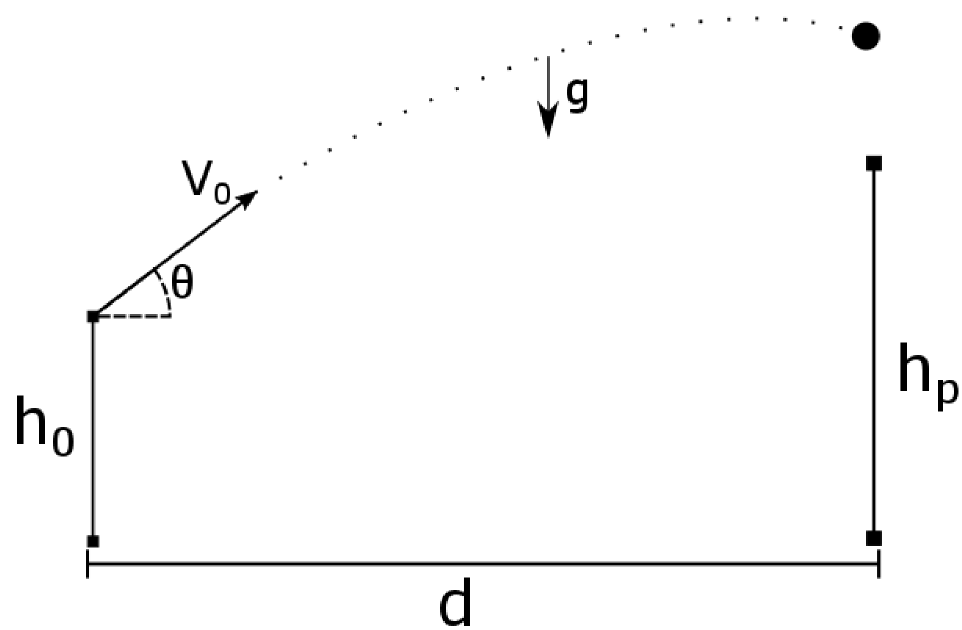

Figure 1:

To solve this problem we first draw a sketch so we can identify the relevant details of the problem. This sketch is shown in figure 1.

1 Motion of the Ball

Now, in order to solve for the time of the jump t_{jump} we need to first determine the motion of the ball. In order to do this we start from the fact that we know the ball is under a constant acceleration, g, in the downwards direction. Note we will define downwards as the negative direction, so we integrate to find $v_y^{ball}(t)$ by

$$a_y^{ball}(t) = \frac{dv_y^{ball}(t)}{dt} = -g$$
$$\Downarrow$$
$$\int dv_y^{ball}(t) = \int -g dt$$
$$v_y^{ball}(t) = -gt + A$$

We determine A by the initial condition of the velocity in the y direction. Before we solve for A we need to determine what the velocity of the ball is at $t = 0$. The problem states that the ball is thrown at a velocity, v_0, at an angle θ. Therefore we need to find the y-component of v_0 which is given by

$$v_{0y}^{ball} = v_0 \sin\theta$$

So, we can determine A by

$$v_y^{ball}(t=0) = v_0 \sin\theta = -g(0) + A$$
$$\Downarrow$$
$$A = v_0 \sin\theta$$

This gives us the equation for the velocity of the ball in the y-direction

$$v_y^{ball}(t) = v_0 \sin\theta - gt. \tag{1}$$

Using equation 1 we can then find the functional form of the y position by the integral

$$\begin{aligned}
v_y^{ball} &= \frac{dy^{ball}(t)}{dt} \\
&\Downarrow \\
\int dy^{ball}(t) &= \int v_y^{ball}(t)dt \\
&= \int v_0 \sin\theta - gtdt \\
&= v_0 \sin\theta t - \frac{1}{2}gt^2 + B
\end{aligned}$$

Again we need to determine the constant of integration by applying the fact that the ball is initially thrown at a height, h_0. This gives the initial condition $y^{ball}(t=0) = h_0$. Which allows us to solve for B by

$$\begin{aligned}
y^{ball}(t=0) = h_0 &= v_0 \sin\theta(0) - \frac{1}{2}g(0)^2 + B \\
&\Downarrow \\
B &= h_0
\end{aligned}$$

Which gives us the equation for the y-component of the ball

$$y^{ball}(t) = h_0 + v_0 \sin\theta t - \frac{1}{2}gt^2 \tag{2}$$

Next we need to find the dynamical equations in the x direction. We assume that there is no air resistance which implies that $a_x^{ball} = 0$. Therefore we can find the equation for the velocity in the x-direction by integrating

$$\begin{aligned}
a_x^{ball}(t) &= \frac{dv_x^{ball}}{dt} \\
&\Downarrow \\
\int dv^{ball}(t)_x &= \int a_x^{ball}(t)dt \\
v_x^{ball}(t) &= \int 0dt \\
v_x^{ball}(t) &= C
\end{aligned}$$

Note that because we have no acceleration we found that the velocity is constant in time. So we need to determine C again by initial conditions. Like before we realize that when a ball is throw with velocity v_0 at an angle θ it has a component in the x direction given by

$$v_x^{ball}(t=0) = v_0 \cos\theta.$$

Now it is easy to see that $C = v_0 \cos\theta$ because as we have said before the velocity is constant for all time. This implies that the initial velocity is the velocity in the x direction for all time. So we have

$$v_x^{ball}(t) = v_0 \cos\theta \tag{3}$$

which we can use to find the function for the x position. Like before we have

$$\begin{aligned}
v_x^{ball}(t) &= \frac{dx^{ball}(t)}{dt} \\
&\Downarrow \\
\int dx^{ball}(t) &= \int v_x^{ball}(t)dt \\
x^{ball}(t) &= \int v_0 \cos\theta dt \\
x^{ball}(t) &= v_0 \cos\theta t + D
\end{aligned}$$

To find D we assume that we throw the ball from an initial position x_0 which implies that $x^{ball}(t = 0) = x_0$. Which solving for D yields

$$\begin{gathered}
x^{ball}(t = 0) = x_0 = v_0 \cos\theta(0) + D \\
\Downarrow \\
D = x_0
\end{gathered}$$

Which gives the equation for the x position of the ball as

$$x^{ball}(t) = x_0 + v_0 \cos\theta t \tag{4}$$

So we can collect the equations that define the motion of the ball

$$\begin{aligned}
v_y^{ball}(t) &= v_0 \sin\theta - gt \\
y^{ball}(t) &= h_0 + v_0 \sin\theta t - \frac{1}{2}gt^2 \\
v_x^{ball}(t) &= v_0 \cos\theta \\
x^{ball}(t) &= x_0 + v_0 \cos\theta t
\end{aligned}$$

Now we can look at the details of the problem as see that we want to determine the height of the ball, $y^{ball}(t)$, when it has traveled a distance d away from the thrower in the x direction. To do this we look at $x^{ball}(t)$ to determine how long it takes the ball to travel a distance d from the thrower. We can write this mathematically as $x^{ball}(t = t_d) = x_0 + d$. So we can solve for the balls travel time, t_d, by

$$\begin{aligned}
x^{ball}(t = t_d) = x_0 + d &= x_0 + v_0 \cos\theta t_d \\
x_0 + d - x_0 &= x_0 + v_0 \cos\theta t_d - x_0 \\
d &= v_0 \cos\theta t_d \\
&\Downarrow \\
t_d &= \frac{d}{v_0 \cos\theta}
\end{aligned}$$

Now that we know the ball's travel time we can find how high the ball is at this time by

$$\begin{aligned}
y^{ball}(t = t_d) &= h_0 + v_0 \sin\theta t_d - \frac{1}{2}gt_d^2 \\
&= h_0 + v_0 \sin\theta \frac{d}{v_0 \cos\theta} - \frac{1}{2}g\left(\frac{d}{v_0 \cos\theta}\right)^2 \\
&= h_0 + d \tan\theta - \frac{1}{2}g\left(\frac{d}{v_0 \cos\theta}\right)^2
\end{aligned}$$

By solving for this we know know how high the catcher is going to need to jump. We can use this result in the next section.

2 Motion of the Catcher

To find how long the catcher is in the air for we first need to derive the equations for the catcher's motion. We assume that the catcher jumps exactly vertically this means we only have to consider the y component of their motion. Because the ball and the catcher are on the same planet we can say that they both feel the same acceleration $a^{catch}(t) = -g$. Using this we can derive equations of motion by integrating

$$a^{catch}(t) = \frac{dv^{catch}(t)}{dt}$$
$$\Downarrow$$
$$\int dv^{catch}(t) = \int a^{catch}(t)dt$$
$$v^{catch}(t) = \int -gdt$$
$$v^{catch}(t) = -gt + E$$

Were we find E by the initial condition $v^{catch}(t=0) = v_0^{catch}$. Note because we split the problem into two different parts we define a new reference for time. So we can solve for E

$$v^{catch}(t=0) = v_0^{catch} = -g0 + E$$
$$\Downarrow$$
$$E = v_0^{catch}$$

So our catchers velocity is given by

$$v^{catch}(t) = v_0^{catch} - gt \tag{5}$$

And like before we can integrate $v^{catch}(t)$ to find the equation for the height of the catcher.

$$v^{catch}(t) = \frac{dy^{catch}(t)}{dt}$$
$$\Downarrow$$
$$\int dy^{catch}(t) = \int v^{catch}(t)dt$$
$$y^{catch}(t) = \int v_0^{catch} - gtdt$$
$$y^{catch}(t) = v_0^{catch}t - \frac{1}{2}gt^2 + F$$

And we find F by apply the initial condition that $y^{catch}(t=0) = h_p$ which gives

$$y^{catch}(t=0) = h_p = v_0^{catch}(0) - g(0)^2 + F$$
$$\Downarrow$$
$$F = h_p$$

which yields the equation of motion

$$y^{catch}(t) = h_p + v_0^{catch}t - \frac{1}{2}gt^2. \tag{6}$$

Now we need to consider what is happening at t_{jump}. We want the catchers height to be the height of the ball we found in the previous part. Which we can write mathematically as the condition

$$y^{catch}(t = t_{jump}) = h_0 + d\tan\theta - \frac{1}{2}g\left(\frac{d}{v_0\cos\theta}\right)^2 = h_p + v_0^{catch}t_{jump} - \frac{1}{2}gt_{jump}^2$$

We are really close to finding t_{jump}, the final missing piece we need is to find velocity the catcher uses to leave the ground, v_0^{catch}. To find this we realize that we want the catcher to be at the top of their jump right when they catch the ball. This implies that the catcher's velocity at the end of his jump is zero. Which we write as the constraint $v^{catch}(t = t_{jump}) = 0$ which allows us to solve

$$v^{catch}(t = t_{jump}) = 0 = v_0^{catch} - gt_{jump}$$
$$\Downarrow$$
$$v_0^{catch} = gt_{jump}$$

Note that t_{jump} is a specific constant time not to be confused with the general t. This distinction keeps v_0^{catch} a constant. So now we can solve for t_{jump} by using $y^{catch}(t)$

$$y^{catch}(t = t_{jump}) = h_0 + d\tan\theta - \frac{1}{2}g\left(\frac{d}{v_0\cos\theta}\right)^2 = h_p + v_0^{catch}t_{jump} - \frac{1}{2}gt_{jump}^2$$
$$\Downarrow$$
$$h_0 + d\tan\theta - \frac{1}{2}g\left(\frac{d}{v_0\cos\theta}\right)^2 = h_p + (gt_jump)t_{jump} - \frac{1}{2}gt_{jump}^2$$
$$h_0 + d\tan\theta - \frac{1}{2}g\left(\frac{d}{v_0\cos\theta}\right)^2 = h_p + gt_{jump}^2 - \frac{1}{2}gt_{jump}^2$$
$$h_0 - h_p + d\tan\theta - \frac{1}{2}g\left(\frac{d}{v_0\cos\theta}\right)^2 = h_p - h_p + \frac{1}{2}gt_{jump}^2$$
$$2\left(h_0 - h_p + d\tan\theta - \frac{1}{2}g\left(\frac{d}{v_0\cos\theta}\right)^2\right) = 2\frac{1}{2}gt_{jump}^2$$
$$\frac{1}{g}2\left(h_0 - h_p + d\tan\theta - \frac{1}{2}g\left(\frac{d}{v_0\cos\theta}\right)^2\right) = \frac{1}{g}gt_{jump}^2$$
$$\Downarrow$$
$$\frac{2\left(h_0 - h_p + d\tan\theta - \frac{1}{2}g\left(\frac{d}{v_0\cos\theta}\right)^2\right)}{g} = t_{jump}^2$$
$$\Downarrow$$
$$t_{jump} = \sqrt{\frac{2\left(h_0 - h_p + d\tan\theta - \frac{1}{2}g\left(\frac{d}{v_0\cos\theta}\right)^2\right)}{g}}$$

As an extra exercise try the solving this problem using this method but with the added complication with a time dependent gravity when $-g \to -Ct$ where C is a positive constant. And see if you can find the expression for t_{jump} as

$$t_{jump} = \sqrt[3]{\frac{3\left(h_0 - h_p + d\tan\theta - \frac{1}{2}g\left(\frac{d}{v_0\cos\theta}\right)^2\right)}{C}}$$