Physics 601

Analytical Mechanics

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Homework #7

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1 Problem #1

For a rotating frame we have *Euler's Equation* given by

$$\frac{d\mathbf{L}}{dt} + \boldsymbol{\omega} \times \mathbf{L} = \boldsymbol{\Gamma}^{ext} \tag{1.1}$$

Which for torque-free motion we have $\boldsymbol{\Gamma}^{ext} = 0$. Given that the angular momentum is given by

$$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \cdot \boldsymbol{\omega}$$

where $\overleftrightarrow{\mathbf{I}}$ is the moment of inertia tensor. For an asymmetric top we have $I_1 < I_2 < I_3$ which for torque-free motion gives us three components to equation 1.1 as

$$\begin{aligned}
I_1\dot{\omega}_1 &= \omega_2\omega_3(I_2 - I_3) \\
I_2\dot{\omega}_2 &= \omega_3\omega_1(I_3 - I_1) \\
I_3\dot{\omega}_3 &= \omega_1\omega_2(I_1 - I_2)
\end{aligned}$$

Given the constraint on the motion $2EI_2 = L^2$ and the constants of motion

$$\begin{aligned}
E &= \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) \\
L^2 &= I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2
\end{aligned}$$

We can solve for ω_1 in terms of ω_2 by

$$\begin{aligned}
2EI_2 &= L^2 \\
&\Downarrow \\
I_1I_2\omega_1^2 + I_2^2\omega_2^2 + I_2I_3\omega_3^2 &= I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 \\
\omega_1^2(I_1I_2 - I_1^2) &= \omega_3^2(I_3^2 - I_2I_3) \\
\omega_1^2 &= \frac{I_3^2 - I_2I_3}{I_1I_2 - I_1^2}\omega_3^2
\end{aligned}$$

Using this relation we can get a relation between ω_1 and ω_2 by

$$\begin{aligned}
L^2 &= I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 \\
&\Downarrow \\
L^2 - I_2^2\omega_2^2 &= \left(I_1^2 + \frac{I_1I_2 - I_1^2}{I_3^2 - I_2I_3}I_3^2\right)\omega_1^2 \\
&\Downarrow \\
\omega_1^2 &= \left(L^2 - I_2^2\omega_2^2\right)\left(I_1^2 + \frac{(I_1I_2 - I_1^2)I_3}{I_3 - I_2}\right)^{-1} \\
&= \left(L^2 - I_2^2\omega_2^2\right)\left(\frac{I_1^2I_3 - I_1^2I_2 + (I_1I_2 - I_1^2)I_3}{I_3 - I_2}\right)^{-1} \\
&= \left(L^2 - I_2^2\omega_2^2\right)\left(\frac{I_1I_2I_3 - I_1^2I_2}{I_3 - I_2}\right)^{-1} \\
&= \left(L^2 - I_2^2\omega_2^2\right)\left(\frac{I_1I_2(I_3 - I_1)}{I_3 - I_2}\right)^{-1} \\
&= \left(L^2 - I_2^2\omega_2^2\right)\frac{I_3 - I_2}{I_1I_2(I_3 - I_1)} \\
&= \left(2E - I_2\omega_2^2\right)\frac{I_3 - I_2}{I_1(I_3 - I_1)}
\end{aligned}$$

Next we can use E to find a relation between ω_3 and ω_2 by

$$
\begin{aligned}
2E &= I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 \\
&\Downarrow \\
2E - I_2\omega_2^2 &= \left(I_3 + I_1\frac{I_3^2 - I_2I_3}{I_1I_2 - I_1^2}\right)\omega_3^2 \\
2E - I_2\omega_2^2 &= \left(I_3 + \frac{I_3^2 - I_2I_3}{I_2 - I_1}\right)\omega_3^2 \\
2E - I_2\omega_2^2 &= \left(\frac{I_2I_3 - I_1I_3 + I_3^2 - I_2I_3}{I_2 - I_1}\right)\omega_3^2 \\
2E - I_2\omega_2^2 &= \left(\frac{I_3(I_3 - I_1)}{I_2 - I_1}\right)\omega_3^2 \\
&\Downarrow \\
\omega_3^2 &= (2E - I_2\omega_2^2)\frac{I_2 - I_1}{I_3(I_3 - I_1)}
\end{aligned}
$$

So now we can solve the integral for ω_2

$$
\begin{aligned}
I_2\dot{\omega}_2 &= \omega_3\omega_1(I_3 - I_1) \\
&\Downarrow \\
\dot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\left((2E - I_2\omega_2^2)^2\frac{I_3 - I_2}{I_1(I_3 - I_1)}\frac{I_2 - I_1}{I_3(I_3 - I_1)}\right)^{1/2} \\
&= \frac{2E - I_2\omega_2^2}{I_2}\left(\frac{(I_3 - I_2)(I_2 - I_1)}{I_1I_3}\right)^{1/2} \\
&= \left(\frac{(2E)^2}{L^2} - \omega_2^2\right)\left(\frac{(I_3 - I_2)(I_2 - I_1)}{I_1I_3}\right)^{1/2} \\
\frac{d\omega_2}{dt} &= (\omega_\infty^2 - \omega_2^2)\left(\frac{(I_3 - I_2)(I_2 - I_1)}{I_1I_3}\right)^{1/2} \\
&\Downarrow \\
\int\frac{d\omega_2}{(1 - (\omega_2/\omega_\infty)^2)} &= \int\omega_\infty^2\left(\frac{(I_3 - I_2)(I_2 - I_1)}{I_1I_3}\right)^{1/2} dt \\
\omega_\infty\tanh^{-1}(\omega_2/\omega_\infty) &= \omega_\infty\frac{t}{\tau} \\
&\Downarrow \\
\omega_2(t) &= \omega_\infty\tanh(t/\tau)
\end{aligned}
$$

Note we defined two new variables by

$$
\begin{aligned}
\omega_\infty &\equiv \frac{2E}{L} \\
\tau^{-1} &\equiv \omega_\infty\left(\frac{(I_3 - I_2)(I_2 - I_1)}{I_1I_3}\right)^{1/2}
\end{aligned}
$$

Now we can use the solution for $\omega_2(t)$ to find the solution for ω_1 by noting that

$$\dot{\omega}_2 = \omega_\infty\tau^{-1}\text{sech}^2(t/\tau)$$

So we can solve for ω_1 by

$$
\begin{gathered}
I_2\dot{\omega}_2 = \omega_3\omega_1(I_3 - I_1) \\
\Downarrow \\
I_2\omega_\infty\tau^{-1}\mathrm{sech}^2(t/\tau) = \left(\frac{I_1(I_2 - I_1)}{I_3(I_3 - I_2)}\right)^{1/2}\omega_1^2(I_3 - I_1) \\
\Downarrow \\
\omega_1^2 = \omega_\infty^2\mathrm{sech}^2(t/\tau)\frac{I_2(I_3 - I_2)}{I_1(I_3 - I_1)} \\
\omega_1(t) = \omega_\infty\left(\frac{I_2(I_3 - I_2)}{I_1(I_3 - I_1)}\right)^{1/2}\mathrm{sech}(t/\tau)
\end{gathered}
$$

And we can find ω_3 from ω_1 by

$$
\begin{aligned}
\omega_3 &= \left(\frac{I_1I_2 - I_1^2}{I_3^2 - I_2I_3}\right)^{1/2}\omega_1(t) \\
&= \omega_\infty\left(\frac{I_2(I_3 - I_2)}{I_1(I_3 - I_1)}\right)^{1/2}\left(\frac{I_1I_2 - I_1^2}{I_3^2 - I_2I_3}\right)^{1/2}\mathrm{sech}(t/\tau) \\
&= \omega_\infty\left(\frac{I_2(I_2 - I_1)}{I_3(I_3 - I_1)}\right)^{1/2}\mathrm{sech}(t/\tau)
\end{aligned}
$$

So the solutions are

$$
\begin{aligned}
\omega_1(t) &= \omega_\infty\left(\frac{I_2(I_3 - I_2)}{I_1(I_3 - I_1)}\right)^{1/2}\mathrm{sech}(t/\tau) \\
\omega_2(t) &= \omega_\infty\tanh(t/\tau) \\
\omega_3(t) &= \omega_\infty\left(\frac{I_2(I_2 - I_1)}{I_3(I_3 - I_1)}\right)^{1/2}\mathrm{sech}(t/\tau)
\end{aligned}
$$

We can use these to write

$$
\omega^2(t) = \omega_\infty^2\left(\tanh^2(t/\tau) + \mathrm{sech}^2(t/\tau)\frac{I_2}{I_3 - I_1}\left(\frac{I_2 - I_1}{I_3} + \frac{I_3 - I_2}{I_1}\right)\right)
$$

We note that the constrant $I_1 < I_2 < I_3$ forces the constant multipling the $\mathrm{sech}^2(t/\tau)$ is

$$
C \equiv \frac{I_2}{I_3 - I_1}\left(\frac{I_2 - I_1}{I_3} + \frac{I_3 - I_2}{I_1}\right) > 0
$$

we see that there are three cases for $\omega^2(t)$ the trivial case when $C = 1$ we have $\omega^2(t) = \omega_\infty^2$ which remains constant in time. The second case where $0 < C < 1$ we have the case where $\omega^2(t)$ starts at C and grows to ω_∞^2 as t increases. The third case is when $C > 1$ we have $\omega^2(t)$ again starting at C but decreases to ω_∞^2 as t increases. This implies that as $t \to \infty$ the only axis of rotation becomes ω_2 rotating at a rate of ω_∞.

2 Problem #2

(a) For the torque-free symmetric top we have the potential energy given as

$$V(\theta) = \frac{1}{2I_1}\left(\frac{p_\phi - p_\psi \cos\theta}{\sin\theta}\right)^2$$

we can find the constant θ solutions by solving the equation $V'(\theta_0) = 0$. So we take a derivative of the potential to get

$$\begin{aligned}
V'(\theta) &= \frac{d}{d\theta}\frac{1}{2I_1}\left(\frac{p_\phi - p_\psi \cos\theta}{\sin\theta}\right)^2 \\
&= \frac{1}{2I_1}2\left(\frac{p_\phi - p_\psi \cos\theta}{\sin\theta}\right)\frac{p_\psi \sin^2\theta - (p_\phi - p_\psi\cos\theta)\cos\theta}{\sin^2\theta} \\
&= \frac{1}{I_1\sin^3\theta}\left(p_\phi - p_\psi\cos\theta\right)\left(p_\psi(\sin^2\theta + \cos^2\theta) - p_\phi\cos\theta\right) \\
&= \frac{1}{I_1\sin^3\theta}\left(p_\phi - p_\psi\cos\theta\right)\left(p_\psi - p_\phi\cos\theta\right)
\end{aligned}$$

So we can see that there are two solutions for θ_0 given by

$$\begin{aligned}
p_\phi - p_\psi\cos\theta_0 &= 0 \\
p_\psi - p_\phi\cos\theta_0 &= 0
\end{aligned}$$

(b) These solutions have corresponding potential energies. For $p_\phi = p_\psi\cos\theta_0$ we have

$$V(\theta) = \frac{1}{2I_1}\left(\frac{p_\phi - p_\psi \cos\theta}{\sin\theta}\right)^2 = 0$$

Which corresponds to free rotation with no perturbation. Then for the second solution $p_\psi = p_\phi\cos\theta_0$ we have

$$\begin{aligned}
V(\theta) &= \frac{1}{2I_1}\left(\frac{p_\phi - p_\psi \cos\theta}{\sin\theta}\right)^2 \\
&= \frac{1}{2I_1}\left(\frac{p_\phi - p_\phi \cos^2\theta}{\sin\theta}\right)^2 \\
&= \frac{1}{2I_1}\left(\frac{p_\phi(1 - \cos^2\theta)}{\sin\theta}\right)^2 \\
&= \frac{1}{2I_1}\left(\frac{p_\phi \sin^2\theta}{\sin\theta}\right)^2 \\
&= \frac{p_\phi^2\sin^2\theta}{2I_1}
\end{aligned}$$

Note that this potential corresponds to the kinetic energy in the angle ϕ this would correspond to circular movement around the $\hat{e}_3^0$ axis.

3 Problem #3

(a) For a symmetric top in a gravitational potential we have an effective potential given by

$$V_{eff}(\theta) = \frac{1}{2I_1}\left(\frac{p_\phi - p_\psi\cos\theta}{\sin\theta}\right)^2 + Mgl\cos\theta$$

Noting the result from problem two we can find the constant θ solutions by solving

$$\begin{aligned}
V'_{eff}(\theta) = 0 &= \frac{d}{d\theta}\left(\frac{1}{2I_1}\left(\frac{p_\phi - p_\psi\cos\theta}{\sin\theta}\right)^2 + Mgl\cos\theta\right) \\
&= \frac{1}{I_1\sin^3\theta}\left(p_\phi - p_\psi\cos\theta\right)\left(p_\psi - p_\phi\cos\theta\right) - Mgl\sin\theta
\end{aligned}$$

We note the equation of motion in the angle ϕ given by

$$\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}$$

This equation is useful because it defines the precession motion as a function of θ. Therefore if we find the solutions for $\dot{\phi}$ we can characterize the motion of the top in a gravitational field. This allows us to replace

$$\begin{aligned}
V'_{eff}(\theta) = 0 &= \frac{1}{I_1\sin^3\theta}\left(p_\phi - p_\psi\cos\theta\right)\left(p_\psi - p_\phi\cos\theta\right) - Mgl\sin\theta \\
&\Downarrow \\
0 &= \frac{\dot{\phi}}{\sin\theta}(p_\psi - (I_1\sin^2\theta\dot{\phi} + p_\psi\cos\theta)\cos\theta) - Mgl\sin\theta \\
0 &= \frac{\dot{\phi}}{\sin\theta}(p_\psi(1-\cos^2\theta) - I_1\sin^2\theta\cos\theta\dot{\phi}) - Mgl\sin\theta \\
0 &= \sin\theta\dot{\phi}(p_\psi - I_1\theta\cos\theta\dot{\phi}) - Mgl\sin\theta \\
0 &= I_1\cos\theta\dot{\phi}^2 - p_\psi\dot{\phi} + Mgl
\end{aligned}$$

So we have a quadratic in $\dot{\phi}$ which we can solve by

$$\dot{\phi} = \frac{p_\psi \pm \sqrt{p_\psi^2 - 4I_1\cos\theta Mgl}}{2I_1\cos\theta} = \frac{p_\psi}{2I_1\cos\theta}\left(1 \pm \sqrt{1 - \frac{4I_1\cos\theta Mgl}{p_\psi^2}}\right)$$

We note the condition for real solutions which is given by

$$p_\psi^2 > 4I_1\cos\theta Mgl$$

which states that there exists a minimum angular momentum we need to be spinning at so that this condition is met. Now we can define a unit-less parameter

$$x \equiv \frac{2I_1\cos\theta Mgl}{p_\psi^2}$$

For a small x we can expand to first order in x to find the solutions for $\dot{\phi}$

$$\begin{aligned}
\dot{\phi} &= \frac{p_\psi}{2I_1\cos\theta}\left(1 \pm \sqrt{1-2x}\right) \\
&= \frac{p_\psi}{2I_1\cos\theta}\left(1 \pm 1 - x\right)
\end{aligned}$$

So we have two solutions the first is

$$\begin{aligned}
\dot{\phi} &= \frac{p_\psi}{2I_1\cos\theta}\,(x) \\
&= \frac{p_\psi}{2I_1\cos\theta}\frac{2I_1\cos\theta Mgl}{p_\psi^2} \\
&= \frac{Mgl}{p_\psi}
\end{aligned}$$

and

$$\begin{aligned}
\dot{\phi} &= \frac{p_\psi}{2I_1\cos\theta}\,(2) \\
&= \frac{p_\psi}{I_1\cos\theta}
\end{aligned}$$

We note that the first solution Mgl/p_ψ corresponds to the zero potential solution from problem two and the second solution $p_\psi/I_1\cos\theta$ corresponds to the constant precession solution.

(b) We note the circular solution from the above part

$$\begin{aligned}
V'_{eff}(\theta) = 0 &= \frac{1}{I_1\sin^3\theta}\left(p_\phi - p_\psi\cos\theta\right)\left(p_\psi - p_\phi\cos\theta\right) - Mgl\sin\theta \\
&\Downarrow \\
\left(p_\phi - p_\psi\cos\theta\right) &= \frac{MglI_1\sin^4\theta}{p_\psi - p_\phi\cos\theta}
\end{aligned}$$

We can use this to evaluate the second derivative of $V_{eff}(\theta)$ by

$$\begin{aligned}
V''_{eff}(\theta) &= \frac{-3\cos\theta}{I_1\sin^4\theta}\left(p_\phi - p_\psi\cos\theta\right)\left(p_\psi - p_\phi\cos\theta\right) - Mgl\cos\theta + \frac{1}{I_1\sin^3\theta}\frac{d}{d\theta}\Big(\left(p_\phi - p_\psi\cos\theta\right)\left(p_\psi - p_\phi\cos\theta\right)\Big) \\
&= \frac{-3\cos\theta}{I_1\sin^4\theta}MglI_1\sin^4\theta - Mgl\cos\theta + \frac{1}{I_1\sin^3\theta}\frac{d}{d\theta}\Big(\left(p_\phi - p_\psi\cos\theta\right)\left(p_\psi - p_\phi\cos\theta\right)\Big) \\
&= -4Mgl\cos\theta + \frac{1}{I_1\sin^3\theta}\Big(\left(p_\phi - p_\psi\cos\theta\right)p_\phi\sin\theta + (p_\psi - p_\phi\cos\theta)p_\psi\sin\theta\Big) \\
&= -4Mgl\cos\theta + \frac{1}{I_1\sin^2\theta}\left(p_\phi^2 + p_\psi^2 - 2p_\phi p_\psi\cos\theta\right) \\
&= -4Mgl\cos\theta + \frac{1}{I_1\sin^2\theta}\left(p_\phi^2 + p_\psi^2(\cos^2\theta + \sin^2\theta) - 2p_\phi p_\psi\cos\theta\right) \\
&= \frac{p_\psi^2}{I_1} - 4Mgl\cos\theta + \frac{1}{I_1\sin^2\theta}\left(p_\phi^2 + p_\psi^2\cos^2\theta - 2p_\phi p_\psi\cos\theta\right) \\
&= \frac{p_\psi^2}{I_1} - 4Mgl\cos\theta + \frac{(p_\phi - p_\psi\cos\theta)^2}{I_1\sin^2\theta}
\end{aligned}$$

4 Problem #4

(a) For a symmetric top in a gravitational field with the initial conditions

$$\dot{\phi} = 2\left(\frac{Mgl}{3I_1}\right)^{1/2}, \qquad \theta = \frac{\pi}{3}, \qquad \dot{\theta} = 0, \qquad \dot{\psi} = (3I_1 - I_3)\left(\frac{Mgl}{3I_1 I_3^2}\right)^{1/2}$$

This system has a Lagrangian that is given by system as

$$L = \frac{1}{2}I_1(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + \frac{1}{2}I_3(\dot{\phi}\cos\theta + \dot{\psi})^2 - Mgl\cos\theta$$

We note the lack of ϕ and ψ dependence which implies that there exists conserved momenta in ϕ and ψ given by

$$\begin{aligned}
p_\psi &= \frac{\partial L}{\partial \dot{\psi}} = I_3(\dot{\phi}\cos\theta + \dot{\psi}) \\
p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = I_1\sin^2\theta\dot{\phi} + I_3(\dot{\phi}\cos\theta + \dot{\psi})\cos\theta \\
&= I_1\sin^2\theta\dot{\phi} + p_\psi\cos\theta
\end{aligned}$$

Using the given initial conditions we can calculate the conserved momenta as

$$\begin{aligned}
p_\psi &= I_3(\dot{\phi}\cos\theta + \dot{\psi}) \\
&= I_3\left(\left(2\frac{Mgl}{3I_1}\right)^{1/2}\cos(\pi/3) + (3I_1 - I_3)\left(\frac{Mgl}{3I_1 I_3^2}\right)^{1/2}\right) \\
&= I_3\left(\frac{Mgl}{3I_1}\right)^{1/2}\left(1 + (3I_1 - I_3)\left(\frac{1}{I_3^2}\right)^{1/2}\right) \\
&= I_3\left(\frac{Mgl}{3I_1}\right)^{1/2}\left(1 + 3\frac{I_1}{I_3} - \frac{I_3}{I_3}\right) \\
&= 3I_1\left(\frac{Mgl}{3I_1}\right)^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
p_\phi &= I_1\sin^2\theta\dot{\phi} + p_\psi\cos\theta \\
&= I_1\sin^2(\pi/3)2\left(\frac{Mgl}{3I_1}\right)^{1/2} + 3I_1\left(\frac{Mgl}{3I_1}\right)^{1/2}\cos(\pi/3) \\
&= I_1\left(\frac{Mgl}{3I_1}\right)^{1/2}\left(\frac{3}{2} + \frac{3}{2}\right) \\
&= 3I_1\left(\frac{Mgl}{3I_1}\right)^{1/2}
\end{aligned}$$

We note that the effective potential is given by the $\dot{\phi}^2$ and the $Mgl\cos\theta$ terms as they are the only functions that depend on the non-dotted coordinates. Neglecting the constant terms this

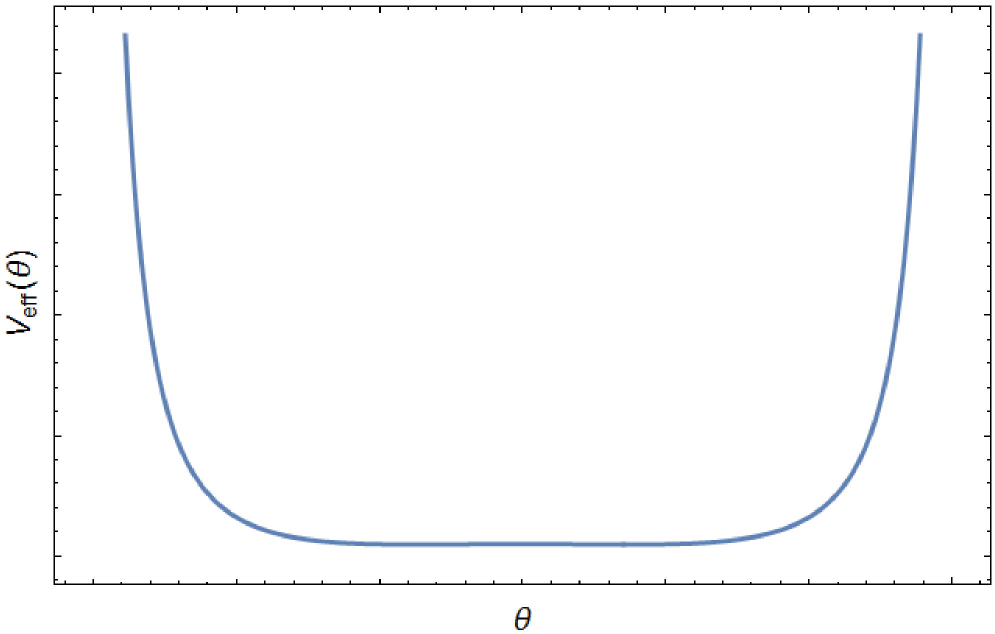

Figure 1: Plot of effective potential normalized to Mgl/I_1 from $0 \leq \theta \leq \pi$.

is given by

$$\begin{aligned}
V_{eff}(\theta) &= \frac{1}{2I_1}\left(\frac{p_\phi - p_\psi \cos\theta}{\sin\theta}\right)^2 + Mgl\cos\theta \\
&= \frac{1}{2I_1\sin^2\theta}\left(3I_1\left(\frac{Mgl}{3I_1}\right)^{1/2} - 3I_1\left(\frac{Mgl}{3I_1}\right)^{1/2}\cos\theta\right)^2 + Mgl\cos\theta \\
&= \frac{9I_1^2}{2I_1\sin^2\theta}\left(\frac{Mgl}{3I_1}\right)(1-\cos\theta)^2 + Mgl\cos\theta \\
&= \frac{3Mgl}{2\sin^2\theta}(1-\cos\theta)^2 + Mgl\cos\theta
\end{aligned}$$

We plot this potential shown in figure 1. We note that this potential implies that θ will stabilize around $\theta = 0$ as the potential goes to infinity at π and $-\pi$.

(b) From the above part we know that the total energy is given by

$$E = \frac{1}{2}I_1\dot{\theta}^2 + \frac{3Mgl}{2\sin^2\theta}(1-\cos\theta)^2 + Mgl\cos\theta$$

we note that this is a constant which we can solve for by using the given initial conditions

$$\begin{aligned}
E &= \frac{1}{2}I_1\dot{\theta}^2 + \frac{3Mgl}{2\sin^2\theta}(1-\cos\theta)^2 + Mgl\cos\theta \\
&\Downarrow \\
&= \cancelto{0}{\frac{1}{2}I_1\dot{\theta}^2} + \frac{3Mgl}{2\sin^2(\pi/3)}(1-\cos(\pi/3))^2 + Mgl\cos(\pi/3) \\
&= \frac{1}{2}Mgl + \frac{1}{2}Mgl = Mgl
\end{aligned}$$

So we can solve for $\dot{\theta}^2$ as

$$\begin{aligned}
\dot{\theta}^2 &= 2\frac{Mgl}{I_1} - \frac{3Mgl}{I_1 \sin^2\theta}\left(1-\cos\theta\right)^2 - 2\frac{Mgl}{I_1}\cos\theta \\
&\Downarrow \\
\sin^2\theta\dot{\theta}^2 &= \frac{Mgl}{I_1}\left(2\sin^2\theta(1-\cos\theta) - 3\left(1-\cos\theta\right)^2\right) \\
\sin^2\theta\dot{\theta}^2 &= \frac{Mgl}{I_1}\left(2(1-\cos^2\theta)(1-\cos\theta) - 3\left(1-\cos\theta\right)^2\right)
\end{aligned}$$

Now we can change variables to $u = \cos\theta$ noting that $\dot{u}^2 = \sin^2\theta\dot{\theta}^2$ this gives us the equation of motion

$$\begin{aligned}
\dot{u}^2 &= \frac{Mgl}{I_1}\left(2(1-u^2)(1-u) - 3\left(1-u\right)^2\right) \\
&= \frac{Mgl}{I_1}\left(2 - 2u - 2u^2 + 2u^3 - 3 - 3u^2 + 6u\right) \\
&= \frac{Mgl}{I_1}\left(-1 + 4u - 5u^2 + 2u^3\right) \\
&= \frac{Mgl}{I_1}(1-u)^2(2u-1)
\end{aligned}$$

Next we can solve the differential equation for u by a separation of variables

$$\begin{aligned}
\frac{du}{(1-u)(2u-1)^{1/2}} &= \left(\frac{Mgl}{I_1}\right)^{1/2} dt \\
&\Downarrow \\
2\tanh^{-1}(\sqrt{2u-1}) &= \left(\frac{Mgl}{I_1}\right)^{1/2} t \\
&\Downarrow \\
2u - 1 &= \tanh^2\left(\frac{1}{2}\left(\frac{Mgl}{I_1}\right)^{1/2} t\right) \\
u &= \frac{1}{2}\left(\frac{\cosh(Mgl/I_1 t) - 1}{\cosh(Mgl/I_1 t) + 1} + 1\right) \\
u &= \frac{\cosh(Mgl/I_1 t)}{\cosh(Mgl/I_1 t) + 1} \\
&\Downarrow \\
\sec\theta &= 1 + \frac{1}{\cosh(Mgl/I_1 t)} \\
\sec\theta &= 1 + \operatorname{sech}\left(\frac{Mgl}{I_1} t\right)
\end{aligned}$$

We see that for large t this solution settles at $\theta = 0$ which agrees with the qualitative result in part (a).