Physics 611

Electromagnetic Theory II

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Homework #3

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1 Problem #1

Given the *Lorentz force equation*

$$m\frac{d^2x^\mu}{d\tau^2} = eF^\mu{}_\nu\frac{dx^\nu}{d\tau} \tag{1.1}$$

in the special case where $\mathbf{E} = (E, 0, 0)$ and $\mathbf{B} = 0$ where E is a constant we can solve for the components of x^μ as functions of the proper time, τ. Note that the field tensor in this special case has a zero value for all components except for

$$F_{01} = -E \qquad F_{10} = E$$

note that when we raise the first index we have

$$F^\mu{}_\nu = \eta^{\mu\sigma}F_{\sigma\nu} \Rightarrow F^1{}_0 = F^0{}_1 = E$$

Using this fact we can see that for each value for the free index, μ, we have the equation

$$m\frac{d^2x^0}{d\tau^2} = eE\frac{dx^1}{d\tau} \qquad m\frac{d^2x^1}{d\tau^2} = eE\frac{dx^0}{d\tau} \qquad m\frac{d^2x^2}{d\tau^2} = 0 \qquad m\frac{d^2x^3}{d\tau^2} = 0$$

This allows us to integrate with respect to $d\tau$ which yields

$$m\frac{dx^0}{d\tau} = p^0 = eEx^1 \qquad m\frac{dx^1}{d\tau} = p^1 = eEx^0 \qquad m\frac{dx^2}{d\tau} = p^2 = A \qquad m\frac{dx^3}{d\tau} = p^3 = B$$

Note that we wrote the equation in terms of the 4-momentum p^μ which is by definition $p^\mu = m\frac{dx^\mu}{d\tau}$. This allows us to use the fact that

$$p^\mu p_\mu = -m^2 \tag{1.2}$$

Now we can choose $B = 0$ without a loss of generality due to the fact that the electric field only lies in the $\hat{e}^1$ direction therefore we can rotate freely about this axis. So we can choose a rotation such that $p^3 = 0$. We can further rename $A = p_0$ as it represents the total momentum initial (when $x^0 = 0$). Therefore by equation 1.2 we have

$$-(p^0)^2 + (p^1)^2 + (p^2)^2 + \cancelto{0}{(p^3)^2} = -m^2$$
$$\Downarrow$$
$$p^0 = \sqrt{m^2 + (p^1)^2 + (p_0)^2}$$
$$\Downarrow$$
$$p^0 = \sqrt{\mathcal{E}_0^2 + (eEx^0)^2}$$

Note that we defined the initial energy as $\mathcal{E}_0^2 \equiv m^2 + p_0^2$. This allows us to solve the differential equation

$$m\frac{dx^0}{d\tau} = p^0 = \sqrt{\mathcal{E}_0^2 + (eEx^0)^2}$$
$$\Downarrow$$
$$m\int\frac{dx^0}{\sqrt{\mathcal{E}_0^2 + (eEx^0)^2}} = \int d\tau$$
$$\frac{m}{eE}\operatorname{arcsinh}\left(\frac{eE}{\mathcal{E}_0}x^0\right) = \tau$$
$$\Downarrow$$
$$x^0(\tau) = \frac{\mathcal{E}_0}{eE}\sinh\left(\frac{eE}{m}\tau\right)$$

Next we can solve for $x^1(\tau)$ using the above result.

$$\begin{aligned}
m\frac{dx^1}{d\tau} &= eEx^0 \\
&\Downarrow \\
\int dx^1 &= \int \frac{eE}{m}\frac{\mathcal{E}_0}{eE}\sinh\left(\frac{eE}{m}\tau\right)d\tau \\
x^1(\tau) &= \frac{\mathcal{E}_0}{m}\cosh\left(\frac{eE}{m}\tau\right)\frac{m}{eE} \\
&= \frac{\mathcal{E}_0}{eE}\cosh\left(\frac{eE}{m}\tau\right)
\end{aligned}$$

Note for x^2 and x^3 we can easily write the equations of motion as

$$\begin{aligned}
x^2(\tau) &= \frac{p_0}{m}\tau \\
x^3(\tau) &= z_0
\end{aligned}$$

Where we define any initial position, z_0, in the $\hat{e}^3$ direction without loss of generality. So we can solve τ in terms of x^2 as $\tau = \frac{m}{p_0}x^2$ which if we replace into the equation $x^1(0)$ we find

$$\begin{aligned}
x^1(x^2) &= \frac{\mathcal{E}_0}{eE}\cosh\left(\frac{eE}{m}\frac{m}{p_0}x^2\right) \\
&= \frac{\mathcal{E}_0}{eE}\cosh\left(\frac{eE}{p_0}x^2\right)
\end{aligned}$$

which recovers the result using the non-convariant Lorentz force equation.

2 Problem #2

(a) Given the definition of the *Hodge dual*

$$ {}^*F_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \tag{2.1}$$

we can write the *Bianchi identity*

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0 \tag{2.2}$$

can be written using equation 2.1 by

$$\partial^\mu\, {}^*F_{\mu\nu} = \frac{1}{2}\partial^\mu \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

We can see how this follows by setting the free index $\nu = 0$ and the only terms that are non-zero are

$$\begin{aligned}
\frac{1}{2}\partial^\mu \epsilon_{\mu 0\rho\sigma}F^{\rho\sigma} &= \frac{1}{2}\partial^1\epsilon_{1023}F^{23} + \frac{1}{2}\partial^1\epsilon_{1032}F^{32} + \frac{1}{2}\partial^2\epsilon_{2013}F^{13} + \frac{1}{2}\partial^2\epsilon_{2031}F^{13} + \frac{1}{2}\partial^3\epsilon_{3021}F^{21} + \frac{1}{2}\partial^3\epsilon_{3012}F^{21} \\
&= \frac{1}{2}\partial^1 F^{23} - \frac{1}{2}\partial^1 F^{32} + \frac{1}{2}\partial^2 F^{13} - \frac{1}{2}\partial^2 F^{13} + \frac{1}{2}\partial^3 F^{21} - \frac{1}{2}\partial^3 F^{21} \\
&= \frac{1}{2}\partial^1 F^{23} + \frac{1}{2}\partial^1 F^{23} + \frac{1}{2}\partial^2 F^{13} + \frac{1}{2}\partial^2 F^{31} + \frac{1}{2}\partial^3 F^{21} + \frac{1}{2}\partial^3 F^{12} \\
&= \partial^1 F^{23} + \partial^2 F^{13} + \partial^3 F^{21}
\end{aligned}$$

So, we see that for a fixed value of ν we cycle through the remaining free index values. Therefore, for all values of ν we cover all 12 possible combinations. This implies that

$$\partial^\mu\, {}^*F_{\mu\nu} = \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu}$$

which allows us to write equation 2.2 as

$$\partial^\mu\, {}^*F_{\mu\nu} = 0$$

(b) Given the vector defined as

$$V^\mu \equiv \epsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma} \tag{2.3}$$

where we define A_ν using the *Lorentz gauge* which implies that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.4}$$

we can calculate

$$\begin{aligned}
\partial_\mu V^\mu &= \epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu F_{\rho\sigma} \\
&= \epsilon^{\mu\nu\rho\sigma}(F_{\mu\nu} + \partial_\nu A_\mu)F_{\rho\sigma} \\
&= \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + \epsilon^{\mu\nu\rho\sigma}\partial_\nu A_\mu F_{\rho\sigma} \\
&= \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + \partial_\mu\epsilon^{\nu\mu\rho\sigma}A_\nu F_{\rho\sigma} \\
&= \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} - \partial_\mu\epsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma} \\
&= \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} - \partial_\mu V^\mu \\
&\Downarrow \\
2\partial_\mu V^\mu &= \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}
\end{aligned}$$

So we see that V^μ has the property that

$$2\partial_\mu V^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

3 Problem #3

(a) For the identity

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\sigma} = -\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\delta^{\rho}_{\gamma} - \delta^{\nu}_{\alpha}\delta^{\rho}_{\beta}\delta^{\mu}_{\gamma} - \delta^{\rho}_{\alpha}\delta^{\mu}_{\beta}\delta^{\nu}_{\gamma} + \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta}\delta^{\rho}_{\gamma} + \delta^{\mu}_{\alpha}\delta^{\rho}_{\beta}\delta^{\nu}_{\gamma} + \delta^{\rho}_{\alpha}\delta^{\nu}_{\beta}\delta^{\mu}_{\gamma} \tag{3.1}$$

we note that the left hand side is antisymmetric in $\mu\nu\rho$ and $\alpha\beta\gamma$ which follows from the properties of the *Levi-Civita symbol*. So the first step to proving equation 3.1 is to show that the right hand side is also antisymmetric in both $\mu\nu\rho$ and $\alpha\beta\gamma$. So we can swap μ and ν and find that

$$-\delta^{\nu}_{\alpha}\delta^{\mu}_{\beta}\delta^{\rho}_{\gamma}-\delta^{\mu}_{\alpha}\delta^{\rho}_{\beta}\delta^{\nu}_{\gamma}-\delta^{\rho}_{\alpha}\delta^{\nu}_{\beta}\delta^{\mu}_{\gamma}+\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\delta^{\rho}_{\gamma}+\delta^{\nu}_{\alpha}\delta^{\rho}_{\beta}\delta^{\mu}_{\gamma}+\delta^{\rho}_{\alpha}\delta^{\mu}_{\beta}\delta^{\nu}_{\gamma} = -(-\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\delta^{\rho}_{\gamma}-\delta^{\nu}_{\alpha}\delta^{\rho}_{\beta}\delta^{\mu}_{\gamma}-\delta^{\rho}_{\alpha}\delta^{\mu}_{\beta}\delta^{\nu}_{\gamma}+\delta^{\nu}_{\alpha}\delta^{\mu}_{\beta}\delta^{\rho}_{\gamma}+\delta^{\mu}_{\alpha}\delta^{\rho}_{\beta}\delta^{\nu}_{\gamma}+\delta^{\rho}_{\alpha}\delta^{\nu}_{\beta}\delta^{\mu}_{\gamma})$$

We see that this follows from the fact that top indices ($\mu\nu\rho$) of the Kronecker deltas are antisymmetric in permutations of $\mu\nu\rho$. The antisymmetry of $\alpha\beta\gamma$ follows from this fact as swapping one of these indices can be considered a swapping of $\mu\nu\rho$. So now we can see for the case when $\mu = \alpha$, $\nu = \beta$, $\rho = \gamma$ we have

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma} = -\delta^{\mu}_{\mu}\delta^{\nu}_{\nu}\delta^{\rho}_{\rho} - \cancel{\delta^{\nu}_{\mu}\delta^{\rho}_{\nu}\delta^{\mu}_{\rho}} - \cancel{\delta^{\rho}_{\mu}\delta^{\mu}_{\nu}\delta^{\nu}_{\rho}} + \cancel{\delta^{\nu}_{\mu}\delta^{\mu}_{\nu}\delta^{\rho}_{\rho}} + \cancel{\delta^{\mu}_{\mu}\delta^{\rho}_{\nu}\delta^{\nu}_{\rho}} + \cancel{\delta^{\rho}_{\mu}\delta^{\nu}_{\nu}\delta^{\mu}_{\rho}} = -1$$

We can see that for permutations we end up with either ± 1. Now in the case where two indices are equal we take $\mu = \nu$ which yields

$$\begin{aligned}
\epsilon^{\mu\mu\rho\sigma}\epsilon_{\alpha\beta\gamma\sigma} &= -\delta^{\mu}_{\alpha}\delta^{\mu}_{\beta}\delta^{\rho}_{\gamma} - \delta^{\mu}_{\alpha}\delta^{\rho}_{\beta}\delta^{\mu}_{\gamma} - \delta^{\rho}_{\alpha}\delta^{\mu}_{\beta}\delta^{\mu}_{\gamma} + \delta^{\mu}_{\alpha}\delta^{\mu}_{\beta}\delta^{\rho}_{\gamma} + \delta^{\mu}_{\alpha}\delta^{\rho}_{\beta}\delta^{\mu}_{\gamma} + \delta^{\rho}_{\alpha}\delta^{\mu}_{\beta}\delta^{\mu}_{\gamma} \\
&= \cancel{-\delta^{\mu}_{\alpha}\delta^{\mu}_{\beta}\delta^{\rho}_{\gamma} + \delta^{\mu}_{\alpha}\delta^{\mu}_{\beta}\delta^{\rho}_{\gamma}} \cancel{-\delta^{\mu}_{\alpha}\delta^{\rho}_{\beta}\delta^{\mu}_{\gamma} + \delta^{\mu}_{\alpha}\delta^{\rho}_{\beta}\delta^{\mu}_{\gamma}} \cancel{-\delta^{\rho}_{\alpha}\delta^{\mu}_{\beta}\delta^{\mu}_{\gamma} + \delta^{\rho}_{\alpha}\delta^{\mu}_{\beta}\delta^{\mu}_{\gamma}} \\
&= 0
\end{aligned}$$

So as we expect if the indices are repeated then we have a zero value. We see that this holds for iterations as before. Therefore, we see that equation 3.1 holds true.

(b) Using the result from part (a) we can see that

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\rho\sigma} &= -\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\delta^{\rho}_{\rho} - \delta^{\nu}_{\alpha}\delta^{\rho}_{\beta}\delta^{\mu}_{\rho} - \delta^{\rho}_{\alpha}\delta^{\mu}_{\beta}\delta^{\nu}_{\rho} + \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta}\delta^{\rho}_{\rho} + \delta^{\mu}_{\alpha}\delta^{\rho}_{\beta}\delta^{\nu}_{\rho} + \delta^{\rho}_{\alpha}\delta^{\nu}_{\beta}\delta^{\mu}_{\rho} \\
&= -4\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\nu}_{\alpha}\delta^{\rho}_{\beta}\delta^{\mu}_{\rho} - \delta^{\rho}_{\alpha}\delta^{\mu}_{\beta}\delta^{\nu}_{\rho} + 2\delta^{\nu}_{\alpha}\delta^{\mu}_{\beta} + \delta^{\mu}_{\alpha}\delta^{\rho}_{\beta}\delta^{\nu}_{\rho} + \delta^{\rho}_{\alpha}\delta^{\nu}_{\beta}\delta^{\mu}_{\rho} \\
&= -4\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta} - \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta} + 4\delta^{\nu}_{\alpha}\delta^{\mu}_{\beta} + \delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} + \delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} \\
&= -4\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} + \delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} + \delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta} - \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta} + 4\delta^{\nu}_{\alpha}\delta^{\mu}_{\beta} \\
&= -2\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} + 2\delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}
\end{aligned}$$

(c) Using the result from part (b) we can take the Hodge dual of a Hodge dual using equation 2.1 to see

$$\begin{aligned}
{}^{*}\left({}^{*}F_{\mu\nu}\right) &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\,{}^{*}F^{\rho\sigma} \\
&= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\frac{1}{2}\epsilon^{\rho\sigma\alpha\beta}F_{\alpha\beta} \\
&= \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\rho\sigma}F_{\alpha\beta} \\
&= \frac{1}{4}\left(-2\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} + 2\delta^{\beta}_{\mu}\delta^{\alpha}_{\nu}\right)F_{\alpha\beta} \\
&= \frac{1}{4}\left(-2F_{\mu\nu} + 2F_{\nu\mu}\right) \\
&= \frac{1}{4}\left(-4F_{\mu\nu}\right) \\
&= -F_{\mu\nu}
\end{aligned}$$