Physics 601

Analytical Mechanics
Professor Siu Chin

Homework #5

Joe Becker
UID: 125-00-4128
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1 Problem #1

(a) We note *Noether's Theorem* which states that for every symmetry in the action there exists a conserved quantity. We can use this theorem to find the conserved quantity that corresponds to the symmetry due to the Lagrangian's invariance under a translation $\mathbf{r}' = \mathbf{r} + \epsilon\mathbf{s}$. This translation transforms our generalized coordinates by

$$q_i \rightarrow q_i' = q_i + \epsilon s_i$$

where the time derivative of q_i is

$$\dot{q}_i \rightarrow \dot{q}_i' = \dot{q}_i + \epsilon\dot{s}_i$$

We can define the difference between the Lagrangian between the two generalized coordinates as a quantity

$$\delta L \equiv L(q_i', \dot{q}_i', t) - L(q_i, \dot{q}_i)$$

we note that $\delta L = 0$ by the assumption that the Lagrangian is invariant under translation. So we expand $L(q_i', \dot{q}_i', t)$ by noting that $q_i' = q_i + \epsilon s_i$ where ϵ is small so we are able to expand about the point q_i with a perturbation ϵs_i. This yields

$$L(q_i', \dot{q}_i', t) = L(q_i, \dot{q}_i, t) + \frac{\partial L}{\partial q_i}\epsilon s_i + \frac{\partial L}{\partial \dot{q}_i}\epsilon\dot{s}_i$$

So we see the δL becomes

$$\delta L = L(q_i', \dot{q}_i', t) - L(q_i, \dot{q}_i)$$
$$\Downarrow$$
$$\delta L = 0 = \underline{L(q_i, \dot{q}_i, t)} + \frac{\partial L}{\partial q_i}\epsilon s_i + \frac{\partial L}{\partial \dot{q}_i}\epsilon\dot{s}_i - \underline{L(q_i, \dot{q}_i)}$$
$$0 = \frac{\partial L}{\partial q_i}\epsilon s_i + \frac{\partial L}{\partial \dot{q}_i}\epsilon\dot{s}_i$$
$$0 = \frac{\partial L}{\partial q_i}s_i + \frac{\partial L}{\partial \dot{q}_i}\dot{s}_i$$

Next we need to use the *generalized momentum* which is defined by

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

where we note that the time derivative of the generalized momentum is

$$\frac{dp_i}{dt} = \frac{\partial L}{\partial q_i}$$

So we see that we have

$$0 = \frac{\partial L}{\partial q_i}s_i + \frac{\partial L}{\partial \dot{q}_i}\dot{s}_i$$
$$\Downarrow$$
$$0 = \frac{dp_i}{dt}s_i + p_i\frac{ds_i}{dt}$$
$$0 = \frac{d}{dt}p_i s_i = \frac{d}{dt}(\mathbf{p} \cdot \mathbf{s})$$

So we see the quantity $\mathbf{p} \cdot \mathbf{s}$ is conserved. This corresponds to the conservation of linear momentum.

1

(b) Next we repeat the process for a perturbation in a rotation given by

$$\mathbf{r}' = \mathbf{r} + \epsilon \mathbf{s} \times \mathbf{r}$$

This yields a generalized coordinate

$$r_i' = r_i + \epsilon(\mathbf{s} \times \mathbf{r})_i$$

with the time derivative given by

$$\dot{r}_i' = \dot{r}_i + \epsilon \frac{d}{dt}(\mathbf{s} \times \mathbf{r})_i$$

Following the same logic as in part (a) we have

$$\delta L = 0 = \frac{\partial L}{\partial r_i}\epsilon(\mathbf{s} \times \mathbf{r})_i + \frac{\partial L}{\partial \dot{r}_i}\epsilon\frac{d}{dt}(\mathbf{s} \times \mathbf{r})_i$$

$$\Downarrow$$

$$0 = \frac{dp_i}{dt}(\mathbf{s} \times \mathbf{r})_i + p_i\frac{d}{dt}(\mathbf{s} \times \mathbf{r})_i$$

$$0 = \frac{d}{dt}p_i(\mathbf{s} \times \mathbf{r})_i$$

$$0 = \frac{d}{dt}\mathbf{p} \cdot (\mathbf{s} \times \mathbf{r})$$

$$0 = \frac{d}{dt}\mathbf{s} \cdot (\mathbf{r} \times \mathbf{p})$$

$$0 = \frac{d}{dt}\mathbf{s} \cdot \mathbf{L}$$

Note we defined the angular momentum by $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ So the quantity $\mathbf{s} \cdot \mathbf{L}$ is conserved. This corresponds to the conservation of angular momentum.

(c) For the Lagrangian for N-particles in 3D

$$L = \sum_{i=1}^{N} \frac{1}{2}m_i\mathbf{v}_i \cdot \mathbf{v}_i - \sum_{i<j}V\left(|\mathbf{r}_i - \mathbf{r}_j|\right)$$

under a uniform translation

$$\mathbf{r}_i' = \mathbf{r}_i + \epsilon\mathbf{s}$$

we see that the potential term remains invariant because it is only dependent on the relative position between particles. Next we note that the vector $\mathbf{s}$ is constant so we have

$$\dot{\mathbf{r}}_i' = \dot{\mathbf{r}}_i.$$

We see that the kinetic energy does not depend on the position so we can easily see that the Lagrangian is invariant. This corresponds to the conservation of energy.
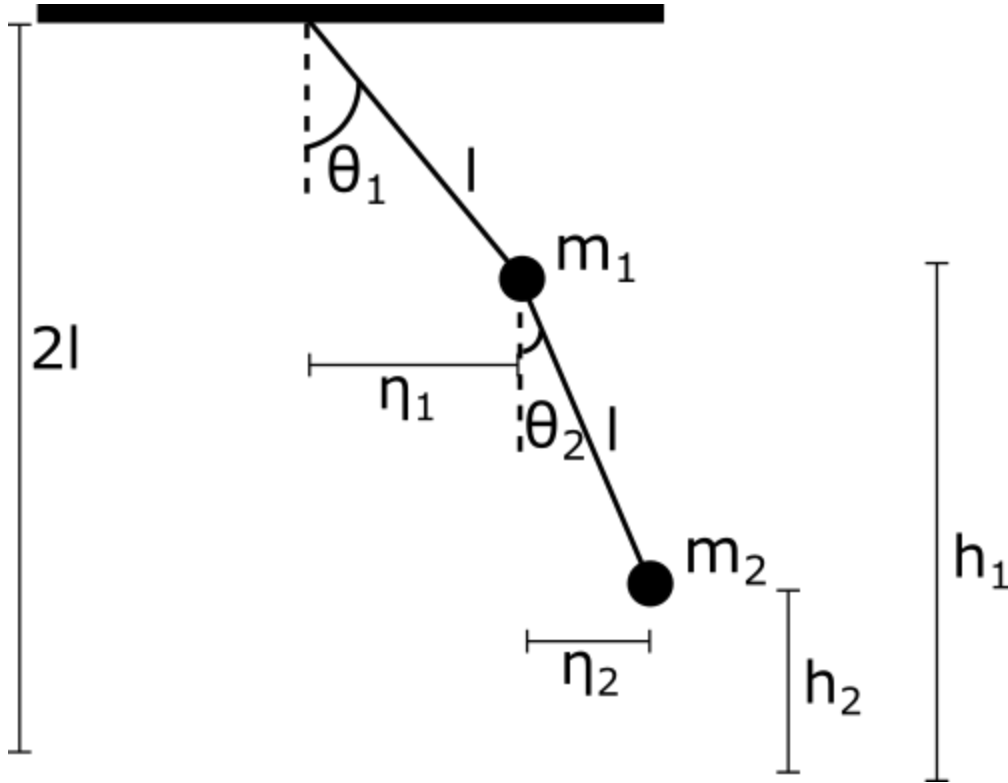
Figure 1: The double pendulum system with labeled lengths and masses.

2   Problem #2

(a) For a double pendulum with equal lengths, l, and different masses m_1 and m_2 as shown in Figure 1. We can use this geometry and the assumption of small oscillations. This allows us to use the transverse displacements from vertical, η_1 and η_2, as the generalized coordinates for the Lagrangian. We see that the displacements are given by

$$\eta_1 = l \sin \theta_1 = l\theta_1 + \mathcal{O}(\theta_1^3); \qquad \eta_2 = l \sin \theta_2 = l\theta_2 + \mathcal{O}(\theta_2^3)$$

We also can see that the displacement for m_2 we have to account for the displacement from the pivot point which is $\eta_1 + \eta_2$. Using this we can say the kinetic energy is given by

$$T = \frac{1}{2}m_1 \dot{\eta}_1^2 + \frac{1}{2}m_2 \left(\dot{\eta}_1 + \dot{\eta}_2 \right)^2$$

In order to determine the potential we note that it is given by

$$V = m_1 g h_1 + m_2 g h_2$$

where we need to find h_1 and h_2 in our generalized coordinates by expanding about θ_1

$$h_1 = 2l - l \cos \theta_1$$
$$= l \left(2 - 1 + \frac{1}{2}\theta_1^2 + \mathcal{O}(\theta_1^3) \right)$$
$$= l \left(1 + \frac{1}{2}\frac{\eta_1^2}{l^2} \right)$$

3

And for h_2 we find

$$h_2 = 2l - l\cos\theta_1 - l\cos\theta_2$$

$$= l\left(2 - 1 + \frac{1}{2}\theta_1^2 - 1 + \frac{1}{2}\theta_2^2 + \mathcal{O}(\theta_1^3, \theta_2^3)\right)$$

$$= \frac{l}{2}\left(\theta_1^2 + \theta_2^2\right)$$

$$= \frac{l}{2}\left(\frac{\eta_1^2}{l^2} + \frac{\eta_2^2}{l^2}\right)$$

$$= \frac{1}{2l}\left(\eta_1^2 + \eta_2^2\right)$$

So we can write the potential in terms of η_1 and η_2 as

$$V = m_1 g h_1 + m_2 g h_2$$

$$\Downarrow$$

$$V = m_1 g l\left(1 + \frac{1}{2}\frac{\eta_1^2}{l^2}\right) + m_2 g\frac{1}{2l}\left(\eta_1^2 + \eta_2^2\right)$$

$$= m_1 g l + \frac{m_1 g \eta_1^2}{2l} + \frac{m_2 g \eta_1^2}{2l} + \frac{m_2 g \eta_2^2}{2l}$$

$$= \frac{g}{2l}\left(m_1\eta_1^2 + m_2\eta_1^2 + m_2\eta_2^2\right)$$

$$= \frac{g}{2l}\left((m_1 + m_2)\eta_1^2 + m_2\eta_2^2\right)$$

Note that we neglected the constant term $m_1 g l$ in the potential as we can shift to a new equilibrium point without loss of generality. So we have the Lagrangian as

$$L = T - V$$

$$\Downarrow$$

$$L = \frac{1}{2}m_1\dot{\eta}_1^2 + \frac{1}{2}m_2\left(\dot{\eta}_1 + \dot{\eta}_2\right)^2 - \frac{g}{2l}\left((m_1 + m_2)\eta_1^2 + m_2\eta_2^2\right)$$

(b) In order to calculate the normal frequencies of the double pendulum. We first write the kinetic energy in the form

$$T = \frac{1}{2}\dot{\eta}_i M_{ij} \dot{\eta}_j$$

Where M_{ij} is a mass matrix that is by noting that

$$T = \frac{1}{2}m_1\dot{\eta}_1^2 + \frac{1}{2}m_2\left(\dot{\eta}_1 + \dot{\eta}_2\right)^2$$

$$\Downarrow$$

$$= \frac{1}{2}\left(m_1\dot{\eta}_1^2 + m_2\dot{\eta}_1^2 + m_2\dot{\eta}_2^2 + m_2\dot{\eta}_1\dot{\eta}_2 + m_2\dot{\eta}_2\dot{\eta}_1\right)$$

$$= \frac{1}{2}\left((m_1 + m_2)\dot{\eta}_1\dot{\eta}_1 + m_2\dot{\eta}_2\dot{\eta}_2 + m_2\dot{\eta}_1\dot{\eta}_2 + m_2\dot{\eta}_2\dot{\eta}_1\right)$$

Which implies that

$$M_{ij} = \begin{pmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{pmatrix}$$

4

Next, we need to write the potential energy in the form

$$V = \frac{1}{2}\eta_i V_{ij}\eta_j$$

where we can see that the potential matrix is given by

$$V_{ij} = \begin{pmatrix} \dfrac{g(m_1 + m_2)}{l} & 0 \\ 0 & \dfrac{gm_2}{l} \end{pmatrix}$$

So, the Lagrangian becomes

$$L = \frac{1}{2}\dot{\eta}_i M_{ij}\dot{\eta}_j - \frac{1}{2}\eta_i V_{ij}\eta_j$$

where if we find the equations of motion by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_i}\right) = \frac{\partial L}{\partial \eta_i}$$

$$\Downarrow$$

$$\frac{d}{dt}\left(M_{ij}\dot{\eta}_j\right) = -V_{ij}\eta_j$$

$$\Downarrow$$

$$M_{ij}\ddot{\eta}_j = -V_{ij}\eta_j$$

We complexify the solution by

$$\eta_j = A_j e^{i\omega t}$$

which gives us

$$\ddot{\eta}_j = -\omega^2 A_j e^{i\omega t}$$

$$\Downarrow$$

$$\ddot{\eta}_j = -\omega^2 \eta_j$$

So we can see that out equation of motion becomes

$$M_{ij}\ddot{\eta}_j + V_{ij}\eta_j = 0$$

$$\Downarrow$$

$$M_{ij}(-\omega^2)\eta_j + V_{ij}\eta_j = 0$$
$$\left(V_{ij} - \omega^2 M_{ij}\right) A_j e^{i\omega t} = 0$$
$$\left(V_{ij} - \omega^2 M_{ij}\right) A_j = 0$$

So we have two equations for the free index i by $i = 1, 2$. So for $i = 1$ we have the sum

$$\left(V_{1j} - \omega^2 M_{1j}\right) A_j = 0$$

$$\Downarrow$$

$$\left(V_{11} - \omega^2 M_{11}\right) A_1 + \left(\cancel{V_{12}}^{\,0} - \omega^2 M_{12}\right) A_2 = 0$$

$$\Downarrow$$

$$\left(\frac{g(m_1 + m_2)}{l} - \omega^2(m_1 + m_2)\right) A_1 - \left(\omega^2 m_2\right) A_2 = 0$$

5

Which we can use to get a ratio between A_1 and A_2 as

$$\frac{A_1}{A_2} = \frac{m_2}{m_1 + m_2} \frac{\omega^2}{\frac{g}{l} - \omega^2}$$

And for $i = 2$ we have

$$\left(V_{2j} - \omega^2 M_{2j}\right) A_j = 0$$
$$\Downarrow$$
$$\left(\cancel{V_{21}}^{\;0} - \omega^2 M_{21}\right) A_1 + \left(V_{22} - \omega^2 M_{22}\right) A_2 = 0$$
$$- \left(\omega^2 m_2\right) A_1 + \left(\frac{g m_2}{l} - \omega^2 m_2\right) A_2 = 0$$
$$\Downarrow$$
$$\left(\omega^2 m_2\right) A_1 = \left(\frac{g m_2}{l} - \omega^2 m_2\right) A_2$$
$$\Downarrow$$
$$\frac{A_1}{A_2} = \frac{\frac{g}{l} - \omega^2}{\omega^2}$$

Now we can solve for the normal frequencies by solving

$$\frac{m_2}{m_1 + m_2} \frac{\omega^2}{\frac{g}{l} - \omega^2} = \frac{\frac{g}{l} - \omega^2}{\omega^2}$$
$$\Downarrow$$
$$(\gamma \omega^2)^2 = \left(\frac{g}{l} - \omega^2\right)^2$$
$$\pm \gamma \omega^2 = \frac{g}{l} - \omega^2$$
$$\omega^2 \pm \gamma \omega^2 = \frac{g}{l}$$
$$\omega^2 (1 \pm \gamma) = \frac{g}{l}$$
$$\Downarrow$$
$$\omega_{\pm}^2 = \frac{g}{l}(1 \pm \gamma)^{-1}$$

Note we defined the quantity γ as

$$\gamma^2 = \frac{m_2}{m_1 + m_2}$$

(c) We can construct the normal-mode eigenvectors by noting for the normal frequency

$$\omega^2 = \frac{g}{l}(1 + \gamma)^{-1}$$

6

we have the ratio between A_1 and A_2 as

$$\frac{A_1}{A_2} = \frac{\frac{g}{l} - \omega^2}{\omega^2}$$

$$\Downarrow$$

$$= \frac{\frac{g}{l} - \frac{g}{l}\frac{1}{1+\gamma}}{\frac{g}{l}\frac{1}{1+\gamma}}$$

$$= \left(1 - \frac{1}{1+\gamma}\right)(1+\gamma)$$

$$= 1 + \gamma - 1 = \gamma$$

So this implies that $A_1 = \gamma A_2$. Then by ω_-^2 we have

$$\frac{A_1}{A_2} = \frac{\frac{g}{l} - \omega^2}{\omega^2}$$

$$\Downarrow$$

$$= \left(1 - \frac{1}{1-\gamma}\right)(1-\gamma)$$

$$= 1 - \gamma - 1 = -\gamma$$

So we have $A_1 = -\gamma A_2$. So we have the normal-mode eigenvectors

$$\vec{\eta}_\pm = A_\pm \begin{pmatrix} 1 \\ \pm\gamma^{-1} \end{pmatrix} e^{i\omega_\pm t}$$

We can note the limiting cases where $m_1 \to 0$ which is the case of a single pendulum of length $2l$ we see that for this limit γ becomes

$$\lim_{m_1 \to 0} \gamma = \lim_{m_1 \to 0} \frac{m_2}{m_1 + m_2} = 1$$

Which yields only a single normal frequency given by

$$\omega^2 = \frac{g}{2l}$$

which is as we expect is the frequency of a pendulum with a length of $2l$. In the other limit as $m_2 \to 0$ we note that the system becomes a single pendulum, but of length l. It is easy to see that

$$\lim_{m_2 \to 0} \gamma = 0$$

Which makes for a single normal mode frequency given by

$$\omega^2 = \frac{g}{l}$$

which is what we expect.

(d) We can calculate the modal matrix, $\mathbf{A}$, by first writing $A_\pm$ as a complex number $A_\pm = \rho_\pm e^{i\phi_\pm}$ which makes the normal eigenmodes become

$$\vec{\eta}_\pm = \rho_\pm \begin{pmatrix} 1 \\ \pm\gamma^{-1} \end{pmatrix} e^{i(\omega_\pm t + \phi_\pm)}$$

7

This allows us to normalize the eigenvectors to the mass matrix M_{ij} by

$$1 = \rho_+(1, \gamma^{-1}) \begin{pmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{pmatrix} \begin{pmatrix} 1 \\ \gamma^{-1} \end{pmatrix} \rho_+$$

$$= \rho_+^2 (m_1 + m_2)(1, \gamma^{-1}) \begin{pmatrix} 1 & \gamma^2 \\ \gamma^2 & \gamma^2 \end{pmatrix} \begin{pmatrix} 1 \\ \gamma^{-1} \end{pmatrix}$$

$$= \rho_+^2 (m_1 + m_2)(1, \gamma^{-1}) \begin{pmatrix} 1 + \gamma \\ \gamma + \gamma^2 \end{pmatrix}$$

$$= \rho_+^2 (m_1 + m_2)(1 + \gamma + 1 + \gamma)$$

$$= 2\rho_+^2 (m_1 + m_2)(1 + \gamma)$$

$$= 2\rho_+^2 m_1 \frac{1 + \gamma}{1 - \gamma^2}$$

$$= 2\rho_+^2 m_1 \frac{1 + \gamma}{(1 + \gamma)(1 - \gamma)}$$

$$= 2\rho_+^2 m_1 \frac{1}{1 - \gamma}$$

$$\Downarrow$$

$$\rho_+ = \sqrt{\frac{1 - \gamma}{2m_1}}$$

Note that we used the fact that $m_1/(m_1 + m_2) = 1 - \gamma^2$. Now for ρ_-

$$1 = \rho_-^2 (m_1 + m_2)(1, -\gamma^{-1}) \begin{pmatrix} 1 & \gamma^2 \\ \gamma^2 & \gamma^2 \end{pmatrix} \begin{pmatrix} 1 \\ -\gamma^{-1} \end{pmatrix}$$

$$= \rho_-^2 (m_1 + m_2)(1, -\gamma^{-1}) \begin{pmatrix} 1 - \gamma \\ \gamma^2 - \gamma \end{pmatrix}$$

$$= \rho_-^2 (m_1 + m_2)(1 - \gamma + 1 - \gamma)$$

$$= 2\rho_-^2 (m_1 + m_2)(1 - \gamma)$$

$$\Downarrow$$

$$\rho_- = -\sqrt{\frac{1 + \gamma}{2m_1}}$$

Note that we choose the negative solution though either are valid. Now that we have normalized eigenvectors we can construct $\mathbf{A}$ by

$$\mathbf{A} = \left(\rho_+ \begin{pmatrix} 1 \\ \gamma^{-1} \end{pmatrix} \quad \rho_- \begin{pmatrix} 1 \\ -\gamma^{-1} \end{pmatrix} \right) = \frac{1}{\sqrt{2m_1}} \begin{pmatrix} (1 - \gamma)^{1/2} & -(1 + \gamma)^{1/2} \\ \gamma^{-1}(1 - \gamma)^{1/2} & \gamma^{-1}(1 + \gamma)^{1/2} \end{pmatrix}$$

Now we can demonstrate that $\mathbf{A}$ diagonalizes the mass matrix by

$$\mathbf{A}^T \mathbf{M} \mathbf{A} = \frac{m_1 + m_2}{2m_1} \begin{pmatrix} (1 - \gamma)^{1/2} & \gamma^{-1}(1 - \gamma)^{1/2} \\ -(1 + \gamma)^{1/2} & \gamma^{-1}(1 + \gamma)^{1/2} \end{pmatrix} \begin{pmatrix} 1 & \gamma^2 \\ \gamma^2 & \gamma^2 \end{pmatrix} \begin{pmatrix} (1 - \gamma)^{1/2} & -(1 + \gamma)^{1/2} \\ \gamma^{-1}(1 - \gamma)^{1/2} & \gamma^{-1}(1 + \gamma)^{1/2} \end{pmatrix}$$

$$= \frac{m_1 + m_2}{2m_1} \begin{pmatrix} (1 - \gamma)^{1/2} & \gamma^{-1}(1 - \gamma)^{1/2} \\ -(1 + \gamma)^{1/2} & \gamma^{-1}(1 + \gamma)^{1/2} \end{pmatrix} \begin{pmatrix} (1 - \gamma)^{1/2}(1 + \gamma) & (1 + \gamma)^{1/2}(\gamma - 1) \\ \gamma(1 - \gamma)^{1/2}(1 + \gamma) & -\gamma(1 + \gamma)^{1/2}(\gamma - 1) \end{pmatrix}$$

$$= \frac{m_1 + m_2}{2m_1} \begin{pmatrix} 2(1 - \gamma^2) & 0 \\ 0 & 2(1 - \gamma^2) \end{pmatrix}$$

$$= \frac{1}{2(1 - \gamma^2)} 2(1 - \gamma^2) \mathbf{I} = \mathbf{I}$$

8

So as we expect we obtain the identity matrix, $\mathbf{I}$. Now for the potential matrix we calculate

$$\mathbf{A}^T\mathbf{V}\mathbf{A} = \frac{g(m_1+m_2)}{2lm_1}\begin{pmatrix} (1-\gamma)^{1/2} & \gamma^{-1}(1-\gamma)^{1/2} \\ -(1+\gamma)^{1/2} & \gamma^{-1}(1+\gamma)^{1/2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \gamma^2 \end{pmatrix}\begin{pmatrix} (1-\gamma)^{1/2} & -(1+\gamma)^{1/2} \\ \gamma^{-1}(1-\gamma)^{1/2} & \gamma^{-1}(1+\gamma)^{1/2} \end{pmatrix}$$

$$= \frac{g}{2l(1-\gamma^2)}\begin{pmatrix} (1-\gamma)^{1/2} & \gamma^{-1}(1-\gamma)^{1/2} \\ -(1+\gamma)^{1/2} & \gamma^{-1}(1+\gamma)^{1/2} \end{pmatrix}\begin{pmatrix} (1-\gamma)^{1/2} & -(1-\gamma)^{1/2} \\ \gamma(1-\gamma)^{1/2} & \gamma(1+\gamma)^{1/2} \end{pmatrix}$$

$$= \frac{g}{2l(1-\gamma^2)}\begin{pmatrix} 2(1-\gamma) & 0 \\ 0 & 2(1+\gamma) \end{pmatrix}$$

$$= \frac{g}{l}\begin{pmatrix} (1+\gamma)^{-1} & 0 \\ 0 & (1-\gamma)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \omega_+^2 & 0 \\ 0 & \omega_-^2 \end{pmatrix}$$

We see that as we expect we get the diagonalized matrix contains the normal frequencies.

(e) We can use the modal matrix, $\mathbf{A}$, to construct the normal coordinates, $\xi(t)$ by

$$\xi(t) = \mathbf{A}^T\mathbf{M}\vec{\eta}(t)$$

Where we take $\vec{\eta}(t)$ as the general solution given by the linear combination

$$\vec{\eta}(t) = \text{Re}(C_+\vec{\eta}_+(t) + C_-\vec{\eta}_-(t))$$

This linear combination is projecting the generalized coordinates η_1 and η_2 into the basis of the normal eigenmodes which form an orthonormal basis by the above work. So we can calculate

$$C_+\eta_+(t) + C_-\eta_-(t) = C_+\rho_+\begin{pmatrix} 1 \\ \gamma^{-1} \end{pmatrix}e^{i(\omega_+t+\phi_+)} + C_-\rho_-\begin{pmatrix} 1 \\ \pm\gamma^{-1} \end{pmatrix}e^{i(\omega_-t+\phi_-)}$$

$$= \frac{1}{\sqrt{2m_1}}\begin{pmatrix} C_+(1-\gamma)^{1/2}e^{i(\omega_+t+\phi_+)} \\ C_+\gamma^{-1}(1-\gamma)^{1/2}e^{i(\omega_+t+\phi_+)} \end{pmatrix} + \frac{1}{\sqrt{2m_1}}\begin{pmatrix} -C_-(1+\gamma)^{1/2}e^{i(\omega_-t+\phi_-)} \\ C_-\gamma^{-1}(1+\gamma)^{1/2}e^{i(\omega_-t+\phi_-)} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2m_1}}\begin{pmatrix} C_+(1-\gamma)^{1/2}e^{i(\omega_+t+\phi_+)} - C_-(1+\gamma)^{1/2}e^{i(\omega_-t+\phi_-)} \\ C_+\gamma^{-1}(1-\gamma)^{1/2}e^{i(\omega_+t+\phi_+)} + C_-\gamma^{-1}(1+\gamma)^{1/2}e^{i(\omega_-t+\phi_-)} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2m_1}}\begin{pmatrix} C_+(1-\gamma)^{1/2}\cos(\omega_+t+\phi_+) - C_-(1+\gamma)^{1/2}\cos(\omega_-t+\phi_-) \\ C_+\gamma^{-1}(1-\gamma)^{1/2}\cos(\omega_+t+\phi_+) + C_-\gamma^{-1}(1+\gamma)^{1/2}\cos(\omega_-t+\phi_-) \end{pmatrix}$$

So we calculate $\vec{\xi}$ as

$$\vec{\xi}(t) = \mathbf{A}^T\mathbf{M}\vec{\eta}_+(t)$$

$$= \frac{m_1+m_2}{2m_1}\begin{pmatrix} (1-\gamma)^{1/2} & \gamma^{-1}(1-\gamma)^{1/2} \\ -(1+\gamma)^{1/2} & \gamma^{-1}(1+\gamma)^{1/2} \end{pmatrix}\begin{pmatrix} 1 & \gamma^2 \\ \gamma^2 & \gamma^2 \end{pmatrix}$$

$$\times \begin{pmatrix} C_+(1-\gamma)^{1/2}\cos(\omega_+t+\phi_+) - C_-(1+\gamma)^{1/2}\cos(\omega_-t+\phi_-) \\ C_+\gamma^{-1}(1-\gamma)^{1/2}\cos(\omega_+t+\phi_+) + C_-\gamma^{-1}(1+\gamma)^{1/2}\cos(\omega_-t+\phi_-) \end{pmatrix}$$

$$= \frac{1}{2(1-\gamma^2)}\begin{pmatrix} (1+\gamma)(1-\gamma)^{1/2} & \gamma(1+\gamma)(1-\gamma)^{1/2} \\ (-1+\gamma)(1+\gamma)^{1/2} & \gamma(1-\gamma)(1+\gamma)^{1/2} \end{pmatrix}$$

$$\times \begin{pmatrix} C_+(1-\gamma)^{1/2}\cos(\omega_+t+\phi_+) - C_-(1+\gamma)^{1/2}\cos(\omega_-t+\phi_-) \\ C_+\gamma^{-1}(1-\gamma)^{1/2}\cos(\omega_+t+\phi_+) + C_-\gamma^{-1}(1+\gamma)^{1/2}\cos(\omega_-t+\phi_-) \end{pmatrix}$$

$$= \frac{1}{2(1-\gamma^2)}\begin{pmatrix} 2C_+(1-\gamma)(1+\gamma)\cos(\omega_+t+\phi_+) \\ 2C_-(1-\gamma)(1+\gamma)\cos(\omega_-t+\phi_-) \end{pmatrix}$$

$$= \begin{pmatrix} C_+\cos(\omega_+t+\phi_+) \\ C_-\cos(\omega_-t+\phi_-) \end{pmatrix}$$

(f) Now in the limiting case where $m_2 << m_1$ where η_1 is slightly displaced we have the initial conditions

$$\eta_1(0) = \delta; \qquad \eta_2(0) = 0; \qquad \dot{\eta}_1(0) = \dot{\eta}_1(0) = 0$$

Applying these initial conditions to $\vec{\xi}(t)$ to determine our coefficients by noting that

$$\vec{\eta}(t) = A\vec{\xi}(t)$$

So we apply the initial condition

$$\dot{\vec{\eta}}(0) = A\dot{\vec{\xi}}(0)$$
$$\Downarrow$$
$$\sqrt{2m_1}\mathbf{A}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} C_+\omega_+ \sin(\omega_+ 0 + \phi_+) \\ C_-\omega_- \sin(\omega_- 0 + \phi_-) \end{pmatrix}$$
$$\Downarrow$$
$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} C_+\omega_+ \sin(\phi_+) \\ C_-\omega_- \sin(\phi_-) \end{pmatrix}$$
$$\Downarrow$$
$$\phi_+ = 0 \qquad \phi_- = 0$$

Now we can apply the initial condition on $\vec{\eta}$

$$\vec{\eta}(0) = A\vec{\xi}(0)$$
$$\Downarrow$$
$$\begin{pmatrix} \delta \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2m_1}} \begin{pmatrix} (1-\gamma)^{1/2} & -(1+\gamma)^{1/2} \\ \gamma^{-1}(1-\gamma)^{1/2} & \gamma^{-1}(1+\gamma)^{1/2} \end{pmatrix} \begin{pmatrix} C_+\cos(\omega_+ 0) \\ C_-\cos(\omega_- 0) \end{pmatrix}$$
$$\begin{pmatrix} \delta \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2m_1}} \begin{pmatrix} (1+\gamma)^{1/2}C_+ - (1-\gamma)^{1/2}C_- \\ \gamma^{-1}(1-\gamma)^{1/2}C_+ + \gamma^{-1}(1+\gamma)^{1/2}C_- \end{pmatrix}$$

We see that the bottom equation yields

$$C_+ = -\left(\frac{1+\gamma}{1-\gamma}\right)^{1/2} C_-$$

Which we can use on our last initial condition to get

$$\delta = (1+\gamma)^{1/2}C_+ - (1-\gamma)^{1/2}C_-$$
$$\Downarrow$$
$$\delta\sqrt{2m_1} = -(1+\gamma)^{1/2}\left(\frac{1+\gamma}{1-\gamma}\right)^{1/2} C_- - (1-\gamma)^{1/2}C_-$$
$$= -C_-\left(\frac{1+\gamma}{(1-\gamma)^{1/2}} - (1-\gamma)^{1/2}\right)$$
$$= -C_-\left(\frac{1+\gamma+1-\gamma}{(1-\gamma)^{1/2}}\right)$$
$$= -C_-\frac{2}{(1-\gamma)^{1/2}}$$
$$\Downarrow$$
$$C_- = -\frac{\delta\sqrt{2m_1}}{2}(1-\gamma)^{1/2}$$

Which implies that C_+ is

$$C_+ = \frac{\delta\sqrt{2m_1}}{2}(1+\gamma)^{1/2}$$

So using this we can find that our $\vec{\eta}(t)$ becomes

$$\vec{\eta}(t) = \frac{1}{\sqrt{2m_1}}\frac{\delta\sqrt{2m_1}}{2}\begin{pmatrix} (1+\gamma)^{1/2}(1-\gamma)^{1/2}\cos(\omega_+ t) - (1-\gamma)^{1/2}(1+\gamma)^{1/2}\cos(\omega_- t) \\ (1+\gamma)^{1/2}\gamma^{-1}(1-\gamma)^{1/2}\cos(\omega_+ t) + (1-\gamma)^{1/2}\gamma^{-1}(1+\gamma)^{1/2}\cos(\omega_- t) \end{pmatrix}$$

$$= \frac{\delta}{2}(1-\gamma^2)^{1/2}\begin{pmatrix} \cos(\omega_+ t) - \cos(\omega_- t) \\ \gamma^{-1}\cos(\omega_+ t) + \gamma^{-1}\cos(\omega_- t) \end{pmatrix}$$

$$= \delta(1-\gamma^2)^{1/2}\begin{pmatrix} \cos((\omega_+ + \omega_-)t/2)\cos((\omega_+ - \omega_-)t/2) \\ \gamma^{-1}\sin((\omega_+ + \omega_-)t/2)\sin((\omega_+ - \omega_-)t/2) \end{pmatrix}$$

Now we have the functions for the generalized coordinates η_1 and η_2

$$\eta_1(t) = \delta(1-\gamma^2)^{1/2}\cos((\omega_+ + \omega_-)t/2)\cos((\omega_+ - \omega_-)t/2)$$
$$\eta_2(t) = \delta(1-\gamma^2)^{1/2}\gamma^{-1}\sin((\omega_+ + \omega_-)t/2)\sin((\omega_+ - \omega_-)t/2)$$

And by the properties of sine and cosine we see that when η_1 is at a max and η_2 is at a minimum. While in the limit as $m_1 \gg m_2$ we see that the amplitude of η_2 reaches a maximum.
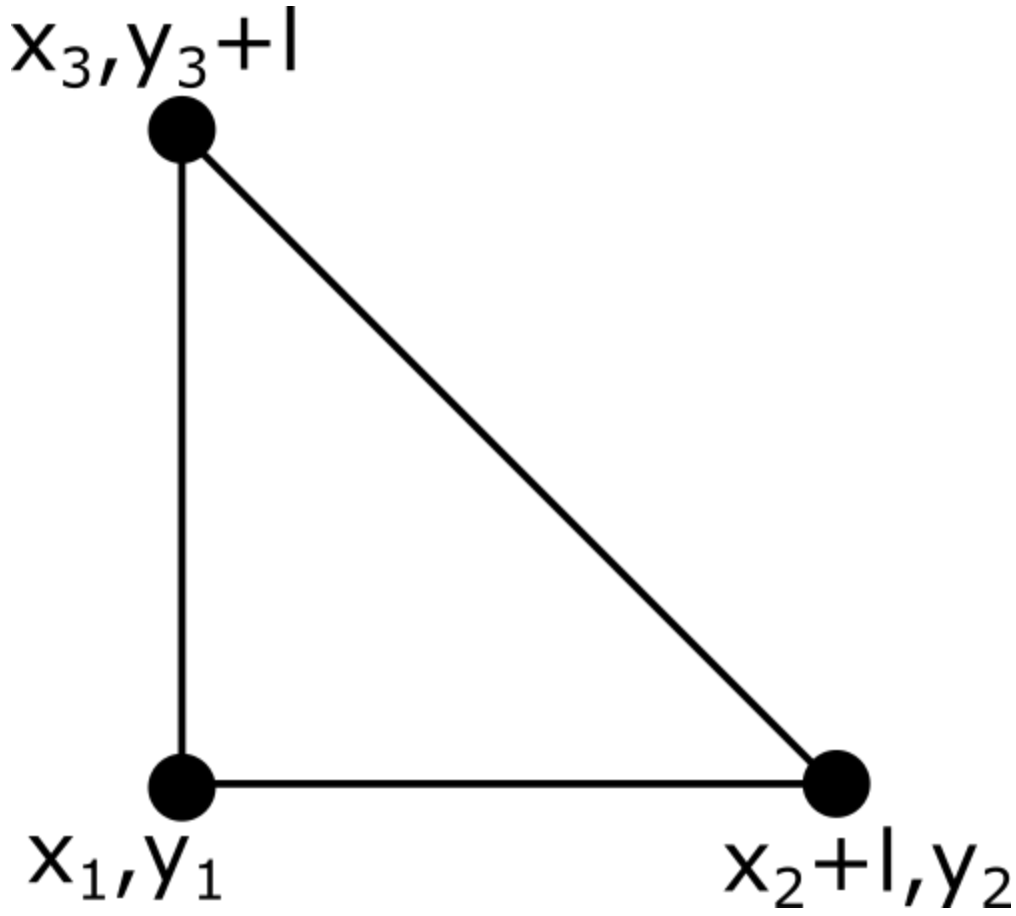
Figure 2: The atomic arrangement.

3 Problem #3

(a) For a molecule that consists of three identical atoms located at vertices of a $45°$ right triangle as shown in Figure 2. The atoms have equal spring constants k between each atoms. For planer motion we define our generalized coordinates by defining the atom with the right angle as (x_1, y_2) which implies if the separation distance is l we have $(x_2 + l, y_2)$ and $(x_3, x_3 + l)$. We note that all three particles share mass m which makes our kinetic energy

$$T = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2 \right)$$

which if we write T in the form

$$T = \frac{1}{2} \dot{\eta}_i M_{ij} \dot{\eta}_j$$

we can easily see that $M_{ij} = m\delta_{ij}$ where δ_{ij} corresponds to the identity matrix. For the potential energy we note that it is dependent on the separation between the atoms. This is given by the distance formula where we assume that the oscillations are small along which implies that the separation distance l dominates. This yields

$$d_{12} = \sqrt{(l + x_2 - x_1)^2 + (y_2 - y_1)^2} \approx l + x_2 - x_1$$

$$d_{13} = \sqrt{(x_3 - x_1)^2 + (l + y_3 - y_1)^2} \approx l + y_3 - y_1$$

$$d_{23} = \sqrt{(x_3 - x_2 - l)^2 + (l + y_3 - y_1)^2} \approx \sqrt{2}l - \frac{\sqrt{2}}{2}(x_3 - x_2) + \frac{\sqrt{2}}{2}(y_3 - y_2)$$

12

So our potential is dependent on the differences between d and the equilibrium length l

$$V = \frac{1}{2}k\left((d_{12} - l)^2 + (d_{13} - l)^2 + (d_{23} - \sqrt{2}l)^2\right)$$

$$= \frac{1}{2}k\left(l + x_2 - x_1 - 1)^2 + (l + y_3 - y_1 - l)^2 + \left(\sqrt{2}l - \frac{\sqrt{2}}{2}(x_3 - x_2) + \frac{\sqrt{2}}{2}(y_3 - y_2) - \sqrt{2}l\right)^2\right)$$

$$= \frac{1}{2}k\left((x_2 - x_1)^2 + (y_3 - y_1)^2 + \frac{1}{2}\left((y_3 - y_2) - (x_3 - x_2)\right)^2\right)$$

$$= \frac{1}{2}k\left((x_2 - x_1)^2 + (y_3 - y_1)^2 + \frac{1}{2}(y_3 - y_2)^2 + \frac{1}{2}(x_3 - x_2)^2 - (x_3 - x_2)(y_3 - y_2)\right)$$

With this potential we can find the matrix elements by

$$V_{ij} = \frac{\partial V}{\partial \eta_i \partial \eta_j}$$

where $\eta_i = (x_1, y_1, x_2, y_2, x_3, y_3)$. We note that this is will create a symmetric matrix So we calculate neglecting the common factor of k we find that

$$\frac{\partial V}{\partial x_1} = -(x_2 - x_1)$$

which allows us to calculate

$$V_{11} = -\frac{\partial}{\partial x_1}(x_2 - x_1) = 1$$

$$V_{12} = -\frac{\partial}{\partial y_2}(x_2 - x_1) = 0$$

$$V_{13} = -\frac{\partial}{\partial x_2}(x_2 - x_1) = -1$$

$$V_{14} = V_{15} = V_{16} = 0$$

Now we see for the first derivative with respect to y_1 we have

$$\frac{\partial V}{\partial y_1} = -(y_3 - y_1)$$

which gives us the unique values

$$V_{22} = -\frac{\partial}{\partial y_1}(y_3 - y_1) = 1$$

$$V_{23} = -\frac{\partial}{\partial x_2}(y_3 - y_1) = 0$$

$$V_{24} = -\frac{\partial}{\partial y_2}(y_3 - y_1) = 0$$

$$V_{25} = -\frac{\partial}{\partial x_3}(y_3 - y_1) = 0$$

$$V_{26} = -\frac{\partial}{\partial y_3}(y_3 - y_1) = -1$$

And for x_2 we have

$$\frac{\partial V}{\partial x_2} = (x_2 - x_1) - \frac{1}{2}(x_3 - x_2) + \frac{1}{2}(y_3 - y_2)$$

$$V_{33} = \frac{\partial}{\partial x_2}(x_2 - x_1) - \frac{1}{2}(x_3 - x_2) + \frac{1}{2}(y_3 - y_2) = \frac{3}{2}$$

$$V_{34} = \frac{\partial}{\partial y_2}(x_2 - x_1) - \frac{1}{2}(x_3 - x_2) + \frac{1}{2}(y_3 - y_2) = -\frac{1}{2}$$

$$V_{35} = \frac{\partial}{\partial x_3}(x_2 - x_1) - \frac{1}{2}(x_3 - x_2) + \frac{1}{2}(y_3 - y_2) = -\frac{1}{2}$$

$$V_{36} = \frac{\partial}{\partial y_3}(x_2 - x_1) - \frac{1}{2}(x_3 - x_2) + \frac{1}{2}(y_3 - y_2) = \frac{1}{2}$$

And for y_2

$$\frac{\partial V}{\partial y_2} = \frac{1}{2}(x_3 - x_2) - \frac{1}{2}(y_3 - y_2)$$

$$V_{44} = \frac{\partial}{\partial y_2}\frac{1}{2}(x_3 - x_2) - \frac{1}{2}(y_3 - y_2) = \frac{1}{2}$$

$$V_{45} = \frac{\partial}{\partial x_3}\frac{1}{2}(x_3 - x_2) - \frac{1}{2}(y_3 - y_2) = \frac{1}{2}$$

$$V_{46} = \frac{\partial}{\partial y_3}\frac{1}{2}(x_3 - x_2) - \frac{1}{2}(y_3 - y_2) = -\frac{1}{2}$$

And for x_3

$$\frac{\partial V}{\partial x_3} = \frac{1}{2}(x_3 - x_2) - \frac{1}{2}(y_3 - y_2)$$

$$V_{55} = \frac{\partial}{\partial x_3}\frac{1}{2}(x_3 - x_2) - \frac{1}{2}(y_3 - y_2) = \frac{1}{2}$$

$$V_{56} = \frac{\partial}{\partial y_3}\frac{1}{2}(x_3 - x_2) - \frac{1}{2}(y_3 - y_2) = -\frac{1}{2}$$

And for y_3

$$V_{66} = \frac{\partial^2 V}{\partial y_3^2} = \frac{\partial}{\partial y_3}\frac{1}{2}(y_3 - y_2) + (y_3 - y_1) - \frac{1}{2}(x_3 - x_2) = \frac{3}{2}$$

Combining these into the matrix $\mathbf{V}$ we have

$$\mathbf{V} = k \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

We can find the eigenfrequencies by solving

$$-\omega^2 M_{ij} + V_{ij} = 0$$

Using a numerical methods we find the eigenvalues of this we get

$$\omega_1^2 = \omega_2^2 = \omega_3^2 = 0$$

$$\omega_4^2 = \frac{k}{m}$$

$$\omega_5^2 = \frac{2k}{m}$$

$$\omega_6^2 = \frac{3k}{m}$$

which corresponds to the eigenmodes of motion. We note the three degenerate modes of $\omega^2 = 0$.

(b) The three degenerate modes are related to the two translations in x and y. The final degenerate mode is given by the rotation in the plane. These modes are described by the eigenvectors

$$
\vec{\eta_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad
\vec{\eta_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \qquad
\vec{\eta_3} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \\ 1/2 \\ -1 \\ -1 \end{pmatrix}
$$

We note that these modes do not have vibrational frequency this is why they have a zero value for ω^2. The non-degenerate solutions have eigenvectors given by

$$
\vec{\eta_4} = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \qquad
\vec{\eta_5} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad
\vec{\eta_6} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \\ -1/2 \\ -1/2 \\ 1 \end{pmatrix}
$$

We see that ω_4^2 is the vibrational mode along the x and y or along the direction of the legs. Note that ω_5^2 is vibration along the hypotenuse and the ω_6^2 is the vibrational mode that expands and contracts the triangle as a whole. Note that each vibrational mode corresponds to a single spring vibration and double spring then a triple spring which explains the factors of 1, 2, and 3 in the eigenfrequencies.
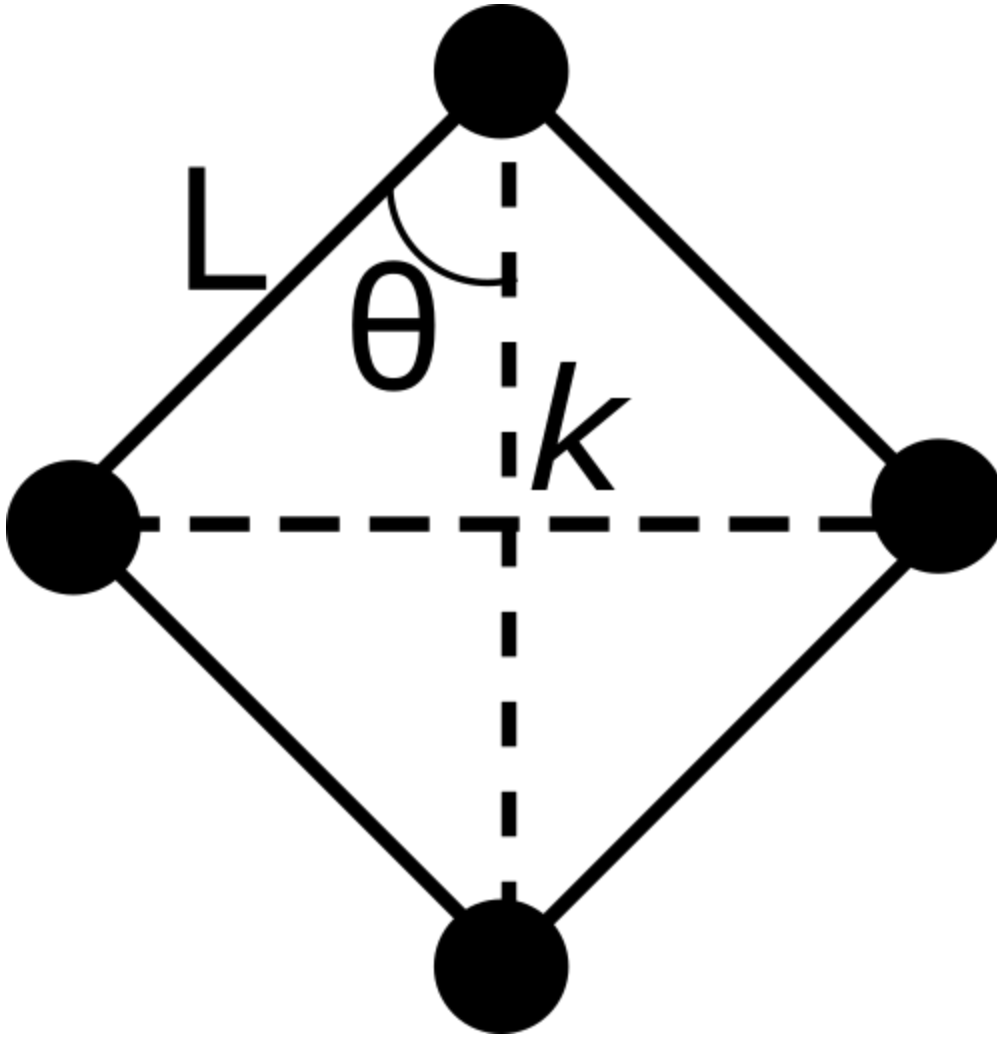
Figure 3: Four masses on a rhombus.

4   Problem #4

We can see that the system shown in Figure 3 of four massless rods of length L are hinged together at their ends to form a rhombus. At each corner there is a particle of mass m with a spring attached to opposite corners with a spring constant, k. By looking at the system a vibration in one spring will constrain the vibration in the other. We define this coordinate as the angle, θ between a spring and the nearest rod. Which gives

$$x = L\cos\theta \qquad y = L\sin\theta$$

and the first derivative yields

$$\dot{x} = -L\sin\theta\dot{\theta} \qquad \dot{y} = L\cos\theta\dot{\theta}$$

Now we can find the kinetic energy by noting that the particles are restricted to move only in the direction of the springs. This simplifies the problems a great deal as the opposite particle will have mirrored movement.

$$T = \frac{1}{2}m(2\dot{x}^2 + 2\dot{y}^2)$$
$$= m(L^2\sin^2\theta\dot{\theta}^2 + L^2\cos^2\theta\dot{\theta}^2)$$
$$= mL^2\dot{\theta}^2$$

Note the factors of 2 arise from the fact that we have two particles moving in x and y And for displacements from the equilibrium length $\sqrt{2}L$ give the potential energy

$$
\begin{aligned}
V &= \frac{1}{2}k\left((\sqrt{2}L - 2x)^2 + (\sqrt{2}L - 2y)^2 \right) \\
&= \frac{1}{2}k\left((\sqrt{2}L - 2L\cos\theta)^2 + (\sqrt{2}L - 2L\sin\theta)^2 \right) \\
&= \frac{1}{2}kL^2\left((\sqrt{2} - 2\cos\theta)^2 + (\sqrt{2} - 2\sin\theta)^2 \right) \\
&= \frac{1}{2}kL^2\left(2 + 4\cos^2\theta - 4\sqrt{2}\cos\theta + 2 - 4\sin^2\theta + 4\sqrt{2}\sin\theta \right) \\
&= kL^2\left(4 - 2\sqrt{2}(\sin\theta + \cos\theta) \right) \\
&= kL^2\left(4 - 2\sqrt{2}\sqrt{2}\sin(\theta + \pi/4) \right) \\
&= 4kL^2\left(1 - \sin(\theta + \pi/4) \right)
\end{aligned}
$$

This yields the Lagrangian

$$
L = mL^2\dot\theta^2 - 4kL^2\left(1 - \sin(\theta + \pi/4) \right)
$$

Now we can calculate the equations of motion by

$$
\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} = 0
$$

$$
\Downarrow
$$

$$
\frac{d}{dt}(2mL^2\dot\theta) + 4kL^2\cos(\theta + \pi/4) = 0
$$

$$
2mL^2\ddot\theta = -4kL^2\cos(\theta + \pi/4)
$$

$$
\Downarrow
$$

$$
\ddot\theta = -\frac{4kL^2}{2mL^2}\cos(\theta + \pi/4)
$$

$$
\ddot\theta = -\frac{2k}{m}\cos(\theta + \pi/4)
$$

We can take this equation of motion for a small oscillation by noting that the equilibrium is when the rhombus is a square which is at $\theta = \pi/4 + \delta\theta$ so we have

$$
\cos(\theta + \pi/4) = \cos(\delta\theta + \pi/2) = \sin(\delta\theta) = \delta\theta + \mathcal{O}(\delta\theta^2)
$$

This reduces our equation of motion into the simple harmonic motion given by

$$
\ddot{\delta\theta} = -\frac{2k}{m}\delta\theta
$$

which we can quickly see gives a frequency of small oscillation

$$
\omega^2 = \frac{2k}{m}
$$