Introduction into linear algebra

Mathematical Preliminaries

Ali Golestani

Department of Mathematics
Iran University of Science and Technology

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Definition 1.1

Let S be a non-empty set and $\phi : S \times S \to S$ a function.
This function is called a Binary Operator and for the image of $(a, b) \in S$ under the function ϕ we write $\phi(a, b)$ or simply $a \cdot b$.

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Definition 1.2

Let S be a non-empty set and ϕ a binary operator. The orbit of $x \in S$ is defined as follows:
$orb(x) = \{x^j; j \in \mathbb{N}\}$

Fields

Corollary 1.1

Note that the set S forms clouser under the operation ϕ.
Meaning, for every $a, b \in S$ the product $a \cdot b \in S$

Fields

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Proof

(Proof by R.A.A) Let ϕ be non-surjective, then there exists $y \in S$ s.t. $\phi^{-1}(y) \notin S$.
From definition (1.2), $orb(y) = \{y, y^2, \ldots, y^m\}$. From piegon hole principle, there exists $n \in \mathbb{N}$ s.t. $y^n = y$. Hence, for every $y \in S$, the pre-image $\phi^{-1}(y) \neq \emptyset$ and thus, ϕ is surjective.■

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Let S be a non-empty set and ϕ and ψ two binary operations on S. The triplet (S, ϕ, ψ) is called a field if the following conditions are satisfied:

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- There exists a unique element $o \in S$ s.t. $\phi(x, o) = x$ for all $x \in S$.

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- For every $x \in S$ there exists a unique element $-x \in S$ s.t. $\phi(x, -x) = o$.

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- ψ distributes over ϕ.

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Fields

Note: From now on, we take ϕ to be addition and ψ to be multiplication; we also show them with the symbols $(+)$ and $(\cdot)$.
Also, we show zero element o as 0 and identity element e as 1 if needed.

Fields

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example 1.3

The set $\mathbb{C}[\sqrt{2}]$ forms a field under the addition and multiplication of complex numbers. (Assignments)

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- V consists of an addition operator $(+)$ satisfying the first four conditions of Addition (Definition 1.3).
- For every $k, k' \in K$ and $v, v_1, v_2 \in V$:
  - $k(k'v) = kk'(v)$
  - $k(v_1 + v_2) = kv_1 + kv_2$

example 1.4

Let K be a field, and V be the set of all n-tuples $(x_1, x_2, \dots, x_n)$ consisting of elements in K.

Define multiplication and addition as below:

- $(+)$: For every $(x_i)_{i=1}^n$ and $(x_j)_{j=1}^n$, $(x_i) + (x_j) = (x_i + x_j)$.
- $(\cdot)$: For every $k \in K$, $k(x_i)_{i=1}^n = (kx_i)_{i=1}^n$

The tuple (V, K) is a linear space.