Introduction into linear algebra

Mathematical Preliminaries

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Definition 1.1

Let S be a non-empty set and $\phi : S \times S \to S$ a function.
This function is called a <u>Binary Operator</u> and for the image of $(a, b) \in S$ under the function ϕ we write $\phi(a, b)$ or simply $a \cdot b$.

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Definition 1.2

Let S be a non-empty set and ϕ a binary operator. The orbit of $x \in S$ is defined as follows:
$orb(x) = \{x^j; j \in \mathbb{N}\}$

Fields

Corollary 1.1

Note that the set S forms clouser under the operation ϕ.
Meaning, for every $a, b \in S$ the product $a \cdot b \in S$

Fields

Definition 1.3

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- For every $x \in S$ there exists a unique element $-x \in S$ s.t. $\phi(x, -x) = o$.

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- ψ distributes over ϕ.

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Note: From now on, we take ϕ to be addition and ψ to be multiplication; we also show them with the symbols $(+)$ and $(\cdot)$.
Also, we show zero element o as 0 and identity element e as 1 if needed.

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The set $\mathbb{C}[\sqrt{2}]$ forms a field under the addition and multiplication of complex numbers. (Assignments)

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- V consists of an addition operator $(+)$ satisfying the first four conditions of Addition (Definition 1.3).
- For every $k, k' \in K$ and $v, v_1, v_2 \in V$:
  - $k(k'v) = kk'(v)$
  - $k(v_1 + v_2) = kv_1 + kv_2$

example 1.4

Let K be a field, and V be the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ consisting of elements in K.

Define multiplication and addition as below:

- $(+)$: For every $(x_i)_{i=1}^{n}$ and $(x_j)_{j=1}^{n}$, $(x_i) + (x_j) = (x_i + x_j)$.
- $(\cdot)$: For every $k \in K$, $k(x_i)_{i=1}^{n} = (kx_i)_{i=1}^{n}$

The tuple (V, K) is a linear space.