Physics 611

Electromagnetic Theory II

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Homework #1

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1 Problem #1

(a) For the tensor in three dimensions

$$M_{ij} = \delta_{ij}\cos\alpha + n_i n_j(1-\cos\alpha) + \epsilon_{ijk}n_k\sin\alpha \tag{1.1}$$

where n_i is a unit vector. Given the identity

$$\epsilon_{ijm}\epsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \tag{1.2}$$

where δ_{ij} is the *Kronecker delta* and ϵ_{ijk} is the *Levi-Civita symbol* we can prove the orthogonality of the tensor given by equation 1.1 by

$$\begin{aligned}
M_{ij}M_{ik} &= (\delta_{ij}\cos\alpha + n_i n_j(1-\cos\alpha) + \epsilon_{ijl}n_l\sin\alpha)(\delta_{ik}\cos\alpha + n_i n_k(1-\cos\alpha) + \epsilon_{ikm}n_m\sin\alpha) \\
&= \delta_{ij}\delta_{ik}\cos^2\alpha + n_i n_j n_i n_k(1-\cos\alpha)^2 + \epsilon_{ijl}\epsilon_{ikm}n_l n_m\sin^2\alpha \\
&\quad + \delta_{ij}n_i n_k(1-\cos\alpha)\cos\alpha + \delta_{ik}n_i n_j(1-\cos\alpha)\cos\alpha \\
&\quad + \delta_{ij}\epsilon_{ikm}n_m\sin\alpha\cos\alpha + \delta_{ik}\epsilon_{ijl}n_l\sin\alpha\cos\alpha \\
&\quad + \epsilon_{ikm}n_i n_j n_m(1-\cos\alpha)\sin\alpha + \epsilon_{ijl}n_i n_k n_l(1-\cos\alpha)\sin\alpha \\
&= \delta_{jk}\cos^2\alpha + n_i n_j n_i n_k(1-\cos\alpha)^2 + (\delta_{jk}\delta_{lm} - \delta_{jm}\delta_{lk})n_l n_m\sin^2\alpha + 2n_j n_k(1-\cos\alpha)\cos\alpha \\
&\quad + \epsilon_{jkm}n_m\sin\alpha\cos\alpha + \epsilon_{kjm}n_m\sin\alpha\cos\alpha + \epsilon_{ikm}n_i n_j n_m(1-\cos\alpha)\sin\alpha + \epsilon_{ijl}n_i n_k n_l(1-\cos\alpha)\sin\alpha \\
&= \delta_{jk}\cos^2\alpha + n_i n_j n_i n_k(1-\cos\alpha)^2 + (\delta_{jk}n_m n_m - n_j n_k)\sin^2\alpha + 2n_j n_k(1-\cos\alpha)\cos\alpha \\
&\quad + \cancel{\epsilon_{jkm}n_m\sin\alpha\cos\alpha} - \cancel{\epsilon_{jkm}n_m\sin\alpha\cos\alpha} + \epsilon_{ikm}n_i n_j n_m(1-\cos\alpha)\sin\alpha + \epsilon_{ijl}n_i n_k n_l(1-\cos\alpha)\sin\alpha \\
&= \delta_{jk}\cancel{\cos^2\alpha} + n_j n_k(1-\cos\alpha)^2 + \delta_{jk}\cancel{\sin^2\alpha} - n_j n_k\sin^2\alpha + 2n_j n_k(1-\cos\alpha)\cos\alpha \\
&\quad + \cancelto{0}{\epsilon_{ikm}n_i n_m}n_j(1-\cos\alpha)\sin\alpha + \cancelto{0}{\epsilon_{ijm}n_i n_m}n_k(1-\cos\alpha)\sin\alpha \\
&= \delta_{jk} + n_j n_k + n_j n_k\cos^2\alpha - n_j n_k\sin^2\alpha - \cancel{2n_j n_k\cos\alpha} + \cancel{2n_j n_k\cos\alpha} - 2n_j n_k\cos^2\alpha \\
&= \delta_{jk} + n_j n_k - n_j n_k(\cos^2\alpha + \sin^2\alpha) \\
&= \delta_{jk}
\end{aligned}$$

Note that we used the fact that n_1 is a unit vector with implies $n_i n_i = 1$ and that $\epsilon_{ijk}n_i n_j = 0$ by the fact that $\epsilon_{ijk}n_i n_j = -\epsilon_{jik}n_i n_j$ for all no zero values of ϵ_{ijk} therefore all non-zero terms will cancel within the sum.

(b) For the special case where the unit vector, n_i points along the $\hat{z}$ direction we note that $n_1 = n_2 = 0$ and $n_3 = 1$ we can see the M_{ij} are

$$\begin{aligned}
M_{11} &= \delta_{11}\cos\alpha + n_1 n_1(1-\cos\alpha) + \epsilon_{11k}n_k\sin\alpha = \cos\alpha \\
M_{12} &= \delta_{12}\cos\alpha + n_1 n_2(1-\cos\alpha) + \epsilon_{12k}n_k\sin\alpha = \sin\alpha \\
M_{13} &= \delta_{13}\cos\alpha + n_1 n_3(1-\cos\alpha) + \epsilon_{13k}n_k\sin\alpha = 0 \\
M_{21} &= \delta_{21}\cos\alpha + n_2 n_1(1-\cos\alpha) + \epsilon_{21k}n_k\sin\alpha = -\sin\alpha \\
M_{22} &= \delta_{22}\cos\alpha + n_2 n_2(1-\cos\alpha) + \epsilon_{22k}n_k\sin\alpha = \cos\alpha \\
M_{23} &= \delta_{23}\cos\alpha + n_2 n_3(1-\cos\alpha) + \epsilon_{23k}n_k\sin\alpha = 0 \\
M_{31} &= \delta_{31}\cos\alpha + n_3 n_1(1-\cos\alpha) + \epsilon_{31k}n_k\sin\alpha = 0 \\
M_{32} &= \delta_{32}\cos\alpha + n_3 n_2(1-\cos\alpha) + \epsilon_{32k}n_k\sin\alpha = 0 \\
M_{33} &= \delta_{33}\cos\alpha + n_3 n_3(1-\cos\alpha) + \epsilon_{33k}n_k\sin\alpha = 1
\end{aligned}$$

Where if we write M_{ij} in matrix form we see that this corresponds to a rotation around the z-axes.

$$M = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2 Problem #2

(a) Given that any two-index Cartesian 3-tensor, T_{ij} can be viewed as a matrix, T, with rows labelled by i columns labelled by j, we can rewrite the following equations

$$D_{ij} = A_{ik}B_{kl}C_{jl} \qquad D_{ij} = A_{ik}B_{jl}C_{kl} \qquad D_{ij} = A_{ik}(B_{kj} + C_{jk}) \tag{2.1}$$

by noting that $M_{ik}N_{kj}$ represents a standard matrix multiplication and that $M_{ik}N_{jk}$ represents matrix multiplication where N is taken as a transpose. This implies that equation 2.2 can be written as

$$D = (A \cdot B) \cdot C^T \qquad D = (A \cdot C) \cdot B^T \qquad D = A \cdot (B + C^T)$$

(b) For any 3×3 matrix, W, with components W_{ij} we can show that

$$W_{il}W_{jm}W_{kn}\epsilon_{lmn} = (\det W)\epsilon_{ijk} \tag{2.2}$$

by first proving the antisymmetry of the left hand side. Note if we interchange the indices by $i \leftrightarrow j$ then we have $W_{jl}W_{im}W_{kn}\epsilon_{lmn}$. Now, now we are free to change the dummy indices freely so we can write

$$W_{jl}W_{im}W_{kn}\epsilon_{lmn} \Rightarrow W_{jm}W_{il}W_{kn}\epsilon_{mln} = -W_{il}W_{jm}W_{kn}\epsilon_{lmn}$$

Note that this follows for any interchange of ijk. For the case where any of the indices ijk are equal we can take the sum of all the non-zero values for lmn as

$$W_{i1}W_{j2}W_{k3} + W_{i3}W_{j1}W_{k2} + W_{i2}W_{j3}W_{k1} - W_{i3}W_{j2}W_{k1} - W_{i1}W_{j3}W_{k2} - W_{i2}W_{j1}W_{k3}$$

we can see that if any of the free index ijk are equal then each positive term will have an exact negative term which implies that for any ijk equal we have a zero value. Therefore we see that the left hand side is antisymmetric which implies that it must be proportional to ϵ_{ijk}. To find the constant of proportionality we can take a non-zero case where ijk are all different which we can write as

$$W_{11}W_{22}W_{33} + W_{13}W_{21}W_{32} + W_{12}W_{23}W_{31} - W_{13}W_{22}W_{31} - W_{11}W_{23}W_{32} - W_{12}W_{21}W_{33}$$
$$\Downarrow$$
$$W_{11}(W_{22}W_{33} - W_{23}W_{32}) + W_{12}(W_{23}W_{31} - W_{21}W_{33}) + W_{13}(W_{21}W_{32} - W_{22}W_{31}) = \det(W)$$

and by the antisymmetry we already have shown we see that for any combination of ijk where they are not equal we will have $\pm\det(W)$ therefore we see that equation 2.2 is true.

(c) Given an antisymmetric 4-tensor, $A_{\mu\nu}$ and a symmetric 4-tensor, $S_{\mu\nu}$ we can preform a *Lorentz transformation* using ${\Lambda^\mu}_\nu$ such that

$$A'_{\mu\nu} = {\Lambda^\sigma}_\mu {\Lambda^\rho}_\nu A_{\sigma\rho}$$
$$S'_{\mu\nu} = {\Lambda^\sigma}_\mu {\Lambda^\rho}_\nu S_{\sigma\rho}$$

we can see that if we interchange the indices $\mu \leftrightarrow \nu$ we have

$$\begin{aligned} A'_{\nu\mu} &= {\Lambda^\sigma}_\nu {\Lambda^\rho}_\mu A_{\sigma\rho} \\ &= {\Lambda^\rho}_\nu {\Lambda^\sigma}_\mu A_{\rho\sigma} \\ &= {\Lambda^\rho}_\nu {\Lambda^\sigma}_\mu (-A_{\sigma\rho}) \\ &= -{\Lambda^\sigma}_\mu {\Lambda^\rho}_\nu A_{\sigma\rho} = A'_{\mu\nu} \end{aligned}$$

Therefore the *Lorentz transformation* preserves antisymmetry. The same follows the symmetric tensor, $A_{\mu\nu}$

$$S'_{\nu\mu} = {\Lambda^\sigma}_\nu {\Lambda^\rho}_\mu S_{\sigma\rho} = {\Lambda^\rho}_\nu {\Lambda^\sigma}_\mu S_{\rho\sigma} = {\Lambda^\rho}_\nu {\Lambda^\sigma}_\mu S_{\sigma\rho} = {\Lambda^\sigma}_\mu {\Lambda^\rho}_\nu S_{\sigma\rho} = S'_{\mu\nu}$$

Note that we renamed the dummy indices ρ and σ.

3 Problem #3

Given the constant 4-vector k_μ such that

$$\phi \equiv e^{ik_\mu x^\mu} \tag{3.1}$$

we can find the condition on k_μ that solves the wave equation

$$\Box\phi = 0$$

where $\Box$ is the *d'Alembertian operator* defined as

$$\Box \equiv \partial_\mu \partial^\mu = -\partial_0^2 + \partial_i^2 \tag{3.2}$$

So if we apply equation 3.2 to equation 3.1 we find that

$$\begin{aligned}
\Box\phi = 0 &= (-\partial_0^2 + \partial_i^2)e^{ik_\mu x^\mu} \\
&= -(ik_0)^2 e^{ik_\mu} + (ik_1)^2 e^{ik_\mu} + (ik_2)^2 e^{ik_\mu} + (ik_3)^2 e^{ik_\mu} \\
&= (k_0^2 - k_1^2 - k_2^2 - k_3^2)e^{ik_\mu} \\
&\Downarrow \\
0 &= -k_0^2 + k_1^2 + k_2^2 + k_3^2 \\
&\Downarrow \\
k_\mu k^\mu &= 0
\end{aligned}$$

Therefore the magnitude of k_μ must be zero in order for equation 3.1 to satisfy the wave equation.

4 Problem #4

(a) For the case where we take two successive Lorentz boots along the x axis, one with velocity, v_1, and the other with velocity, v_2 we use the transformation for a pure boost in x as

$$x' = \gamma(x - vt), \qquad y' = y, \qquad z' = z, \qquad t' = \gamma(t - vx) \tag{4.1}$$

where we define

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2}} \tag{4.2}$$

note that we are using natural units. Using equation 4.1 we can see that the first boost at v_1 transforms as

$$x' = \gamma_1(x - v_1 t), \qquad y' = y, \qquad z' = z, \qquad t' = \gamma_1(t - v_1 x)$$

Now if we boost again at v_2 we have the transformation in x

$$\begin{aligned}
x'' &= \gamma_2(x' - v_2 t') \\
&= \gamma_2((\gamma_1(x - v_1 t) - v_2\gamma_1(t - v_1 x)) \\
&= \gamma_2\gamma_1(x - v_1 t - v_2 t + v_1 v_2 x) \\
&= \gamma_2\gamma_1((1 + v_1 v_2)x - (v_1 + v_2)t)
\end{aligned}$$

and t

$$\begin{aligned}
t'' &= \gamma_2(t' - v_2 x') \\
&= \gamma_2(\gamma_1(t - v_1 x) - v_2\gamma_1(x - v_1 t)) \\
&= \gamma_2\gamma_1(t - v_1 x - v_2 x + v_2 v_1 t) \\
&= \gamma_2\gamma_1((1 + v_1 v_2)t - (v_1 + v_2)x)
\end{aligned}$$

Now if we repeat the same boost but in the reverse order we first take a boost of v_2 which yields

$$x' = \gamma_2(x - v_2 t), \qquad y' = y, \qquad z' = z, \qquad t' = \gamma_2(t - v_2 x)$$

next we take another boost in v_1 which yields a transform in x

$$\begin{aligned}
x'' &= \gamma_1(x' - v_1 t') \\
&= \gamma_1((\gamma_2(x - v_2 t) - v_1\gamma_2(t - v_2 x)) \\
&= \gamma_1\gamma_2(x - v_2 t - v_1 t + v_1 v_2 x) \\
&= \gamma_1\gamma_2((1 + v_1 v_2)x - (v_1 + v_2)t)
\end{aligned}$$

and t

$$\begin{aligned}
t'' &= \gamma_1(t' - v_2 x') \\
&= \gamma_1(\gamma_2(t - v_2 x) - v_2\gamma_2(x - v_2 t)) \\
&= \gamma_1\gamma_2(t - v_2 x - v_2 x + v_2 v_1 t) \\
&= \gamma_1\gamma_2((1 + v_1 v_2)t - (v_1 + v_2)x)
\end{aligned}$$

We see that two successive boosts of v_1 and v_2 commute.

(b) Taking the results from part (a) we see that we can write the total combined boost as

$$\begin{aligned} t'' &= \gamma_1\gamma_2((1+v_1v_2)t-(v_1+v_2)x) \\ &= \frac{1}{\sqrt{1-v_1^2}}\frac{1}{\sqrt{1-v_2^2}}(1+v_1v_2)\left(t-\frac{v_1+v_2}{1+v_1v_2}x\right) \\ &= \sqrt{\frac{1+2v_1v_2+(v_1v_2)^2}{1-v_2^2-v_1^2+(v_1v_2)^2}}\left(t-\frac{v_1+v_2}{1+v_1v_2}x\right) \\ &= \frac{1}{1-((v_1+v_2)/(1+v_1v_2))^2}\left(t-\frac{v_1+v_2}{1+v_1v_2}x\right) \qquad = \frac{1}{1-v_3^2}(t-v_3x) \end{aligned}$$

We see that we can write two successive boosts as a single boost with a velocity of v_3 where we define v_3 with the velocity addition formula given by

$$v_3 \equiv \frac{v_1+v_2}{1+v_1v_2} \tag{4.3}$$

(c) Now if we take two successive boost, but now we take the first in the x direction, $\vec{v}_1 = (v_1, 0, 0)$, then the second in the y direction, $\vec{v}_2 = (0, v_2, 0)$ the first boost transforms as

$$x' = \gamma_1(x - v_1t), \qquad y' = y, \qquad z' = z, \qquad t' = \gamma_1(t - v_1x)$$

now the second transformation transforms as

$$x'' = \gamma_1(x - v_1t), \qquad y'' = \gamma_2(y - v_2t'), \qquad z'' = z, \qquad t'' = \gamma_1(t' - v_2y)$$

we note that for t'' and y'' we need to account for the x boost by

$$\begin{aligned} y'' &= \gamma_2(y - v_2t') \\ &= \gamma_2(y - v_2\gamma_1(t - v_1x)) \end{aligned}$$

and for t'' we have

$$\begin{aligned} t'' &= \gamma_2(t' - v_2y) \\ &= \gamma_2(\gamma_1(t - v_1x) - v_2y) \end{aligned}$$

Now, if we reverse the boosts we see that the transformation becomes

$$x' = x, \qquad y' = \gamma_1(y - v_1t), \qquad z' = z, \qquad t' = \gamma_1(t - v_1y)$$

which in turn makes the second boost into the transformation

$$x'' = \gamma_2(x - v_2\gamma_1(t - v_1y)), \qquad y'' = \gamma_1(y - v_1t), \qquad z'' = z, \qquad t'' = \gamma_2(\gamma_1(t - v_1y) - v_2x)$$

We see that these two transformations mix the coordinates x and y into the double boosted frame which depends on the order of the boosts. Therefore the boosts do no commute.

(d) Given the results from part (c)

$$y'' = \gamma_2(y - v_2\gamma_1(t - v_1x)), \qquad t'' = \gamma_2(\gamma_1(t - v_1x) - v_2y)$$

we see that we cannot write these in form of equation 4.1 because we have both x and y coordinates mixed. Therefore no matter what velocity we choose we cannot make place it in the form of a pure boost. The only way to make it a pure boost is too mix the x and y coordinates into a new coordinate y' this describes a rotation.