Physics 624

Quantum Mechanics II

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Homework #1

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1 Problem #1

For the case of a time-independent Hamiltonian the propagator, $K(q,t;q',t'=0)$ satisfies the equation

$$\hat{q}(-t)K(q,t;q',t'=0) = cK(q,t;q',t'=0),$$

where $\hat{q}(t)$ is an operator in the Heisenberg representation. We can find c by noting that the Heisenberg representation of the operator $\hat{q}(t)$ is given by

$$\hat{q}(t) = S^{-1}(t)\hat{q}S(t)$$

where S is the unitary transformation given by

$$S = \exp\left(-\frac{i}{\hbar}\hat{H}t\right)$$

Therefore we see that the operator becomes $\hat{q}(-t) = S^{-1}(-t)\hat{q}S(-t)$. This allows us to treat the propagator as a wave function dependent on the parameter q and t that is being acted on by the transformation $S(-t)$. We note that $S(-t)$ evolves the wave function that it acts upon backwards $-t$ in time. This implies that

$$S(-t)K(q,t;q',t'=0) = K(q,0;q',0) = \delta(q'-q)$$

we note that the delta function arises by the property of the propagator when $t'=t$. Using the fact that $\delta(q'-q)$ is an eigenfunction of the operator $\hat{q}$ we note that

$$\begin{aligned}
\hat{q}(-t)K(q,t;q',t'=0) &= S^{-1}(-t)\hat{q}S(-t)K(q,t;q',t'=0) \\
&= S^{-1}(-t)\hat{q}\delta(q'-q) \\
&= q'S(t)\delta(q'-q)
\end{aligned}$$

We note that the inverse transformation on the negative time step $-t$ is the same as the transformation with a positive time step t. This transformation acting on the delta function, $\delta(q'-q)$, recovers the propagator with implies that

$$\hat{q}(-t)K(q,t;q',t'=0) = q'K(q,t;q',t'=0)$$

or that $c=q'$. Therefore when we act $\hat{q}(-t)$ on the propagator we recover the initial condition q'.

2 Problem #2

(a) We can find the Green's Function, $G_E(x, x')$ of the Schrödinger's Equation for a free particle with $E < 0$ vanishing at $|x - x'| \to \infty$ by noting that $G_E(x, x')$ solves Schrödinger's Equation by

$$(\hat{H} - E)G_E(x, x') \equiv -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}G_E(x, x') - EG_E(x, x') = \delta(x - x').$$

We can solve this equation by using the general solution of G_E given by

$$G_E(x, x') = A(x')e^{k|x-x'|} + B(x')e^{-k|x-x'|}$$

where $k = \sqrt{-2mE}/\hbar$ note that we assume that $E < 0$ which makes the solution an exponential. Applying the assumption that the particle vanishes for $|x - x'| \to \infty$ we see that $A(x') = 0$ must be true. In order to find $B(x')$ we note that G_E can be considered as the solution to the delta function potential which yields a discontinuity in the derivative of G_E at $x = x'$ which implies that

$$\frac{dG_E(x', x')}{dx} = -\frac{B(x')}{k}e^{-k|x-x'|} = -\frac{2m}{\hbar^2}$$
$$\Downarrow$$
$$B = \frac{m}{k\hbar^2}$$

Note the factor of $2m/\hbar^2$ follows from the solution to the delta function potential. Therefore, the Green function is

$$G_E(x, x') = \frac{m}{k\hbar^2}e^{-k|x-x'|}$$

(b) We can use this Green's function to represent the Schrödinger's equation with a short-range potential $U(x)$ where $U(x) \to 0$ as $x \to \infty$. This makes the Schrödinger's equation become

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) - E\psi(x) = U(x)\psi(x)$$

which allows us to use the general fact that a Green's function gives the solution to an inhomogeneous differential equation by an integral

$$y(x) = \int_{-\infty}^{\infty} G(x, x')f(x')dx' \tag{2.1}$$

So for this potential we have equation 2.1 become

$$\psi(x) = \int_{-\infty}^{\infty} -\frac{m}{k\hbar^2}e^{-k|x-x'|}U(x')\psi(x')dx'$$

(c) In order to find the momentum representation of G_E we note that for any linear operator there is an associated Green's function given by

$$\hat{L}\psi(\xi) \equiv \int L(\xi, \xi')\psi(\xi')d\xi'$$

this implies that there exists an operator associated with G_E which we will represent with $\hat{G}_E$. Given that the Green's function is a solution to the equation

$$(\hat{H} - E)G_E(x - x') = \delta(x - x')$$

we can write this relation independent of representation using the fact that the delta function is the position space representation of the identity operator, $\hat{I}$. Therefore,

$$(\hat{H} - E)\hat{G}_E = \hat{I}$$
$$\Downarrow$$
$$\hat{G}_E = (\hat{H} - E)^{-1}$$

Now we can simply write the Hamiltonian in momentum representation as $\hat{H} = p^2/2m$ which yields

$$G_E(p) = \frac{1}{p^2/2m - E}$$

3 Problem #3

(a) Given a particle in the field generated by a uniform force given by

$$U = -\mathbf{F_0} \cdot \mathbf{r}$$

we can find the coordinate representation of the propagator, $K(\mathbf{r}, t, \mathbf{r}', t_0)$ by first using the relation found in Problem 1 which states in three dimensions

$$\hat{\mathbf{r}}(-t)K(\mathbf{r}, t, \mathbf{r}', t_0) = \mathbf{r}'K(\mathbf{r}, t, \mathbf{r}', t_0)$$

where $\hat{\mathbf{r}}(t)$ is the Heisenberg representation of the position operator given by

$$\begin{aligned}
\hat{\mathbf{r}}(t) = \hat{S}^{-1}(t)\hat{\mathbf{r}}\hat{S} &= \exp\left(i\frac{\hat{H}}{\hbar}t\right)\hat{\mathbf{r}}\exp\left(-i\frac{\hat{H}}{\hbar}t\right) \\
&= \left(1 + \frac{i\hat{H}}{\hbar}t + ...\right)\hat{\mathbf{r}}\left(1 - \frac{i\hat{H}}{\hbar}t + ...\right) \\
&= \hat{\mathbf{r}} + \frac{i}{\hbar}[\hat{H}, \hat{\mathbf{r}}]t - \frac{1}{2\hbar^2}[\hat{H}, [\hat{H}, \hat{\mathbf{r}}]]t^2
\end{aligned}$$

Note that the commutation relation between $\hat{H}$ and $\hat{\mathbf{r}}$ is given as

$$[\hat{H}, \hat{\mathbf{r}}] = -\frac{i\hbar}{m}\hat{\mathbf{p}}$$

and

$$[\hat{H}, \hat{\mathbf{p}}] = -i\hbar\frac{\partial U}{\partial d\mathbf{r}}$$

therefore we have

$$\hat{\mathbf{r}}(t) = \hat{\mathbf{r}} + \frac{\hat{\mathbf{p}}}{m}t + \frac{\mathbf{F_0}}{2m}t^2$$

so if we use the relation we find that

$$\left(\hat{\mathbf{r}} - \frac{\mathbf{p}}{m}t + \frac{\mathbf{F_0}}{2m}t^2\right)K(\mathbf{r}, t, \mathbf{r}', t_0) = \mathbf{r}'K(\mathbf{r}, t, \mathbf{r}', t_0)$$

which if we generalize to a free particle in three dimensions but where $\mathbf{r}' = \mathbf{r} - \mathbf{F_0}t^2/2m$ which yields

$$K(\mathbf{r}, t, \mathbf{r}', t_0) = \left(\frac{m}{2\pi i\hbar(t - t_0)}\right)^{3/2}\exp\left(\frac{i}{\hbar}\left(\frac{1}{2m(t - t_0)}\left(\mathbf{r} - \mathbf{r}' - \frac{\mathbf{F_0}t^2}{2m}\right)^2 + \mathbf{F_0} \cdot \mathbf{r}t - \frac{F_0^2 t^3}{6m}\right)\right)$$

(b) We follow the same process for the momentum representation noting that

$$\begin{aligned}\hat{\mathbf{p}}(t) &= \hat{\mathbf{p}} + \frac{i}{\hbar}[\hat{H}, \hat{\mathbf{p}}]t - \frac{1}{2\hbar^2}\cancelto{0}{[\hat{H}, [\hat{H}, \hat{\mathbf{p}}]]}t^2 \\ &= \hat{\mathbf{p}} + \frac{\mathbf{F_0}}{m}t\end{aligned}$$

which yields

$$K(\mathbf{p}, t, \mathbf{r}', t_0) = \exp\left(-\frac{i(t-t_0)}{2m\hbar}\left(p^2 - \mathbf{F_0}\cdot\mathbf{p}(t-t_0) + \frac{1}{3}F_0^2(t-t_0)^2\right)\right)\delta(\mathbf{p} - \mathbf{p}' - \mathbf{F_0}(t-t_0))$$