Physics 601

Analytical Mechanics

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Homework #8

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1 Problem #1

(a) For a mass, m, moving in a central potential, $V(r)$, we have the Lagrangian in spherical coordinates

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - V(r)$$

this allows us to calculate the canonical momenta p_r, p_θ, and p_ϕ by taking the derivative of the Lagrangian with respect to the generalized coordinates $\dot{r}$, $\dot{\theta}$, and $\dot{\phi}$. So we calculate

$$\begin{aligned}
p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} \\
p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \\
p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\dot{\phi}
\end{aligned}$$

(b) Using the canonical momenta found in part (a) we can derive the Hamiltonian, H, for a central potential in spherical coordinates by

$$H = \sum_i p_i\dot{q}_i - L$$

where we write $\dot{q}_i$ in terms of the generalized momenta

$$\begin{aligned}
H &= p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) + V(r) \\
&\Downarrow \\
H &= p_r\frac{p_r}{m} + p_\theta\frac{p_\theta}{mr^2} + p_\phi\frac{p_\phi}{mr^2\sin^2\theta} - \frac{1}{2}m\left(\left(\frac{p_r}{m}\right)^2 + r^2\left(\frac{p_\theta}{mr^2}\right)^2 + r^2\sin^2\theta\left(\frac{p_\phi}{mr^2\sin^2\theta}\right)^2\right) + V(r) \\
&= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + V(r)
\end{aligned}$$

(c) Now that we have the Hamiltonian we can use *Hamilton's Equations*

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{1.1}$$

to find the equations of motion. So for θ we have

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}$$

Which, as expected, is the equation from the canonical momentum. The second equation of motion in θ is given by

$$\begin{aligned}
\dot{p}_\theta = -\frac{\partial H}{\partial \theta} &= -\frac{p_\phi^2}{2mr^2}\frac{d}{d\theta}\left(\frac{1}{\sin^2\theta}\right) \\
&= \frac{p_\phi^2}{mr^2}\frac{\cos\theta}{\sin^3\theta}
\end{aligned}$$

Now we can calculate the equations of motion for ϕ as

$$\dot{\phi} = \frac{\partial H}{\partial p_\theta} = \frac{p_\phi}{mr^2\sin^2\theta}$$

and

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$$

this implies that p_ϕ is a constant of motion. Now for the radial equation of motion

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

and

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} + \frac{p_\phi^2}{mr^3 \sin^2\theta} + \frac{dV(r)}{dr}$$

(d) We note that in general there is only a single conserved quantity which follows from $\dot{p}_\phi = 0$ which implies that there the canonical momentum, p_ϕ, is conserved.

2 Problem #2

For a spherical pendulum in which a particle of mass, m, in a gravitational field constrained to move on the surface of a sphere of radius, l. As we found in problem 1 we have a Hamiltonian in spherical coordinates as

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2\theta} + V(r, \theta, \phi)$$

where we note that $p_r = 0$ because we are fixed to the surface of the sphere. Note that the height of particle in the potential is given by $l - l\cos\theta$ which implies that our potential is

$$V(\theta) = mgl(1 - \cos\theta) = -mgl\cos\theta$$

note that we shifted the zero potential point down by mgl without loss of generality. So our Hamiltonian is

$$H = \frac{p_\theta^2}{2ml^2} + \frac{p_\phi^2}{2ml^2 \sin^2\theta} - mgl\cos\theta$$

We note that like in problem one there is no ϕ dependence therefore p_ϕ is a conserved quantity. Which allows us to find the equations of motion in θ and ϕ by

$$\begin{aligned}
\dot{p}_\phi &= -\frac{\partial H}{\partial \phi} = 0 \\
\dot{\phi} &= \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{ml^2 \sin^2\theta} \\
\dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2}{ml^2}\frac{\cos\theta}{\sin^3\theta} - mgl\sin\theta \\
\dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}
\end{aligned}$$

Now we can solve this Hamiltonian by expanding about a constant angle θ_0 to second order where we note the solution for $\dot{p}_\theta = 0$ which implies that

$$p_\phi^2 = \frac{m^2 g l^3 \sin^4\theta_0}{\cos\theta_0}$$

for constant circular motion where we note the expansion

$$f(\theta) = f(\theta_0) + f'(\theta_0)(\theta - \theta_0) + \frac{1}{2}f''(\theta_0)(\theta - \theta_0)^2$$

So we expand $\sin^{-2}\theta$ about a small perturbation $\theta = \theta_0 + \delta\theta$ noting that $\delta\theta = \theta - \theta_0$

$$\sin^{-2}\theta = \sin^{-2}\theta_0 - 2\frac{\cos\theta_0}{\sin^3\theta_0}\delta\theta + \frac{\cos(2\theta_0) + 2}{\sin^4\theta_0}\delta\theta^2 + \mathcal{O}(\delta\theta^3)$$

and $\cos\theta$ as

$$\cos\theta = \cos\theta_0 - \sin\theta_0\delta\theta - \cos\theta_0\delta\theta^2 + \mathcal{O}(\delta\theta^3)$$

So our Hamiltonian becomes

$$H = \frac{p_\theta^2}{2ml^2} + \frac{p_\phi^2}{2ml^2}\left(\sin^{-2}\theta_0 - 2\frac{\cos\theta_0}{\sin^3\theta_0}\delta\theta + \frac{\cos(2\theta_0) + 2}{\sin^4\theta_0}\delta\theta^2\right) - mgl(\cos\theta_0 - \sin\theta_0\delta\theta - \cos\theta_0\delta\theta^2)$$

So we can find the equation of motion for the perturbation $\delta\theta$ by noting that $p_\theta = ml^2\delta\dot{\theta}$. So we can find the equation of motion

$$\begin{aligned}
\dot{p}_\theta = ml^2\delta\ddot{\theta} = -\frac{\partial H}{\partial\delta\theta} &= -\frac{p_\phi^2}{2ml^2}\left(-2\frac{\cos\theta_0}{\sin^3\theta_0} + 2\frac{\cos(2\theta_0)+2}{\sin^4\theta_0}\delta\theta\right) + mgl(\sin\theta_0 + \cos\theta_0\delta\theta) \\
&\Downarrow \\
\delta\ddot{\theta} &= -\frac{p_\phi^2}{2m^2l^4}\left(-2\frac{\cos\theta_0}{\sin^3\theta_0} + 2\frac{\cos(2\theta_0)+2}{\sin^4\theta_0}\delta\theta\right) + \frac{g}{l}(\sin\theta_0 + \cos\theta_0\delta\theta) \\
&= -\frac{g\sin^4\theta_0}{2l\cos\theta_0}\left(-2\frac{\cos\theta_0}{\sin^3\theta_0} + 2\frac{\cos(2\theta_0)+2}{\sin^4\theta_0}\delta\theta\right) + \frac{g}{l}(\sin\theta_0 + \cos\theta_0\delta\theta) \\
&= -\frac{g}{l}\left(\frac{\cos(2\theta_0)+2}{\cos\theta_0} + \cos\theta_0\right)\delta\theta \\
&= -\frac{g}{l}\left(\frac{2\cos^(\theta_0) - 1 + 2 + \cos^2\theta_0}{\cos\theta_0}\right)\delta\theta \\
&= -\frac{g}{l\cos\theta_0}\left(1 + 3\cos\theta_0\right)\delta\theta
\end{aligned}$$

So we have simple harmonic motion with an oscillation frequency

$$\omega^2 = \frac{g}{l\cos\theta_0}\left(1 + 3\cos\theta_0\right)$$

3 Problem #3

(a) For a relativistic particle in a static potential $V(\mathbf{r})$ we have a Lagrangian

$$L = -mc^2\sqrt{1 - v^2/c^2} - V(\mathbf{r})$$

where we note that $v = \dot{\mathbf{r}}$ so our equations of motion from the Lagrangian equations are

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial v} &= \frac{d}{dt}\left(-mc^2\frac{1}{2}(1 - v^2/c^2)^{-1/2}(-2v/c^2)\right)\\ &= m\left(\dot{v}(1 - v^2/c^2)^{-1/2} + v^2/c^2(1 - v^2/c^2)^{-3/2}\dot{v}\right) \quad = m\dot{v}(1 - v^2/c^2)^{-1/2}\left(1 + v^2/c^2(1 - v^2/c^2)^{-1}\right)\end{aligned}$$

And the derivative with respect to $\mathbf{r}$ is

$$\frac{\partial L}{\partial \mathbf{r}} = -\frac{dV}{d\mathbf{r}}$$

so we have the equation of motion

$$m\dot{v}(1 - v^2/c^2)^{-1/2}\left(1 + v^2/c^2(1 - v^2/c^2)^{-1}\right) = -\frac{dV}{d\mathbf{r}}$$

note in the non-relativistic limit where $v << c$ this equation becomes the classical result.

(b) We can find the canonical momentum $\mathbf{p}$ as

$$\mathbf{p} = \frac{\partial L}{\partial v} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}$$

which allows us to calculate the Hamiltonian by first solving for $\dot{\mathbf{r}}$ as

$$\begin{aligned}\dot{\mathbf{r}} = \mathbf{v} &= \frac{\mathbf{p}\sqrt{1 - v^2/c^2}}{m}\\ &\Downarrow\\ v^2 &= \frac{p^2}{m^2}\left(1 - \frac{v^2}{c^2}\right)\\ &\Downarrow\\ v^2 + v^2\frac{p^2}{m^2c^2} &= \frac{p^2}{m^2}\\ &\Downarrow\\ \mathbf{v} &= \frac{\mathbf{p}}{m\sqrt{1 + (p/mc)^2}}\end{aligned}$$

So now we can solve the Hamiltonian noting that $(1 - v^2/c^2)^{1/2} = (1 + (p/mc)^2)^{-1/2}$

$$\begin{aligned}H &= \mathbf{p}\cdot\mathbf{v} - L\\ &\Downarrow\\ &= \mathbf{p}\cdot\frac{\mathbf{p}}{m\sqrt{1 + (p/mc)^2}} + \frac{mc^2}{\sqrt{1 + (p/mc)^2}} + V(\mathbf{r})\\ &= \frac{p^2mc^2}{m\sqrt{m^2c^4 + p^2c^2}} + \frac{m^2c^4}{\sqrt{m^2c^4 + p^2c^2}} + V(\mathbf{r})\\ &= \frac{m^2c^4 + p^2c^2}{\sqrt{m^2c^4 + p^2c^2}} + V(\mathbf{r})\\ &= \sqrt{m^2c^4 + p^2c^2} + V(\mathbf{r})\end{aligned}$$

We see that H does not depend explicitly on time, t. Therefore it is a constant of motion.

(c) For a spherically symmetric potential we have $V(\mathbf{r}) \to V(r)$. This implies that our potential is independent of θ and ϕ. This implies that the motion of the particle is constrained to planer motion. This reduces our Hamiltonian to

$$H = c^2\sqrt{m^2c^4 + p_r^2 + r^{-2}p_\theta^2} + V(\mathbf{r})$$

We note that the Hamiltonian is cyclic in θ which implies that p_θ is a conserved quantity. We note that for planer motion $\mathbf{r} \times \mathbf{p} = p_\theta$ therefore we know the angular momentum, $\mathbf{r} \times \mathbf{p}$, is conserved.

4 Problem #4

(a) For a particle of charge, e, moving in a electromagnetic field $\Phi = 0$ and $\mathbf{A} = \hat{z}A_z(x, y, t)$ we can note that

$$\mathbf{E} = \nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\frac{1}{c}\frac{\partial A_z}{\partial t}$$

and

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\partial A_z}{\partial y}\hat{x} - \frac{\partial A_z}{\partial x}\hat{y}$$

which allows us to construct the explicit first integral through *the Lorentz-Force Equation*

$$m\ddot{\mathbf{r}} = e\left[\mathbf{E} + \frac{1}{c}\dot{\mathbf{r}} \times \mathbf{B}\right] \tag{4.1}$$

where we can calculate

$$\dot{\mathbf{r}} \times \mathbf{B} = \frac{\partial A_z}{\partial x}\dot{z}\hat{x} + \frac{\partial A_z}{\partial y}\dot{z}\hat{y} + \left(-\frac{\partial A_z}{\partial dx}\dot{x} - \frac{\partial A_z}{\partial dy}\dot{y}\right)\hat{z}$$

So we can take the z component of equation 4.1 as

$$\begin{aligned}
m\ddot{z} &= -\frac{e}{c}\left(\frac{\partial A_z}{\partial t} + \frac{\partial A_z}{\partial x}\dot{x} + \frac{\partial A_z}{\partial y}\dot{y}\right)\\
\ddot{z} &= -\frac{e}{cm}\frac{dA_z}{dt}\\
&\Downarrow\\
\frac{d}{dt}\left(\dot{z} + \frac{e}{cm}A_z\right) &= 0\\
&\Downarrow\\
\dot{z} + \frac{e}{cm}A_z &= C
\end{aligned}$$

This gives us a equation of motion in a propagation in the z direction.

(b) Now we can use this result to get the motion in the directions perpendicular to the propagation x and y. By grouping the $\hat{x}$ and $\hat{y}$ terms of equation 4.1

$$\begin{aligned}
m\ddot{\mathbf{r}}_\perp &= \frac{e}{c}\left(\frac{\partial A_z}{\partial x}\dot{z}\hat{x} + \frac{\partial A_z}{\partial y}\dot{z}\hat{y}\right)\\
&\Downarrow\\
\ddot{\mathbf{r}}_\perp &= \frac{e}{cm}\left(\frac{\partial A_z}{\partial x}\left(C - \frac{e}{cm}A_z\right)\hat{x} + \frac{\partial A_z}{\partial y}\left(C - \frac{e}{cm}A_z\right)\hat{y}\right)\\
&= \frac{e}{cm}\left(\frac{1}{2}\frac{cm}{e}\frac{\partial}{\partial x}\left(C - \frac{e}{cm}A_z\right)^2\hat{x} + \frac{1}{2}\frac{cm}{e}\frac{\partial}{\partial y}\left(C - \frac{e}{cm}A_z\right)^2\hat{y}\right)\\
&= \frac{1}{2}\nabla_\perp\left(C - \frac{e}{cm}A_z\right)^2
\end{aligned}$$

(c) For a uniform magnetic field given by $\mathbf{B} = B_0\hat{x}$ we can apply the results from the above parts by looking at

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\partial A_z}{\partial y}\hat{x} - \frac{\partial A_z}{\partial x}\hat{y}$$

which implies that

$$\frac{\partial A_z}{\partial y} = B_0$$
$$\frac{\partial A_z}{\partial x} = 0$$

So up to an additive constant which we can neglect we have

$$A_z = B_0 y$$

Therefore our perpendicular motion becomes

$$\begin{aligned}\ddot{\mathbf{r}}_\perp &= \frac{1}{2}\nabla_\perp \left(C - \frac{eB_0}{cm} y\right)^2 \\ &= \frac{eB_0}{cm}\left(C - \frac{eB_0}{cm} y\right)\hat{y}\end{aligned}$$

This yields the equations of motion in x and y as

$$\begin{aligned}\ddot{x} &= 0 \\ \ddot{y} &= \frac{CeB_0}{cm} - \left(\frac{eB_0}{cm}\right)^2 y\end{aligned}$$

This implies that we have a constant motion in x and oscillatory motion in y which has a result

$$\begin{aligned}x(t) &= v_0 t \\ y(t) &= \frac{C}{\omega} + A\cos(\omega t)\end{aligned}$$

where we define the angular frequency as

$$\omega \equiv \frac{eB_0}{cm}$$

we can use this result to find the motion in z as

$$\begin{aligned}\dot{z} &= C - \frac{eB_0}{cm}\left(\frac{C}{\omega} + A\cos(\omega t)\right) \\ &= -A\omega\cos(\omega t) \\ &\Downarrow \\ z(t) &= -A\sin(\omega t)\end{aligned}$$

So we have the total equations of motion

$$x(t) = v_0 t, \qquad y(t) = \frac{C}{\omega} + A\cos(\omega t), \qquad z(t) = -A\sin(\omega t)$$

which corresponds to helical motion about x shifted to an new equilibrium position at $y = -C/\omega$.