Physics 607

Statistical Physics and Thermodynamics

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Homework #11

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1 Problem #1

(1) We can find the second critical field, H_{c2} assuming that the area of the normal vortex reaches one half of the area of the elementary cell in the vortex lattice by noting that the critical field goes by

$$H_{c2} = \frac{\Phi_0}{A}$$

where A is the area of the normal vortex. We see that is this area is half of the elementary cell of length, a, we have

$$A = \frac{1}{2}\frac{\sqrt{3}a^2}{2} \Rightarrow H_{c2} = \frac{4\Phi_0}{\sqrt{3}a^2}$$

(2) Given the *Ginzburg-Landau equation* in the external field

$$\frac{1}{4m}\left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)^2 \psi + a\psi + b|\psi|^2 = 0 \tag{1.1}$$

where we linearize the wave function near the transition temperature, T_c as

$$|\psi|^2 = \begin{cases} -\frac{a}{b} & a < 0 \\ 0 & a > 0 \end{cases}$$

where we define $a = \alpha(T - T_c)$. Now we use the Landau gauge where we take $\mathbf{A} = Bx\hat{y}$ which allows us to express equation 1.1 near the transition as the linear equation

$$-\frac{\hbar^2}{4m^*}\nabla^2\psi + a\psi$$

which yields the solution

$$\psi = \psi_0 e^{-x/\xi} \qquad \text{with } \xi = \sqrt{\frac{\hbar^2}{-4m^*a}}$$

Note that $a < 0$ in this regime so we can see that $\xi \in \mathbb{R}$. We take ξ to be the characteristic length of coherence. This gives us the fact that the quantized vortices within the super conductors have a radius of ξ. If we recall that the second critical field is when the lattice of vortices reaches the point of overlap we see that $A = 2\pi\xi^2$ which yields the result

$$H_{c2} = \frac{\Phi_0}{2\pi\xi^2}$$

(3) We can find the energy of an individual superconducting vortex in the superconductor of the II-kind by noting the free energy of a vortex filament per unit length is given as

$$f_v^0 = \int \left(\frac{m^* n_s v_s}{2} + \frac{B^2}{8\pi}\right)\rho d\rho d\varphi$$

Where we take B to be the *MacDonald function* solving this integral in the II-kind case we have the energy per filament as

$$\varepsilon = \frac{\pi n_s \hbar^2}{m^*}\ln\left(\frac{\lambda}{\xi}\right)$$

where we take λ as the penetration length defined as

$$\lambda = \sqrt{\frac{m^* c}{4\pi n_s e^2}}$$

note that for II-kind superconductors we have $\lambda < \xi$ so that this energy is negative which implies that the creation of vortices is energy favorable.

(4) We see that the free energy of a vortex for a fixed external field, H, goes as

$$f_v = f_v^0 - \int \frac{\mathbf{B}\cdot\mathbf{H}}{4\pi}\rho d\rho$$

we see that the for a fixed H the integral just is the flux of an individual vortex, Φ_0. Therefore,

$$f_v \approx f_v^0 - H\Phi_0$$

We see that as H increases we have a decreasing free energy, this implies that at the point when $f_v = 0$ we are at the first critical field. This implies that

$$H_{c1} = \frac{f_v^0}{\Phi_0} = \frac{\Phi_0}{2\pi\lambda^2}$$

2 Problem #2

(1) Assuming that the radius of the normal core of the vortex is equal to ξ and no pinning forces for vortices we can take the current of the vortex by *London's Equation*

$$\mathbf{j}_s = -\frac{en_s}{m}\mathbf{A}$$

We note that due to the *Lorentz Force* we have an electric field $E = vB$ this results in a resistivity of the form

$$\rho = \frac{vB}{j_s}$$

which we can approximate as a relation to the normal resistivity, ρ_n, by

$$\rho = \rho_n\frac{H}{H_{c2}}$$

(2) Using the result from part (1) we see that for an electric field in the x direction E_x and magnetic field in the z direction, H, we calculate the electric field in the y direction by noting that E_x generates a current

$$j_x = \rho E_x = \rho_n E_x\frac{H}{H_{c2}}$$

this current with H generates an electric field by $\mathbf{j}_x \times \mathbf{H}$ which yields

$$E_y = \rho_n E_x\frac{H^2}{H_{c2}}$$

3 Problem #3

(1) We can find the partition function of the 1d Ising model at zero magnetic field by taking the general energy of the spin configuration as

$$E = -J \sum_{nn} \boldsymbol{\sigma}_x \boldsymbol{\sigma}_x$$

which in 1d we have

$$E(\sigma_x) = -J \sum_{n=0}^{N-1} \sigma_n \sigma_{n+1} - J\sigma_N \sigma_1$$

where we have N spins in the chain. So the partition follows as

$$Z(T) = \sum_{\{\sigma_x\}} \exp\left[-\frac{E(\sigma_x)}{T}\right]$$

which we can represent as

$$Z(T) = \sum_{\{\sigma_x\}} \prod_b \exp\left(K\sigma_b\right)$$

Using this we can take the free energy of the system as

$$F = -J - T\ln\left[\cosh(J/T) + \sqrt{\sinh^2(J/T) + e^{-4J/T}}\right]$$

which allows us to calculate the entropy as

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N} = \ln\left[\cosh(J/T) + \sqrt{\sinh^2(J/T) + e^{-4J/T}}\right] + \frac{J}{T}\left(\cosh(J/T) + \sqrt{e^{-4J/T} + \sinh^2(J/T)}\right)^{-1}\left(\frac{2e^{-4J/T}}{\sqrt{e^{-4J/T} + \sinh^2(J/T)}} - \sinh(J/T)\right)$$

(2) We can calculate the correlation function of two spins, $\langle \sigma_n \sigma_m \rangle$, as a function of the distance $|n-m|$ and temperature by noting that

$$\langle \sigma_n \sigma_m \rangle = \cos^2(2\phi) + \left(\frac{\lambda_-}{\lambda_+}\right)^{n-m} \sin^2(2\phi)$$

where $\lambda_\pm$ are the eigenvalues of the transfer matrix. This result reduces to

$$\langle \sigma_n \sigma_m \rangle = (\tanh(J/T))^{n-m}$$

(3) For a Ising chain of N spins placed in an external magnetic field, H, we have a energy

$$E = -J\sum_{i=1}^{N} \sigma_i \sigma_{i+1} - H\sum_{i=1}^{N} \sigma_i$$

which we can write in the symmetric form

$$E = \sum_{i=1}^{N} \left(-J\sigma_i\sigma_{i+1} - H(\sigma_i + \sigma_{i+1})/2\right)$$

this yields the partition function

$$
\begin{aligned}
Z &= \sum_{\sigma_1=\pm1}\sum_{\sigma_2=\pm1}...\sum_{\sigma_N=\pm1}\prod_{i=1}^{N}\exp[J\sigma_i\sigma_{i+1}/T + H(\sigma_i+\sigma_{i+1})/2T] \\
&= \sum_{\sigma_1=\pm1}\sum_{\sigma_2=\pm1}...\sum_{\sigma_N=\pm1} T(\sigma_1,\sigma_2)T(\sigma_2,\sigma_3)...T(\sigma_i,\sigma_{i+1})...T(\sigma_N,\sigma_1)
\end{aligned}
$$

Note the periodic boundary condition $\sigma_{N+1} = \sigma_1$. We define

$$
T(\sigma_i,\sigma_{i+1}) = \exp[J\sigma_i\sigma_{i+1}/T + H(\sigma_i+\sigma_{i+1})/2T]
$$

which we take as a 2×2 transfer matrix as

$$
T(\sigma_i,\sigma_{i+1}) = \begin{pmatrix} T(+,+) & T(+,-) \\ T(-,+) & T(-,-) \end{pmatrix} = \begin{pmatrix} \exp(J/T+H/T) & \exp(-J/T) \\ \exp(-J/T) & \exp(J/T-H/T) \end{pmatrix}
$$

Which correspond to the four different spin configurations. If we diagonalize T such that

$$
U^{-1}TU = \Lambda = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}
$$

we see that the partition function becomes simply

$$
Z = \mathrm{Tr}\Lambda^N
$$

where

$$
\Lambda^N = \begin{pmatrix} \lambda_+^N & 0 \\ 0 & \lambda_-^N \end{pmatrix}
$$

where the eigenvalues are given as

$$
\lambda_\pm = \exp(J/T)\left(\cosh(H/T) \pm \sqrt{\sinh^2(H/T) + \exp(-4J/T)}\right)
$$

so we can see that

$$
Z = \lambda_+^N + \lambda_-^N
$$

Which yields the free energy as

$$
F = -\frac{T}{N}\lim_{N\to\infty}\ln(Z) = -T\ln(\lambda_+)
$$

from this it follows that the magnetization per spin is

$$
m = \frac{\sinh(H/T)}{\sqrt{\sinh^2(H/T)+\exp(-4J/T)}}
$$