Physics 601

Analytical Mechanics

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Homework #12

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1 Problem #1

(a) For the generator

$$F_2(q,P) = qP + \epsilon H(q,P)$$

where the Hamiltonian is separable and of the form

$$H(q,p) = \frac{p^2}{2m} + V(q)$$

we have the Canonical Transformations given by

$$\begin{aligned} Q &= q + \epsilon\frac{p}{m} - \frac{\epsilon^2}{m}\frac{dV}{dq} \\ P &= p - \epsilon\frac{dV}{dq} \end{aligned}$$

This allows us to calculate the *Jacobian matrix of transformation*, M, by

$$M = \begin{pmatrix} \frac{dQ}{dq} & \frac{dQ}{dp} \\ \frac{dP}{dq} & \frac{dP}{dp} \end{pmatrix} = \begin{pmatrix} 1 - \frac{\epsilon^2}{m}\frac{d^2V}{dq^2} & \frac{\epsilon}{m} \\ -\epsilon\frac{d^2V}{dq^2} & 1 \end{pmatrix}$$

This allows us to calculate the determinant of M as

$$\begin{aligned} \det(M) &= \left(1 - \frac{\epsilon^2}{m}\frac{d^2V}{dq^2}\right) - \frac{\epsilon}{m}\left(-\epsilon\frac{d^2V}{dq^2}\right) \\ &= 1 - \frac{\epsilon^2}{m}\frac{d^2V}{dq^2} + \frac{\epsilon^2}{m}\frac{d^2V}{dq^2} = 1 \end{aligned}$$

Therefore $\det(M) = 1$ as we expect for a canonical transformation.

(b) We can repeat this for the generator

$$F_3(Q,P) = -pQ + \epsilon H(Q,p)$$

which has the canonical transformation given by

$$\begin{aligned} Q &= q + \epsilon\frac{p}{m} - \frac{\epsilon^2}{m}\frac{dV}{dq} \\ P &= p - \epsilon\frac{dV}{dq} \end{aligned}$$

We note that the *Poisson Brackets* give the $\det(M)$ so that we can calculate

$$\begin{aligned} \det(M) = \{Q,P\} &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \\ &= 1 - \frac{\epsilon^2}{m}\frac{d^2V}{dq^2} + \frac{\epsilon^2}{m}\frac{d^2V}{dq^2} \\ &= 1 \end{aligned}$$

2 Problem #2

(a) For a Lie operator $\hat{S}$ we are given the identity

$$\hat{S}(fg) = (\hat{S}f)g + f(\hat{S}g)$$

which we can use to show that

$$e^{\epsilon\hat{S}}(fg) = (e^{\epsilon\hat{S}}f)(e^{\epsilon\hat{S}}g).$$

First we expand the exponential to get

$$e^{\epsilon\hat{S}} = 1 + \epsilon\hat{S} + \frac{1}{2}(\epsilon\hat{S})^2 + \frac{1}{3!}(\epsilon\hat{S})^3 + ...$$

So we act the expansion on the product (fg) to get

$$\begin{aligned}
e^{\epsilon\hat{S}}(fg) &= \left(1 + \epsilon\hat{S} + \frac{1}{2}(\epsilon\hat{S})^2 + \frac{1}{3!}(\epsilon\hat{S})^3 + ...\right)(fg) \\
&= fg + \epsilon\hat{S}(fg) + \frac{1}{2}(\epsilon\hat{S})^2(fg) + \frac{1}{3!}(\epsilon\hat{S})^3(fg) + ... \\
&= fg + \epsilon(\hat{S}f)g + \epsilon f(\hat{S}g) + \frac{1}{2}\epsilon^2\hat{S}\left((\hat{S}f)g + f(\hat{S}g)\right) + ... \\
&= fg + \epsilon(\hat{S}f)g + \epsilon f(\hat{S}g) + \frac{1}{2}\epsilon^2\left((\hat{S}^2f)g + 2\hat{S}f(\hat{S}g) + f(\hat{S}^2g)\right) + ... \\
&= fg + \epsilon(\hat{S}f)g + \epsilon f(\hat{S}g) + \epsilon^2\hat{S}f(\hat{S}g) + \frac{1}{2}\epsilon^2(\hat{S}^2f)g + \frac{1}{2}\epsilon^2 f(\hat{S}^2g) + ... \\
&= \left(f + \epsilon\hat{S}f + \frac{1}{2}\epsilon^2\hat{S}^2f + ...\right)\left(g + \epsilon\hat{S}g + \frac{1}{2}\epsilon^2\hat{S}^2g + ...\right) \\
&= (e^{\epsilon\hat{S}}f)(e^{\epsilon\hat{S}}g)
\end{aligned}$$

(b) Given

$$\hat{S}\{f,g\} = \{\hat{S}f, g\} + \{f, \hat{S}g\}$$

we can show that

$$\begin{aligned}
e^{\epsilon\hat{S}}\{f,g\} &= \left(1 + \epsilon\hat{S} + \frac{1}{2}(\epsilon\hat{S})^2 + \frac{1}{3!}(\epsilon\hat{S})^3 + ...\right)\{f,g\} \\
&= \left(\{f,g\} + \epsilon\hat{S}\{f,g\} + \frac{1}{2}(\epsilon\hat{S})^2\{f,g\} + \frac{1}{3!}(\epsilon\hat{S})^3\{f,g\} + ...\right) \\
&= \{f,g\} + \epsilon\{\hat{S}f,g\} + \epsilon\{f,\hat{S}g\} + \frac{1}{2}\epsilon^2\{f,\hat{S}^2g\} + \frac{1}{2}\epsilon^2\{\hat{S}^2f,g\} + ... \\
&= \left\{\left(f + \epsilon\hat{S}f + \frac{\epsilon^2}{2}\hat{S}^2f + ...\right), \left(g + \epsilon\hat{S}g + \frac{\epsilon^2}{2}\hat{S}^2g + ...\right)\right\} \\
&= \{e^{\epsilon\hat{S}}f, e^{\epsilon\hat{S}}g\}
\end{aligned}$$

3 Problem #3

For the harmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2$$

we note that the Lie operator is given as

$$\hat{H} = \{\cdot, H\} = \frac{p}{m}\frac{\partial}{\partial q} - kq\frac{\partial}{\partial p}$$

This allows us to calculate the transformations

$$q(t) = e^{t\hat{H}}q = \left(1 + t\hat{H} + \frac{1}{2}t^2\hat{H}^2 + \frac{1}{3!}t^2\hat{H}^3 + ...\right)q$$
$$p(t) = e^{t\hat{H}}p = \left(1 + t\hat{H} + \frac{1}{2}t^2\hat{H}^2 + \frac{1}{3!}t^2\hat{H}^3 + ...\right)p$$

Where we take q and p to be the initial position and momentum. So we can calculate $q(t)$ as

$$\begin{aligned}
q(t) &= q + t\hat{H}q + \frac{1}{2}t^2\hat{H}^2q + \frac{1}{3!}t^3\hat{H}^3q + ... \\
&= q + t\frac{p}{m} + \frac{1}{2}t^2\hat{H}\frac{p}{m} + \frac{1}{3!}t^3\hat{H}^2\frac{p}{m} + ... \\
&= q + t\frac{p}{m} - \frac{1}{2}t^2\frac{k}{m}q - \frac{1}{3!}t^3\hat{H}\frac{k}{m}q + ... \\
&= q + t\frac{p}{m} - \frac{1}{2}t^2\frac{k}{m}q - \frac{1}{3!}t^3\frac{k}{m}\frac{p}{m} + ... \\
&= q\left(1 - \frac{1}{2}\left(\sqrt{\frac{k}{m}}t\right)^2 + \frac{1}{4!}\left(\sqrt{\frac{k}{m}}t\right)^4 + ...\right) + \frac{p}{m}\sqrt{\frac{m}{k}}\left(t - \frac{1}{3!}\left(\sqrt{\frac{k}{m}}t\right)^3 + ...\right) \\
&= q\cos(\omega t) + \frac{p}{m\omega}\sin(\omega t)
\end{aligned}$$

Where we define $\omega^2 = k/m$. We repeat for $p(t)$ as

$$\begin{aligned}
p(t) &= p + t\hat{H}p + \frac{1}{2}t^2\hat{H}^2p + \frac{1}{3!}t^3\hat{H}^3p + ... \\
&= p - tkq - \frac{1}{2}t^2\hat{H}kq - \frac{1}{3!}t^3\hat{H}^2kq + ... \\
&= p - tkq - \frac{1}{2}t^2\omega^2p - \frac{1}{3!}t^3\hat{H}\omega^2p + ... \\
&= p - tkq - \frac{1}{2}t^2\omega^2p + \frac{1}{3!}t^3\omega^2kq + ... \\
&= p\left(1 - \frac{1}{2}(\omega t)^2 + ...\right) - \frac{qk}{\omega}\left(t - \frac{1}{3!}(\omega t)^2 + ...\right) \\
&= p\cos(\omega t) - qm\omega\sin(\omega t)
\end{aligned}$$

So we note that the transformation given by $e^{t\hat{H}}$ yields the exact solution.

4 Problem #4

For the same harmonic oscillator in problem #3 we can use the transformation

$$
\begin{aligned}
q(t) &= \mathcal{T} q \\
p(t) &= \mathcal{T} p
\end{aligned}
$$

Where we take

$$
\mathcal{T} = \exp\left(\frac{1}{2}t\hat{T}\right)\exp\left(t\hat{V}\right)\exp\left(\frac{1}{2}t\hat{T}\right)
$$

we note that the transformation $e^{t\hat{H}}$ yields the exact solution. We can write this to second order in t as

$$
\begin{aligned}
e^{t\hat{H}} = e^{t(\hat{V}+\hat{T})} &= 1 + t(\hat{V}+\hat{T}) + \frac{1}{2}t^2(\hat{V}+\hat{T})^2 + \mathcal{O}(t^3) \\
&= 1 + t(\hat{V}+\hat{T}) + \frac{1}{2}t^2(\hat{V}^2 + \hat{V}\hat{T} + \hat{T}\hat{V} + \hat{T}^2) + \mathcal{O}(t^3)
\end{aligned}
$$

We can see that if we expand $\mathcal{T}$ to second order we see that

$$
\begin{aligned}
\mathcal{T} &= \left(1 + t\frac{1}{2}\hat{T} + \frac{1}{2}t^2\frac{1}{2}\hat{T}^2\right)\left(1 + t\hat{V} + \frac{1}{2}t^2\hat{V}^2\right)\left(1 + t\frac{1}{2}\hat{T} + \frac{1}{2}t^2\frac{1}{2}\hat{T}^2\right) + \mathcal{O}(t^3) \\
&= \left(1 + t\frac{1}{2}\hat{T} + \frac{1}{2}t^2\frac{1}{2}\hat{T}^2\right)\left(1 + \frac{1}{2}t\hat{T} + \frac{1}{4}t^2\hat{T}^2 + t\hat{V} + \frac{1}{2}t^2\hat{V}\hat{T} + \frac{1}{2}t^2\hat{V}^2\right) + \mathcal{O}(t^3) \\
&= 1 + t\frac{1}{2}\hat{T} + \frac{1}{2}t^2\frac{1}{2}\hat{T}^2 + t\hat{V} + \frac{1}{2}t^2\hat{V}\hat{T} + \frac{1}{2}t^2\hat{V}^2 + \frac{1}{2}t\hat{T} + \frac{1}{4}t\hat{T}^2 + \frac{1}{2}t^2\hat{T}\hat{V} + \mathcal{O}(t^3) \\
&= 1 + t(\hat{T}+\hat{V}) + \frac{1}{2}t^2\left(\frac{3}{2}\hat{T}^2 + \hat{V}\hat{T} + \hat{T}\hat{V} + \hat{V}^2\right) + \mathcal{O}(t^3)
\end{aligned}
$$

We note that the $\hat{T}^2$ term has an additional factor, but this term is zero when acting on q or p. Therefore we can say that

$$
\begin{aligned}
e^{t\hat{H}} q &= \mathcal{T} q + \mathcal{O}(t^3) \\
e^{t\hat{H}} p &= \mathcal{T} p + \mathcal{O}(t^3)
\end{aligned}
$$

which implies that the transformation $\mathcal{T}$ solves the simple harmonic motion to second order in t.

Bonus Problem

For the transformation given by

$$\mathcal{T}\left(\frac{t}{2-s}\right)\mathcal{T}\left(\frac{-st}{2-s}\right)\mathcal{T}\left(\frac{t}{2-s}\right)$$

we can show that this solves the harmonic oscillator to fourth order in t by noting that the operators $\hat{T}^n$ and $\hat{V}^n$ are zero when acting on p or q for this system. Therefore we can expand neglecting those terms to get

$$\begin{aligned}
e^{t\hat{H}} &= 1 + t(\hat{V}+\hat{T}) + \frac{1}{2}t^2(\hat{V}\hat{T}+\hat{T}\hat{V}) + \frac{1}{3!}t^3(\hat{V}\hat{T}+\hat{T}\hat{V})(\hat{T}+\hat{V}) + \frac{1}{4!}t^4(\hat{V}\hat{T}+\hat{T}\hat{V})(\hat{T}+\hat{V})^2 + \mathcal{O}(t^5)\\
&= 1 + t(\hat{V}+\hat{T}) + \frac{1}{2}t^2(\hat{V}\hat{T}+\hat{T}\hat{V}) + \frac{1}{3!}t^3(\hat{V}\hat{T}\hat{V}+\hat{T}\hat{V}\hat{T}) + \frac{1}{4!}t^4(\hat{V}\hat{T}\hat{V}+\hat{T}\hat{V}\hat{T})(\hat{T}+\hat{V}) + \mathcal{O}(t^5)\\
&= 1 + t(\hat{V}+\hat{T}) + \frac{1}{2}t^2(\hat{V}\hat{T}+\hat{T}\hat{V}) + \frac{1}{3!}t^3(\hat{V}\hat{T}\hat{V}+\hat{T}\hat{V}\hat{T}) + \frac{1}{4!}t^4(\hat{V}\hat{T}\hat{V}\hat{T}+\hat{T}\hat{V}\hat{T}\hat{V}) + \mathcal{O}(t^5)\\
&= 1 + t\hat{V} + \frac{1}{2}t^2\hat{T}\hat{V} + \frac{1}{3!}t^3\hat{V}\hat{T}\hat{V} + \frac{1}{4!}t^4\hat{T}\hat{V}\hat{T}\hat{V} + \mathcal{O}(t^5)
\end{aligned}$$

Note we neglected terms with $\hat{T}$ on the right as those terms go to zero for the q transformation. Now we can expand our new transformation again neglecting the higher power terms to get

$$\begin{aligned}
\mathcal{T}\left(\frac{t}{2-s}\right) &= 1 + \frac{t}{2-s}(\hat{T}+\hat{V}) + \frac{t^2}{2(2-s)^2}(\hat{V}\hat{T}+\hat{T}\hat{V}) + \frac{t^3}{4(2-s)^3}\hat{T}\hat{V}\hat{T}\\
\mathcal{T}\left(\frac{-st}{2-s}\right) &= 1 - \frac{st}{2-s}(\hat{T}+\hat{V}) + \frac{s^2t^2}{2(2-s)^2}(\hat{V}\hat{T}+\hat{T}\hat{V}) - \frac{s^3t^3}{4(2-s)^3}\hat{T}\hat{V}\hat{T}
\end{aligned}$$

So the product of these to fourth order in t yields

$$\begin{aligned}
&\mathcal{T}\left(\frac{-st}{2-s}\right)\mathcal{T}\left(\frac{t}{2-s}\right)\\
&= \left(1 - \frac{st}{2-s}(\hat{T}+\hat{V}) + \frac{s^2t^2}{2(2-s)^2}(\hat{V}\hat{T}+\hat{T}\hat{V}) - \frac{s^3t^3}{4(2-s)^3}\hat{T}\hat{V}\hat{T}\right)\left(1 + \frac{t}{2-s}\hat{V} + \frac{t^2}{2(2-s)^2}\hat{T}\hat{V}\right)\\
&= 1 + t\left(\frac{1}{(2-s)}\hat{V} - \frac{s}{(2-s)}\hat{V}\right) + t^2\left(\frac{s^2}{2(2-s)^2}\hat{T}\hat{V} + \frac{1}{2(2-s)^2}\hat{T}\hat{V} - \frac{2s}{2(2-s)^2}\hat{T}\hat{V}\right)\\
&\qquad + t^3\left(\frac{s^2}{2(2-s)^3}\hat{V}\hat{T}\hat{V} - \frac{s}{2(2-s)^3}\hat{V}\hat{T}\hat{V}\right) + t^4\left(\frac{s^2}{4(2-s)^4}\hat{T}\hat{V}\hat{T}\hat{V} - \frac{s^3}{4(2-s)^4}\hat{T}\hat{V}\hat{T}\hat{V}\right) + \mathcal{O}(t^5)\\
&= 1 + \frac{(1-s)t}{2-s}\hat{V} + \frac{(1-s)^2t^2}{(2-s)^2}\hat{T}\hat{V} + \frac{(s^2-s)t^3}{2(2-s)^3}\hat{V}\hat{T}\hat{V} - \frac{(s^3+s^2)t^4}{4(2-s)^4}\hat{T}\hat{V}\hat{T}V + \hat{\mathcal{O}}(t^5)
\end{aligned}$$

Again we neglected the terms with $\hat{T}$ on the right. Next we forward multiply $\mathcal{T}(t/(2-s))$ to yield

$$\begin{aligned}
&\mathcal{T}\left(\frac{t}{2-s}\right)\mathcal{T}\left(\frac{-st}{2-s}\right)\mathcal{T}\left(\frac{t}{2-s}\right)\\
&= \left(1 + \frac{t}{2-s}(\hat{T}+\hat{V}) + \frac{t^2}{2(2-s)^2}(\hat{V}\hat{T}+\hat{T}\hat{V}) + \frac{t^3}{4(2-s)^3}\hat{T}\hat{V}\hat{T}\right)\\
&\qquad \times\left(1 + \frac{t(1-s)}{2-s}\hat{V} + \frac{(1-s)^2t^2}{(2-s)^2}\hat{T}\hat{V} + \frac{(s^2-s)t^3}{2(2-s)^3}\hat{V}\hat{T}\hat{V} - \frac{(s^3+s^2)t^4}{4(2-s)^4}\hat{T}\hat{V}\hat{T}\hat{V}\right) + \mathcal{O}(t^5)\\
&= 1 + t\left(\frac{1-s}{2-s}\hat{V} + \frac{1}{2-s}\hat{V}\right) + t^2\left(\frac{2(1-s)^2}{2(2-s)^2}\hat{T}\hat{V} + \frac{1}{2(2-s)^2}\hat{T}\hat{V} + \frac{2(1-s)}{(2-s)^2}\hat{T}\hat{V}\right)\\
&\qquad + t^3\left(\frac{(1-s)^2}{(2-s)^4}\hat{V}\hat{T}\hat{V} + \frac{1-s}{4(2-s)^4}\hat{V}\hat{T}\hat{V} + \frac{(1-s)}{(2-s)^2}\hat{V}\hat{T}\hat{V}\right) + t^4\left(\frac{(s^2-s)}{2(2-s)^4}\hat{T}\hat{V}\hat{T}\hat{V} - \frac{(s^3+s^2)}{4(2-s)^4}\hat{T}\hat{V}\hat{T}\hat{V}\right) + \mathcal{O}(t^5)\\
&= 1 + t\hat{V} + \frac{1}{2}t^2\hat{T}\hat{V} + \frac{1}{3!}t^3\hat{V}\hat{T}\hat{V} + \frac{1}{4!}t^4\hat{T}\hat{V}\hat{T}\hat{V} + \mathcal{O}(t^5) \qquad \text{for} \quad s = 2^{1/3}
\end{aligned}$$