Physics 607

Statistical Physics and Thermodynamics

Professor Valery Pokrovsky

Homework #6

Joe Becker
UID: 125-00-4128
March 3rd, 2016

1 Problem #1

(1) To calculate the change in energy of an electron gas at zero temperature due to a magnetic field, H, we note that the Fermi-Dirac distribution becomes

$$f(\varepsilon) = \frac{1}{e^{(\varepsilon \pm \mu_B H - \mu)/T} + 1}$$

where μ_B is the Bohr magneton which is given by

$$\mu_B = \frac{e\hbar}{2mc}$$

Note that the positive $\mu_B H$ is related to the state where the spin is parallel to the magnetic field and the negative $\mu_B H$ is the anti-parallel case. Using $f(\varepsilon)$ and the density of states, $\nu(\varepsilon)$, for a spin $\frac{1}{2}$ particle given as

$$\nu(\varepsilon) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\varepsilon^{1/2}$$

we can calculate the energy of the spin up state as temperature goes to zero noting that $f(\varepsilon)$ becomes the step function. Therefore

$$\begin{aligned}
E_\uparrow = \int_0^\infty \varepsilon\nu(\varepsilon)f(\varepsilon)d\varepsilon &= \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{\varepsilon^{3/2}}{e^{(\varepsilon+\mu_B H-\mu)/T}}d\varepsilon \\
&= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^{\mu-\mu_B H}\varepsilon^{3/2}d\varepsilon \\
&= \frac{V}{5\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}(\mu-\mu_B H)^{5/2}
\end{aligned}$$

The same follows for the spin down energy just with the change in sign.

$$E_\downarrow = \frac{V}{5\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}(\mu+\mu_B H)^{5/2}$$

This implies that the total energy is

$$E = E_\uparrow + E_\downarrow = \frac{V}{5\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left((\mu-\mu_B H)^{5/2} + (\mu+\mu_B H)^{5/2}\right)$$

Therefore we can find the magnetization, M, by taking the derivative with respect to H

$$M = \frac{\partial E}{\partial H} = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\mu_B\left((\mu+\mu_B H)^{3/2} - (\mu-\mu_B H)^{3/2}\right)$$

Note we can take $\mu = \varepsilon_F$ and assume that $\mu_B H << \varepsilon_F$ which allows us to expand about $B = 0$ to yield

$$\begin{aligned}
M &= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\mu_B\left((\varepsilon_F+\mu_B H)^{3/2} - (\varepsilon_F-\mu_B H)^{3/2}\right) \\
&= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\mu_B\left(\mu_B\varepsilon_F^{1/2}H + \mu_B\varepsilon_F^{1/2}H\right) \\
&= 3\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\varepsilon_F^{1/2}\mu_B^2 H \\
&= 3\nu(\varepsilon_F)\mu_B^2 H
\end{aligned}$$

Note this matches from the direct calculation by using number of states except for the factor of 3.

(2) We can repeat this process for low temperature $T << \varepsilon_F$ by noting the approximation to second order in T is as

$$E_\uparrow = \frac{V}{5\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left((\mu-\mu_B H)^{5/2}+\frac{5\pi T^2}{8}(\mu-\mu_B H)^{1/2}\right)$$
$$E_\downarrow = \frac{V}{5\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left((\mu+\mu_B H)^{5/2}+\frac{5\pi T^2}{8}(\mu+\mu_B H)^{1/2}\right)$$

We can see that the first term yields the $T = 0$ solution found in part (a). So to find the correction for low temperature we take the term of order T^2 to find

$$\begin{aligned}
E^{(2)} &= \frac{VT^2}{8\pi}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left((\mu+\mu_B H)^{1/2}+(\mu-\mu_B H)^{1/2}\right)\\
&\Downarrow\\
M^{(2)} = \frac{\partial E^{(2)}}{\partial H} &= \frac{VT^2\mu_B}{16\pi}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left((\mu+\mu_B H)^{-1/2}-(\mu-\mu_B H)^{-1/2}\right)\\
&\Downarrow\\
&\approx -\frac{VT^2\mu_B^2 H}{16\pi}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon_F^{-3/2}\\
&-\frac{\pi}{8}\left(\frac{T}{\epsilon_F}\right)^2\nu(\epsilon_F)\mu_B^2 H
\end{aligned}$$

So the corrected magnetization is

$$M = \nu(\epsilon_F)\mu_B H\left(3-\frac{\pi T^2}{8\epsilon_F^2}\right)$$

which yields the corrected susceptibility as

$$\chi = \frac{\partial M}{\partial H} = \nu(\epsilon_F)\mu_B\left(3-\frac{\pi T^2}{8\epsilon_F^2}\right)$$

Note as we expect an increase in temperature decreases the magnetization and susceptibility.

2 Problem #2

(1) For a degenerate electronic gas in an external magnetic field there exists orbital motion. We assume without loss of generality that the magnetic field points in the $\hat{z}$ direction. This implies that there are two types of motion one being in the direction of the magnetic field and the other being the circular precession within the perpendicular (xy) plane. Using this fact we can calculate the density of states by noting that in the presence of a magnetic field the energy energy levels are

$$\varepsilon = \frac{p_z^2}{2m} + \hbar\omega_c\left(l + \frac{1}{2}\right)$$

where $l = 0, 1, 2, ...$ are called the *Landau Levels* and ω_c is the cyclotron frequency given as

$$\omega_c = \frac{eH}{mc}$$

This allows us to split the calculation into the two separate motions. For the perpendicular motion we note that the momentum in y for a fixed container is

$$p_y = \frac{2\pi\hbar l}{L_y}$$

so for each Landau level we have $\delta p_y = 2\pi\hbar/L_y$. This directly relates to the energy per Landau level by

$$N_\perp = \frac{L_x L_y eH}{2\pi\hbar c}$$

Next if we take the momentum in z to be continuous we can integrate to find

$$\int_{-p_{zF}}^{p_{zF}} \frac{dp_z}{2\pi\hbar} = \frac{p_{zF}}{\pi\hbar}$$

It directly follows that the total number of particles is

$$N = N_\perp \sum_{l=0}^{\mu/\hbar\omega_c} \frac{p_{zF}}{\pi\hbar}$$

where the Fermi momentum in z follows from the total energy

$$p_{zF} = \sqrt{2m\left(\varepsilon - \hbar\omega_c\left(l + \frac{1}{2}\right)\right)}$$

So it follows that the density of states is

$$\begin{aligned}
\nu(\varepsilon) &= 2N_\perp L_z \sum_{l=0}^{\mu/\hbar\omega_c} \int \sqrt{2m\left(\varepsilon - \hbar\omega_c\left(l + \frac{1}{2}\right)\right)} \frac{dp_z}{2\pi\hbar} \\
&= 2\frac{\sqrt{2m}VeH}{(2\pi\hbar)^2 c} \sum_{l=0}^{\mu/\hbar\omega_c} \left(\varepsilon - \hbar\omega_c\left(l + \frac{1}{2}\right)\right)^{-1/2} \\
&= \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \hbar\omega_c \sum_{l=0}^{\mu/\hbar\omega_c} \left(\varepsilon - \hbar\omega_c\left(l + \frac{1}{2}\right)\right)^{-1/2}
\end{aligned}$$

If we define the dimensionless variable ξ and $\nu_0(\varepsilon)$ as the density of states with no magnetic field as

$$\xi \equiv \frac{\varepsilon}{\hbar\omega_c} \qquad \nu_0(\varepsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2}$$

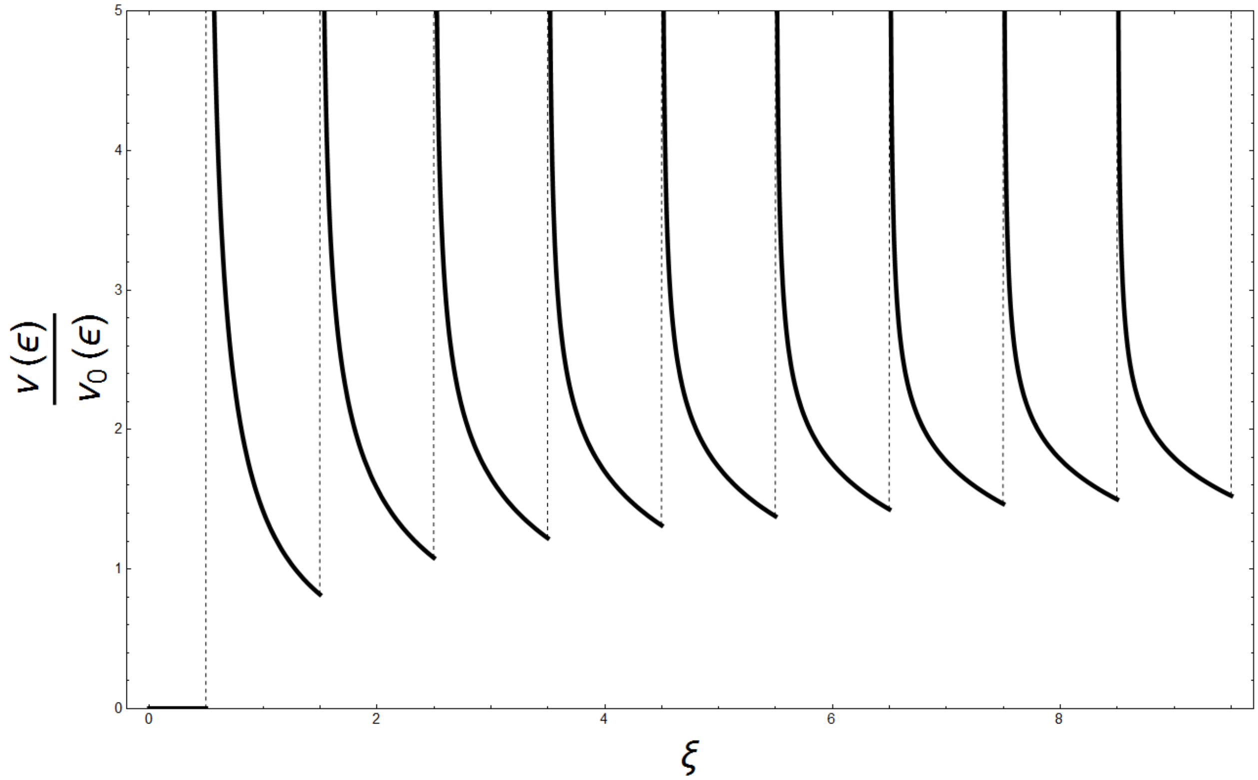

Figure 1: A plot of the ratio of density of states for diamagnetism, $\nu(\varepsilon)$, to a free electron density of states, $\nu_0(\varepsilon)$, versus the quantity $\xi = \varepsilon/\hbar\omega_c$.

we can our density of states into the relation

$$\frac{\nu(\varepsilon)}{\nu_0(\varepsilon)} = \frac{1}{2\sqrt{\xi}} \sum_{l=0}^{\xi} \left(\xi - l - \frac{1}{2}\right)^{-1/2}$$

we plot the ratio $\nu(\varepsilon)/\nu_0(\varepsilon)$ verses ξ is figure 1. Note the asymptotic behavior of the plot which correspond to singularities within the sum.

(2) Using the result from part (a) we are able to calculate the density of the system by integrating where we approximate the temperature to be small so that

$$\begin{aligned}
n = \frac{N}{V} &= \frac{1}{V}\int_0^{\infty} \nu(\varepsilon) f(\varepsilon) d\varepsilon \\
&= \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \hbar\omega_c \sum_{l=0}^{\mu/\hbar\omega_c} \int_0^{\infty} \left(\varepsilon - \hbar\omega_c\left(l + \frac{1}{2}\right)\right)^{-1/2} \frac{1}{e^{(\varepsilon - \hbar\omega_c(l+1/2)-\mu)/T}+1} d\varepsilon \\
&= \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \hbar\omega_c \sum_{l=0}^{\mu/\hbar\omega_c} \left[2\left(\mu - \hbar\omega_c\left(l + \frac{1}{2}\right)\right)^{1/2} - \frac{\pi^2 T^2}{12}\left(\mu - \hbar\omega_c\left(l + \frac{1}{2}\right)\right)^{-3/2}\right] \\
&= \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \hbar\omega_c \sum_{l=0}^{\mu/\hbar\omega_c} \left(\mu - \hbar\omega_c\left(l + \frac{1}{2}\right)\right)^{1/2} \left[1 - \frac{\pi^2 T^2}{24}\left(\mu - \hbar\omega_c\left(l + \frac{1}{2}\right)\right)^{-2}\right]
\end{aligned}$$

We see that there exists a singularity for $\mu = \hbar\omega_c(l + 1/2)$

(3) To find the energy we calculate the integral taking $T = 0$ so that

$$
\begin{aligned}
E &= \int_0^\infty \varepsilon \nu(\varepsilon) f(\varepsilon) d\varepsilon \\
&= \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \hbar\omega_c \sum_{l=0}^{\mu/\hbar\omega_c} \int_0^{\mu+\hbar\omega_c(1+1/2)} \varepsilon \left(\varepsilon - \hbar\omega_c\left(l+\frac{1}{2}\right)\right)^{-1/2} d\varepsilon \\
&= \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \hbar\omega_c \int_0^\mu \frac{2\varepsilon^{3/2}}{\hbar\omega_c} + \frac{\hbar\omega_c}{48\varepsilon^{1/2}} d\varepsilon \\
&= \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \hbar\omega_c \left(\frac{4\mu^{5/2}}{5\hbar\omega_c} + \frac{\hbar\omega_c\mu^{1/2}}{24}\right)
\end{aligned}
$$

Where we approximated the sum using the *Euler Summation Formula*

$$
\sum_{n=0}^\infty f(n+1/2) = \int_0^\infty f(x)dx + \frac{1}{24} f'(0) + \ldots
$$

How taking $d\omega_c/dH = e/mc$ we calculate the magnetism as

$$
\begin{aligned}
M = \frac{\partial E}{\partial H} = \frac{\partial E}{\partial \omega_c}\frac{\partial \omega_c}{\partial H} &= \frac{e}{mc}\frac{\partial}{\partial \omega_c}\left[\frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \hbar\omega_c \left(\frac{4\mu^{5/2}}{5\hbar\omega_c} + \frac{\hbar\omega_c\mu^{1/2}}{24}\right)\right] \\
&= \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{\hbar^2\omega_c\mu^{1/2}}{12} \\
&= \frac{1}{12}\nu_0(\mu)\mu_B\hbar\omega_c \\
&= \frac{1}{12}\nu_0(\mu)\mu_B^2 H
\end{aligned}
$$

3 Problem #3

(1) Given that the cosmic background microwave radiation (CBMR) has a thermal black body spectrum at $T = 2.72\,\mathrm{K}$ we can find the frequency that the spectrum is at a maximum by taking *Planck Distribution*

$$E_\omega = \frac{V\hbar}{\pi^2 c^3}\frac{\omega^3}{e^{-\hbar\omega/T} - 1}$$

and finding ω_{max} which maximizes the distribution. We take a derivative to find ω_{max} as

$$\begin{aligned}
\frac{dE}{d\omega} &= \frac{V\hbar}{\pi^2 c^3}\left(\frac{3\omega^2}{e^{\hbar\omega/T} - 1} - \frac{\hbar}{T}\frac{\omega^3 e^{\hbar\omega/T}}{(e^{\hbar\omega/T} - 1)^2}\right) \\
&\Downarrow \\
0 &= \frac{3\omega_{max}^2}{e^{\hbar\omega_{max}/T} - 1} - \frac{\hbar}{T}\frac{\omega_{max}^3 e^{\hbar\omega_{max}/T}}{(e^{\hbar\omega_{max}/T} - 1)^2} \\
&\Downarrow \\
3(e^{\hbar\omega_{max}/T} - 1) &= \frac{\hbar\omega_{max}}{T} e^{\hbar\omega_{max}/T} \\
&\Downarrow \\
3(1 - e^{-\zeta}) &= \zeta
\end{aligned}$$

Where we take

$$\zeta = \frac{\hbar\omega_{max}}{kT}$$

note that we added the constant k so that we can calculate ω_{max} from a temperature in kelvin. Numerically we find that $\zeta \approx 2.822$ so

$$f_{max} = \frac{\omega_{max}}{2\pi} = \zeta\frac{kT}{\hbar} = \frac{2.822}{2\pi}\frac{(1.38\times10^{-23}\,\mathrm{J\ K^{-1}})(2.72\,\mathrm{K})}{1.05\times10^{-34}\,\mathrm{J\ s^{-1}}} = 160\,\mathrm{GHz}$$

(2) If we assume that the radius of the universe changes with time linearly and that the expansion for the CBMR is adiabatic which implies that the photon gas undergoes expansion by

$$TV^{1/3} = C$$

where C is a constant. Using this fact we can relate the current temperature of the CBMR given as $T_c = 2.73\,\mathrm{K}$ to the temperature of the universe at one year. Using the fact that the radius of the universe changes linearly we see that

$$\begin{aligned}
V_0 &= \frac{4}{3}\pi(r_0 t_0)^3 \\
V_c &= \frac{4}{3}\pi r_c^3 = \frac{4}{3}\pi(r_0 t_c)^3
\end{aligned}$$

Where t_c is the current age of the universe. So

$$\begin{aligned}
T_0 V_0^{1/3} &= T_c V_c^{1/3} \\
&\Downarrow \\
T_0 = T_c t_c &= (2.73\,\mathrm{K})\left(\frac{1.4\times10^{10}\,\mathrm{yr}}{1\,\mathrm{yr}}\right) = 3.8\times10^{10}\,\mathrm{K}
\end{aligned}$$

Note we took $t_0 = 1\,\mathrm{yr}$.

(3) If we consider the Sun as a black body with a radius of $R = 7.0 \times 10^5\,\mathrm{km}$ and temperature $T = 6 \times 10^3\,\mathrm{K}$ we can calculate the emitted radiation through the Stefan-Boltzmann constant, σ, as

$$
\begin{aligned}
J_{sun} = \int \sigma T^4 d\Omega &= \sigma T^4 4\pi R^2 \\
&= (5.67 \times 10^{-8}\,\mathrm{J\ m^{-2}\ K^{-4}\ s^{-1}})(6 \times 10^3\,\mathrm{K})(4\pi)(7.0 \times 10^8\,\mathrm{m}) \\
&= 4.5 \times 10^{26}\,\mathrm{J\ s^{-1}}
\end{aligned}
$$

(4) Assuming that the earth is also a black body with the distance between the earth and sun is $R_o = 5 \times 10^8\,\mathrm{km}$ and radius of $R_e = 6 \times 10^3\,\mathrm{km}$ we can calculate the amount of energy that is absorbed by the earth by noting we assumed the sun was also a black body in part (3) so the amount of energy propagates without changing in phase space. This implies that the earth sees it's cross section a distance R_o

$$
\begin{aligned}
A_{earth} = J_{sun}\frac{\pi R_E^2}{4\pi R_o^2} &= 4.5 \times 10^{26}\,\mathrm{J\ s^{-1}}\frac{\pi(6 \times 10^6\,\mathrm{m})}{4\pi(5 \times 10^{11}\,\mathrm{m})^2} \\
&= 1.35 \times 10^{21}\,\mathrm{J\ s^{1-}}
\end{aligned}
$$

Now, due to the assumption that earth is a black body we can take all this energy absorbed is emitted. Where we integrated over the solid angle

$$
\begin{aligned}
A_{earth} = J_{earth} &= \sigma T^4 4\pi R_e^2 \\
&\Downarrow \\
T &= \frac{A_{earth}}{4\pi\sigma R_e^2} = \frac{1.35 \times 10^{21}\,\mathrm{J\ s^{-1}}}{4\pi(5.67 \times 10^{-8}\,\mathrm{J\ m^{-2}\ K^{-4}\ s^{-1}})(6 \times 10^6\,\mathrm{m})^2} \\
&= 287\,\mathrm{K}
\end{aligned}
$$