

Introduction into linear algebra

Mathematical Preliminaries

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Mathematical Preliminaries

Fields

Definition 1.1

Let S be a non-empty set and $\phi : S \times S \rightarrow S$ a function.

This function is called a Binary Operator and for the image of $(a, b) \in S$ under the function ϕ we write $\phi(a, b)$ or simply $a \cdot b$.



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Definition 1.2

Let S be a non-empty set and ϕ a binary operator. The orbit of $x \in S$ is defined as follows:

$$\text{orb}(x) = \{x^j; j \in \mathbb{N}\}$$



Corollary 1.1

Note that the set S forms clouser under the operation ϕ .

Meaning, for every $a, b \in S$ the product $a \cdot b \in S$



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- For every $x \in S$ there exists a unique element $-x \in S$ s.t. $\phi(x, -x) = o$.



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- ψ distributes over ϕ .



Fields

Note: From now on, we take ϕ to be addition and ψ to be multiplication; we also show them with the symbols $(+)$ and (\cdot) .

Also, we show zero element o as 0 and identity element e as 1 if needed.



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example 1.3

The set $\mathbb{C}[\sqrt{2}]$ forms a field under the addition and multiplication of complex numbers. (Assignment)



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Let K be a field and V be a non-empty set of objects (aka vectors). The tuple (V, K) is called a linear space iff:



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Let K be a field and V be a non-empty set of objects (aka vectors). The tuple (V, K) is called a linear space iff:

- V consists of an addition operator $(+)$ satisfying the first four conditions of Addition (Definition 1.3).
- For every $k, k' \in K$ and $v, v_1, v_2 \in V$:
 - $k(k'v) = kk'(v)$
 - $k(v_1 + v_2) = kv_1 + kv_2$



example 1.4

Let K be a field, and V be the set of all n -tuples (x_1, x_2, \dots, x_n) consisting of elements in K .

Define multiplication and addition as below:

- $(+)$: For every $(x_i)_{i=1}^n$ and $(x_j)_{j=1}^n$, $(x_i) + (x_j) = (x_i + x_j)$.
- (\cdot) : For every $k \in K$, $k(x_i)_{i=1}^n = (kx_i)_{i=1}^n$

The tuple (V, K) is a linear space.

