Physics 607

Statistical Physics and Thermodynamics

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Homework #5

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February 22nd, 2016

1 Problem #1

(1) For a degenerate ($T << \mu$) Fermi gas we have the number of particles given by the integral

$$N = V\int_0^\infty f(\varepsilon)\nu(\varepsilon)d\varepsilon \tag{1.1}$$

where $\nu(\varepsilon)$ is the density of states given by

$$\nu(\varepsilon) = \frac{g_s m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\varepsilon^{1/2} \tag{1.2}$$

and $f(\varepsilon)$ is the Fermi-Dirac distribution

$$f(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/T}+1} \tag{1.3}$$

This implies that we can calculate the energy E of the system by the integral

$$E = V\int_0^\infty \varepsilon f(\varepsilon)\nu(\varepsilon)d\varepsilon \tag{1.4}$$

We note that both equations 1.1 and 1.4 have integrals of the form

$$I = \int_0^\infty \frac{\varepsilon^n}{e^{(\varepsilon-\mu)/T}+1}$$

therefore we wish to approximate the solution for I in the limit of small temperatures where we take $1 << \mu/T$. So we apply a change of variable by $(\varepsilon-\mu)/T = z$ so that the integral becomes

$$\begin{aligned}
I &= \int_{-\mu/T}^\infty \frac{(\mu+Tz)^n}{e^z+1}Tdz \\
&= -T\int_0^{-\mu/T} \frac{(\mu+Tz)^n}{e^z+1}dz + T\int_0^\infty \frac{(\mu+Tz)^n}{e^z+1}dz \\
&= T\int_0^{\mu/T} \frac{(\mu-Tz)^n}{e^{-z}+1}dz + T\int_0^\infty \frac{(\mu+Tz)^n}{e^z+1}dz
\end{aligned}$$

Note that we convert the first integral by $z \to -z$ also note that

$$\frac{1}{e^-z+1} = 1 - \frac{1}{e^z+1}$$

So we have

$$\begin{aligned}
I &= \int_0^{\mu/T}(\mu-Tz)^n Tdz - T\int_0^{\mu/T}\frac{(\mu-Tz)^n}{e^z+1}dz + T\int_0^\infty \frac{(\mu+Tz)^n}{e^z+1}dz \\
&= \int_0^\mu \varepsilon^n d\varepsilon + T\int_0^\infty \frac{(\mu+Tz)^n-(\mu-Tz)^n}{e^z+1}dz
\end{aligned}$$

Where we take the limit of $\mu/T \to \infty$ which follows by our low temperature assumption. Note the Taylor expansion of a general function of the form $f(a+bx)$ about $x=0$

$$\begin{aligned}
f(a+bx) &\approx f(a) + bf'(a)x + \frac{b^2}{2}f^{(2)}(a)x^2 + \frac{b^3}{3!}f^{(3)}(a)x^3 + \mathcal{O}(x^4) \\
f(a-bx) &\approx f(a) - bf'(a)x + \frac{b^2}{2}f^{(2)}(a)x^2 - \frac{b^3}{3!}f^{(3)}(a)x^3 + \mathcal{O}(x^4)
\end{aligned}$$

We note that for $f(a+bx)-f(a-bx)$ all the even powers cancel. So for our integral to leading order we have in T

$$I = \int_0^\mu \varepsilon^n d\varepsilon + 2n\mu^{n-1}T^2 \int_0^\infty \frac{z}{e^z+1}dz + \mathcal{O}(T^4)$$

Note that we can calculate the integral by

$$\begin{aligned}
\int_0^\infty \frac{z}{e^z+1}dz &= 2\int_0^\infty \frac{ze^{-z}}{e^{-z}+1}dz \\
&= 2\sum_{k=1}^\infty \int_0^\infty (-1)^k z e^{-kz} dz \\
&= 2\sum_{k=1}^\infty \frac{(-1)^k}{k^2} \int_0^\infty e^{-u} u du \\
&= \Gamma(2)\sum_{k=1}^\infty \frac{1}{k^2} \\
&= \Gamma(2)\zeta(2) = \frac{\pi^2}{6}
\end{aligned}$$

Where we use the *Gamma Function*

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du$$

and the *Riemann Zeta Function*

$$\zeta(x) = \sum_{k=1}^\infty \frac{1}{k^x}$$

So to second order in T we have

$$I = \frac{\mu^{n+1}}{n+1} + \frac{\pi^2}{6}n\mu^{n-1}T^2 + \mathcal{O}(T^4)$$

So we can approximate N where $n = 1/2$ as

$$\begin{aligned}
N = V\int_0^\infty f(\varepsilon)\nu(\varepsilon)d\varepsilon &= \frac{Vg_s m^{3/2}}{\sqrt{2}\pi^2\hbar^3} I(n=1/2) \\
&= \frac{Vg_s m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\left(\frac{2}{3}\mu^{3/2} + \frac{\pi^2}{12}\frac{T^2}{\mu^{1/2}}\right)
\end{aligned}$$

and for E we approximate using $n = 3/2$ as

$$\begin{aligned}
E = V\int_0^\infty \varepsilon f(\varepsilon)\nu(\varepsilon)d\varepsilon &= \frac{Vg_s m^{3/2}}{\sqrt{2}\pi^2\hbar^3} I(n=3/2) \\
&= \frac{Vg_s m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\left(\frac{2}{5}\mu^{5/2} + \frac{\pi^2}{4}T^2\mu^{1/2}\right)
\end{aligned}$$

Given that the thermodynamic potential is related to energy by $\Omega = -2/3E$ we can see that

$$\Omega = -\frac{2}{3}E = -\frac{\sqrt{2}Vg_s m^{3/2}}{3\pi^2\hbar^3}\left(\frac{2}{5}\mu^{5/2} + \frac{\pi^2}{4}T^2\mu^{1/2}\right)$$

Now we can find the free energy to second order in T by

$$
\begin{aligned}
F = \Omega + \mu N &= -\frac{\sqrt{2}Vg_s m^{3/2}}{3\pi^2\hbar^3}\left(\frac{2}{5}\mu^{5/2} + \frac{\pi^2}{4}T^2\mu^{1/2}\right) + \mu\left(\frac{Vg_s m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\left(\frac{2}{3}\mu^{3/2} + \frac{\pi^2}{12}\frac{T^2}{\mu^{1/2}}\right)\right) \\
&= \frac{Vg_s m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\left[-\frac{4}{15}\mu^{5/2} - \frac{\pi^2}{6}T^2\mu^{1/2} + \frac{2}{3}\mu^{5/2} + \frac{\pi^2}{12}T^2\mu^{1/2}\right] \\
&= \frac{Vg_s m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\frac{2}{5}\mu^{5/2}\left[1 - \frac{5\pi^2}{24}\left(\frac{T}{\mu}\right)^2\right]
\end{aligned}
$$

Now we take $\mu = \varepsilon_F$ where ε_F is the Fermi energy given by

$$
\varepsilon_F = \frac{p_F^2}{2m} = \frac{\hbar^2(6\pi^2 N)^{2/3}}{2m(Vg_s)^{2/3}} = \gamma N^{2/3}
$$

So we have the free energy as a function of the number of particles given by

$$
\begin{aligned}
F &= \frac{Vg_s m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\frac{2}{5}\mu^{5/2}\left[1 - \frac{5\pi^2}{24}\left(\frac{T}{\mu}\right)^2\right] \\
&\Downarrow \\
&= \frac{3N}{5\varepsilon_F^{3/2}}\left[\varepsilon_F^{5/2} - \frac{5\pi^2}{12}T^2\varepsilon_F^{1/2}\right] \\
&= \frac{3N}{5}\left[\varepsilon_F - \frac{5\pi^2}{12}\frac{T^2}{\varepsilon_F}\right] \\
&= \frac{3}{5}\left[\gamma N^{5/3} - \frac{5\pi^2}{12\gamma}T^2 N^{1/3}\right]
\end{aligned}
$$

So we can find μ by the derivative

$$
\begin{aligned}
\mu = \left(\frac{\partial F}{\partial N}\right)_{V,T} &= \frac{3}{5}\left[\frac{5}{3}\gamma N^{2/3} - \frac{5\pi^2}{12}\frac{1}{3}\frac{T^2}{\gamma N^{2/3}}\right] \\
&\Downarrow \\
&= \left(\varepsilon_F - \frac{\pi^2}{12}\frac{T^2}{\varepsilon_F}\right) \\
&= \varepsilon_F\left(1 - \frac{\pi^2}{12}\left(\frac{T}{\varepsilon_F}\right)^2\right) + \mathcal{O}(T^3)
\end{aligned}
$$

(2) We can use the result from part (1) to find the correction to the energy by taking the ratio and keeping only terms to second order in T

$$
\begin{aligned}
\frac{E}{N} &= \left(\frac{2}{5}\mu^{5/2} + \frac{\pi^2}{4}T^2\mu^{1/2}\right)\left(\frac{2}{3}\mu^{3/2} + \frac{\pi^2}{12}\frac{T^2}{\mu^{1/2}}\right)^{-1} \\
&= \frac{3}{5}\mu\left(1 + \frac{5\pi^2}{8}\left(\frac{T}{\mu}\right)^2\right)\left(1 + \frac{\pi^2}{8}\left(\frac{T}{\mu}\right)^2\right)^{-1} \\
&\approx \frac{3}{5}\varepsilon_F\left(1 - \frac{\pi^2}{12}\left(\frac{T}{\varepsilon_F}\right)^2\right)\left(1 + \frac{5\pi^2}{8}\left(\frac{T}{\varepsilon_F}\right)^2\right)\left(1 - \frac{\pi^2}{8}\left(\frac{T}{\varepsilon_F}\right)^2\right) + \mathcal{O}(T^3) \\
&\approx \frac{3}{5}\varepsilon_F\left(1 + \frac{5\pi^2}{12}\left(\frac{T}{\varepsilon_F}\right)^2\right) + \mathcal{O}(T^3)
\end{aligned}
$$

Now we see that only the second term depends on T this implies that the specific heat at fixed volume to second order in temperature is

$$\left(\frac{\partial E}{\partial T}\right)_{N,V} = \frac{\partial}{\partial T}\left(\frac{N\pi^2}{4\varepsilon_F}T^2\right)$$
$$= \frac{N\pi^2}{2\varepsilon_F}T$$

(3) Given the isothermal and adiabatic compressibility

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T \qquad \kappa_S = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_S$$

we can calculate κ_T and κ_S by first noting that

$$p = \left(\frac{\partial E}{\partial V}\right)_{N,S} = -\left(\frac{\partial F}{\partial V}\right)_{N,T}$$

We found the approximations for F and E already where we note that the Fermi energy is proportional volume by

$$\varepsilon_F = \frac{\hbar^2(6\pi^2 N)^{2/3}}{2m(Vg_s)^{2/3}} = \gamma' V^{-2/3}$$

so we have

$$\kappa_S^{-1} = -V\frac{\partial^2 E}{\partial V^2} = -V\frac{\partial^2}{\partial V^2}\left[\frac{3}{5}N\varepsilon_F\left(1+\frac{5\pi^2}{12}\left(\frac{T}{\varepsilon_F}\right)^2\right)\right]$$
$$= -\frac{3}{5}NV\frac{\partial^2}{\partial V^2}\left[\gamma' V^{-2/3} + \frac{5\pi^2}{12}\frac{T^2}{\gamma'}V^{2/3}\right]$$
$$= -\frac{3}{5}NV\frac{\partial}{\partial V}\left[-\frac{2}{3}\gamma' V^{-5/3} + \frac{2}{3}\frac{5\pi^2}{12}\frac{T^2}{\gamma'}V^{-1/3}\right]$$
$$= -\frac{2}{3}NV\left(\gamma' V^{-8/3} - \frac{\pi^2}{12}\frac{T^2}{\gamma'}V^{-4/3}\right)$$
$$= -\frac{2}{3}N\left(\frac{\varepsilon_F}{V} - \frac{\pi^2}{12}\frac{T^2}{\epsilon_F V}\right)$$
$$\Downarrow$$
$$\kappa_S = -\frac{3V}{2N\epsilon_F}\left(1+\frac{\pi^2}{12}\left(\frac{T}{\epsilon_F}\right)^2\right) + \mathcal{O}(T^3)$$

Now we repeat the process for the isothermal case

$$\kappa_T^{-1} = V\frac{\partial^2 F}{\partial V^2} = V\frac{\partial^2}{\partial V^2}\left[\frac{3N}{5}\left(\varepsilon_F - \frac{5\pi^2}{12}\frac{T^2}{\varepsilon_F}\right)\right]$$
$$= \frac{3}{5}NV\frac{\partial^2}{\partial V^2}\left[\gamma' V^{-2/3} - \frac{5\pi^2}{12}\frac{T^2}{\gamma'}V^{2/3}\right]$$
$$= \frac{2}{3}\frac{N\varepsilon_F}{V}\frac{\partial^2}{\partial V^2}\left[1+\frac{\pi^2}{12}\left(\frac{T}{\varepsilon_F}\right)^2\right)$$
$$\Downarrow$$
$$\kappa_T = -\frac{3V}{2N\epsilon_F}\left(-1+\frac{\pi^2}{12}\left(\frac{T}{\epsilon_F}\right)^2\right) + \mathcal{O}(T^3)$$

We see that for $T = 0$ we have equal and opposite expansions for each case.

(4) For an electron-proton plasma that has turned into a dense neutron gas at zero temperature in which we assume this is an ideal gas. We state that the density of this gas must be so great that it overcomes the degeneracy pressure given as

$$pV = \frac{2}{5}N\varepsilon_F$$

so that the electrons combine with the protons in the nucleus. This yields the relation

$$n >> \frac{5}{2}\frac{p}{\varepsilon_F}$$

2 Problem #2

(1) For a Bose gas of ^{39}K atoms whose density is $10^{15}\,\mathrm{cm}^{-3}$ we can find the condensation temperature by the formula

$$T_{BEC} = 3.31\frac{\hbar^2 n^{2/3}}{mk} = 3.31\frac{(1.05\times10^{-34}\,\mathrm{m}^2\ \mathrm{kg}\ \mathrm{s}^{-1})^2(10^{15}\,\mathrm{cm}^{-3})^{2/3}}{(6.47\times10^{-26}\,\mathrm{kg})(1.38\times10^{-23}\,\mathrm{J}\ \mathrm{K}^{-1})} = 4.1\times10^{-6}\,\mathrm{K}$$

(2) For a Bose gas of particles with the mass, m, is placed into an anisotropic oscillatory potential of the following form

$$V(x,y,z) = \frac{m\omega^2\left(x^2+y^2\right)}{2} + \frac{m\Omega^2 z^2}{2}$$

where we assume that $\omega << \Omega$ and both oscillator lengths $I_\omega = \sqrt{\hbar/m\omega}$ and $I_\Omega = \sqrt{\hbar/m\Omega}$ are much larger than the distance between particles in the gas, $n^{-1/3}$. We can find the temperature of it's Bose-Einstein condensate by taking the distribution function

$$f(\varepsilon) = \frac{1}{e^{(\varepsilon(\mathbf{r},\mathbf{p})-\mu)/T}-1}$$

where the energy is given by the harmonic oscillator

$$\varepsilon = \hbar\omega\left(n_x+\frac{1}{2}\right) + \hbar\omega\left(n_y+\frac{1}{2}\right) + \hbar\Omega\left(n_z+\frac{1}{2}\right)$$

Using this relation we can calculate the density of states $\nu(\varepsilon)$ by integrating over the energy space to find that

$$\nu(\varepsilon) = \frac{\varepsilon^2}{2\hbar^3\omega^2\Omega}$$

This allows us to calculate the number of particles as

$$N = \int_0^\infty f(\varepsilon)\nu(\varepsilon)d\varepsilon = \frac{1}{2\hbar^3\omega^2\Omega}\int_0^\infty \frac{\varepsilon^2}{e^{(\varepsilon-\mu)/T}-1}d\varepsilon$$

We note that we reach condensation at $T = T_{BEC}$ and $\mu = 0$ so we have

$$\begin{aligned} N &= \frac{1}{2\hbar^3\omega^2\Omega}\int_0^\infty \frac{\varepsilon^2}{e^{\varepsilon/T_{BEC}}-1}d\varepsilon \\ &= \frac{2T_{BEC}^3\zeta(3)}{2\hbar^3\omega^2\Omega} \\ &= \frac{(1.20)T_{BEC}^3}{\hbar^3\omega^2\Omega} \\ &\Downarrow \\ T_{BEC} &= \left(\frac{N\hbar^3\omega^2\Omega}{1.20}\right)^{1/3} \end{aligned}$$

(3) To find the specific heat near the condensation temperature we calculate the energy by

$$E = \int_0^\infty \varepsilon f(\varepsilon)\nu(\varepsilon)d\varepsilon = \frac{1}{2\hbar^3\omega^2\Omega}\int_0^\infty \frac{\varepsilon^3}{e^{(\varepsilon-\mu)/T}-1}d\varepsilon$$

$$\Downarrow$$

$$= \frac{1}{2\hbar^3\omega^2\Omega}\int_0^\infty \frac{\varepsilon^3}{e^{\varepsilon/T_{BEC}}-1}d\varepsilon$$

$$= \frac{1}{2\hbar^3\omega^2\Omega}\frac{\pi^4 T_{BEC}^4}{15}$$

So the specific heat is given by

$$C_V = \frac{\partial E}{\partial T} = \frac{2\pi^2 T_{BEC}^3}{15\hbar^2\omega^2\Omega}$$