Physics 615

Methods of Theoretical Physics I

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Homework #12

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1 Problem #1

To prove the identity

$$\exp(i\vec{\sigma}\cdot\vec{n}\omega) = \cos\omega\cdot\mathbf{1} + i\vec{\sigma}\vec{n}\sin\omega \tag{1.1}$$

where σ_i are the Pauli matrices, $\mathbf{1}$ is the 2×2 identity matrix, and $\vec{n}$ is a unit vector in $\mathbb{R}^3$, we take the exponential of a matrix, A, to be defined as

$$e^A = \sum_{n=0}^{\infty}\frac{1}{n!}A^n \tag{1.2}$$

Therefore using equation 1.2 on equation 1.1 where $\vec{\sigma}\cdot\vec{n}$ is taken to be the matrix, A

$$\begin{aligned}
\exp(i\vec{\sigma}\cdot\vec{n}\omega) &= \sum_{n=0}^{\infty}\frac{1}{n!}(i\omega\vec{\sigma}\cdot\vec{n})^n \\
&= \sum_{n=0}^{\infty}\frac{i^n}{n!}(\vec{\sigma}\cdot\vec{n})^n \\
&= \sum_{n=0}^{\infty}\frac{(-1)^{2n}\omega^{2n}}{(2n)!}(\vec{\sigma}\cdot\vec{n})^{2n} + i\sum_{n=0}^{\infty}\frac{(-1)^{2n+1}\omega^{2n+1}}{(2n+1)!}(\vec{\sigma}\cdot\vec{n})^{2n+1}
\end{aligned}$$

We note that the Pauli matrices are given as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Which implies that

$$\vec{\sigma}\cdot\vec{n} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}$$

Note the factor of $\sqrt{3}$ comes from the normalization of n in $\mathbb{R}^3$. Using this we can see that

$$\begin{aligned}
(\vec{\sigma}\cdot\vec{n})^2 &= \frac{1}{3}\begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}\begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 1+(1-i)(1+i) & (1-i)-(1-i) \\ (1+i)-(1+i) & (1+i)(1-i)+1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}
\end{aligned}$$

And we can generalize to all even powers to say

$$(\vec{\sigma}\cdot\vec{n})^{2n} = \mathbf{1}$$

Therefore

$$\begin{aligned}
\exp(i\vec{\sigma}\cdot\vec{n}\omega) &= \sum_{n=0}^{\infty}\frac{(-1)^{2n}\omega^{2n}}{(2n)!}\mathbf{1} + i\sum_{n=0}^{\infty}\frac{(-1)^{2n+1}\omega^{2n+1}}{(2n+1)!}(\vec{\sigma}\cdot\vec{n})\mathbf{1} \\
&= \cos\omega\cdot\mathbf{1} + i\vec{\sigma}\vec{n}\sin\omega
\end{aligned}$$

2 Problem #2

Given Hermitian matrices A, B and unitary matrices C, D. Which implies that

$$A = A^\dagger, \qquad B = B^\dagger, \qquad C^\dagger C = CC^\dagger = 1 \qquad , D^\dagger D = DD^\dagger = 1$$

1) We can show that

$$\begin{aligned}
(C^{-1}AC)^\dagger &= C^\dagger A^\dagger \left(C^{-1}\right)^\dagger \\
&= C^{-1}A^\dagger C \\
&= C^{-1}AC
\end{aligned}$$

Therefore $C^{-1}AC$ is Hermitian.

2) We can show that

$$\begin{aligned}
C^{-1}DC(C^{-1}DC)^\dagger &= C^{-1}D\cancelto{1}{CC^\dagger}D^\dagger(C^{-1})^\dagger \\
&= C^{-1}\cancelto{1}{DD^\dagger}(C^{-1})^\dagger \\
&= C^{-1}(C^{-1})^\dagger \\
&= C^\dagger C = 1
\end{aligned}$$

Therefore C^1DC is a unitary matrix.

3) We can show that

$$\begin{aligned}
(i(AB - BA))^\dagger &= -i((AB)^\dagger - (BA)^\dagger) \\
&= -i(B^\dagger A^\dagger - A^\dagger B^\dagger) \\
&= -i(BA - AB) = i(AB - BA)
\end{aligned}$$

Therefore $i(AB - BA)$ is Hermitian.

3 Problem #3

Given the unitary matrix $\mathbf{1}$ we can find the eigenvalues λ of this matrix by enforcing the condition

$$\det(\mathbf{A} - \lambda\mathbf{1}) = 0$$

where we take $\mathbf{A}$ to be the unitary matrix. This implies that

$$\det(\mathbf{1} - \lambda\mathbf{1}) = \det((1-\lambda)\mathbf{1}) = (1-\lambda)^n$$

where n is the rank of the unitary matrix. So the equation

$$(1-\lambda)^n = 0$$

implies that $|\lambda| = 1$ or that λ is unimodular.

4 Problem #4

For the center, Z, of a group, G, defined as the set of elements $z \in G$ which commute with all elements within G

$$Z = \{z \in G|\ zg = gz,\ \forall g \in G\}$$

We note that the group Z is an Abelian group by definition. This is due to the fact for elements $f, g \in Z$ by definition we have $fg = gf$. Therefore we have a commutative operator (multiplication).