Physics 611

Electromagnetic Theory II

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Homework #7

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1 Problem #1

(a) We can solve for the TE modes in a waveguide whose cross section is a rectangle of side a along the x axis and side b along the y axis by solving the two dimensional *Helmholtz Equation*

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \Omega^2\psi = 0. \tag{1.1}$$

Note for the TE modes we take $E_z = 0$ and $B_z = \psi$ where we apply the boundary condition

$$\left.\frac{\partial\psi}{\partial n}\right|_S = 0$$

where n is normal to the surface S that defines the waveguide cross section. We can solve equation 1.1 through the standard separation of variable technique where we take $\psi(x, y) = X(x)Y(y)$ which implies

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \Omega^2\psi = 0$$
$$\Downarrow$$
$$\frac{1}{X(x)}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y}{\partial y^2} + \Omega^2 = 0$$

This allows us to take the partial differential equation into two ordinary differential equations

$$\frac{\partial^2 X}{\partial x^2} = k_x^2 X(x)$$
$$\frac{\partial^2 Y}{\partial y^2} = k_y^2 Y(y)$$

Where $k_y^2 = \Omega^2 - k_x^2$. So we see that we can solve each of these through a combination of sine and cosine.

$$X(x) = A\sin k_x x + B\cos k_x x$$
$$Y(y) = C\sin k_y y + D\cos k_y y$$

Now we apply the boundary conditions, first for $x = 0$ we see that we have

$$\left.\frac{dX}{dx}\right|_{x=0} = 0 = Ak_x\cos 0 - Bk_x\sin 0 \Rightarrow A = 0$$

And that for a non trivial solution for $x = a$ we must have

$$\left.\frac{dX}{dx}\right|_{x=a} = 0 = -Bk_x\sin k_x a \Rightarrow k_x = \frac{m\pi}{a}$$

The same follows for $y = 0$ and $y = b$ which gives us the solution

$$\psi_{mn}(x, y) = C_{mn}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

note we combine the constants B and D into C_{mn}. This gives us the TE modes

$$E_z = 0 \qquad B_z(x, y) = C_{mn}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

where $\Omega_{mn}^2 = m^2\pi^2/a^2 + n^2\pi^2/b^2$.

(b) Recall that by definition $k^2 = \omega^2 - \Omega_{mn}^2$ this relation gives a cutoff frequency in which we have a real wave vector. We can see from the solution to part (a) that the smallest value of Ω that results in an oscillating field is when $m = 1$ and $n = 0$. The TE_{10} mode gives a value for Ω as

$$\Omega = \frac{\pi}{a}$$

This implies that $\omega_{\min}^{TE} = \pi/a$ If we compare this result to the result for the TM mode which is $\omega_{\min}^{TM} = \pi\sqrt{1/a^2 + 1/b^2}$ we can see that the ratio of the two cutoff frequencies is

$$\frac{\omega_{\min}^{TE}}{\omega_{\min}^{TM}} = \left(1 + \frac{a^2}{b^2}\right)^{-1/2}$$

2 Problem #2

(a) For a waveguide made out of an isosceles right-triangle with sides a, a, and $a\sqrt{2}$ we can use the special case of the rectangular waveguide where $a = b$, but with a linear combination of the eigenfunctions of the form $\psi_{mn} + \alpha\psi_{nm}$. This gives the TM solution of the form

$$E_z = C_{mn} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{a}y\right) + \alpha C_{nm} \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{a}y\right)$$

Note for the legs $x = a$ and $y = 0$ we already satisfy the boundary condition $E_z(a, y) = E_z(x, 0) = 0$. So, we just need to satisfy the boundary condition on the hypotenuse which is defined by the line $y = x$. Note for this case we have

$$\begin{aligned} C_{mn} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{a}x\right) + \alpha C_{nm} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{a}x\right) &= 0 \\ &\Downarrow \\ C_{mn} + \alpha C_{nm} &= 0 \\ \alpha &= -\frac{C_{mn}}{C_{nm}} \end{aligned}$$

So we have the solution of the form

$$E_z = C_{mn}\left[\sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{a}x\right) - \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{a}x\right)\right]$$

(b) Note for the general right-triangle we have a solution of the form

$$E_z = C_{mn} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) + \alpha C_{nm} \sin\left(\frac{m\pi}{a}y\right) \sin\left(\frac{n\pi}{b}x\right)$$

Note this follows from the same argument from part (a). We constructed a linear combination of the solutions where we take the $E_z(x, y)$ rectangular solution then solve for the case where we swap the coordinates ($E_z(y, x)$). The problem with this approach is that there is no longer the symmetry of the system that is present in the isosceles triangle case. This is why we cannot write the solution as $\psi_{nm} + \alpha\psi_{mn}$. This becomes clear when we apply the boundary condition on the line $y = \frac{b}{a}x$ this yields

$$\begin{aligned} 0 &= C_{mn} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) + \alpha C_{nm} \sin\left(\frac{m\pi}{a}y\right) \sin\left(\frac{n\pi}{b}x\right) \\ &\Downarrow \\ 0 &= C_{mn} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{a}x\right) + \alpha C_{nm} \sin\left(\frac{m\pi b}{a^2}x\right) \sin\left(\frac{n\pi}{b}x\right) \end{aligned}$$

which we cannot for α for any x and y.

(c) Note for the special case where we take a right-triangle with $b = a/\sqrt{3}$ we can verify that the function

$$\psi_{nm} = \sin\frac{l\pi x}{a}\sin\frac{(m-n)\pi y}{a\sqrt{3}} + \sin\frac{m\pi x}{a}\sin\frac{(n-l)\pi y}{a\sqrt{3}} + \sin\frac{n\pi x}{a}\sin\frac{(l-m)\pi y}{a\sqrt{3}}$$

where $l \equiv -m-n$ is an eigenfunction of the *Helmholtz Equation*.

$$\begin{aligned}
&\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -\Omega^2\psi \\
&\qquad\qquad\Downarrow \\
-\Omega^2\psi &= -\frac{\pi^2}{a^2}\left(\left(l^2 + \frac{(m-n)^2}{3}\right)\sin\frac{l\pi x}{a}\sin\frac{(m-n)\pi y}{a\sqrt{3}}\right. \\
&\quad + \left(m^2 + \frac{(n-l)^2}{3}\right)\sin\frac{m\pi x}{a}\sin\frac{(n-l)\pi y}{a\sqrt{3}} \\
&\quad \left. + \left(n^2 + \frac{(l-m)^2}{3}\right)\sin\frac{n\pi x}{a}\sin\frac{(l-m)\pi y}{a\sqrt{3}}\right) \\
&= -\frac{\pi^2}{a^2}\left(\left(m^2 + n^2 + 2nm + \frac{m^2+n^2-2mn}{3}\right)\sin\frac{l\pi x}{a}\sin\frac{(m-n)\pi y}{a\sqrt{3}}\right. \\
&\quad + \left(m^2 + \frac{n^2+l^2-2ln}{3}\right)\sin\frac{m\pi x}{a}\sin\frac{(n-l)\pi y}{a\sqrt{3}} \\
&\quad \left. + \left(n^2 + \frac{m^2+l^2-2lm}{3}\right)\sin\frac{n\pi x}{a}\sin\frac{(l-m)\pi y}{a\sqrt{3}}\right) \\
&= -\frac{\pi^2}{a^2}\left(\left(\frac{4m^2+4n^2+4mn}{3}\right)\sin\frac{l\pi x}{a}\sin\frac{(m-n)\pi y}{a\sqrt{3}}\right. \\
&\quad + \left(\frac{3m^2+n^2+n^2+m^2+2nm+2n(n+m)}{3}\right)\sin\frac{m\pi x}{a}\sin\frac{(n-l)\pi y}{a\sqrt{3}} \\
&\quad \left. + \left(\frac{3n^2+m^2+n^2+m^2+2nm+2m(n+m)}{3}\right)\sin\frac{n\pi x}{a}\sin\frac{(l-m)\pi y}{a\sqrt{3}}\right) \\
&= -\frac{\pi^2}{a^2}\left(\left(\frac{4m^2+4n^2+4mn}{3}\right)\sin\frac{l\pi x}{a}\sin\frac{(m-n)\pi y}{a\sqrt{3}}\right. \\
&\quad + \left(\frac{4m^2+4n^2+4nm}{3}\right)\sin\frac{m\pi x}{a}\sin\frac{(n-l)\pi y}{a\sqrt{3}} \\
&\quad \left. + \left(\frac{4m^2+4n^2+4nm)}{3}\right)\sin\frac{n\pi x}{a}\sin\frac{(l-m)\pi y}{a\sqrt{3}}\right) \\
&= -\frac{4\pi^2}{3a^2}(m^2+n^2+mn)\psi
\end{aligned}$$

Therefore we see that ψ is an eigenfunction of the *Helmholtz Equation* with the eigenvalue

$$\Omega^2_{mn} = -\frac{4\pi^2}{3a^2}(m^2+n^2+mn)$$

(d) Note using the eigenfunction from part (c) we can verify that the boundary conditions for the TM modes hold for each leg of the triangle. For $x = a$ we see that we have

$$\psi_{nm}(a,y) = \cancelto{0}{\sin l\pi}\sin\frac{(m-n)\pi y}{a\sqrt{3}} + \cancelto{0}{\sin m\pi}\sin\frac{(n-l)\pi y}{a\sqrt{3}} + \cancelto{0}{\sin n\pi}\sin\frac{(l-m)\pi y}{a\sqrt{3}} = 0$$

for any integer value of m and n. Next for $y = 0$ we see that each term gets a $\sin 0$ so every term goes to zero or $\psi_{nm}(x, 0) = 0$. The final boundary condition is on the line $y = x/\sqrt{3}$ this implies that

$$\begin{aligned}
\psi_{nm} &= \sin\frac{l\pi x}{a}\sin\frac{(m-n)\pi x}{3a} + \sin\frac{m\pi x}{a}\sin\frac{(n-l)\pi x}{3a} + \sin\frac{n\pi x}{a}\sin\frac{(l-m)\pi x}{3a} \\
&= \frac{1}{2}\Big[\cos\Big(\frac{\pi x}{3a}(3l - m + n)\Big) - \cos\Big(\frac{\pi x}{3a}(3l + m - n)\Big) \\
&\quad + \cos\Big(\frac{\pi x}{3a}(3m - n + l)\Big) - \cos\Big(\frac{\pi x}{3a}(3m + n - l)\Big) \\
&\quad + \cos\Big(\frac{\pi x}{3a}(3n - l + m)\Big) - \cos\Big(\frac{\pi x}{3a}(3n + l - m)\Big)\Big] \\
&= \frac{1}{2}\Big[\cos\Big(\frac{\pi x}{3a}(-4m - 2n)\Big) - \cos\Big(\frac{\pi x}{3a}(-2m - 4n)\Big) \\
&\quad + \cos\Big(\frac{\pi x}{3a}(2m - 2n)\Big) - \cos\Big(\frac{\pi x}{3a}(4m + 2n)\Big) \\
&\quad + \cos\Big(\frac{\pi x}{3a}(4n + 2m)\Big) - \cos\Big(\frac{\pi x}{3a}(2n - 2m)\Big)\Big] \\
&= \frac{1}{2}\Big[\cos\Big(\frac{\pi x}{3a}(4m + 2n)\Big) - \cos\Big(\frac{\pi x}{3a}(4m + 2n)\Big) \\
&\quad + \cos\Big(\frac{\pi x}{3a}(2m - 2n)\Big) - \cos\Big(\frac{\pi x}{3a}(2m - 2n)\Big) \\
&\quad + \cos\Big(\frac{\pi x}{3a}(4n + 2m)\Big) - \cos\Big(\frac{\pi x}{3a}(4n + 2m)\Big)\Big] \\
&= 0
\end{aligned}$$

So $\psi_{nm} = 0$ on all boundaries.