Physics 606

Quantum Mechanics I

Professor Aleksei Zheltikov

Homework #7

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1 Problem #1

To find the expectation value, $< \hat{L}^2 >$, in the state with a wave function

$$\psi(\theta, \varphi) = A \sin\theta \cos\varphi$$

we note that $\psi(\theta, \varphi)$ is a spherical harmonic with $l = 1$. This implies that

$$\begin{aligned}
< \hat{L}^2 > &= \int_{d\Omega} \psi^*(\theta, \varphi) \hat{L}^2 \psi(\theta, \varphi) d\Omega \\
&= \int_{d\Omega} \psi^*(\theta, \varphi) \hbar^2 (1)(1+1) \psi(\theta, \varphi) d\Omega \\
&= 2\hbar^2
\end{aligned}$$

2 Problem #2

To find the eigenvalue of the $\hat{L}^2$ operator corresponding to the eigenfunction

$$Y(\theta, \varphi) = A(3\cos^2\theta - 1 + \sin(2\theta)\cos\varphi)$$

we note that $Y(\theta, \varphi)$ is the eigenfunction that corresponds to the sum of two spherical harmonics

$$Y(\theta, \varphi) = A(Y_{20} + 2Y_{21})$$

note both are in the $l = 2$ state. This implies

$$\begin{aligned}
\hat{L}^2 Y(\theta, \varphi) &= A(\hat{L}^2 Y_{20} + 2\hat{L}^2 Y_{21}) \\
&= A(\hbar^2 (2)(2+1) Y_{20} + \hbar^2 (2)(2+1) 2Y_{21}) \\
&= 6\hbar^2 Y(\theta, \varphi)
\end{aligned}$$

3 Problem #3

For a s-state ($l = 0$) particle with at mass, m_0, in an spherically symmetric infinitely deep rectangular potential well of radius a is described by the wave function

$$\psi_{klm} = A j_l(kr) Y_{lm}(\theta, \varphi)$$

for $l = 0$ we have the state

$$\psi_{k00} = A\frac{\sin(kr)}{kr}$$

Where we can find the normalization constant by noting the continuity condition that $\psi_{k00}(r = 0) = \psi_{k00}(r = a) = 0$ which implies that $k = n\pi/a$. Where n becomes the principle quantum number

$$\begin{aligned}
1 &= \int_V \psi^*_{k00}\psi_{k00} dV \\
&= 4\pi \int_0^a A^2 \left(\frac{\sin(kr)}{kr}\right)^2 r^2 dr \\
&= \frac{4\pi a^2}{n^2\pi^2} A^2 \int_0^a \sin^2(kr) dr \\
&= \frac{4a^2}{n^2\pi} A^2 \frac{a}{2} \\
&\Downarrow \\
A &= n\sqrt{\frac{\pi}{2a^3}}
\end{aligned}$$

we can find the expectation value of r of a particle in this state by

$$\begin{aligned}
< r > &= \int_V \psi^*_{k00} r \psi_{k00} dV \\
&= 4\pi \int_0^a A\frac{\sin(kr)}{kr} r A\frac{\sin(kr)}{kr} r^2 dr \\
&= 4\pi \frac{n^2\pi}{2a^3}\frac{a^2}{n^2\pi^2} \int_0^a r\sin^2(kr) dr \\
&= \frac{2}{a}\frac{a^2}{4} = \frac{a}{2}
\end{aligned}$$

which is the same for one dimensional infinite square well which is what we expect. The same follows for $< r^2 >$

$$\begin{aligned}
< r^2 > &= \int_V \psi^*_{k00} r \psi_{k00} dV \\
&= \frac{2}{a}\int_0^a r^2 \sin^2(kr) dr \\
&= \frac{a^2}{6}\left(2 - \frac{3}{(n\pi)^2}\right)
\end{aligned}$$

Which means we can find the variance as

$$< (\Delta r)^2 > = < r >^2 - < r^2 > = \frac{a^2}{12}\left(1 - \frac{6}{(n\pi)^2}\right)$$

4 Problem #4

For a particle of mass, m_0, in the ground state of a spherically symmetric infinitely deep square potential we a wave function given by

$$\psi_{100}(r) = \frac{1}{\sqrt{2a\pi}} \frac{\sin(\pi r/a)}{r}$$

which in order to find the momentum probability distribution we find ψ in momentum representation as

$$\begin{aligned}
\phi_{100}(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int_V \psi_{100}(r) \exp\left(\frac{-ipr}{\hbar}\right) dV \\
&= \frac{4\pi}{2\pi\sqrt{\hbar}} \int_0^a \sin\left(\frac{\pi r}{a}\right) \exp\left(\frac{-ipr}{\hbar}\right) dr \\
&= \frac{2}{\sqrt{\hbar a}} a\pi \frac{1 + e^{-ipa/\hbar}}{\pi^2 - (pa/\hbar)^2} \\
&= 2\pi \sqrt{\frac{a}{\hbar}} \frac{1 + e^{-ipa/\hbar}}{\pi^2 - (pa/\hbar)^2}
\end{aligned}$$

So the momentum probability distribution is given by

$$\begin{aligned}
|\phi(p)|^2 &= \frac{4\pi^2 a}{\hbar} \frac{1 + e^{ipa/\hbar}}{\pi^2 - (pa/\hbar)^2} \frac{1 + e^{-ipa/\hbar}}{\pi^2 - (pa/\hbar)^2} \\
&= \frac{4\pi^2 a}{\hbar} \frac{2 + e^{-ipa/\hbar} + e^{ipa/\hbar}}{(\pi^2 - (pa/\hbar)^2)^2} \\
&= \frac{8\pi^2 a}{\hbar} \frac{1 + \cos(pa/\hbar)}{(\pi^2 - (pa/\hbar)^2)^2}
\end{aligned}$$