Introduction into linear algebra Mathematical Preliminaries

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Mathematical Preliminaries

Fields

Definition 1.1

Let S be a non-empty set and $\phi: S \times S \to S$ a function.

This function is called a Binary Operator and for the image of $(a, b) \in S$ under the function ϕ we write $\phi(a, b)$ or simply $a \cdot b$.



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Definition 1.2

Let S be a non-empty set and ϕ a binary operator. The orbit of $x \in S$ is defined as follows:

$$orb(x) = \{x^j; j \in \mathbb{N}\}$$



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Corollary 1.1

Note that the set ${\it S}$ forms clouser under the operation $\phi.$

Meaning, for every $a, b \in S$ the product $a \cdot b \in S$



Theorem 1.1

If S has finite number of elements, say m, ϕ must be **Surjective**.



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Proof

(Proof by R.A.A) Let ϕ be non-surjective, then there exists $y \in S$ s.t. $\phi^{-1}(y) \notin S$.

From definition (1.2), $orb(y) = \{y, y^2, \dots, y^m\}$. From piegon hole principle, there exists $n \in \mathbb{N}$ s.t. $y^n = y$. Hence, for every $y \in S$, the pre-image $\phi^{-1}(y) \neq \emptyset$ and thus, ϕ is surjective.



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Definition 1.3

Let S be a non-empty set and ϕ and ψ two binary operations on S. The triplet (S, ϕ, ψ) is called a field if the following conditions are satisfied:



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- There exists a unique element $o \in S$ s.t. $\phi(x, o) = x$ for all $x \in S$.





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- There exists a unique element $o \in S$ s.t. $\phi(x, o) = x$ for all $x \in S$.
- For every $x \in S$ there exists a unique element $-x \in S$ s.t. $\phi(x,-x)=o.$



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Conditions on ψ

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- For every non-zero element $x \in S$ there exists a <u>unique</u> element $x^{-1} \in S$ s.t. $\psi(x, x^{-1}) = e$.



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- ψ distributes over ϕ .



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Note: From now on, we take ϕ to be addition and ψ to be multiplication; we also show them with the symbols (+) and (\cdot) .

Also, we show zero element o as 0 and identity element e as 1 if needed.





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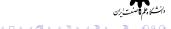
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The set of rational numbers ${\mathbb Q}$ forms a field under the ordinary addition and multiplication.

example 1.3

The set $\mathbb{C}[\sqrt{2}]$ forms a field under the addition and multiplication of complex numbers. (Assignments)



Vector Spaces

Definition 1.4

Let K be a field and V be a non-empty set of objects (aka vectors). The tuple (V, K) is called a linear space iff:





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Let K be a field and V be a non-empty set of objects (aka vectors). The tuple (V, K) is called a linear space iff:

- V consists of an addition operator (+) satisfying the first four conditions of Addition (Definition 1.3).
- For every $k, k' \in K$ and $v, v_1, v_2 \in V$:
 - k(k'v) = kk'(v)
 - $k(v_1 + v_2) = kv_1 + kv_2$



Vector Spaces

example 1.4

Let K be a field, and V be the set of all n-tuples (x_1, x_2, \ldots, x_n) consisting of elements in K.

Define multiplication and addition as below:

- (+): For every $(x_i)_{i=1}^n$ and $(x_j)_{j=1}^n$, $(x_i) + (x_j) = (x_i + x_j)$.
- (·): For every $k \in K$, $k(x_i)_{i=1}^n = (kx_i)_{i=1}^n$

The tuple (V, K) is a linear space.



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