

Physics 615  
Methods of Theoretical Physics I  
Professor Katrin Becker

Homework #6

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UID: 125-00-4128  
October 14th, 2015

## 1 Problem #1

For the function

$$f(z) = \frac{e^z}{z(z^2 - 16)}$$

we can see that  $f(z)$  is analytic everywhere except  $z = 0$  and  $|z| = 4$ . So we can calculate the integral

$$I = \oint_C \frac{e^z}{z(z^2 - 16)}$$

where the contour  $C$  is defined as the boundary of the annulus between the circles  $|z| = 1$  and  $|z| = 3$ . We note that  $f(z)$  is analytic everywhere along  $C$  which implies that

$$I = \oint_C \frac{e^z}{z(z^2 - 16)} = 0$$

due to the fact that  $C$  is a closed contour.

## 2 Problem #2

1) For the integral

$$\oint_C e^{z^2} \left( \frac{1}{z^2} - \frac{1}{z^3} \right)$$

where  $C$  is the unit circle centered at the origin. We first expand  $e^{z^2}$  by

$$e^{z^2} = 1 + z^2 + \frac{z^4}{2} + \dots$$

Which makes our integral become

$$\begin{aligned} \oint_C e^{z^2} \left( \frac{1}{z^2} - \frac{1}{z^3} \right) &= \oint_C \left( 1 + z^2 + \frac{z^4}{2} + \dots \right) \left( \frac{1}{z^2} - \frac{1}{z^3} \right) \\ &= \oint_C \frac{1}{z^2} - \frac{1}{z^3} + 1 - \frac{1}{z} + \dots \end{aligned}$$

We note that the only integral in this sum that is nonzero is the  $z^{-1}$ . We can use the fact that  $z$  on the unit circle is given by  $z = e^{i\phi}$  which allows us to integrate over the circle by

$$\begin{aligned} - \oint_C \frac{1}{z} dz &= - \oint_C \frac{1}{e^{i\phi}} d(e^{i\phi}) \\ &= - \int_0^{2\pi} \frac{1}{e^{i\phi}} i e^{i\phi} d\phi \\ &= - \int_0^{2\pi} i d\phi \\ &= -2\pi i \end{aligned}$$

So the solution to the integral is

$$\oint_C e^{z^2} \left( \frac{1}{z^2} - \frac{1}{z^3} \right) = -2\pi i$$

2) For the integral

$$\oint_C \frac{(\sin z)^6}{(z - \frac{\pi}{6})^3}$$

we note the derivative formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w - z)^{n+1}}$$

where we can rearrange

$$\begin{aligned} f^{(n)}(z) &= \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w - z)^{n+1}} \\ &\Downarrow \\ \oint_C \frac{(\sin z)^6}{(z - \pi/6)^3} &= \frac{n!}{2\pi i} f^{(2)}(z) \\ &= \frac{2\pi i}{2!} \frac{d^2}{dz^2} (\sin z)^6 \Big|_{\pi/6} \\ &= \frac{2\pi i}{2} \frac{d}{dz} 6(\sin z)^5 \cos z \Big|_{\pi/6} \\ &= i\pi \left( 30(\sin z)^4 (\cos z)^2 - 6(\sin z)^6 \right) \Big|_{\pi/6} \\ &= i\pi (30(\sin \pi/6)^4 (\cos \pi/6)^2 - 6(\sin \pi/6)^6) \\ &= i\pi \left( 30 \frac{1}{2^4} \frac{3}{4} - 6 \frac{1}{2^6} \right) \\ &= i\pi \left( \frac{90}{64} - \frac{6}{64} \right) \\ &= i\pi \left( \frac{84}{64} \right) \\ &= \frac{21}{16} \pi i \end{aligned}$$

### 3 Problem #3

Given *Rodrigues' formula*

$$P_n(t) = \frac{1}{2^n n!} \left( \frac{d}{dx} \right)^n [(x^2 - 1)^n] \quad (3.1)$$

we can write the derivative as a complex integral by the formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w - z)^{n+1}}$$

where we note that

$$f^{(n)}(z) = \left( \frac{d}{dz} \right)^n [(z^2 - 1)^n]$$

which implies that  $f(z) = (z^2 - 1)^n$ . We can take the nth derivative of  $f(z)$  by the integral

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{(z^2 - 1)^n}{(z - t)^{n+1}}$$

And by replacing into equation 3.1 we get

$$\begin{aligned} P_n(t) &= \frac{1}{2^n n!} \left( \frac{d}{dx} \right)^n [(x^2 - 1)^n] \\ &\Downarrow \\ P_n(t) &= \frac{1}{2^n n!} \frac{n!}{2\pi i} \oint_C \frac{(z^2 - 1)^n}{(z - t)^{n+1}} \\ &= \frac{1}{2\pi i 2^n} \oint_C \frac{(z^2 - 1)^n}{(z - t)^{n+1}} \end{aligned}$$

### 4 Problem #4

For the function

$$f(z) = \frac{1}{e^z - 1}$$

we can calculate the first three non-zero terms in the *Laurent expansion* where we expand about the point  $z = 0$ . We expand

$$e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \dots$$

which makes  $f(z)$  become

$$f(z) = \frac{1}{z + z^2/2 + z^3/6 + \dots} = \frac{1}{z} \frac{1}{1 + z/2 + z^2/6 \dots}$$

Now we can expand

$$\frac{1}{1 + z/2 + z^2/6} = 1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^3}{24}$$

so the first four terms of the Laurent expansion are

$$f(z) = \frac{1}{e^z - 1} = \frac{1}{z} \frac{1}{1 + z/2 + z^2/6} = z^{-1} - \frac{1}{2} + \frac{1}{12}z - \frac{1}{24}z^2$$