A train with an initial velocity v_0 starts breaking at position x_0. It experiences a deceleration of the form

$$a(t) = -ct^2. \tag{1}$$

For safety reasons we wish to keep the magnitude of deceleration under a maximum value given by $a_{max} = g/3$. With this information we can calculate the distance we need to start breaking such that the train comes to a stop without the magnitude of the deceleration exceeding a_{max}. Note that we made $a(t)$ be a negative quantity because the breaks slow down the train. This is an arbitrary choice that we can make either way without changing the final solution.

First we need to determine the initial conditions of the system. An initial condition is the state of the system at the start of the time or when $t = 0$. So we are told that we start with an initial velocity of v_0 at position x_0 therefore we say

$$v(t = 0) = v_0$$
$$x(t = 0) = x_0.$$

Here we are explicitly assuming that velocity, $v(t)$, and position, $x(t)$, are functions of time, the function form of which we will need to determine.

First to determine the functional form of velocity, $v(t)$, we use the fact that acceleration is the change of velocity with respect to time. Mathematically we write this as

$$a(t) = \frac{dv(t)}{dt} \tag{2}$$

where d/dt is the derivative with respect to time. Therefore we can rearrange equation 2 by multiplying both sides by dt

$$a(t) = \frac{dv(t)}{dt}$$
$$\Downarrow$$
$$a(t)dt = \frac{dv(t)}{\cancel{dt}}\cancel{dt}$$
$$\Downarrow$$
$$a(t)dt = dv(t)$$

Now to we can solve for $v(t)$ by integrating both sides where we use the given function of acceleration shown in equation 1.

$$\int dv(t) = \int a(t)dt$$
$$\Downarrow$$
$$v(t) = \int -ct^2 dt$$
$$= -\frac{c}{3}t^3 + A$$

Note that we gained a new constant, A, from integrating. This is called a constant of integration. We can solve for A by using the initial condition $v(0) = v_0$. This yields

$$v(t = 0) = v_0 = -\frac{c}{3}\cancel{(0)^3}^{\,0} + A$$
$$v_0 = A$$

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We note that the integration constant in this case was just our initial velocity. So now we have a function that describes our velocity as a function of time given by

$$v(t) = -\frac{c}{3}t^3 + v_0. \tag{3}$$

Next we need to determine the functional form of position, $x(t)$. This process follows in the same was we determined $v(t)$. Namely we use the fact that velocity is the change of position with respect to time or mathematically

$$v(t) = \frac{dx(t)}{dt} \tag{4}$$

Like before, we rearrange equation 4 by multiplying both sides by dt to get

$$v(t) = \frac{dx(t)}{dt}$$
$$\Downarrow$$
$$v(t)dt = \frac{dx(t)}{dt}dt$$
$$\Downarrow$$
$$v(t)dt = dx(t)$$

Now to solve for $x(t)$ we integrate both sides of this equations using the formula for $v(t)$ we found above that is shown in equation 3. So,

$$\int dx(t) = \int v(t)dt$$
$$\Downarrow$$
$$x(t) = \int \left(-\frac{c}{3}t^3 + v_0\right)dt$$
$$= \int -\frac{c}{3}t^3 dt + \int v_0 dt$$
$$= -\frac{c}{12}t^4 + v_0 t + B$$

Again we gained a constant of integration, B, and like before we use our initial condition $x(0) = x_0$ to determine B by

$$x(t = 0) = x_0 = -\frac{c}{12}(0)^4 {}^{0} + v_0(0) {}^{0} + B$$
$$x_0 = B$$

So we found that $B = x_0$. Now we have found another function this time that function describes our position as a function of time. This is given by

$$x(t) = -\frac{c}{12}t^4 + v_0 t + x_0. \tag{5}$$

Grouping the equations that describe the motion of our train together we have

$$x(t) = -\frac{c}{12}t^4 + v_0 t + x_0$$
$$v(t) = -\frac{c}{3}t^3 + v_0$$
$$a(t) = -ct^2$$

2

Now we are ready to start finding the distance we need to start breaking at! To do this we need to realize ultimately the train needs to come to a stop. This means at a final time we will call t_f the velocity of the train needs to be zero. Mathematically we say this by $v(t = t_f) = 0$. We realize that t_f is an important quantity to finding the distance we need to stop breaking at, because the distance the train has traveled once we have come to a stop is $x(t = t_f) - x_0$. So we need to determine the final time, t_f. We can do this by taking the velocity function we found to be equation 3 and evaluating it at $t = t_f$, and solving for t_f.

$$v(t = t_f) = 0 = -\frac{c}{3}(t_f)^3 + v_0$$

$$\Downarrow$$

$$-v_0 = -\frac{c}{3}t_f^3$$

$$v_0\frac{3}{c} = \cancel{-}\frac{\cancel{c}}{3}t_f^3\cancel{-}\frac{\cancel{3}}{c}$$

$$\Downarrow$$

$$t_f^3 = \frac{3v_0}{c}$$

$$\left(t_f^3\right)^{1/3} = \left(\frac{3v_0}{c}\right)^{1/3}$$

$$\Downarrow$$

$$t_f = \left(\frac{3v_0}{c}\right)^{1/3}$$

So we have found t_f but we are not done yet! We see that the final time, t_f, depends on the constant c that was given as part of equation 1. To find c we need to use the safety constraint that states that $a_{max} = g/3$. We reason that if we want to ensure that we do not exceed a_{max} we need to reach a_{max} at time final. This means that $a(t = t_f) = g/3$. So using equation 1, which is the given functional form of acceleration, evaluated at $t = t_f$ we can solve for c.

$$a(t = t_f) = -\frac{g}{3} = -ct_f^2$$

$$\Downarrow$$

$$\frac{g}{3} = ct_f^2$$

$$\frac{g}{3}\frac{1}{t_f^2} = \cancel{c}t_f^2\frac{1}{\cancel{t_f^2}}$$

$$\Downarrow$$

$$c = \frac{g}{3t_f^2}$$

We note that c has a t_f^2 in it. This means we are going to need to do some extra work when we

replace c in our first solution of t_f.

$$t_f = \left(\frac{3v_0}{c}\right)^{1/3}$$

$$\Downarrow$$

$$t_f = \left(\frac{3v_0}{\frac{g}{3t_f^2}}\right)^{1/3} = \left(\frac{3v_0}{1}\frac{3t_f^2}{g}\right)^{1/3}$$

$$t_f^3 = \left(\left(\frac{9v_0t_f^2}{g}\right)^{1/3}\right)^3$$

$$t_f^3 \frac{1}{t_f^2} = \frac{9v_0 \cancel{t_f^2}}{g}\frac{1}{\cancel{t_f^2}}$$

$$\Downarrow$$

$$t_f = \frac{9v_0}{g}$$

Finally we have an expression for t_f that we can plug into equation 5! Now we find the distance it takes to come to a stop, $x(t = t_f) - x_0$. But first we need to replace c into $x(t)$.

$$x(t) = -\frac{c}{12}t^4 + v_0 t + x_0$$

$$\Downarrow$$

$$x(t) = -\frac{\frac{g}{3t_f^2}}{12}t^4 + v_0 t + x_0$$

$$x(t) = -\frac{g}{36t_f^2}t^4 + v_0 t + x_0$$

Now we can find the stopping distance $x(t = t_f) - x_0$ by

$$x(t = t_f) - x_0 = -\frac{g}{36t_f^2}(t_f)^4 + v_0(t_f) + \cancel{x_0} - \cancel{x_0}$$

$$\Downarrow$$

$$x(t = t_f) - x_0 = -\frac{g}{36}t_f^2 + v_0 t_f$$

$$\Downarrow$$

$$x(t = t_f) - x_0 = -\frac{g}{36}\left(\frac{9v_0}{g}\right)^2 + v_0\frac{9v_0}{g}$$

$$= -\frac{g}{36}\frac{81v_0^2}{g^2} + \frac{9v_0^2}{g}$$

$$= -\frac{81v_0^2}{36g} + \frac{9v_0^2}{g}$$

$$= \frac{v_0^2}{g}\left(9 - \frac{81}{36}\right)$$

$$= \frac{27}{4}\frac{v_0^2}{g}$$

4

We see that in the end result the distance the train travels to a stop does not depend on x_0! This is because we are interested in the distance traveled relative to x_0. So we can choose any value for x_0 and it would not effect the final solution. Note that we can check to make sure that the units work out. We know that the units of x are in meters. And the units of v_0^2/g are $\frac{(m^2/s^2)}{(m/s^2)} = m$ so our units work! Now we can plug in our values where $v_0 = 89\,\mathrm{m/s}$ and $g = 9.8\,\mathrm{m/s^2}$ we get

$$x(t = t_f) - x_0 = \frac{27}{4}\frac{v_0^2}{g}$$

$$\Downarrow$$

$$x(t = t_f) - x_0 = \frac{27}{4}\frac{(89)^2}{9.8} \approx 5500\,\mathrm{m}$$

Now for extra practice try solving for the more general case where $a(t) = ct^m$ and $a_{max} = g/k$ and see if you can recreate the general solution

$$x(t_f) - x_0 = \frac{v_0^2(m+1)^2 k}{g(m+2)}$$