Physics 606

Quantum Mechanics I

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Homework #1

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1 Problem #1

Given the *translation operator*, $\hat{T}_a$ defined as

$$\hat{T}_a \Psi(x) = \Psi(x+a)$$

we can find the Hermitian conjugate of $\hat{T}_a$ by using the definition of a Hermitian conjugate

$$\int \psi^* \hat{A} \varphi d\tau = \int \varphi \hat{A}^\dagger \psi^* d\tau \tag{1.1}$$

where ψ and φ are arbitrary functions. So we can go about finding $\hat{T}_a^\dagger$ by solving equation 1.1 in one dimension for $\psi^*(x)$ and $\varphi(x)$. Which yields

$$\int \psi^* \hat{A} \varphi d\tau = \int \varphi \hat{A}^\dagger \psi^* d\tau$$
$$\Downarrow$$
$$\int \psi^*(x) \hat{T}_a \varphi(x) dx = \int \varphi(x) \hat{T}_a^\dagger \psi^*(x) dx$$
$$\Downarrow$$
$$\int \psi^*(x) \varphi(x+a) dx = \int \varphi(x) \hat{T}_a^\dagger \psi^*(x) dx$$

We note that we need to get the left hand side of the above equation to be equal to the right hand side. This implies that we need to get $\varphi(x+a)$ to equal $\varphi(x)$. So we can set a change of variables on the left hand side such that $x \to x-a$ which yields

$$\int \psi^*(x) \varphi(x+a) dx = \int \varphi(x) \hat{T}_a^\dagger \psi^*(x) dx$$
$$\Downarrow$$
$$\int \psi^*(x-a) \varphi(x-a+a) dx = \int \varphi(x) \hat{T}_a^\dagger \psi^*(x) dx$$
$$\int \psi^*(x-a) \varphi(x) dx = \int \varphi(x) \hat{T}_a^\dagger \psi^*(x) dx$$
$$\Downarrow$$
$$\hat{T}_a^\dagger \psi^*(x) = \psi^*(x-a)$$

Therefore, the Hermitian conjugate of the translation operator, $\hat{T}_a$ is the translation in the opposite direction or

$$\hat{T}_a^\dagger = \hat{T}_{-a}$$

where

$$\hat{T}_a^\dagger \Psi(x) = \Psi(x-a)$$

2 Problem #2

For an operator of the form $\hat{F} = F\left(\hat{f}\right)$ given that $F(z)$ is expandable by

$$F(z) = \sum_{n=0}^{\infty} c_n z^n$$

we can expand the operator as

$$\hat{F} = \sum_{n=0}^{\infty} c_n \hat{z}^n \tag{2.1}$$

(a) For the given operator

$$\hat{G}_a = \exp\left(a\frac{d}{dx}\right)$$

we can apply equation 2.1 by noting that e^z can be expanded into the power series

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

by a Taylor expansion. This implies that

$$\hat{G}_a = \sum_{n=0}^{\infty} \frac{1}{n!}\left(a\frac{d}{dx}\right)^n.$$

So we can apply this expansion to the function

$$\Phi(x) = \hat{G}_a \Psi(x)$$

which yields

$$\begin{aligned}
\Phi(x) &= \hat{G}_a \Psi(x) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\left(a\frac{d}{dx}\right)^n \Psi(x) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n \Psi(x)}{dx^n} a^n \\
&= \Psi(x+a)
\end{aligned}$$

Therefore we can see that $\hat{G}_a$ is the translation operator with distance a. This makes sense because we know $\exp(d/dx)$ is related to the wave vector which is related to the velocity. We can therefore infer that this operator would be related to some sort of translation.

(b) For a new operator we are given

$$\hat{G}_a = \exp\left(ax\frac{d}{dx}\right)$$

which we can expand using the expansion of e^z from before this yields the result

$$\hat{G}_a = \sum_{n=0}^{\infty} \frac{1}{n!}\left(ax\frac{d}{dx}\right)^n.$$

which we can use to find the function $\Phi(x)$ by

$$\begin{aligned}\Phi(x) &= \hat{G}_a \Psi(x) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(ax \frac{d}{dx} \right)^n \Psi(x)\end{aligned}$$

Next we note that we can expand $\Psi(x)$ by

$$\Psi(x) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m \Psi(0)}{dx^m} x^m$$

replacing this into our series over n we have

$$\begin{aligned}\Rightarrow &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(ax \frac{d}{dx} \right)^n \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m \Psi(0)}{dx^m} x^m \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a^n}{n!} \frac{1}{m!} \frac{d^m \Psi(0)}{dx^m} x^n \frac{d^n}{dx^n} x^m \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a^n}{n!} \frac{1}{m!} \frac{d^m \Psi(0)}{dx^m} x^n \frac{m!}{(m-n)!} x^{m-n} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a^n}{m!} \frac{d^m \Psi(0)}{dx^m} \frac{m!}{(m-n)!n!} x^m \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a^n}{m!} \frac{d^m \Psi(0)}{dx^m} \binom{m}{n} x^m \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m \Psi(0)}{dx^m} x^m \sum_{n=0}^{\infty} \binom{m}{n} a^n \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m \Psi(0)}{dx^m} x^m (1+a)^m \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m \Psi(0)}{dx^m} (x(1+a))^m \\ \Phi(x) &= \Psi(x(1+a))\end{aligned}$$

3 Problem #3

Given the Hermitian operator $\hat{f}$ which follows the relation

$$\hat{f}^3 = c^2 \hat{f} \tag{3.1}$$

we can determine the eigenvalues of $\hat{f}$ by noting that because $\hat{f}$ is Hermitian it follows that

$$\hat{f}|\psi\rangle = f|\psi\rangle$$

where f is a real valued eigenvalue. So we need to determine the value for f. We are given the equality from equation 3.1 we can act both sides of a wave function $|\psi\rangle$

$$\begin{gathered} \hat{f}^3 = c^2 \hat{f} \\ \Downarrow \\ \hat{f}^3|\psi\rangle = c^2 \hat{f}|\psi\rangle \end{gathered}$$

We can calculate the right hand side to get

$$\begin{aligned} \hat{f}^3|\psi\rangle &= \hat{f}^2 \hat{f}|\psi\rangle \\ &= f\hat{f}^2|\psi\rangle \\ &= f^2 \hat{f}|\psi\rangle \\ &= f^3|\psi\rangle \end{aligned}$$

and the left hand side gives

$$c^2 \hat{f}|\psi\rangle = c^2 f|\psi\rangle$$

So by equation 3.1 we get

$$\begin{gathered} \hat{f}^3|\psi\rangle = c^2 \hat{f}|\psi\rangle \\ \Downarrow \\ f^3|\psi\rangle = c^2 f|\psi\rangle \end{gathered}$$

This implies that

$$\begin{aligned} \left[f^3 - c^2 f\right]|\psi\rangle &= 0 \\ &\Downarrow \\ f^3 - c^2 f &= 0 \\ &\Downarrow \\ f &= \pm c, 0 \end{aligned}$$

Therefore the eigenvalues for $\hat{f}$ are $\pm c$ and 0.