Physics 615

Methods of Theoretical Physics I

Professor Katrin Becker

Homework #13

Joe Becker
UID: 125-00-4128
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1 Problem #1

Given the monomials

$$1, x, x^2, x^3, ..,$$

we can construct four orthogonal functions in the range $-1 < x < 1$ by taking the inner product of each function with the condition

$$\langle f_n(x)|f_m(x)\rangle = 0 \text{ for } n \neq m$$

the first functions we can pick as $f_0(x) = 1$ and $f_1(x) = x$ such that

$$\begin{aligned}\int_{-1}^{1} f_0(x) f_1(x) dx &= \int_{-1}^{1} x dx \\ &= \frac{1}{2}x^2\Big|_{-1}^{1} \\ &= \frac{1}{2} - \frac{1}{2} = 0\end{aligned}$$

Now we use a general form $f_2(x) = a + bx + cx^2$ and find a, b, c by the orthogonal condition

$$\begin{aligned}\int_{-1}^{1} f_0(x) f_2(x) dx = 0 &= \int_{-1}^{1} a + bx + cx^2 dx \\ &= ax + \frac{b}{2}x^2 + \frac{c}{3}x^3\Big|_{-1}^{1} \\ &= a + \frac{b}{2} + \frac{c}{3} + a - \frac{b}{2} + \frac{c}{3} \\ &\Downarrow \\ -a &= \frac{c}{3}\end{aligned}$$

and

$$\begin{aligned}\int_{-1}^{1} f_1(x) f_2(x) dx = 0 &= \int_{-1}^{1} ax + bx^2 + cx^3 dx \\ &= \frac{a}{2}x^2 + \frac{b}{3}x^3 + \frac{c}{4}x^4\Big|_{-1}^{1} \\ &\Downarrow \\ b &= 0\end{aligned}$$

So we pick $a = 1$ which yields

$$f_2(x) = 3x^2 - 1$$

now we repeat for $f_3(x) = a + bx + cx^2 + dx^3$ and we find

$$\begin{aligned}\int_{-1}^{1} f_0(x) f_3(x) dx = 0 &= \int_{-1}^{1} a + bx + cx^2 + dx^3 dx \\ &= ax + \frac{b}{2}x^2 + \frac{c}{3}x^3 + \frac{d}{4}x^4\Big|_{-1}^{1} \\ &= a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4} + a - \frac{b}{2} + \frac{c}{3} - \frac{d}{4} \\ &\Downarrow \\ -a &= \frac{c}{3}\end{aligned}$$

and

$$\begin{aligned}
\int_{-1}^{1} f_1(x)f_3(x)dx = 0 &= \int_{-1}^{1} ax + bx^2 + cx^3 + dx^4 dx \\
&= \frac{a}{2}x^2 + \frac{b}{3}x^3 + \frac{c}{4}x^4 + \frac{d}{5}x^5 \Big|_{-1}^{1} \\
&= \frac{2b}{3} + \frac{2d}{5} \\
-5b &= 3d
\end{aligned}$$

and

$$\int_{-1}^{1} f_2(x)f_3(x)dx = 0 = \int_{-1}^{1} (a + bx + cx^2 + dx^3)(3x^2 - 1)dx$$

$$\Downarrow$$

$$c = 0$$

So we pick $b = -3$ and we have

$$f_3(x) = 5x^3 - 3x$$

we note that these polynomials are the Legendre polynomials without the normalization constant.

2 Problem #2

Given the differential equation

$$p(x)y'' + r(x)y' + q(x)y + \lambda\rho(x)y = 0$$

we can multiply it by a integrating factor

$$F(x) = \exp\left[\int^x \frac{r(u) - p'(u)}{p(u)}\right] du$$

where we note that

$$\frac{d}{dx}\big(F(x)p(x)y'\big) = F(x)p(x)y'' + F(x)r(x)y'$$

which yields

$$F(x)p(x)y'' + F(x)r(x)y' + F(x)q(x)y + \lambda F(x)\rho(x)y = \frac{d}{dx}\left(F(x)p(x)\frac{dy}{dx}\right) + F(x)q(x)y + \lambda F(x)\rho(x)y$$

Which is is Strum-Liouville form.