Disorderly Escape

a group theoretic investigation by

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1. Introduction

We start by setting up the problem in a group-theoretic framework. Start with a finite set of states R, and construct the set of m by n matrices with entries in R, $X = M_{m\times n}(R)$. Now we define an action of the group $G = S_m \times S_n$ on X by

$$(g \cdot x)_{ij} = x_{\sigma(i)\tau(j)} \tag{1}$$

where $g = (\sigma, \tau)$ for $\sigma \in S_m, \tau \in S_n$.

It's then easy to see that we're interested in counting the number of orbits of this group action, $|X/G|$. But we know by Burnside's lemma that this is given by

$$|X/G| = \frac{1}{|G|} \sum_{g\in G} |X^g|, \tag{2}$$

where X^g is the set of elements of X that are invariant under the action by g. Our main goals now are to find an efficient way to index the elements of G, as well as an expression for $|X^g|$ that goes well with that indexing. We'll approach this by studying how the group action induces a natural monomorphism $\Phi : S_m \times S_n \hookrightarrow S_{mn}$.

2. The inclusion $S_m \times S_n \leq S_{mn}$

Let $\phi : \{1 \dots m\} \times \{1 \dots n\} \hookrightarrow\!\!\!\!\!\rightarrow \{1 \dots mn\}$ be some bijection. It then induces a monomorphism $\Phi : S_m \times S_n \hookrightarrow S_{mn}$ by

$$\Phi(\sigma, \tau)(k) = \phi((\sigma, \tau)(\phi^{-1}(k))). \tag{3}$$

The point being, this allows us to think of $X = R^{mn}$ and $G \leq S_{mn}$. The invariance condition is now

$$x_k = x_{g(k)}\, \forall k \in \{1 \dots mn\}, \tag{4}$$

which immediately implies $x_k = x_{g^p(k)}$ for all $p \in \mathbb{N}$. From that, it's easy to see that we have freedom in choosing, for each cycle in g, an element of R. This yields the simple formula

$$|X^g| = |R|^{C(g)} \tag{5}$$

where $C : G \to \mathbb{N}$ counts the number of cycles in the cycle decomposition of g as an element of S_{mn}.

3. Cycles

In the following argument we'll take $k = \phi(i, j)$. Take $g = (\sigma, \tau) \in S_m \times S_n$, and suppose that i and j are respectively in $p-$cycle of σ and a $q-$cycle of τ. Then (i, j) must be in some $r-$cycle of (σ, τ). We now compute r. Since Φ is a homomorphism we can write the cycle condition as

$$k = \Phi(\sigma, \tau)^r(k) = \Phi(\sigma^r, \tau^r)(k),$$

therefore $\phi(i, j) = \phi((\sigma^r, \tau^r)(\phi^{-1}(k)))$, so

$$(i, j) = (\sigma^r, \tau^r)(i, j) = (\sigma^r(i), \tau^r(j))$$

and we conclude $p|r$ and $q|r$. The smallest such number r is $r = \mathrm{lcm}(p, q)$ and is the desired cycle length. Since this pair of cycles involves a total of pq elements, it's then easy to see that they must correspond to $pq/\mathrm{lcm}(p, q) = \gcd(p, q)$ cycles in g.

This leads us to indexing elements of S_n by the lengths of cycles in their decomposition. It's well known that these are the group's conjugacy classes, corresponding to integer partitions λ of n, which we denote by $\lambda \vdash n$, and have sizes

$$N(\lambda) = \prod_{p=1}^{n} \frac{n!}{p^{c_p(\lambda)} c_p(\lambda)!}. \tag{6}$$

An aside on notation for this equation, an integer partition $\lambda \vdash n$ is a finite non-decreasing sequence of positive integers $\lambda = (\lambda_1, \cdots, \lambda_k)$ such that $n = \sum_{i=i}^{k} \lambda_i$. We then define the count representation of λ to be $c_p(\lambda)$, counting how many times the part p appears in λ, and we get $n = \sum_{p=1}^{n} p c_p(\lambda)$.

To use the combination property of cycles that we discussed in the beginning of the section we now define two operations on the set $\mathcal{P} = \{\lambda \vdash n : n \in \mathbb{N}\}$, giving it a semiring structure. For partitions $\lambda \vdash m$ and $\mu \vdash n$ we define a commutative associative addition $\lambda \frown \mu \vdash m + n$ by

$$c_p(\lambda \frown \mu) := c_p(\lambda) + c_p(\mu), \tag{7}$$

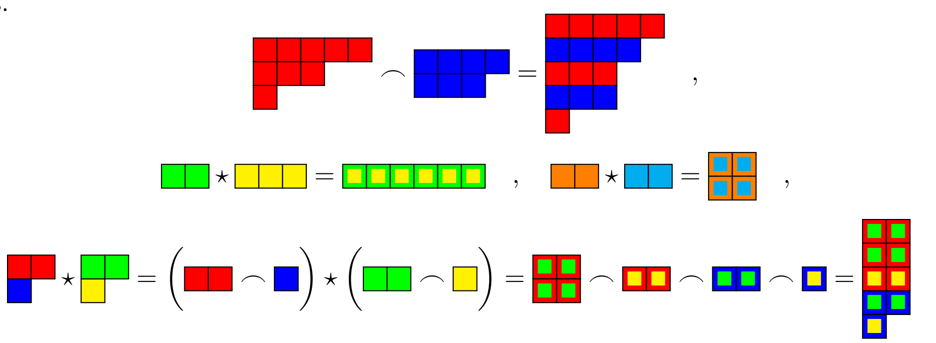
as well as a commutative associative product $\lambda \star \mu \vdash mn$ by the distributive property

$$\lambda \star (\mu \frown \nu) := (\lambda \star \mu) \frown (\lambda \star \nu) \quad \forall \lambda, \mu, \nu \in \mathcal{P} \tag{8}$$

and its action on single parts as

$$(p) \star (q) = (\underbrace{\mathrm{lcm}(p, q), \ldots, \mathrm{lcm}(p, q)}_{\gcd(p,q) \text{ times}}). \tag{9}$$

These operations can be particularly well illustrated by Young diagrams. These represent integer partitions by giving each part p a row of length p. We now show some example computations.



4. Conclusion

We now put all of the previous results into the expression of Burnside's lemma (2). his gives us our final expression, which gives a simple prescription for the computation of the count we're looking for,

$$|X/G| = \frac{1}{m!n!} \sum_{\substack{\lambda \vdash m \\ \mu \vdash n}} N(\lambda) N(\mu) |R|^{\sum_{p=1}^{mn} c_p(\lambda \star \mu)}. \tag{10}$$

Efficient calculation of this result relies mostly on efficient generation of the partitions and computation of N, since the $\star$ and c_p have simple and fast algorithms, and the latter could even be combined with the exponentiation of $|R|$.

Finally, as a side note, since $\star$ is associative, this result can be generalized to arbitrary dimensions: $X = M_{n_1 \times \cdots \times n_d}(R)$ immediately has

$$|X/G| = \sum_{\substack{\lambda_i \vdash n_i \\ \forall i = \{1 \ldots d\}}} \prod_{j=1}^{d} \left(\frac{N(\lambda_j)}{n_j!} \right) |R|^{\sum_{p=1}^{n_1 \cdots n_d} c_p(\lambda_1 \star \cdots \star \lambda_d)} \tag{11}$$

orbits under the action of the group $G = S_{n_1} \times \cdots \times S_{n_d}$.