Physics 611

Electromagnetic Theory II

Professor Christopher Pope

Homework #4

Joe Becker
UID: 125-00-4128
October 19th, 2016

1 Problem #1

(a) Given the energy-momentum tensor for the electromagnetic field

$$T_{\mu\nu} = \frac{1}{8\pi}\left(F_{\mu\rho}F_{\nu}{}^{\rho} + {}^{*}F_{\mu\rho}\,{}^{*}F_{\nu}{}^{\rho}\right) \tag{1.1}$$

where ${}^{*}F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. We can see that equation 1.1 can be reduced by noting that

$$\begin{aligned}
{}^{*}F_{\mu\rho}\,{}^{*}F_{\nu}{}^{\rho} &= \left(\frac{1}{2}\epsilon_{\mu\rho\pi\sigma}F^{\pi\sigma}\right)\eta_{\nu\lambda}\,{}^{*}F^{\lambda\rho} \\
&= \left(\frac{1}{2}\epsilon_{\mu\rho\pi\sigma}F^{\pi\sigma}\right)\eta_{\nu\lambda}\frac{1}{2}\epsilon^{\lambda\rho\phi\omega}F_{\phi\omega} \\
&= \frac{1}{4}\eta_{\nu\lambda}F^{\pi\sigma}F_{\phi\omega}\left(-\delta^{\lambda}_{\mu}\delta^{\phi}_{\pi}\delta^{\omega}_{\sigma} - \delta^{\phi}_{\mu}\delta^{\omega}_{\pi}\delta^{\lambda}_{\sigma} - \delta^{\omega}_{\mu}\delta^{\lambda}_{\pi}\delta^{\phi}_{\sigma} + \delta^{\phi}_{\mu}\delta^{\lambda}_{\pi}\delta^{\omega}_{\sigma} + \delta^{\lambda}_{\mu}\delta^{\omega}_{\pi}\delta^{\phi}_{\sigma} + \delta^{\omega}_{\mu}\delta^{\phi}_{\pi}\delta^{\lambda}_{\sigma}\right) \\
&= \frac{1}{4}\left(-\eta_{\nu\mu}F^{\pi\sigma}F_{\pi\sigma} - \eta_{\nu\sigma}F^{\pi\sigma}F_{\mu\pi} - \eta_{\nu\pi}F^{\sigma\pi}F_{\mu\sigma} + \eta_{\nu\pi}F^{\pi\sigma}F_{\mu\sigma} + \eta_{\nu\mu}F^{\pi\sigma}F_{\sigma\pi} + \eta_{\nu\sigma}F^{\pi\sigma}F_{\pi\mu}\right) \\
&= \frac{1}{4}\left(-2\eta_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} + 4\eta_{\nu\sigma}F^{\sigma\rho}F_{\mu\rho}\right) \\
&= -\frac{1}{2}\left(\eta_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} - 2F_{\nu}{}^{\rho}F_{\mu\rho}\right)
\end{aligned}$$

We see that equation 1.1 becomes

$$\begin{aligned}
T_{\mu\nu} &= \frac{1}{8\pi}\left(F_{\mu\rho}F_{\nu}{}^{\rho} + {}^{*}F_{\mu\rho}\,{}^{*}F_{\nu}{}^{\rho}\right) \\
&\Downarrow \\
&= \frac{1}{8\pi}\left(F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{2}\eta_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} + F_{\nu}{}^{\rho}F_{\mu\rho}\right) \\
&= \frac{1}{4\pi}\left(F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}\eta_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right)
\end{aligned}$$

Note this is the typical form of the electromagnetic field energy momentum tensor.

(b) We can find the expression for $T_{\mu\rho}T^{\nu\rho}$ by noting that

$$\begin{aligned}
T^{\nu\rho} &= \eta^{\nu\sigma}\eta^{\rho\lambda}T_{\sigma\lambda} \\
&= \frac{1}{4\pi}\left(\eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}{}^{\phi} - \frac{1}{4}\eta^{\nu\sigma}\eta^{\rho\lambda}\eta_{\sigma\lambda}F^{\phi\omega}F_{\phi\omega}\right) \\
&= \frac{1}{4\pi}\left(\eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}{}^{\phi} - \frac{1}{4}\eta^{\nu\rho}F^{\sigma\lambda}F_{\sigma\lambda}\right)
\end{aligned}$$

So we can calculate defining $F^2 \equiv F^{\mu\nu}F_{\mu\nu}$

$$\begin{aligned}
T_{\mu\rho}T^{\nu\rho} &= \frac{1}{(4\pi)^2}\left(F_{\mu\sigma}F_{\rho}{}^{\sigma} - \frac{1}{4}\eta_{\mu\rho}F^{\sigma\lambda}F_{\sigma\lambda}\right)\left(\eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}{}^{\phi} - \frac{1}{4}\eta^{\nu\rho}F^{\sigma\lambda}F_{\sigma\lambda}\right) \\
&= \frac{1}{(4\pi)^2}\left(\eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}{}^{\phi}F_{\mu\omega}F_{\rho}{}^{\omega} + \frac{1}{16}\eta_{\mu\rho}\eta^{\nu\rho}(F^2)^2 - \frac{1}{4}F^2\eta^{\nu\rho}F_{\mu\sigma}F_{\rho}{}^{\sigma} - \frac{1}{4}F^2\eta_{\mu\rho}\eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}{}^{\phi}\right) \\
&= \frac{1}{(4\pi)^2}\left(\eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}{}^{\phi}F_{\mu\omega}F_{\rho}{}^{\omega} + \frac{1}{16}\delta^{\nu}_{\mu}(F^2)^2 - \frac{1}{4}F^2\eta^{\nu\rho}F_{\mu\sigma}F_{\rho}{}^{\sigma} - \frac{1}{4}F^2\eta^{\nu\rho}F_{\rho\sigma}F_{\mu}{}^{\sigma}\right) \\
&= \frac{1}{(4\pi)^2}\left(\eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}{}^{\phi}F_{\mu\omega}F_{\rho}{}^{\omega} + \frac{1}{16}\delta^{\nu}_{\mu}(F^2)^2 - \cancel{\frac{1}{4}F^2\eta^{\nu\rho}F_{\mu\sigma}F_{\rho}{}^{\sigma} + \frac{1}{4}F^2\eta^{\nu\rho}F_{\mu\sigma}F_{\rho}{}^{\sigma}}\right)
\end{aligned}$$

Next we consider the term

$$\begin{aligned}\eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}{}^{\phi}F_{\mu\omega}F_{\rho}{}^{\omega} &= F^{\nu}{}_{\sigma}F_{\lambda}{}^{\sigma}F_{\mu\rho}F^{\lambda\rho} \\ &= F_{\mu\lambda}F_{\sigma\rho}F^{\sigma\lambda}F^{\nu\rho}\end{aligned}$$

Note that if we swap the dummy indices λ and ρ we see

$$F_{\mu\lambda}F_{\sigma\rho}F^{\sigma\lambda}F^{\nu\rho} = F_{\mu\rho}F_{\sigma\lambda}F^{\sigma\rho}F^{\nu\lambda}$$

which is only non zero if $\mu = \nu$ which implies that this term must be proportional to δ^{ν}_{μ}. We can also note that each product of two field tensors must be antisymmetric which allows us to say

$$F_{\mu\lambda}F_{\rho\lambda}F^{\nu\sigma}F^{\rho\sigma} = \left(\epsilon^{\nu\lambda\sigma\rho}F^{\nu\lambda}F^{\sigma\rho}\right)\left(\epsilon^{\mu\rho\sigma\lambda}F_{\mu\rho}F_{\sigma\lambda}\right)\delta^{\nu}_{\mu} = (2\mathbf{E}\cdot\mathbf{B})^2\delta^{\nu}_{\mu}$$

So ultimately we have

$$T_{\mu\rho}T^{\nu\rho} = \frac{1}{(4\pi)^2}\left(\eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}{}^{\phi}F_{\mu\omega}F_{\rho}{}^{\omega} + \frac{1}{16}\delta^{\nu}_{\mu}(F^2)^2\right) = \frac{1}{(8\pi)^2}\left((E^2-B^2)^2 + (2\mathbf{E}\cdot\mathbf{B})^2\right)\delta^{\nu}_{\mu}$$

2 Problem #2

(a) Given the Lagrangian density

$$\mathcal{L} = -\frac{1}{16}F^{\mu\nu}F_{\mu\nu} - \frac{m^2}{8\pi}A^\mu A_\mu + J^\mu A_\mu \tag{2.1}$$

we can derive the equations of motion from the *Euler-Lagrange equations*

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \right) = 0 \tag{2.2}$$

Noting that $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ we have

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial A_\mu} &= -\frac{m^2}{8\pi}A^\mu + J^\mu \\
\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} &= \frac{\partial}{\partial(\partial_\nu A_\mu)} \left(-\frac{1}{16}F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) \right) = \frac{1}{4}F^{\mu\nu}
\end{aligned}$$

So equation **??** yields the equation of motion

$$-\partial_\nu F^{\mu\nu} + m^2 A^\mu = 4\pi J^\mu$$

(b) Using the result from part (a) we can find the solution for the scalar potential $\phi \equiv A^0$ for a point charge q located at the origin. This implies that we have $J^0 = q\delta^3(\mathbf{r})$ which allows us to solve for $\mu = 0$

$$\begin{aligned}
-\partial_\nu F^{0\nu} + m^2 A^0 &= 4\pi q \delta^3(\mathbf{r}) \\
&\Downarrow \\
-\nabla^2 \phi + m^2 \phi &= 4\pi q \delta^3(\mathbf{r}) \\
&\Downarrow \\
-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r} \right) + m^2 \phi &= 4\pi q \delta(r) \\
-\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r} \right) + r^2 m^2 \phi &= 4\pi r^2 q \delta(r) \\
&\Downarrow \\
r\phi(r) &= Ce^{-mr}
\end{aligned}$$

Where we can solve for the constant C by noting that for *Gauss' law* to hold $C = q$ therefore

$$\phi(r) = \frac{qe^{-mr}}{r}$$

3 Problem #3

(a) Given the potentials describing an electric charge, e, moving with constant velocity, $\mathbf{v}$

$$\phi(\mathbf{r}, t) = \frac{e\gamma}{r'}, \qquad \mathbf{A}(\mathbf{r}, t) = \mathbf{v}\phi$$

we note that the magnetic field is given as

$$\begin{aligned}
\mathbf{B} &= \nabla \times \mathbf{A} \\
&= \nabla \times (\mathbf{v}\phi) \\
&= \nabla\phi \times \mathbf{v} + \phi\cancelto{0}{(\nabla \times \mathbf{v})} \\
&= -\mathbf{E} \times \mathbf{v} \\
&= \mathbf{v} \times \mathbf{E}
\end{aligned}$$

(b) Using the expressions we were given in part (a) we can find the electric field by

$$\begin{aligned}
\mathbf{E} &= -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \\
&= -\nabla\frac{e\gamma}{r'} - \mathbf{v}\frac{\partial}{\partial t}\frac{e\gamma}{r'}
\end{aligned}$$

we note that we can assume that $\mathbf{v}$ is pointing along the $\hat{x}$ direction this gives us

$$r' = \sqrt{\gamma^2(x - vt)^2 + y^2 + z^2}$$

which allows us to calculate

$$\begin{aligned}
\frac{\partial r'}{\partial t} &= \gamma\frac{\partial}{\partial t}\left(\sqrt{(x - vt)^2 + y^2 + z^2}\right) = \gamma\left((x - vt)^2 + y^2 + z^2\right)^{-1/2} 2(x - vt)(-v) \\
&= -\frac{2\gamma^2(x - vt)v}{r'} \\
\frac{\partial r'}{\partial x} &= \frac{2\gamma^2(x - vt)}{r'} \\
\frac{\partial r'}{\partial y} &= \frac{2y}{r'} \\
\frac{\partial r'}{\partial z} &= \frac{2z}{r'}
\end{aligned}$$

So this gives the resulting electric field

$$\begin{aligned}
\mathbf{E} &= -\nabla\frac{e\gamma}{r'} - \mathbf{v}\frac{\partial}{\partial t}\frac{e\gamma}{r'} \\
&= \frac{e\gamma}{r'^2}\nabla r' + v\hat{x}\frac{e\gamma}{r'^2}\frac{\partial r'}{\partial t} \\
&= \frac{2e\gamma}{r'^3}\mathbf{r}' + v^2\frac{2e\gamma}{r'^3} \\
&= \frac{e(1 - v^2)\mathbf{R}}{R_*^3}
\end{aligned}$$

Note this matches the result derived in the notes where $R_*^2 \equiv (x - vt)^2 + (1 - v^2)(y^2 + z^2)$