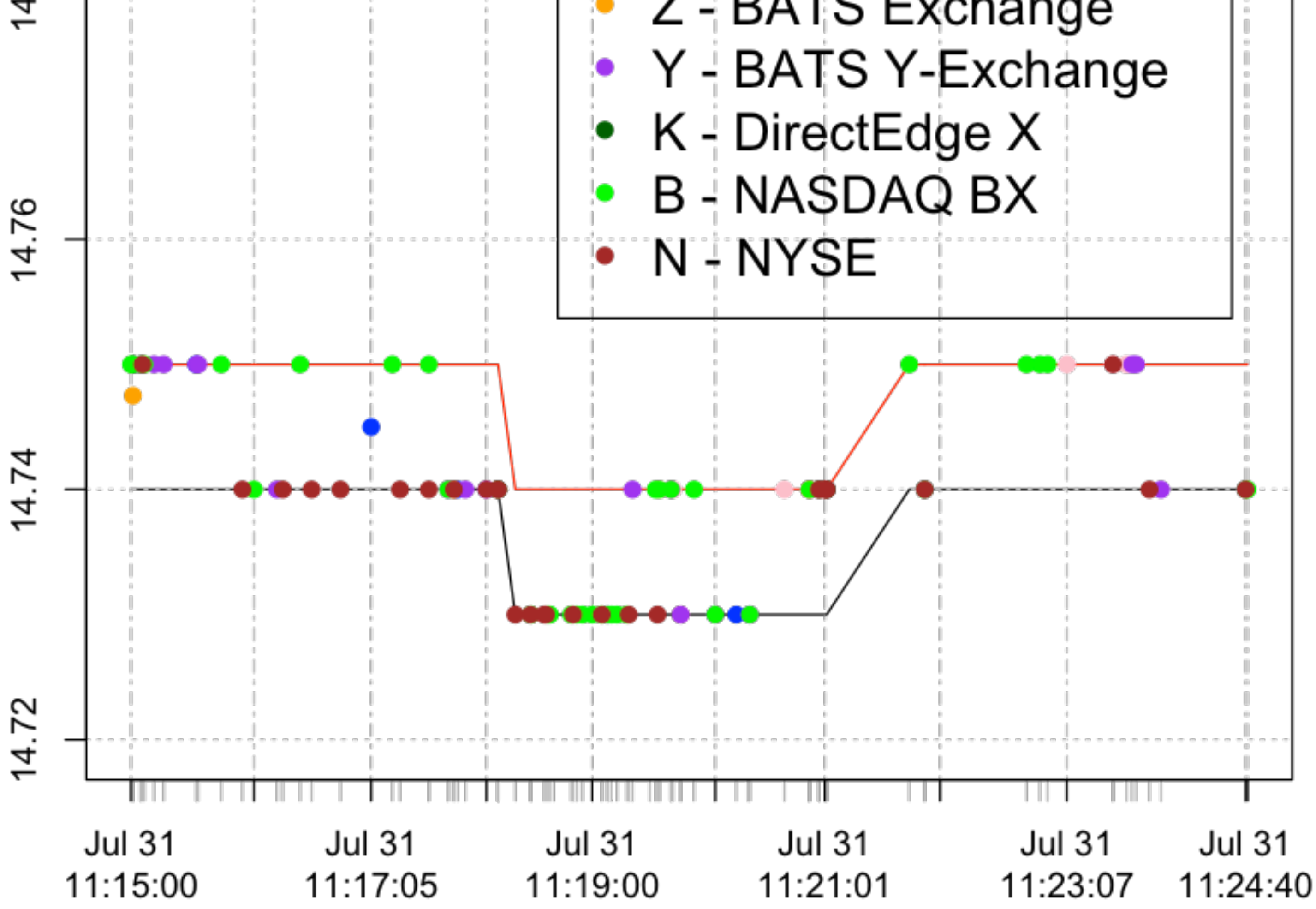


Figure 3: BAC trades (31-Jul-2013)

Signing trades: The Lee and Ready tick test

Lee and Ready devised a simple method to infer trade direction in cases when quote data (and so mid-price) is not available:

- If a trade is on an uptick, it is a buy.
- If the trade is on a downtick, it is a sell.
- If the trade is on a zero-uptick, it is a buy.
- If the trade is on a zero-downtick, it is a sell.

It can be verified that this algorithm classifies most trades in the same way as the classification based on comparison with the mid-price. However, note that according to Bandi and Russell “the Lee and Ready method, for example, is known to categorize incorrectly about 15% of the trades”.

Bessembinder recommendation

- If quotes are available, Lee and Ready (1991) recommend comparing current trade prices with quotes lagged by around 5 seconds.
- Ellis, Michaely, and O’Hara (EMO) assign trades executed at the ask (bid) quote as buys (sells), while using the tick test for all other trades.

However, [Bessembinder][2] (in a 2003 paper) recommends as follows:

1. Use the EMO technique in preference to the LR method to sign trades
2. Implement the EMO technique on the basis of contemporaneous rather than earlier quotations
3. Use quotation midpoints in effect somewhat prior to the trade report time as the benchmark quote when measuring effective bid-ask spreads.

TAQ data cleaning

Data cleaning is critical. Here follows a recipe for TAQ cleaning due to [Barndorff-Nielsen et al.][1] and implemented in the R-package highfrequency:

All data

- Delete entries with a timestamp outside the 9:30am–4pm window when the exchange is open.
- Delete entries with a bid, ask or transaction price equal to zero.
- Retain entries originating from a single exchange (NYSE in our application). Delete other entries.

Quote data only

- When multiple quotes have the same time stamp, we replace all these with a single entry with the median bid and median ask price.
- Delete entries for which the spread is negative.
- Delete entries for which the spread is more that 50 times the median spread on that day.
- Delete entries for which the mid-quote deviated by more than 10 mean absolute deviations from a rolling centered median (excluding the observation under consideration) of 50 observations (25 observations before and 25 after).

Trade data only

- Delete entries with corrected trades. (Trades with a Correction Indicator, CORR $\neq$ 0).
- Delete entries with abnormal Sale Condition. (Trades where COND has a letter code, except for E and F). See the TAQ 3 Users Guide for additional details about sale conditions.
- If multiple transactions have the same time stamp, use the median price.
- Delete entries with prices that are above the ask plus the bid–ask spread. Similar for entries with prices below the bid minus the bid–ask spread.

Autocorrelation of BAC trade signs

In [20]:

```
load("tq.rData")
tqdata <- tqBAC

# Compute trade signs
tradeSigns <- getTradeDirection(tqdata)
```

In [21]:

```
# Autocorrelation of trade signs
acTS <- acf(tradeSigns,main=NA,plot=FALSE)
plot(acTS,main=NA,col="red")
```

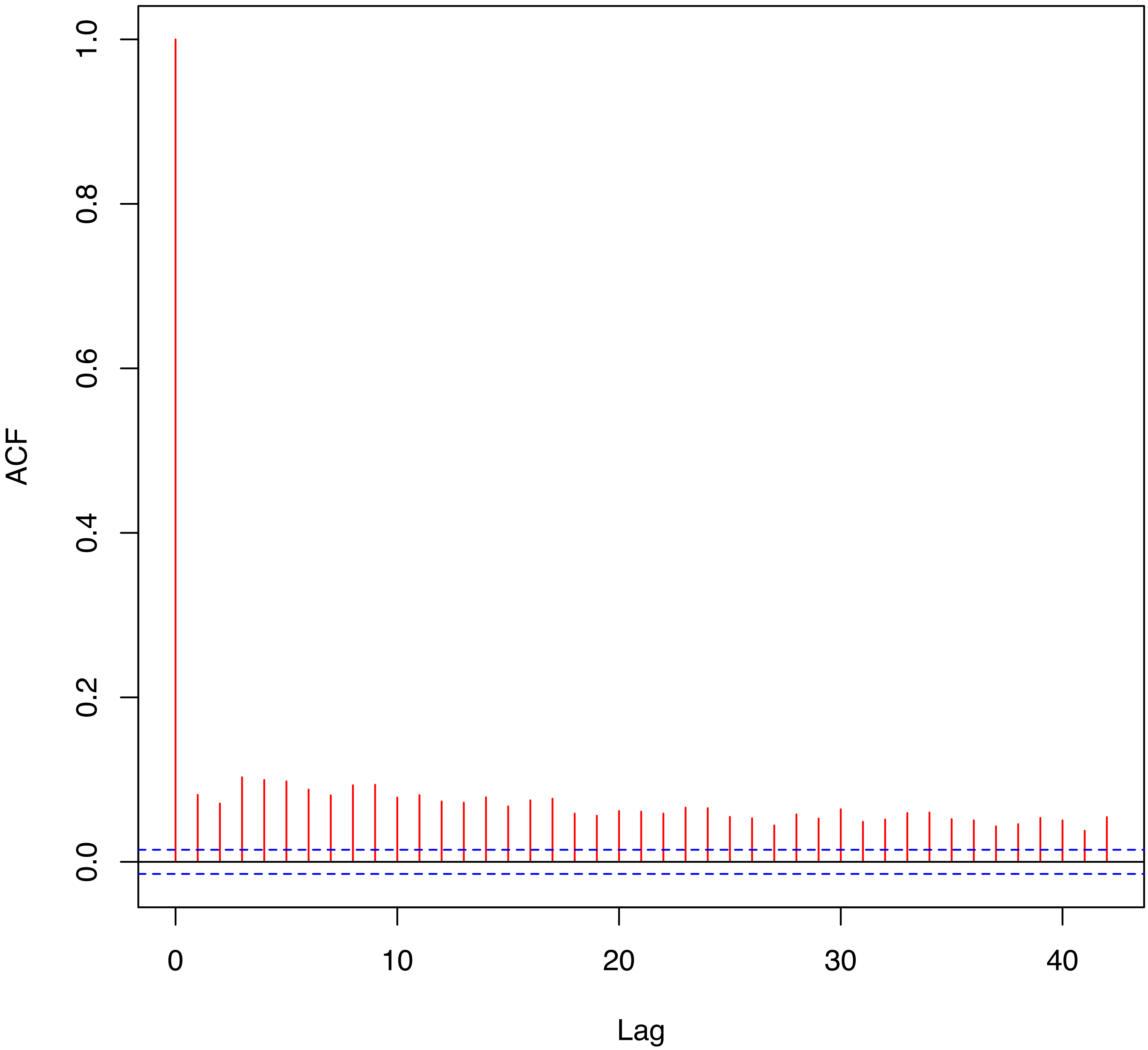


Figure 4: Autocorrelation of trade signs (04-May-2012)

Estimate of Roll model using trade signs

- Recall the Roll model:
$$p_t = m_t + c\,\epsilon_t.$$

- Recall that the Roll half-spread estimate just from trade data was 0.295 cents.

- Now that we have trade signs, we may use these to estimate the half-spread from the regression
$$\Delta p_t = \Delta m_t + c\,(\epsilon_t - \epsilon_{t-1})$$

In [22]:

```
deps <- diff(tradeSigns)
mids <- (as.numeric(tqBAC$OFR) + as.numeric(tqBAC$BID))/2
dm <- diff(mids)
fit.lm <- lm(dp ~ dm + deps)
fit.lm$coeff[3]
```

deps: 0.00264198285675995

In [23]:

```
summary(fit.lm)
```

```
Call:
lm(formula = dp ~ dm + deps)

Residuals:
       Min         1Q     Median         3Q        Max 
-0.0247128 -0.0009968  0.0000032  0.0010032  0.0208947 

Coefficients:
              Estimate Std. Error t value Pr(>|t|)    
(Intercept) -3.213e-06  1.832e-05  -0.175    0.861    
dm           6.175e-01  1.800e-02  34.299   <2e-16 ***
deps         2.642e-03  1.358e-05 194.535   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.002448 on 17854 degrees of freedom
Multiple R-squared:  0.6817,	Adjusted R-squared:  0.6816 
F-statistic: 1.912e+04 on 2 and 17854 DF,  p-value: < 2.2e-16
```

- The resulting estimate of the half-spread in this case is 0.264 cents.

- Only the first order autocorrelation coefficient of price changes is significant, consistent with the Roll model.

- Trade signs are very autocorrelated with all coefficients significant.

A generalization of the Roll model

- As before, we model trade prices p_t as
$$p_t = m_t + c\,\epsilon_t$$
but we allow the evolution of the efficient price to depend on the trade sign:

(2)
$$\Delta m_t = m_t - m_{t-1} = \lambda\,\epsilon_t + u_t$$

with $\mathrm{Cov}[\epsilon_t, u_t] = 0$.
  - This could reflect market impact and/or asymmetric information.

- Autocovariances are:

$$\begin{aligned}\gamma_0 &= \mathbb{E}[\Delta p^2] = c^2 + (c+\lambda)^2 + \sigma_u^2 \\ \gamma_1 &= \mathbb{E}[\Delta p_t\,\Delta p_{t-1}] = -c\,(c+\lambda)\end{aligned}$$

- All higher order autocovariances are zero.

Estimation of parameters

- The variance of the efficient price is given by
$$\mathrm{Var}[\Delta m_t] = \lambda^2 + \sigma_u^2 = \gamma_0 + 2\,\gamma_1$$

- Thus γ_0 and γ_1 are not sufficient to identify c and λ separately.

- With synchronized quotes or alternatively trade sign data, we can identify λ and c separately by regression. Specifically,

$$\Delta p_t = \lambda\,\epsilon_t + c\,(\epsilon_t - \epsilon_{t-1}) + u_t.$$

  - With our BAC data, we get $\lambda = 0.054$ cents and $c = 0.234$ cents.

Estimation of Generalized Roll model

In [24]:

```
(fit.lmG <- lm(dp ~ tradeSigns[-1] + deps))
```

```
Call:
lm(formula = dp ~ tradeSigns[-1] + deps)

Coefficients:
   (Intercept)  tradeSigns[-1]            deps
    -4.025e-05       5.402e-04       2.337e-03
```

Empirical notions of spread

- **The effective spread**

$$s^E = \langle 2\,\epsilon_t\,(p_t - m_t)\rangle$$

- **The realized spread**

$$s^R = \langle 2\,\epsilon_t\,(p_t - m_{t+\tau})\rangle$$

  for some $\tau > 0$ (5 minutes say).

- In the Roll model, the two notions of spread coincide but if trading affects the mid-price, the two notions will not in general agree.
  - Note that we don't need to believe in asymmetric information to believe in market impact. Recall that there is market impact in the zero-intelligence model.

- In general, empirically, the realized spread is smaller than the effective spread.
  - With our BAC data, the effective spread is 0.283 cents and the realized spread after 500 trades is 0.146 cents.

In [13]:

```
# Effective spread
(es <- mean(tradeSigns*(p-mids))) # This is the effective half-spread
```

Out[13]:

```
[1] 0.002826028
```

In [14]:

```
# Realized spread
n <- length(p)
mids <- (as.numeric(tqdata$OFR) + as.numeric(tqdata$BID))/2
rs <- function(lag){mean(tradeSigns[1:(n-lag)]*(p[1:(n-lag)]-mids[-(1:lag)]))}
(rs(500))
```

Out[14]:

```
[1] 0.001464515
```

BAC realized spread over time

In [15]:

```
rsLag <- sapply(1:500,rs)
plot(1:500,rsLag,col="red",type="l",xlab=expression(tau), ylab = "Realized spread")
```

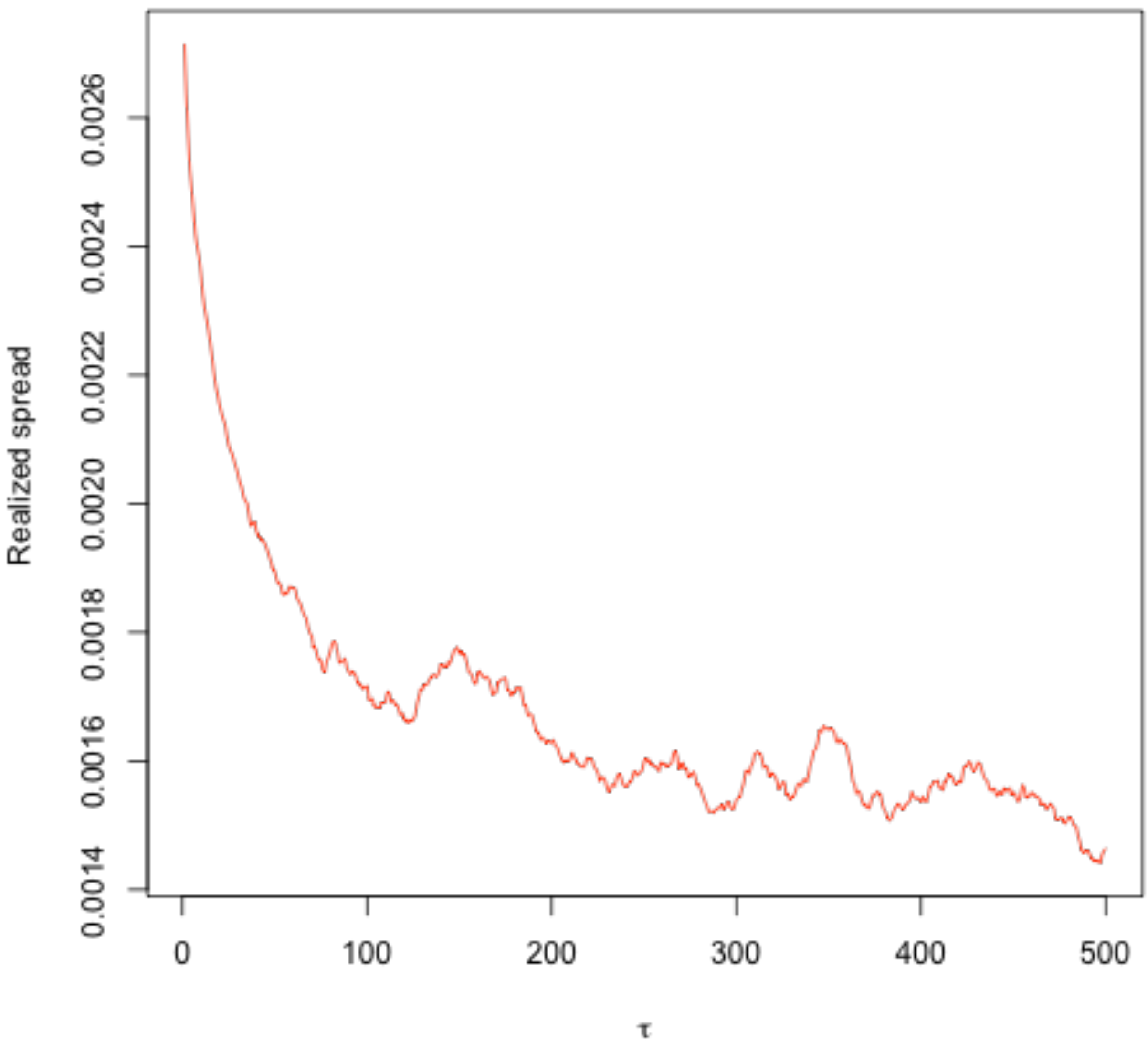


Figure 5: Realized BAC spread vs number of trades after execution (04-May-2012)

BAC realized spread by reporting exchange

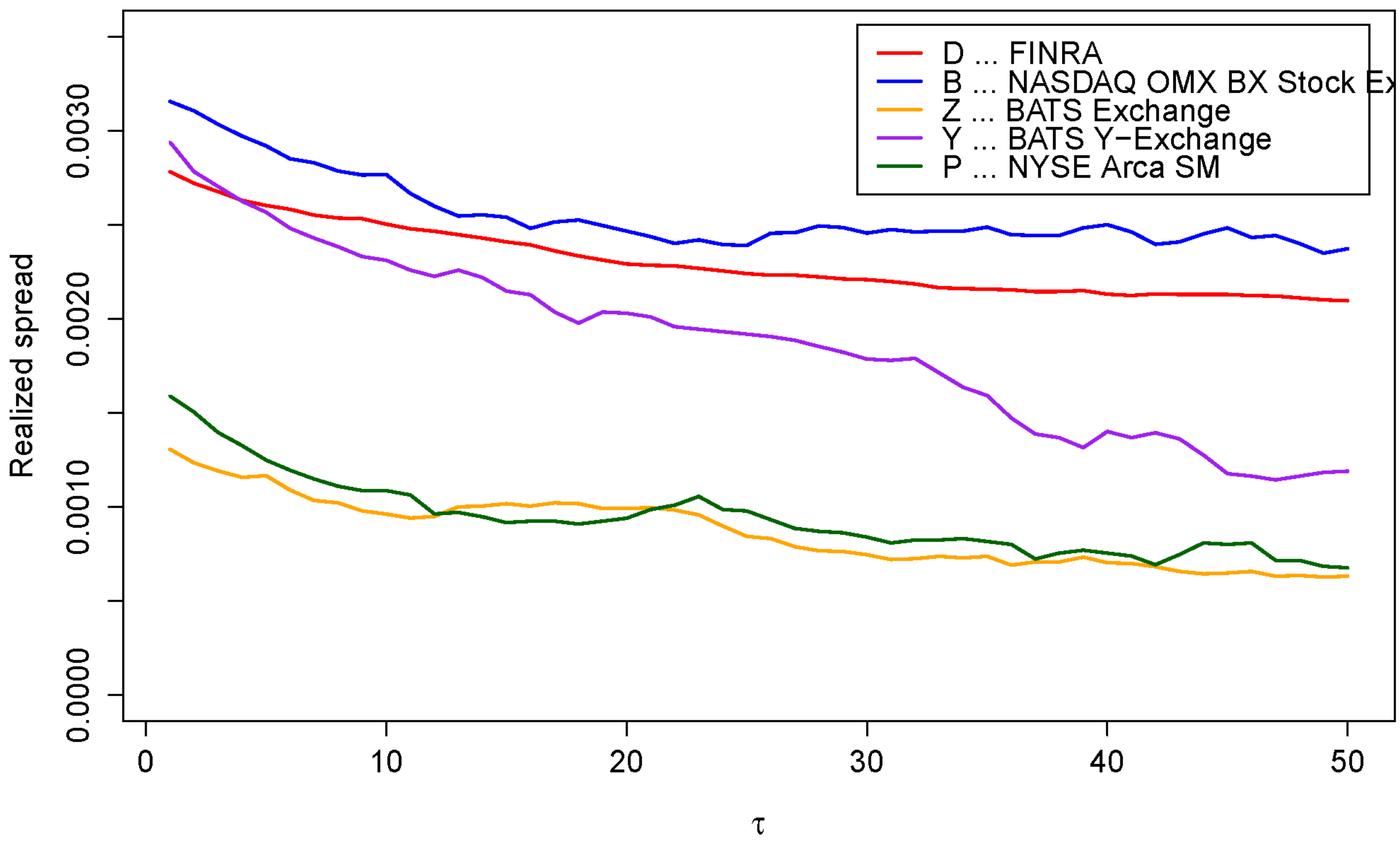


Figure 6: Different exchanges have different realized spread profiles depending on fee structure.

Determinants of the bid-ask spread

Recall that there are three main determinants of the bid-ask spread:

- Processing costs (which include the profit of the market maker)
- Inventory costs such as cost of risk capital
- Adverse selection

In the Roll model, only processing costs are taken into account. In the generalized Roll model, adverse selection is also taken into account.

Models with asymmetric information

There are two main classes of model with asymmetric information:

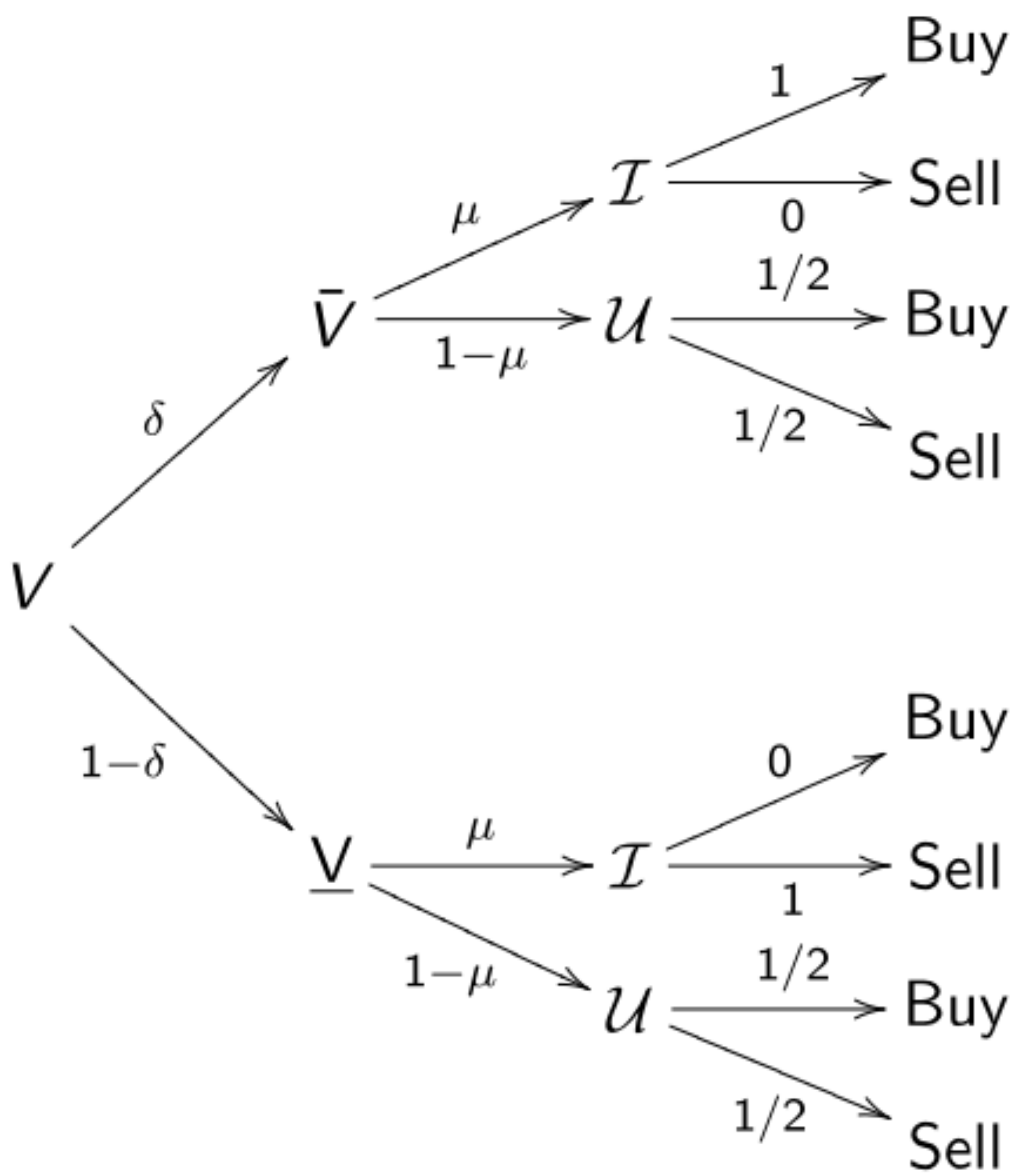
- Strategic trader models where typically a single informed trader optimizes his use of information.
  - A good example is the Kyle model which we will cover in the next lecture.
- Sequential trader models where randomly selected traders arrive one after the other, independently.
  - The prototypical example is Glosten and Milgrom (1985).
- In sequential trader models, the spread set by the market maker compensates for adverse selection.

A sequential trade model (Glosten and Milgrom)

- In the Glosten and Milgrom model, the market maker $\mathcal{M}$ learns the informed trader $\mathcal{I}$'s information by observing the order flow.
  - If there are more buys than sells over time, $\mathcal{M}$ sets the price higher.
- $\mathcal{M}$ posts bid and ask prices B and A.
- The value of the security is either high ($\bar{V}$) with probability δ or low ($\underline{V}$).
- A trader is drawn at random: either informed $\mathcal{I}$ with probability μ or uninformed $\mathcal{U}$.

Sequential trade model event tree

(3)



Unconditional probabilities

The unconditional probability of a buy is given by

$$\Pr[Buy] = \frac{1}{2}(1-\mu) + \delta\,\mu = \frac{1}{2}\,\{1 - \mu\,(1 - 2\,\delta)\}\,.$$

Similarly, the unconditional probability of a sell is given by

$$\Pr[Sell] = \frac{1}{2}(1-\mu) + (1-\delta)\,\mu = \frac{1}{2}\,\{1 + \mu\,(1 - 2\,\delta)\}\,.$$

If $\delta = \frac{1}{2}$, $\Pr[Buy] = \Pr[Sell] = \frac{1}{2}$.

Optimal bid price

Under *perfect competition*, the optimal ask price is set as

$$\begin{aligned} A &= \mathbb{E}[V|Buy] \\ &= \bar{V}\ \Pr(\bar{V}|Buy) + \underline{V}\ \Pr(\underline{V}|Buy) \\ &= \frac{\bar{V}\ \Pr(\bar{V}\cap Buy) + \underline{V}\ \Pr(\underline{V}\cap Buy)}{\Pr(Buy)} \end{aligned}$$

By inspection of the Glosten and Milgrom tree (3), we have

$$\begin{aligned} \Pr(\bar{V}\cap Buy) &= \delta\left\{\mu + \frac{1}{2}(1-\mu)\right\} = \frac{1}{2}\delta(1+\mu) \\ \Pr(\underline{V}\cap Buy) &= (1-\delta)\left\{0 + \frac{1}{2}(1-\mu)\right\} = \frac{1}{2}(1-\delta)(1-\mu) \end{aligned}$$

and we already computed that $\Pr(Buy) = \frac{1}{2}\,\{1 - \mu\,(1 - 2\,\delta)\}$. So

$$A = \mathbb{E}[V|Buy] = \frac{\underline{V}\,(1-\delta)\,(1-\mu) + \bar{V}\,\delta\,(1+\mu)}{1 - \mu\,(1 - 2\,\delta)}.$$

The spread

Similarly,

$$B = \mathbb{E}[V|Sell] = \frac{\underline{V}\,(1-\delta)\,(1+\mu) + \bar{V}\,\delta\,(1-\mu)}{1 + \mu\,(1 - 2\,\delta)}$$

so the spread is

$$s = A - B = \frac{4\,(\bar{V} - \underline{V})\,\mu\,\delta\,(1-\delta)}{1 - \mu^2\,(1 - 2\,\delta)^2}.$$

When $\delta = 1/2$,

$$s = \mu\,(\bar{V} - \underline{V})$$

and in this case, the spread is proportional to the probability of informed trading.

Observations

- The mid-price equals the fair price $\mathbb{E}[V] = \delta\,\underline{V} + (1-\delta)\,\bar{V}$ only when $\delta = 1/2$.
  - The bid and ask prices are not set symmetrically around the efficient price.
- The maximum spread is at $\delta = 1/2$ when uncertainty on V is maximized. The closer δ is to 1 or 0, the smaller the spread.
- As the probability of informed trading $\mu \to 1$, $B \to \underline{V}$ and $A \to \bar{V}$.
- The more informed trading is, the wider the spread. In the limit where for example insider trading is allowed, we may see $\mu = 1$ and subsequently market failure with no dealer willing to post quotes.

Model dynamics

- The dealer now updates his belief about the efficient price by taking the sign of the previous trade into account.

Let $\delta_k = \Pr(\bar{V}|\delta_{k-1}, \epsilon_{k-1})$ where ϵ_i denotes the sign of the ith trade.

Then if $\epsilon_{k-1} = +1$,

$$\delta_k = \frac{\Pr(\bar{V} \cap (\epsilon_{k-1} = +1)|\delta_{k-1})}{\Pr(\epsilon_{k-1} = +1|\delta_{k-1})} = \frac{(1+\mu)\,\delta_{k-1}}{1 - \mu\,(1 - 2\,\delta_{k-1})}.$$

and if $\epsilon_{k-1} = -1$,

$$\delta_k = \frac{(1-\mu)\,\delta_{k-1}}{1 + \mu\,(1 - 2\,\delta_{k-1})}.$$

This may be summarized as:

(4)

$$\delta_k = \frac{(1 + \mu\,\epsilon_{k-1})\,\delta_{k-1}}{1 - \mu\,\epsilon_{k-1}\,(1 - 2\,\delta_{k-1})}.$$

Thus, δ_k may be computed recursively from the time series $\{\epsilon_i : i < k\} =: \mathcal{F}_k$ and δ_0.

Dynamical properties of the model

- The trade price series is a martingale.

- Both bid and ask prices are expectations conditioned on an expanding information set (the time series of trade signs):

$$\begin{aligned} B_k &= \mathbb{E}[V|\mathcal{F}_k, \epsilon_k = -1] \\ A_k &= \mathbb{E}[V|\mathcal{F}_k, \epsilon_k = +1] \end{aligned}$$

- Order signs are predictable: $\mathbb{E}[\epsilon_k|\mathcal{F}_k] \neq 0$ in general.

- The spread declines over time. Trading reveals the efficient price.

- Orders are serially correlated because informed traders always trade in the same direction.

- There is market impact in this model. A buy causes both the bid and the offer to increase.

Further observations

- If processing costs are c, the bid decreases by c and the ask increases by c.
  - Bid and ask prices are still martingales but the time series of trade prices is no longer a martingale.

- From the Roll model perspective, information asymmetry has broken the independence between the trade signs ϵ_i and the innovation to the efficient price.

- It is implicit that there is no risk aversion and unlimited capital in the Glosten and Milgrom model so there is no inventory component to the bid and ask prices.

Market efficiency

According to the classical definitions, prices are

- *strong-form efficient* if they reflect all private information
- *semi-strong-form efficient* if they reflect all public information
- *weak-form efficient* if they reflect all information in their past values.

In the Glosten and Milgrom model, prices converge to the true value eventually becoming strong-form efficient. However at any given time, as a martingale reflecting all information available to the market maker, the price is only semi-strong-form efficient,

- Note that prices can depart substantially from the true value, depending on order flow. So it's not clear how interesting it is to know that the market price is semi-strong-form efficient.

The Foucault (1999) model

- Trading starts at time $t = 1$ and ends at $t = T$ where T is a random stopping time.
  - At each time t, the probability that trading will terminate is $1 - \rho$.

- If trading terminates, the payoff is

$$V_T = v_0 + \sum_{t=1}^{T} \epsilon_t$$

where ϵ_t is an iid sequence of random variables with $\epsilon_t = \pm\sigma$ with probability $1/2$.

- At each t, a trader arrives with a private valuation (or *reservation*) price $R_t = v_t + y_t$ with $y_t = \pm L$ (plus sign biases towards buying).
    - This trader may either submit a market order or quote a two-way price.

- Limit orders expire after one period so at any time t either the book is empty or it has one share on each side posted by the trader arriving at $t - 1$.

Bid and ask prices under perfect competition

Assuming perfect competition, the trader will set bid and offer prices so that he has zero utility at time $t + 1$:

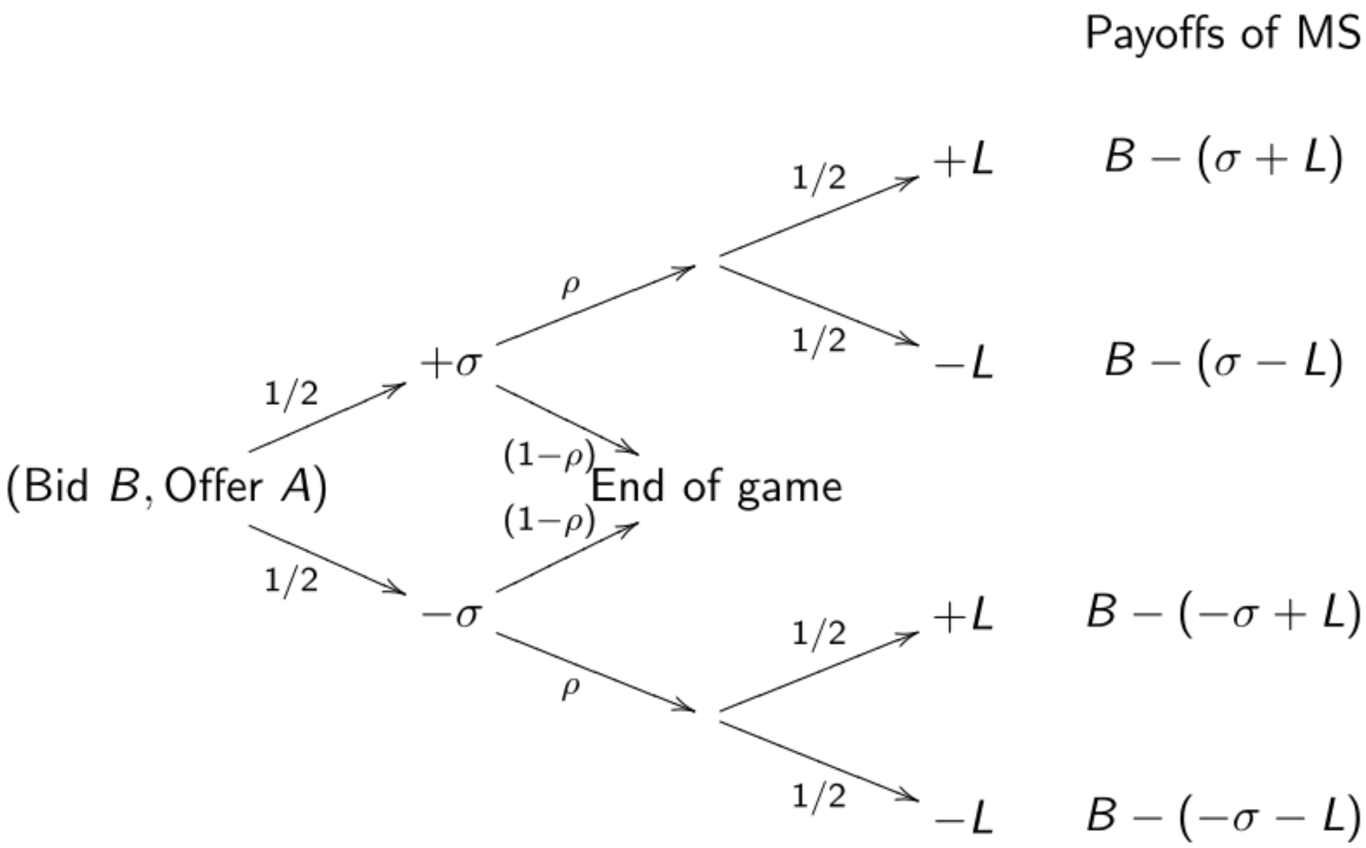
$$\begin{aligned} \mathbb{E}[V_T + y_t|\text{Bid hit}] - B &= 0 \\ A - \mathbb{E}[V_T + y_t|\text{Offer lifted}] &= 0 \end{aligned}$$

and $\mathbb{E}[V_T|v_{t+1}] = v_{t+1}$ so

$$\begin{aligned} B &= R_t + \mathbb{E}[\Delta v_{t+1}|\text{Bid hit}] \\ A &= R_t + \mathbb{E}[\Delta v_{t+1}|\text{Offer lifted}] \end{aligned}$$

The time t trader knows the statistics of the process v_t and can compute the probabilities of execution at B and A.

A generic timestep in the Foucault model



Computing the optimal bid price B

Consider the bid price B:

- As B increases, the probability of execution increases because MS payoffs increase.
- As B increases, the market maker's payoff decreases.

At some point, the expected market maker payoff will be zero and that will be the equilibrium level of B.

To proceed, we assume that $\sigma > 3\,L/2$.

- It is nearly always the case in practice that the signal is smaller than the noise!
- Even in the ZI market studied in HW1, the standard deviation of the final price after 100 events was much greater than the mean price.

Computing the equilibrium bid price B

We can use the tree diagram to compute the conditional expectations of Δv. Wlog, we set $v_t = 0$ and consider the case of a trader with private valuation $+L$. We denote the probability of the trader's bid being hit conditional on B by $\mathbb{P}(\text{hit}|B)$.

Bid price range	$\mathbb{P}$**(hit**$|B$**)**	$\mathbb{E}$**[**Δv_{t+1} **|Bid hit]**	**Breakeven** B
$> L + \sigma$	ρ	0	L
$\in (\sigma - L, L + \sigma]$	$\frac{3}{4}\rho$	$-\sigma/3$	$L - \sigma/3$

$\in (L-\sigma, \sigma-L]$	$\frac{1}{2}\rho$	$-\sigma$	$L-\sigma$
$\leq L-\sigma$	$\frac{1}{4}\rho$	$-\sigma$	$L-\sigma$

Only on the fourth line do we see a match between the bid price range in the first column and the breakeven bid price (from perfect competition) in the last column. The equilibrium bid price for a trader with private valuation $+L$ is thus $+L-\sigma$.

Summary and conclusions

- The equilibrium solution of the Foucault (1999) model (assuming $\sigma > 3\,L/2$) has the form
$$\{B, A\} = \pm L + \{-\sigma, +\sigma\}$$
so the spread $A-B$ is always $2\,\sigma$.
  - $s/2 = \sigma$: "The half-spread equals volatility per trade".
- Bids and offers are biased according to the private valuation of the market maker.

- The bid-offer spread compensates for adverse selection.
  - Our bid is hit when the price is falling - against us!
  - Our bid is not hit when the price is rising - against us again!
  - Higher volatility means greater adverse selection which is compensated by a wider spread.

Wyart et. al.

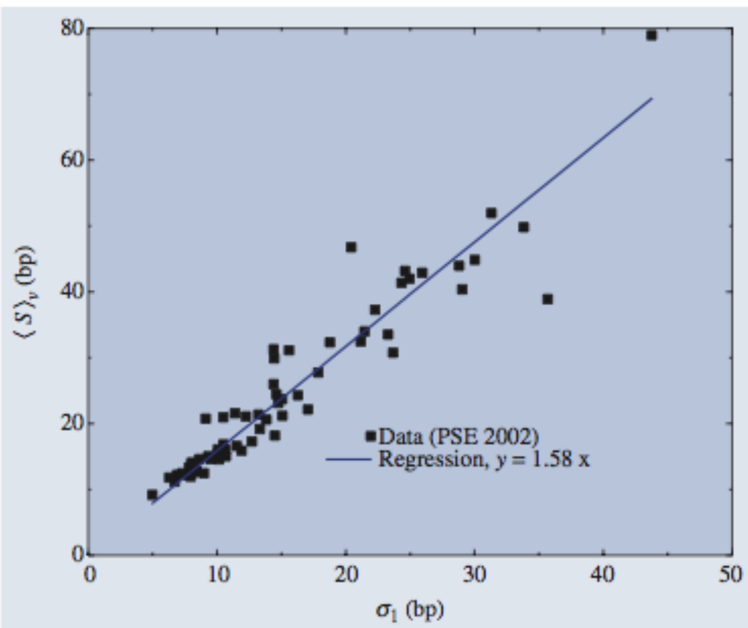


Figure 10. Test of equation (32) for 68 stocks from the Paris Stock Exchange in 2002, averaged over the entire year. The value of the linear regression slope is $c \approx 1.58$, with $R^2 = 0.96$.

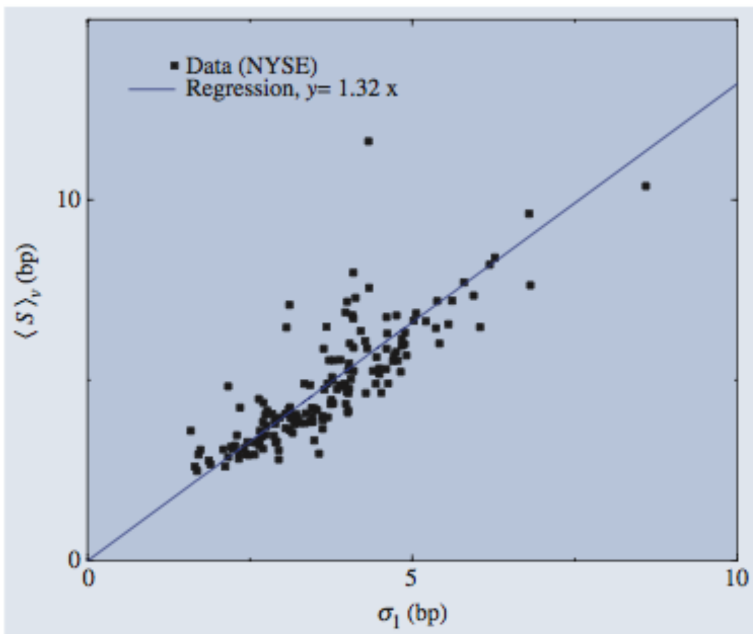


Figure 12. Test of equation (32) for stocks from the NYSE in 2005. Each point corresponds to a pair $(\langle S \rangle, \sigma_1)$, computed by averaging over the entire year. Both quantities are expressed in basis points. From a linear fit, we find $c \approx 1.32$, with $R^2 = 0.91$.

Figure 7: Graphs showing that spread is proportional to volatility in real markets

Wyart et. al.

[Wyart et al.][8] find that

- $s \approx 1.58\sigma$ for the Paris Stock Exchange in 2002.
- $s \approx 1.32\sigma$ for the NYSE in 2005.

[Wyart et al.][8], show that the following assumptions imply proportionality of volatility per trade and spread:

1. Diffusion of market prices (*i.e.* $\langle (S_\tau - S_0)^2 \rangle \propto \tau$).
2. The costs of market orders and limit orders are roughly equal.

Strategic uninformed trading

- It is possible for an uninformed trader to push up the price by buying.
  - Is market manipulation possible in this model?
  - It certainly would be better for the uninformed trader to trade strategically.

- For everyone except the actual buyer, the expectation of the efficient price increases after a trade.
  - However, the expectation of the trader himself cannot change. He has no more information after the trade than he had before.

- Sequential trader models are thus fundamentally unrealistic as uninformed traders never learn and cannot behave strategically.

Further extensions of the Roll model

- Sequential trader models suggest how private information may get impounded into the price.
  - The spread must include a component that compensates for adverse selection.

- We now proceed to further extend the Roll model to investigate empirically the components of the bid-ask spread.

The Glosten and Harris model

- Order processing costs and adverse selection costs are allowed to depend on trade size.

- A simplified version of their dynamics is:

(5)

$$p_t = m_t + c\,\epsilon_t + \lambda\,x_t$$

(6)

$$m_t = m_{t-1} + \lambda\, x_{t-1} + u_t$$

where x_t is (signed) trade size and λ may be identified with the Kyle lambda.

- The half-spread in this model is now given by

$$\frac{s_t}{2} = c + \lambda\, |x_t|$$

We now estimate Glosten and Harris equation using (5) and (6)

In [16]:

```
x <- as.numeric(tqBAC$SIZE)*tradeSigns # Signed volume
n <- length(x)

(fit.gh6 <- lm(dm ~ x[-n]))
lambda <- fit.gh6$coeff[2] # lambda is very tiny in our data

lhs <- p - mids - lambda*x
(fit.gh5 <- lm(lhs ~ tradeSigns)) # We get c = 0.271 cents
```

Out[16]:

```
Call:
lm(formula = dm ~ x[-n])

Coefficients:
(Intercept)        x[-n]
  1.959e-06    9.119e-09
```

Out[16]:

```
Call:
lm(formula = lhs ~ tradeSigns)

Coefficients:
(Intercept)   tradeSigns
  0.0004257    0.0027092
```

This gives $c = 0.27$ cents and λ very small.

The Glosten and Harris model

- Note that p_t may be reexpressed as

(7)

$$p_t = \lambda \sum_{i \leq t} x_i + c\, \epsilon_t + \sum_{i \leq t} u_i.$$

Thus λ may be estimated by regressing the daily price change against net order flow.

- The first term may be identified with permanent impact and the second with temporary impact.

We now estimate Glosten and Harris equation using (7)

In [17]:

```
x <- as.numeric(tqBAC$SIZE)*tradeSigns # Signed volume
cumx <- cumsum(x)
(fit.gh7 <- lm(p ~ cumx + tradeSigns)) # We get c = 0.209 cents
lambda7 <- fit.gh7$coeff[2] # lambda7 is 0.000169 cents per 1,000 shares
c <- fit.gh7$coeff[3]
c(c,lambda7)
```

Out[17]:

```
Call:
lm(formula = p ~ cumx + tradeSigns)

Coefficients:
(Intercept)         cumx   tradeSigns
  7.798e+00    1.690e-09    2.087e-03
```

Out[17]:

```
  tradeSigns         cumx
2.086820e-03 1.690072e-09
```

This gives $c = 0.21$ cents and λ very small.

- Glosten and Harris estimated their model on NYSE stock data from 01-Dec-1981 to 31-Jan-1983 finding:

$$\begin{aligned} c &= 4.44 \text{ cents} \\ \lambda &= 1.13 \text{ cents per 1,000 shares} \end{aligned}$$

- From our BAC data, $c = 0.271$ cents, $\lambda = 0.0009$ cents per $1,000$ shares.

The ILLIQ measure (using daily data)

- The net order flow $\sum_i x_i$ is typically not easily available.
- This led Amihud to propose an alternative proxy for transactions costs:

$$\text{ILLIQ} = \left\langle \frac{|\Delta P_t|}{V_t} \right\rangle$$

  where V_t is the absolute volume traded on day t. $\langle \cdot \rangle$ = sample mean.
- It turns out that ILLIQ has a higher rank correlation with effective spread than the Roll estimator.

The Madhavan Richardson and Roomans (MRR) model

The Madhavan Richardson and Roomans (MRR) model

- In the Glosten and Milgrom model, the price depends on order flow.
- In the MRR model, as in Glosten-Milgrom, the revision in beliefs is positively correlated with the innovation in the order flow:

  (8)

  $$\Delta V_t = \lambda\,(\epsilon_t - \mathbb{E}[\epsilon_t|\mathcal{F}_{t-1}]) + e_t$$

  where V_t is the efficient price and e_t represents for example news.

- As in Glosten and Milgrom, the market maker posts bid and ask prices that are conditioned on the trade being a buy or a sell.
- This leads to the empirical model

  (9)

  $$p_t = V_t + \phi\,\epsilon_t + \xi_t$$

  where ξ_t is iid noise that could represent rounding for example.

Autocorrelation of order flow

- From (8) and (9),

  (10)

  $$p_t = V_{t-1} + \lambda\,(\epsilon_t - \mathbb{E}[\epsilon_t|\mathcal{F}_{t-1}]) + \phi\,\epsilon_t + e_t + \xi_t$$

- Let γ be the probability of a continuation (in trade sign)
  $$\gamma = \Pr[\epsilon_t = \epsilon_{t-1}].$$
- Because of trade-splitting, continuations are more likely than reversals so $\gamma > 1/2$.
- The first order autocorrelation ρ_1 of trade signs is then $2\,\gamma - 1$.

Expected trade sign

- Now model the order sign process as AR(1) so that
  $$\epsilon_t = \rho\,\epsilon_{t-1} + \eta_t$$
  where η_t is independent noise. Then $\mathbb{E}[\epsilon_t|\epsilon_{t-1}] = \rho\,\epsilon_{t-1}$.
- From (8) and (9) we then obtain

  (11)

  $$\Delta p_t = (\phi + \lambda)\,\epsilon_t - (\phi + \rho\,\lambda)\,\epsilon_{t-1} + e_t + \Delta\xi_t$$

  which does not involve the fair price V_t and may be estimated from data.

- Note here the assumption that autocorrelation coefficients of order higher than one are zero.

- Thus, in this model, price changes reflect:
  - The unexpected component of order flow
  - News
  - Noise due to tick size and so on.

MRR results

Table 2
Summary statistics of GMM model parameter estimates

	9:30–10:00	10:00–11:30	11:30–2:00	2:00–3:30	3:30–4:00
θ					
Mean	0.0415	0.0318	0.0275	0.0274	0.0287
(Avg. Std.Er.)	(0.0057)	(0.0023)	(0.0019)	(0.0022)	(0.0038)
Std. Dev.	0.0277	0.0212	0.0190	0.0190	0.0200
Median	0.0355	0.0274	0.0234	0.0236	0.0241
ϕ					
Mean	0.0344	0.0402	0.0437	0.0450	0.0461
(Avg. Std.Er.)	(0.0053)	(0.0021)	(0.0017)	(0.0021)	(0.0036)
Std. Dev.	0.0166	0.0125	0.0109	0.0111	0.0119
Median	0.0368	0.0419	0.0450	0.0469	0.0485
ρ					
Mean	0.4073	0.3676	0.3684	0.3789	0.3847
(Avg. Std.Er.)	(0.0370)	(0.0184)	(0.0166)	(0.0203)	(0.0330)
Std. Dev.	0.0724	0.0657	0.0720	0.0763	0.0884
Median	0.4021	0.3663	0.3700	0.3838	0.3871
λ					
Mean	0.3360	0.3086	0.2893	0.2874	0.2825
(Avg. Std.Er.)	(0.0218)	(0.0108)	(0.0097)	(0.0118)	(0.0184)
Std. Dev.	0.0984	0.0971	0.0984	0.0949	0.0920
Median	0.3411	0.3105	0.2888	0.2898	0.2886

Table 2 presents summary statistics of the GMM model estimates of the parameters for the 274 NYSE-listed stocks in the 1990 sample period over five intraday trading intervals. The table presents the mean coefficient estimate across the stocks, the mean standard error of the mean estimates, the standard deviation of the estimates across the 274 stocks, and the median estimate for the four main parameters of interest: θ, the asymmetric information component; ϕ, the transaction cost component; ρ, the autocorrelation coefficient of the order flow; and λ, the probability a trade takes place between the quotes.

Two quotes from MRR

> The mean value of θ (which we called λ) falls by over a third from the opening to the middle of the day (from 4.15 to 2.75 cents) and remains at this level until the final period where it increases slightly.
>
> The transaction cost element (ϕ) is approximately 3.4 cents in the first half-hour and increases steadily over the day to 4.6 cents in the final half-hour interval, a rise of about 30%.

- So, according to MRR, contributions from fixed costs and adverse selection are roughly equal.
  - From our BAC data, $\lambda = 0.059$ cents, $\phi = 0.229$ cents so nearly all of the effective spread is market maker profit.

The Huang and Stoll model

- In this model, the unobservable efficient price is first modeled as

(12)

$$V_t = V_{t-1} + \alpha \frac{s}{2} \epsilon_{t-1} + e_t$$

where s is the spread and α is how much of the spread may be attributed to informed trading (adverse selection).

- Also, according to the inventory model of Ho and Stoll, the mid-price m_t is given by

(13)

$$m_t = V_t + \beta \frac{s}{2} \sum_{i<t} \epsilon_i$$

so

$$\Delta m_t = \Delta V_t + \beta \frac{s}{2} \epsilon_{t-1} = (\alpha + \beta) \frac{s}{2} \epsilon_{t-1} + e_t$$

Changes in trade price

- Now we assume

$$p_t = m_t + \frac{s}{2} \epsilon_t + \xi_t$$

- This gives

$$\Delta p_t = (\alpha + \beta) \frac{s}{2} \epsilon_{t-1} + \frac{s}{2} (\epsilon_t - \epsilon_{t-1}) + e_t + \Delta \xi_t$$

which is amenable to estimation because it does not depend on unobservables.

  - Note that the effects of α and β cannot be distinguished.

- The Roll model may be recovered by setting $\alpha = \beta = 0$ to get

$$\Delta p_t = \frac{s}{2}(\epsilon_t - \epsilon_{t-1}) + \eta_t$$

for some noise η_t.

Distinguishing adverse selection and inventory effects

- In inventory models, a change in the quote changes the arrival rate of trades.
  - This induces negative serial covariance in mid-quotes, separate from and in addition to the bid-ask bounce in Δp_t.
- Let π be the probability of a reversal. Then

$$\mathbb{E}[\epsilon_{t-1}|\epsilon_{t-2}] = (1 - 2\,\pi)\,\epsilon_{t-2}$$

- We modify (12) accordingly:

(14)

$$\begin{aligned} \Delta V_t &= \alpha\,\frac{s}{2}\,(\epsilon_{t-1} - \mathbb{E}[\epsilon_{t-1}|\epsilon_{t-2}]) + e_t \\ &= \alpha\,\frac{s}{2}\,\epsilon_{t-1} - \alpha\,\frac{s}{2}\,(1 - 2\,\pi)\,\epsilon_{t-2} + e_t \end{aligned}$$

- Note that

$$\mathbb{E}[\Delta V_t|V_{t-1}, \epsilon_{t-2}] = 0$$

Distinguishing adverse selection and inventory effects

- Combining (13) and (14) gives

(15)

$$\Delta m_t = (\alpha + \beta)\,\frac{s}{2}\,\epsilon_{t-1} - \alpha\,\frac{s}{2}\,(1 - 2\,\pi)\,\epsilon_{t-2} + e_t$$

- Note that what matters for inventory is not the unexpected portion of the trade sign but the actual trade sign.
- Also, mid-quote changes are predictable:

$$\mathbb{E}[\Delta m_t|m_{t-1}, \epsilon_{t-2}] = \beta\,\frac{s}{2}\,(1 - 2\,\pi)\,\epsilon_{t-2}$$

- Equation (15) may be estimated directly or combined with (13) to give

(16)

$$\Delta p_t = \frac{s}{2}\,\epsilon_t + (\alpha + \beta - 1)\,\frac{s}{2}\epsilon_{t-1} - \alpha\,\frac{s}{2}\,\epsilon_{t-2} + e_t$$

Comments

- Why stop at ϵ_{t-2}? The trade sign process is supposedly long-memory and so all higher order autocovariances are positive.

- Equation (14) would then become
$$\Delta V_t = \alpha \frac{s}{2} (\epsilon_{t-1} - \mathbb{E}[\epsilon_{t-1} | \mathcal{F}_{t-1}]) + e_t$$
- We could use any forecasting model we liked to compute $\mathbb{E}[\epsilon_{t-1} | \mathcal{F}_{t-1}]$.

Empirical results

- Huang and Stoll data is from all trading days in 1992.
- It is notable that the most highly traded stock in their sample is MO with 751 trades per day.
  - 1992 MO volume traded per day was around 5.6 million shares.
  - Recent MO volume traded per day is around 10 million shares over roughly 35,000 trades. Times have changed!

Quote from Huang and Stoll:

> Under the adjusted results, the average order processing component of the traded spread is 61.8%, the average adverse information component is 9.6% (α), and the average inventory cost component (β) is 28.7%.

With our BAC data, $\frac{s}{2} = 0.288$ cents, $\alpha = 1.0\%$, $\beta = 17.9\%$.

R-code to estimate Huang and Stoll equation (16)

In [19]:

```
eps0 <- tradeSigns[-c(1,2)]
epsm1 <- tradeSigns[-c(1,n)]
epsm2 <- tradeSigns[-c(n-1,n)]
dp1 <- dp[-1]

(fit.hs16 <- lm(dp1 ~ eps0 + epsm1 + epsm2))

s2 <- fit.hs16$coeff[2] # The effective half-spread
alpha <- -fit.hs16$coeff[4]/s2
beta <- fit.hs16$coeff[3]/s2 -alpha+1
data.frame(s2,alpha,beta)
```

Out[19]:

```
Call:
lm(formula = dp1 ~ eps0 + epsm1 + epsm2)

Coefficients:
(Intercept)         eps0        epsm1        epsm2
 -3.852e-05    2.879e-03   -2.335e-03   -2.888e-05
```

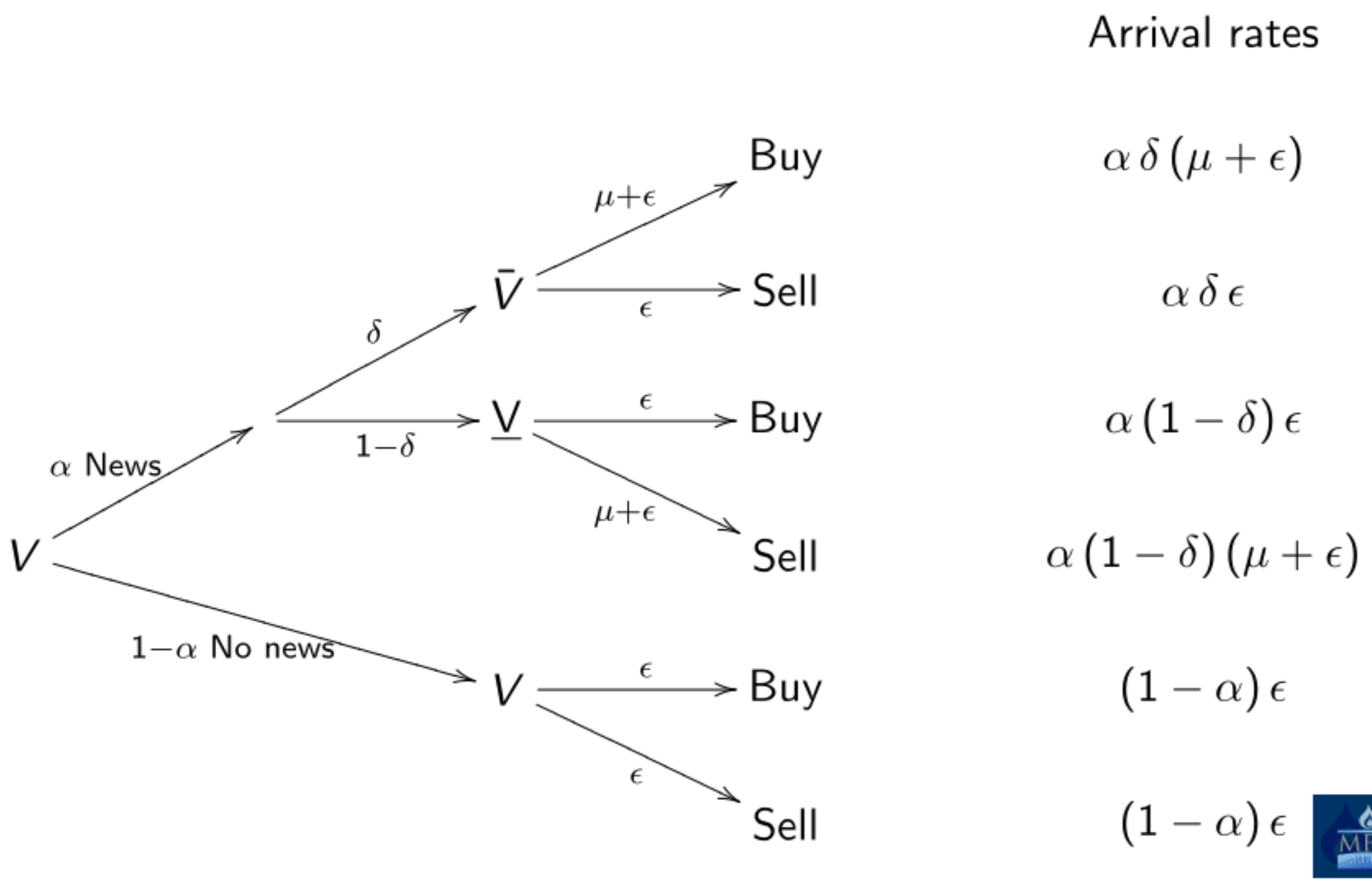
Out[19]:

```
              s2      alpha      beta
eps0 0.002878567 0.01003296 0.1788732
```

The PIN model

- The Glosten and Milgrom model is extended to include the arrival of news.
- Agents are not sequentially drawn in discrete time but arrive randomly in continuous time. Events are modeled as Poisson processes.
  - News arrives with intensity α. It is good news with probability δ, bad news with probability $1-\delta$.
  - Informed trades arrive with intensity μ.
  - Uninformed buys arrive with intensity ϵ.
  - Uninformed sells arrive with intensity ϵ.

PIN model event tree



Unconditional arrival rates

The total arrival rate of buy orders is given by

$$\lambda_{Buy} = \alpha\,\delta\,(\epsilon + \mu) + \alpha\,(1-\delta)\,\epsilon + (1-\alpha)\,\epsilon = \alpha\,\delta\,\mu + \epsilon$$

Similarly, the total arrival rate of sell orders is given by

$$\lambda_{Sell} = (1-\delta)\,\alpha\,\mu + \epsilon$$

and the total arrival rate by

$$\lambda = \alpha\,\mu + 2\,\epsilon.$$

PIN

The *probability of informed trading* (PIN) is the unconditional probability that a randomly chosen trader on a randomly chosen day is informed:

$$PIN = \frac{\alpha\,\mu}{\alpha\,\mu + 2\,\epsilon}$$

- PIN is estimated by maximum likelihood.

Maximum likelihood estimation of PIN

- Assume we can sign trades to give a sequence of buys and sells.
- The likelihood function is the relative probability of observing a given number of buys and sells given the parameters $\psi = \{\alpha, \delta, \mu, \epsilon\}$ of the model.
- The probability of observing n events in time T when the underlying process is Poisson with rate λ is

$$e^{-\lambda T} \frac{(\lambda T)^n}{n!}$$

Maximum likelihood estimation of PIN

- We take μ and ϵ to be daily rates and consider the number of buys B and sells S over one day.
- Then

$$\begin{aligned} L(B, S; \psi) &= \alpha \, \delta \, e^{-(\epsilon+\mu)} \frac{(\epsilon+\mu)^B}{B!} \, e^{-\epsilon} \frac{\epsilon^S}{S!} \\ &+ \alpha \, (1-\delta) \, e^{-\epsilon} \frac{\epsilon^B}{B!} \, e^{-(\epsilon+\mu)} \frac{(\epsilon+\mu)^S}{S!} \\ &+ (1-\alpha) \, e^{-2\,\epsilon} \frac{\epsilon^{B+S}}{B!\,S!} \end{aligned}$$

- The parameters ψ are estimated by maximizing $L(B, S; \psi)$.

Criticisms of PIN

- Every day is a new day: Each day's trading is independent of the previous day.
- Because of multiple trade reports, the trade-signing process itself is suspect.
- News events occur at the start of trading.
- There are only two possible results, high and low.
- α and μ enter into PIN only through the product $\alpha\,\mu$: Estimates for α and μ separately are less precise.

Interpretation of PIN

- Subsequent papers by Easley, O'Hara et al. have shown that PIN is an explanatory variable for the cross-section of returns.
  - In fact, it can apparently replace momentum in the Fama-French-plus-momentum factor model favored by for example AQR.
- Perhaps because of this, PIN is used by equity long-short traders.
- Given the highly stylized assumptions, it's not at all clear that PIN measures informed trading. It does probably give a good indication of order-flow one-sidedness (or momentum).
  - This is termed *information risk* by the authors.

Features of models studied so far

- The component of the bid-ask spread relating to order processing costs is an immediate cost. This cost of trading is temporary and has no price impact.
- The component of the bid-ask spread due to inventory effects has temporary impact on the price that takes some time to dissipate.
- The component of the bid-ask spread due to information effects (adverse selection) is permanent.
  - In the more sophisticated models, only the unexpected component of order flow impacts the price.

Adverse selection

- In the traditional microstructure literature, adverse selection is typically conflated with the effect of informed trading.
- Recall the quote from [Hasbrouck][5]:

> Orders do not impact prices. It is more accurate to say that orders forecast prices.

- It could be that prices move because of the impact of orders, a purely mechanical explanation.
  - It could be that the main information that traders have is the ultimate size of their own order (or *metaorder*). The market sees only the part of the metaorder that has already been revealed.

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