Physics 601

Analytical Mechanics
Professor Siu Chin

Homework #11

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1 Problem #1

(a) For a free falling object with the Hamiltonian

$$H = \frac{p^2}{2m} + mgq$$

with initial conditions q_0 and $p_0 = 0$. We know from kinematics that the time from the initial height to the ground is given by

$$t = \sqrt{\frac{2q_0}{g}}$$

this implies the period of oscillation is twice this time or

$$T = 2\sqrt{\frac{2q_0}{g}}$$

(b) We can compute the action variable, J, for this system by noting that the Hamiltonian is equal to a constant energy, E, and solving for p as

$$\begin{gathered} E = \frac{p^2}{2m} + mgq \\ \Downarrow \\ p = \sqrt{2mE - 2m^2gq} \end{gathered}$$

This allows us to integrate p over a cycle of q by

$$\begin{aligned} J &= \frac{1}{2\pi}\oint pdq \\ &= \frac{1}{2\pi}2\int_0^{q_0}\sqrt{2mE - 2m^2gq}dq \\ &= -\frac{2}{3\pi}\left.\frac{(2mE - 2m^2gq)^{3/2}}{2m^2g}\right|_0^{q_0} \\ &= -\frac{1}{3\pi m^2 g}\Big((2mE - 2m^2gq_0)^{3/2} - (2mE)^{3/2}\Big) \end{aligned}$$

We note that at q_0 we have $E = mgq_0$ so $q_0 = E/mg$ which replacing yields

$$J = \frac{(2mE)^{3/2}}{3\pi m^2 g}$$

(c) Using the result from part (b) we can calculate the angular frequency by

$$\omega = \frac{\partial E}{\partial J}$$

where we first solve for $E(J)$ as

$$\begin{gathered} J = \frac{(2mE)^{3/2}}{3\pi m^2 g} \\ \Downarrow \\ E = \frac{(3\pi m^2 gJ)^{2/3}}{2m} \end{gathered}$$

So we can calculate the derivative with respect to J as

$$\omega = \frac{\partial E}{\partial J} = \frac{(3\pi m^2 g)^{2/3}}{3m} J^{-1/3}$$

but if we replace J in terms of q_0 we have

$$\begin{aligned}
\omega &= \frac{(3\pi m^2 g)^{2/3}}{3m} \left(\frac{3\pi m^2 g}{(2m^2 g q_0)^{3/2}} \right)^{1/3} \\
&= \frac{\pi m g}{(2m^2 g q_0)^{1/2}} \\
&= \pi \sqrt{\frac{g}{2q_0}}
\end{aligned}$$

(d) Now we know that period is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} \sqrt{\frac{2q_0}{g}} = 2\sqrt{\frac{2q_0}{g}}$$

which agrees with the result from part (a).

2 Problem #2

We can calculate the integral

$$I = \int_a^b \frac{dx}{x}\sqrt{(x-a)(b-x)}$$

by expanding the product and bringing in the x^{-1} term to yield

$$I = \int_a^b dx\sqrt{-1+(a+b)/x-ab/x^2}$$

which allows us to use integration by parts which states

$$\int udv = uv - \int vdu$$

where we take

$$\begin{aligned} dv &= dx \qquad & u &= \sqrt{-1+(a+b)/x-ab/x^2} \\ v &= x \qquad & du &= \frac{1}{2}\left(-1+\frac{a+b}{x}-\frac{ab}{x^2}\right)^{-1/2}\left(-\frac{a+b}{x^2}+\frac{2ab}{x^3}\right) \end{aligned}$$

Which implies that

$$\begin{aligned} I =& \sqrt{(x-a)(b-x)}\Big|_a^b - \frac{1}{2}\int_a^b x\left(-1+\frac{a+b}{x}-\frac{ab}{x^2}\right)^{-1/2}\left(-\frac{a+b}{x^2}+\frac{2ab}{x^3}\right) \\ &= 0 - \frac{1}{2}\int_a^b x\left(-1+\frac{a+b}{x}-\frac{ab}{x^2}\right)^{-1/2}\left(-\frac{a+b}{x^2}+\frac{2ab}{x^3}\right) \\ &= -\frac{1}{2}\int_a^b \frac{dx}{\sqrt{-1+(a+b)/x-ab/x^2}}\left(-\frac{a+b}{x}+\frac{2ab}{x^2}\right) \\ &= \frac{a+b}{2}\int_a^b \frac{dx}{\sqrt{-x^2+(a+b)x-ab}} - ab\int_a^b \frac{dx}{\sqrt{-x^4+(a+b)x^3-abx^2}} \\ &= \frac{a+b}{2}\int_a^b \frac{dx}{\sqrt{-(x-(a+b)/2)^2+((a-b)/2)^2}} - ab\int_a^b \frac{dx}{\sqrt{-x^4+(a+b)x^3-abx^2}} \end{aligned}$$

We can calculate the first integral using a substitution

$$\frac{a-b}{2}\sin u = x - \frac{a+b}{2}$$

which has the infinitesimal of

$$\frac{a-b}{2}\cos u du = dx$$

So the first term becomes

$$\begin{aligned} \frac{a+b}{2}\int_a^b \frac{dx}{\sqrt{-(x-(a+b)/2)^2+((a-b)/2)^2}} &= (a+b)\int_{u(a)}^{u(b)} \frac{(a-b)/2\cos u}{\sqrt{((a-b)/2)^2(1-\sin^2 u)}}du \\ &= \frac{a+b}{2}\int_{u(a)}^{u(b)} \frac{(a-b)/2\cos u}{((a-b)/2\cos u}du \\ &= \frac{a+b}{2}\int_{u(a)}^{u(b)} du \\ &= \frac{a+b}{2}\arcsin\left(\frac{2x-a-b}{a-b}\right)\Big|_a^b = \frac{a+b}{2}\pi \end{aligned}$$

Now for the second integral we use the same substitution so that we are left with

$$
\begin{aligned}
-ab \int_a^b \frac{dx}{\sqrt{-x^4 + (a+b)x^3 - abx^2}} &= -ab \int_{u(a)}^{u(b)} \frac{du}{x(u)} \\
&= -ab \int_{u(a)}^{u(b)} \frac{2}{(a-b)\sin(u) + a + b} \\
&= -2\frac{ab}{\sqrt{ab}} \arctan\left(\frac{(a+b)\tan(u/2) + a - b}{2\sqrt{ab}} \right)\Bigg|_{-\pi}^{\pi} \\
&= -2\frac{ab}{\sqrt{ab}}\frac{\pi}{2} = -\pi\sqrt{ab}
\end{aligned}
$$

Therefore we combine terms to find I as

$$
I = \pi \left(\frac{a+b}{2} - \sqrt{ab} \right)
$$

3 Problem #3

(a) For the isotropic two dimensional harmonic oscillator we have the Hamiltonian

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{1}{2}m\omega_0^2 r^2$$

We can calculate J_θ by noting for a central force problem $p_\theta = L$ where L is a constant. Therefore

$$J_\theta = \frac{1}{2\pi}\oint p_\theta d\theta = \frac{L}{2\pi}\int_0^{2\pi} d\theta = L$$

Using this result and solving for p_r as

$$p_r = \sqrt{2m(E - L^2/2mr^2 - 1/2m\omega_0^2 r^2)}$$

we can calculate by using the result from problem (2)

$$\begin{aligned}
J_\theta = \frac{1}{2\pi}\oint p_r dr &= \frac{1}{2\pi}\oint dr\sqrt{2m(E - L^2/2mr^2 - 1/2m\omega_0^2 r^2)} \\
&= \frac{1}{2\pi}\oint dr\frac{1}{r}\sqrt{2mEr^2 - L^2 - m^2\omega_0^2 r^4} \\
&= \frac{m\omega_0}{4\pi}\oint dr'\frac{1}{r'}\sqrt{-m^2\omega_0^2 r'^2 + (2mE/m^2\omega_0^2)r' - L^2/m^2\omega_0^2} \\
&= \frac{m\omega_0}{4\pi}\pi\left(\frac{mE}{m^2\omega_0^2} - \frac{L}{m\omega_0}\right) \\
&= \frac{E}{4\omega_0} - \frac{L}{4}
\end{aligned}$$

Note that we used a change of variables $r' = r^2$ with $dr' = dr/2r$

(b) Solving the result from part (b) we have

$$E = 4\omega_0 J_r + \omega_0 L$$

So we can calculate the angular frequencies as

$$\omega_r = \frac{\partial E}{\partial J_r} = 4\omega_0, \qquad \omega_\theta = \frac{\partial E}{\partial L} = \omega_0$$

We can see that this results in a closed orbit due to the fact that

$$\frac{\omega_r}{\omega_\theta} = 4$$

which implies that we have an integer multiple of ω_θ in radial oscillations.

(c) The correct period is given by $\omega_\theta/2\pi$ because the frequency in the θ direction is the motion in which the radial motion precesses about.

4 Problem #4

(a) We can calculate a canonical perturbation given by the form

$$H(p,q,\epsilon) = H_0(p,q) + \epsilon H_1(p,q), \qquad \text{where} \qquad H_0(p,q) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$

for a perturbation of the form

$$H_1(p,q) = \frac{1}{2}m\omega_1^2 q^2$$

we can calculate the first order term to the corrected Kamiltonian given by

$$K(J) = \omega_0 J + \epsilon K_1(J)$$

where $K_1(J)$ is given by the average over ϕ_0 or

$$K_1(J) = \Big\langle H_1 \Big\rangle_{\phi_0}$$

So we use the solution of the unperturbed state

$$q = \sqrt{\frac{2J}{m\omega_0}} \sin\phi_0$$

So we calculate

$$\begin{aligned} \Big\langle H_1 \Big\rangle_{\phi_0} &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}m\omega_1^2 \frac{2J}{m\omega_0}\sin^2\phi_0 d\phi_0 \\ &= \frac{1}{2\pi}\frac{J\omega_1^2}{\omega_0}\int_0^{2\pi} \sin^2\phi_0 d\phi_0 \\ &= \frac{J\omega_1^2}{2\omega_0} \end{aligned}$$

Which yields

$$K(J) = \omega_0 J + \epsilon\frac{J\omega_1^2}{2\omega_0}$$

which gives us the corrected angular frequency

$$\omega = \frac{\partial K}{\partial J} = \omega_0 + \epsilon\frac{\omega_1^2}{2\omega_0}$$

We can compare this to the exact solution $\omega = \sqrt{\omega_0^2 + \omega_1^2}$ by expanding to first order

$$\begin{aligned} \sqrt{\omega_0^2+\omega_1^2} &= \omega_0\sqrt{1+\frac{\omega_1^2}{\omega_0^2}} \\ &= \omega_0\left(1 + \frac{1}{2}\frac{\omega_1^2}{\omega_0^2}\right) \\ &= \omega_0 + \frac{1}{2}\frac{\omega_1^2}{\omega_0} \end{aligned}$$

So we see that it agrees to first order for ω_1/ω_0 small.

(b) We can repeat this process for the perturbation

$$H_1(p,q) = \frac{1}{6}mq^6$$

which yields

$$\begin{aligned}
\Big\langle H_1 \Big\rangle_{\phi_0} &= \frac{1}{2\pi}\int_0^{2\pi} H_1 d\phi_0 \\
&= \frac{1}{2\pi}m\frac{1}{6}\left(\frac{2J}{m\omega_0}\right)^3 \int_0^{2\pi} \sin^6\phi_0 d\phi_0 \\
&= \frac{1}{2\pi}m\frac{1}{6}\left(\frac{2J}{m\omega_0}\right)^3 \frac{5}{8}\pi \\
&= \frac{5}{12}\frac{J^3}{m^2\omega_0^3}
\end{aligned}$$

So we have the corrected Kamiltonian as

$$K(J) = \omega_0 J + \epsilon\frac{5}{12}\frac{J^3}{m^2\omega_0^3}$$

which yields

$$\omega = \omega_0 + \epsilon\frac{5}{4}\frac{J^2}{m^2\omega_0^3}$$

where we note that

$$J = \frac{E}{\omega_0} = \frac{1}{2}m\omega_0 q_0^2$$

which allows us to say that

$$\omega = \omega_0 + \epsilon\frac{5}{16}\frac{q_0^4}{\omega_0^2}$$