Physics 601
Analytical Mechanics
Professor Siu Chin

Homework #9

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1   Problem #1

For a one-dimensional harmonic oscillator of mass, m, and spring constant k we can construct a Hamiltonian by finding the Lagrangian as

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2$$

where $\omega = \sqrt{k/m}$. We find the canonical momentum of the system as

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}$$

which allows us to construct the Hamiltonian by

$$
\begin{aligned}
H = p\dot{q} - L &= \frac{p^2}{m} - \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \\
&= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \\
&= \frac{1}{2}m\left(\left(\frac{p}{m}\right)^2 + (\omega q)^2\right)
\end{aligned}
$$

For the *Canonical Transformation* of a one-dimensional harmonic oscillator given by

$$Q = C(p + im\omega q), \qquad P = C(p - im\omega q)$$

We can find the constant C by using the fact that the determinant of the Jacobian is unity. This implies that

$$
\begin{aligned}
1 &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \\
&= (Cim\omega)C - C(-Cim\omega) \\
&\quad 2C^2 im\omega \\
&\Downarrow \\
C &= \frac{1}{\sqrt{2im\omega}}
\end{aligned}
$$

So our transformations become

$$Q = \frac{1}{\sqrt{2im\omega}}(p + im\omega q), \qquad P = \frac{1}{\sqrt{2im\omega}}(p - im\omega q)$$

Now we can find the generating function $F_2(q, P)$ for this transformation by noting that

$$\frac{\partial F_2}{\partial q} = p, \qquad \frac{\partial F_2}{\partial P} = Q$$

Where we can solve the transformations such that

$$p(q, P) = \sqrt{2im\omega}P + im\omega q$$

and

$$Q(q, P) = \frac{1}{\sqrt{2im\omega}}(p(q, P) + im\omega q) = P + \sqrt{2im\omega}q$$

Which allows us to find

$$\frac{\partial F_2}{\partial P} = Q = P + \sqrt{2im\omega}q$$

$$\Downarrow$$

$$F_2(q, P) = \frac{1}{2}P^2 + \sqrt{2im\omega}qP + f(q)$$

Now we can find the function of q by

$$\frac{\partial F_2}{\partial q} = P = \sqrt{2im\omega}P + f'(q)$$

$$\Downarrow$$

$$\sqrt{2im\omega}P + im\omega q = \sqrt{2im\omega}P + f'(q)$$

$$\Downarrow$$

$$f(q) = \frac{1}{2}im\omega q^2$$

So our generator function is

$$F_2(q, P) = \frac{1}{2}P^2 + \sqrt{2im\omega}qP + \frac{1}{2}im\omega q^2$$

which allows us see that

$$QP = \frac{1}{2im\omega}(p + im\omega q)(p - im\omega q)$$

$$= -\frac{i}{2m\omega}(p^2 + (m\omega q)^2)$$

$$= -\frac{i}{2}m\left(\frac{1}{\omega}\left(\frac{p}{m}\right)^2 + \omega q^2\right)$$

Which implies that the Hamiltonian is

$$H = i\omega QP$$

so the equations of motion are given by

$$\dot{P} = -\frac{\partial H}{\partial Q} = -i\omega P, \qquad \dot{Q} = \frac{\partial H}{\partial P} = i\omega Q$$

Which easily allows us to solve the equations of motion as

$$P(t) = Ae^{-i\omega t}, \qquad Q(t) = Be^{i\omega t}$$

2   Problem #2

We note that the generator $F(q, Q) = F_1(q, Q)$ follows from the Legendre transform that states

$$\delta \int_{t_1}^{t_2} (p\dot{q} - H) - (P\dot{Q} - K)dt = 0$$

which implies that the difference in the integrand can only differ by a total time derivative

$$(p\dot{q} - H) - (P\dot{Q} - K) = \frac{dF}{dt}$$

which implies that

$$dF = (K - H)dt + pdq - PdQ$$

We note that F is a function of q and Q. We can use this to find $F_3(p, Q)$ by noting that

$$d(pq) = pdq + qdp$$

which allows us to write

$$dF = (K - H)dt + pdq - PdQ$$
$$\Downarrow$$
$$dF = (K - H)dt + d(pq) - qdp - PdQ$$
$$\Downarrow$$
$$d(F - pq) = (K - H)dt - qdp - PdQ$$

So $F_3(p, Q) = F - pq$. Next we repeat the process to find $F_4(p, P)$ by noting that

$$d(PQ) = PdQ + QdP$$

which implies

$$dF = (K - H)dt + pdq - PdQ$$
$$\Downarrow$$
$$dF = (K - H)dt + pdq - (d(PQ) - QdP)$$
$$\Downarrow$$
$$d(F + PQ) = (K - H)dt + pdq + QdP$$

which implies that $F_4(p, P) = F + PQ$.

3 Problem #3

(a) For the infinitesimal transformation with the generator given by

$$F_2(q, P) = qP + \epsilon H(q, P)$$

where H is the Hamiltonian given by

$$H(q, p) = \frac{p^2}{2m} + V(q)$$

we can find the canonical transformations associated with $F_2(q, P)$ by

$$Q = \frac{\partial F_2}{\partial P} = q + \epsilon \frac{\partial}{\partial P} \left(\frac{P^2}{2m} + V(q) \right)$$

$$= q + \epsilon \frac{P}{m}$$

Then we can find p and invert by

$$p = \frac{\partial F_2}{\partial q} = P + \epsilon \frac{dV}{dq}$$

$$\Downarrow$$

$$P = p - \epsilon \frac{dV}{dq}$$

So replacing P in the Q term we get the canonical transformations to first order in ϵ

$$P = p - \epsilon \frac{dV}{dq}$$
$$Q = q + \epsilon \frac{p}{m}$$

(b) If we take ϵ to be a small time step we can see that for $q = q(t)$ and $p = p(t)$ we have the transformations

$$q(t + \epsilon) = Q = q(t) + \epsilon \frac{p(t)}{m}$$
$$p(t + \epsilon) = P = p(t) - \epsilon \frac{dV}{dq}$$

If we Taylor expand $q(t + \epsilon)$ about ϵ to first order we see that

$$q(t + \epsilon) = q(t) + \dot{q}\epsilon$$
$$p(t + \epsilon) = p(t) + \dot{p}\epsilon$$

We note that the velocity $\dot{q}$ is related to momentum p by $\dot{q} = p/m$ which implies that The q expansion is the same as the canonical transformation. We also note that $\dot{p} = F = -dV/dq$ so the expansion of p also is the same to first order.

(c) Using the results from above we can find the transformed Hamiltonian $K(P, Q)$ by noting

$$H(q, p) = \frac{p^2}{2m} + V(q)$$

$$\Downarrow$$

$$K(Q, P) = \frac{1}{2m} \left(P + \epsilon \frac{dV}{dq} \right)^2 + V \left(Q - \epsilon \frac{P}{m} \right)$$

$$= \frac{1}{2m} \left(P^2 + P\epsilon \frac{dV}{dq} + \epsilon^2 \left(\frac{dV}{dq} \right)^2 \right) + V(Q) - \epsilon \frac{P}{m} \frac{dV}{dQ} + \epsilon^2 \frac{p^2}{m^2} \frac{d^2V}{dQ^2}$$

$$= \frac{P^2}{2m} + V(Q) + \epsilon \left(P \frac{dV}{dq} - \frac{P}{m} \frac{dV}{dQ} \right) + \epsilon^2 \left(\left(\frac{dV}{dq} \right)^2 + \frac{p^2}{m^2} \frac{d^2V}{dQ^2} \right)$$

4   Problem #4

(a) For the generator
$$F_3(p,Q) = -pQ + \epsilon H(Q,p)$$
where H is the same Hamiltonian from problem 3 this allows up to find the transformations by
$$q = -\frac{\partial F_3}{\partial p} = Q - \epsilon\frac{p}{m}$$

and
$$P = -\frac{\partial F_3}{\partial Q} = p - \epsilon\frac{dV}{dQ}$$

So solving for Q we have the canonical transformations by noting that
$$\frac{\partial V}{\partial q} = \frac{\partial V}{\partial Q}\frac{\partial V}{\partial q}$$

but as we can see $dQ/dq = 1$ so
$$Q = q + \epsilon\frac{p}{m}$$
$$P = p - \epsilon\frac{dV}{dq}$$

which are the same canonical transformations as problem 3.

(b) For ϵ as a time step we can find the same result that
$$q(t+\epsilon) = Q = q(t) + \epsilon\frac{p(t)}{m}$$
$$p(t+\epsilon) = P = p(t) - \epsilon\frac{dV}{dq}$$

This again is the same as the Taylor expansion about ϵ to first order. Where $\dot{p} = -dV/dq$ and $\dot{q} = p/m$.

(c) We note that the canonical transformation found in part (a) is the same as the transformation from problem 3. Therefore the transformed Hamiltonian to second order is the same given by
$$K(Q,P) = \frac{P^2}{2m} + V(Q) + \epsilon\left(P\frac{dV}{dq} - \frac{P}{m}\frac{dV}{dQ}\right) + \epsilon^2\left(\left(\frac{dV}{dq}\right)^2 + \frac{p^2}{m^2}\frac{d^2V}{dQ^2}\right)$$

(d)

5   Problem #5

For the Kepler problem with a three dimensional Hamiltonian given by

$$H = \frac{\mathbf{p}^2}{2m} - \frac{k}{|\mathbf{r}|}.$$

Using this we can show that

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

is a constant of motion by taking the Poisson bracket with H and noting that the l component of $\mathbf{L}$ is

$$\left(\mathbf{r} \times \mathbf{p} \right)_l = \epsilon_{lmn} r_m p_n$$

$$\frac{d\mathbf{L}}{dt} = \{\mathbf{L}, H\} = \frac{\partial \mathbf{L}}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial \mathbf{L}}{\partial p_i} \frac{\partial H}{\partial q_i}$$

$$= (\epsilon_{lin} p_n) \frac{p_i}{m} - (\epsilon_{lmi} r_m) \left(-\frac{r_i}{|\mathbf{r}|^3} \right)$$

$$= 0$$

Note the above equation is equal to zero due to the permutations about the Levi-Civita symbol. Therefore the angular momentum is a conserved quantity as we expected. We can repeat this process for

$$\mathbf{A} = \frac{\mathbf{p} \times \mathbf{L}}{km} - \frac{\mathbf{r}}{|\mathbf{r}|}$$

where we note that the cross product is given by

$$(\mathbf{p} \times \mathbf{L})_m = \epsilon_{mni} p_n L_i = \epsilon_{mni} \epsilon_{ijk} p_n r_j p_k$$

$$= (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}) p_n r_j p_k$$

$$= r_m p_k p_k - p_m r_j p_j$$

This allows us to calculate the Poisson bracket by

$$\frac{d\mathbf{A}}{dt} = \{\mathbf{A}, H\} = \frac{\partial \mathbf{A}}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial \mathbf{A}}{\partial p_i} \frac{\partial H}{\partial q_i}$$

$$= \left(\frac{p_k^2 - p_m p_i}{km} - \frac{1}{|\mathbf{r}|} + \frac{r_i^2}{|\mathbf{r}|^3} \right) \frac{p_i}{m} - \left(\frac{2r_m p_i - p_i r_j - p_m r_i}{km} \right) \left(-\frac{r_i}{|\mathbf{r}|^3} \right)$$

$$= \left(\frac{p_k^2 - p_m p_i}{km} - \frac{1}{|\mathbf{r}|} + \frac{|\mathbf{r}|^2}{|\mathbf{r}|^3} \right) \frac{p_i}{m} - \left(\frac{2r_m p_i - 2p_i r_j}{km} \right) \left(-\frac{r_i}{|\mathbf{r}|^3} \right)$$

$$= \frac{p_i p_k^2 - p_m p_i^2}{km^2} \qquad\qquad\qquad = 0$$

So under a central potential the LRL-vector is conserved as we expect.