Physics 624

Quantum Mechanics II

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Homework #2

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UID: 125-00-4128
February 15th, 2016

1 Problem #1

Given a particle in a uniform time-dependent field with a force $\mathbf{F}(t) \to 0$ for $|t| \to \infty$ we can find the change in the average value of the energy caused by interaction with the field, $U = -\mathbf{F}(t) \cdot \mathbf{r}(t)$ by noting that the time evolution of an operator in the Heisenberg representation is given by the commutator with the Hamiltonian

$$\frac{d\hat{F}}{dt} = \frac{1}{i\hbar}[\hat{F}, \hat{H}]$$

which allows us to gain the Heisenberg equations of motion

$$\frac{d\hat{\mathbf{p}}(t)}{dt} = -\nabla U = \mathbf{F}(t)$$
$$\frac{d\hat{\mathbf{r}}(t)}{dt} = \frac{\mathbf{p}}{m}$$

Solving for $\hat{\mathbf{p}}(t)$ we integrate to get

$$\hat{\mathbf{p}}(t) + C = \int_{-\infty}^{t} \mathbf{F}(t')dt'$$
$$\Downarrow$$
$$\hat{\mathbf{p}}(t) = \int_{-\infty}^{t} \mathbf{F}(t')dt' + \mathbf{p}_0$$

Note we use an initial condition to find the integration constant because we assume $\mathbf{F}(t = -\infty) = 0$ which implies that

$$\mathbf{p}_0 \equiv \mathbf{p}(t = -\infty)$$

. We also apply this assumption to see that at $|t| \to \infty$ the particle acts like a free particle with the Hamiltonian

$$\lim_{|t|\to\infty} \hat{H}(t) = \frac{\hat{\mathbf{p}}^2(t = \pm\infty)}{2m}$$

therefore we can find the change in the average value of energy from $t = -\infty$ to $t = \infty$ through the Hamiltonian by noting that

$$\langle E(\pm\infty)\rangle = \langle H\rangle(\pm\infty) = \frac{\langle \hat{\mathbf{p}}^2(\pm\infty)\rangle}{2m}$$

where

$$\hat{\mathbf{p}}^2(t) = \mathbf{p}_0^2 + 2\mathbf{p}_0 \int_{-\infty}^{t} \mathbf{F}(t')dt' + \left(\int_{-\infty}^{t} \mathbf{F}(t')dt'\right)^2$$

we note that at $t = -\infty$ we have $\hat{\mathbf{p}}^2(-\infty) = \mathbf{p}_0^2$ as all the integrals go to zero. So this implies that

$$\langle E(-\infty)\rangle = \frac{\langle \mathbf{p}_0^2\rangle}{2m}$$

and that

$$\langle E(\infty)\rangle = \frac{\langle \mathbf{p}_0^2\rangle}{2m} + \frac{\langle \mathbf{p}_0\rangle}{m} \int_{-\infty}^{\infty} \mathbf{F}(t')dt' + \left(\int_{-\infty}^{\infty} \mathbf{F}(t')dt'\right)^2$$
$$\langle E(\infty)\rangle = \langle E(-\infty)\rangle + \frac{\langle \mathbf{p}_0\rangle}{m} \int_{-\infty}^{\infty} \mathbf{F}(t')dt' + \left(\int_{-\infty}^{\infty} \mathbf{F}(t')dt'\right)^2$$
$$\Downarrow$$
$$\langle E(\infty)\rangle - \langle E(-\infty)\rangle = \frac{\langle \mathbf{p}_0\rangle}{m} \int_{-\infty}^{\infty} \mathbf{F}(t')dt' + \left(\int_{-\infty}^{\infty} \mathbf{F}(t')dt'\right)^2$$

2 Problem #2

(a) For a ground state harmonic oscillator under an applied external force $\mathbf{F}(t)$, such that $\mathbf{F}(t) \to 0$ for $|t| \to \infty$ we can find the time evolution of the creation and annihilation operators, $\hat{a}$ and $\hat{a}^\dagger$, by taking the Heisenberg representation

$$\frac{d\hat{\mathbf{p}}(t)}{dt} = -m\omega^2\mathbf{r} + \mathbf{F}(t)$$
$$\frac{d\hat{\mathbf{r}}(t)}{dt} = \frac{\mathbf{p}}{m}$$

This allows us to use the definition of the creation and annihilation operators to say

$$\begin{aligned}
\frac{d\hat{a}}{dt} &= \sqrt{\frac{m\omega}{2\hbar}}\left(\frac{d\hat{\mathbf{r}}}{dt} + \frac{i}{m\omega}\frac{d\hat{\mathbf{p}}}{dt}\right) \\
&= \sqrt{\frac{m\omega}{2\hbar}}\left(\frac{\hat{\mathbf{p}}}{m} - i\omega\hat{\mathbf{r}} + \frac{i\mathbf{F}(t)}{m\omega}\right) \\
&= -i\omega\sqrt{\frac{m\omega}{2\hbar}}\left(\frac{i\hat{\mathbf{p}}}{m\omega} + \hat{\mathbf{r}}\right) + \sqrt{\frac{m\omega}{2\hbar}}\frac{i\mathbf{F}(t)}{m\omega} \\
&= -i\omega\hat{a} + i\frac{\mathbf{F}(t)}{\sqrt{2\hbar m\omega}}
\end{aligned}$$

And the same follows for $\hat{a}^\dagger$ as

$$\frac{d\hat{a}^\dagger}{dt} = i\omega\hat{a}^\dagger - i\frac{\mathbf{F}(t)}{\sqrt{2\hbar m\omega}}$$

Now we can solve the differentials by direct integration using an integrating factor which yields

$$\begin{aligned}
\hat{a}(t) &= a(0)e^{-i\omega t} + \frac{ie^{-i\omega t}}{\sqrt{2\hbar m\omega}}\int_{-\infty}^{t}\mathbf{F}(t')e^{i\omega t'}dt' \\
\hat{a}^\dagger(t) &= a^\dagger(0)e^{i\omega t} - \frac{ie^{i\omega t}}{\sqrt{2\hbar m\omega}}\int_{-\infty}^{t}\mathbf{F}(t')e^{-i\omega t'}dt'
\end{aligned}$$

Where we have $a(0)$ and $a^\dagger(0)$ follow from the integration constants which we take to be initial conditions. Note that the fact that $\hat{a}$ and $\hat{a}^\dagger$ are Hermitian conjugates still holds.

(b) To find the average energy as $t \to +\infty$ we note that in this limit $(F)(t) = 0$ so we have an unforced harmonic oscillator which has a Hamiltonian

$$\lim_{t\to\infty}\hat{H}(t) = \hbar\omega\left(\hat{a}^\dagger(\infty)\hat{a}(\infty) + \frac{1}{2}\right)$$

where we find

$$\hat{a}(t)\hat{a}^\dagger(t) = a(0)a^\dagger(0) - \frac{ia(0)}{\sqrt{2\hbar m\omega}}\int_{-\infty}^{t}\mathbf{F}(t')e^{-i\omega t'}dt' + \frac{ia^\dagger(0)}{\sqrt{2\hbar m\omega}}\int_{-\infty}^{t}\mathbf{F}(t')e^{i\omega t'}dt' + \left|\frac{i}{\sqrt{2\hbar m\omega}}\int_{-\infty}^{t}\mathbf{F}(t')e^{i\omega t'}dt'\right|^2$$

note that in the $-\infty$ limit the integrals disappear which implies that

$$\langle E(-\infty)\rangle = \hbar\omega\left\langle\hat{a}^\dagger(-\infty)\hat{a}(-\infty) + \frac{1}{2}\right\rangle = \hbar\omega\left\langle a^\dagger(0)a(0) + \frac{1}{2}\right\rangle$$

which allows us to say that for $t \to \infty$ we have the average energy

$$\langle E(\infty)\rangle = \langle E(-\infty)\rangle - \frac{i\hbar\omega a(0)}{\sqrt{2\hbar m\omega}}\int_{-\infty}^{\infty}\mathbf{F}(t')e^{-i\omega t'}dt' + \frac{i\hbar\omega a^\dagger(0)}{\sqrt{2\hbar m\omega}}\int_{-\infty}^{\infty}\mathbf{F}(t')e^{i\omega t'}dt' + \left|\frac{i}{\sqrt{2m}}\int_{-\infty}^{\infty}\mathbf{F}(t')e^{i\omega t'}dt'\right|^2$$

(c) Next we can find the excitation probabilities of a stationary state for $t \to \infty$ using the fact that the system is in the ground state for $t \to -\infty$ which implies that

$$\hat{H}(-\infty)|\psi\rangle \left(a^{\dagger}(0)a(0) + \frac{1}{2}\right)|\psi\rangle = \frac{\hbar\omega}{2}|\psi\rangle$$

or that $|\psi\rangle = |0\rangle_{-\infty}$. Next we note that at positive infinity we are again in a stationary state as $\mathbf{F} = 0$. Therefore we can generate the n^{th} excited state by acting the creation operator on $|0\rangle$ n-times, but due to the fact that we are operating in Heisenberg representation we need to use the creation operator at $t = \infty$

$$|n\rangle_{+\infty} = \frac{\left(\hat{a}^{\dagger}(\infty)\right)^n}{\sqrt{n!}}|0\rangle_{+\infty}$$

Note that the set of eigenkets $|n\rangle_{+\infty}$ forms an orthonormal basis, which allows us to write the initial state in this basis as

$$|0\rangle_{-\infty} = \sum_{n=0}^{\infty} c_n |n\rangle_{+\infty}$$

where the coefficient $|c_n|^2$ is the transition probability from $|0\rangle_{-\infty}$ to $|n\rangle_{+\infty}$. We can determine c_n by noting that if we act the ladder operator we get

$$\hat{a}(\infty)|n\rangle_{+\infty} = \sqrt{n}|n-1\rangle_{+\infty}$$

and we note that if we act $a(0)$ onto the expansion we get

$$0 = a(0)\sum_{n=0}^{\infty} c_n|n\rangle_{+\infty}$$

noting that $a(0)$ is also contained in $\hat{a}(\infty)$ as found in part (a)

$$\hat{a}(\infty) = e^{-i\omega t}\left(a(0) + \frac{i}{\sqrt{2\hbar m\omega}}\int_{-\infty}^{\infty} \mathbf{F}(t')e^{i\omega t'}dt'\right)$$

. Therefore it must follow that

$$c_n = \frac{c_{n-1}}{\sqrt{n}}\frac{i}{\sqrt{2\hbar m\omega}}\int_{-\infty}^{\infty} \mathbf{F}(t')e^{i\omega t'}dt'$$

Which implies that the n^{th} state is related to the ground state by

$$c_n = c_0\frac{1}{\sqrt{n!}}\left(\frac{i}{\sqrt{2\hbar m\omega}}\int_{-\infty}^{\infty} \mathbf{F}(t')e^{i\omega t'}dt'\right)^n$$

therefore we can apply the normalization condition to yeild

$$\sum_{n=0}^{\infty}|c_n|^2 = 1 = |c_0|^2\sum_{n=0}^{\infty}\frac{1}{n!}\left|\frac{i}{\sqrt{2\hbar m\omega}}\int_{-\infty}^{\infty} \mathbf{F}(t')e^{i\omega t'}dt'\right|^{2n}$$

$$= |c_0|^2 \exp\left[\left|\frac{i}{\sqrt{2\hbar m\omega}}\int_{-\infty}^{\infty} \mathbf{F}(t')e^{i\omega t'}dt'\right|^2\right]$$

$$\Downarrow$$

$$|c_0|^2 = \exp\left[-\frac{1}{2\hbar m\omega}\left|\int_{-\infty}^{\infty} \mathbf{F}(t')e^{i\omega t'}dt'\right|^2\right]$$

So the transition probability follows

$$|c_n|^2 = \frac{1}{n!}\left(\frac{i}{\sqrt{2\hbar m\omega}}\int_{-\infty}^{\infty} \mathbf{F}(t')e^{i\omega t'}dt'\right)^{2n}\exp\left[-\frac{1}{2\hbar m\omega}\left|\int_{-\infty}^{\infty} \mathbf{F}(t')e^{i\omega t'}dt'\right|^2\right]$$

3 Problem #3

(a) For a transformation from a stationary frame of reference to a frame of reference that uniformly rotates with an angular velocity, $\boldsymbol{\omega}$, we can find a unitary operator, $\mathscr{D}(\omega)$, by noting that for a constant angular velocity we move at a time-dependent angle $\boldsymbol{\phi}(t) = \boldsymbol{\omega} t$. Then we treat this like a infinitesimal rotation through the angle $-\boldsymbol{\phi}(t)$ which yields

$$\mathscr{D}(\boldsymbol{\omega}) = \lim_{N\to\infty}\left[1 + i\left(\frac{\mathbf{J}}{\hbar}\right)\cdot\left(\frac{\boldsymbol{\omega} t}{N}\right)\right]^N = \exp\left(\frac{i\mathbf{J}\cdot\boldsymbol{\omega} t}{\hbar}\right)$$

Note we rotate through a negative angle because we are rotating the frame of reference with can be though of rotating the vectors in the opposite direction. We also are projecting the angular momentum $\mathbf{J}$ to be along the rotation $\boldsymbol{\omega}$. Without loss of generality we assume that both $\mathbf{J}$ and $\boldsymbol{\omega}$ point along the z direction.

(b) So we can use transformation $\mathscr{D}$ we found in part (a) to transform the coordinate operator into the rotating frame. This follows like any unitary transformation of an operator where

$$\begin{aligned}
\hat{x}(t) &= \mathscr{D}(\boldsymbol{\omega}) x \mathscr{D}^\dagger(\boldsymbol{\omega}) \\
&= \exp\left(\frac{iJ\omega t}{\hbar}\right) x \exp\left(-\frac{iJ\omega t}{\hbar}\right) \\
&= x + \frac{i\omega t}{\hbar}[J_z, x] - \frac{1}{2}\left(\frac{\omega t}{\hbar}\right)^2 [J_z,[J_z,x]] + \ldots \\
&= x + \frac{i\omega t}{\hbar}(i\hbar y) + \frac{1}{2}\left(\frac{\omega t}{\hbar}\right)^2 (\hbar^2 x) + \ldots \\
&= x\left(1 + \frac{1}{2}(\omega t)^2 + \ldots\right) - y\left(\omega t + \frac{1}{3!}(\omega t)^3\right) \\
&= x\cos(\omega t) - y\sin(\omega t)
\end{aligned}$$

Note this is due to the commutation relation between coordinates and angular momentum. Using the same commutation relation it follows that

$$\begin{aligned}
\hat{y}(t) &= x\sin(\omega t) + y\cos(\omega t) \\
\hat{z}(t) &= z
\end{aligned}$$

Note the $\hat{z}(t)$ remains untransformed due to the fact that $[J_z, z] = 0$.

(c) We repeat this process for the momentum noting that

$$\begin{aligned}
[J_z, p_x] &= i\hbar p_y \\
[J_z, p_y] &= -i\hbar p_x \\
[J_z, p_z] &= 0
\end{aligned}$$

Which yields the same transformation as in part (b), but with momentum. Therefore

$$\begin{aligned}
\hat{p}_x(t) &= p_x\cos(\omega t) - p_y\sin(\omega t) \\
\hat{p}_y(t) &= p_x\sin(\omega t) + p_y\cos(\omega t) \\
\hat{p}_z(t) &= p_z
\end{aligned}$$

(d) To find the Hamiltonian of the particle in this frame we note that the unrotated Hamiltonian is in the general form

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + U(\mathbf{r}, t)$$

note that we require the potential to be dependent on time in order to account for the rotation. So we can apply the unitary transformation

$$\hat{H}(t) = \mathscr{D}\hat{H}\mathscr{D}^{\dagger}$$

Which must still satisfy the Schr\'odinger equation

$$i\hbar\frac{d\psi}{dt} = \hat{H}\psi$$

. So we can transform both sides to yield

$$\mathscr{D}i\hbar\frac{d\psi)}{dt} = \mathscr{D}\hat{H}\psi$$
$$\Downarrow$$
$$i\hbar\left(\frac{d(\mathscr{D}\psi)}{dt} - \psi\frac{d\mathscr{D}}{dt}\right) = \hat{H}(\mathscr{D}\psi)$$
$$\Downarrow$$
$$i\hbar\frac{d(\mathscr{D}\psi)}{dt} = \left(\mathscr{D}\hat{H}\mathscr{D}^{\dagger} + i\hbar\frac{d\mathscr{D}}{dt}\mathscr{D}^{\dagger}\right)(\mathscr{D}\psi)$$

So our new Hamiltonian operator is the term given in the parenthesis. Which for the rotation is

$$\hat{H}(t) = -\frac{\hat{\mathbf{p}}^2}{2m} + U'(\mathbf{r}) - \frac{\mathbf{J}\cdot\boldsymbol{\omega}}{\hbar}$$