Physics 615

Methods of Theoretical Physics I

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Homework #2

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1 Problem #1

We can derive the *Rodrigues' formula* given by

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l \tag{1.1}$$

where $P_l(x)$ represent *Legendre's polynomials.* We first take note of the derivative

$$\frac{d}{dx}(x^2-1)^l = l(x^2-1)^{l-1}(2x)$$

Which leads to the following identity

$$(x^2-1)\frac{d}{dx}(x^2-1)^l = 2lx(x^2-1)^l \tag{1.2}$$

Now we need to calculate the $l+1$th derivative of both sides of equation 1.2 by *Leibnitz rule* which is given by

$$\frac{d^{l+1}}{dx^{l+1}}\Big[u(x)v(x)\Big] = \sum_{k=0}^{l+1}\binom{l+1}{k}\frac{d^{l+1-k}u}{dx^{l+1-k}}\frac{d^k v}{dx^k}. \tag{1.3}$$

So if we can see that using equation 1.3 on the left hand side of equation 1.2 to get

$$\frac{d^{l+1}}{dx^{l+1}}\left[(x^2-1)\frac{d}{dx}(x^2-1)^l\right] = \sum_{k=0}^{l+1}\binom{l+1}{k}\frac{d^{l+1-k}}{dx^{l+1-k}}\left[\frac{d}{dx}(x^2-1)^l\right]\frac{d^k}{dx^k}\left[x^2-1\right]$$

We see that the term

$$\frac{d^k}{dx^k}\left[x^2-1\right]$$

goes to zero for $k \geq 0$, therefore we our sum only goes to $k=2$. Which makes the sum become

$$\begin{aligned}
\Rightarrow &= \binom{l+1}{0}\frac{d^{l+1}}{dx^{l+1}}\left[\frac{d}{dx}(x^2-1)^l\right](x^2-1) + \binom{l+1}{1}\frac{d^l}{dx^l}\left[\frac{d}{dx}(x^2-1)^l\right](2x) + \binom{l+1}{2}\frac{d^{l-1}}{dx^{l-1}}\left[\frac{d}{dx}(x^2-1)^l\right](2) \\
&= \frac{d^2}{dx^2}\left[\frac{d^l}{dx^l}(x^2-1)^l\right](x^2-1) + (l+1)2x\frac{d}{dx}\left[\frac{d^l}{dx^l}(x^2-1)^l\right] + l(l+1)\left[\frac{d^l}{dx^l}(x^2-1)^l\right]
\end{aligned}$$

Then we apply equation 1.3 to the right hand side of equation 1.2 to get

$$\begin{aligned}
\frac{d^{l+1}}{dx^{l+1}}\left[2lx(x^2-1)^l\right] &= \sum_{k=0}^{l+1}\binom{l+1}{k}\frac{d^{l+1-k}}{dx^{l+1-k}}\left[(x^2-1)^l\right]\frac{d^k}{dx^k}\left[2lx\right] \\
&= \binom{l+1}{0}\frac{d^{l+1}}{dx^{l+1}}\left[(x^2-1)^l\right](2lx) + \binom{l+1}{1}\frac{d^l}{dx^l}\left[(x^2-1)^l\right](2l) \\
&= 2lx\frac{d}{dx}\left[\frac{d^l}{dx^l}(x^2-1)^l\right] + 2l(l+1)\left[\frac{d^l}{dx^l}(x^2-1)^l\right]
\end{aligned}$$

We can define

$$y \equiv \left(\frac{d}{dx}\right)^l (x^2-1)^l$$

which when we combine both sides of 1.2

$$\left(x^2-1\right)\frac{d^2}{dx^2}y+(l+1)2x\frac{d}{dx}y+l(l+1)y=2lx\frac{d}{dx}y+2l(l+1)y$$
$$\Downarrow$$
$$\left(x^2-1\right)\frac{d^2}{dx^2}y+(l+1)2x\frac{d}{dx}y+l(l+1)y-2lx\frac{d}{dx}y-2l(l+1)y=0$$
$$\left(x^2-1\right)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-l(l+1)y=0$$
$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+l(l+1)y=0$$

We see that this is the *Legendre Differential Equation* which we know has the solutions $P_l(x)$. This implies that y are the Legendre Polynomials with a normalization factor. To find the normalization factor we impose the condition $P_n(1)=1$. We note that this condition makes every term with x^2-1 go to zero. We can infer that the only terms that do not have this term is the term that is derived l times this gives us a factor of $2^l l!$. So for $x=1$ we have

$$y=2^l l!$$

which implies that

$$P_l(x)=\frac{1}{2^l l!}y=\frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l\left(x^2-1\right)^l$$

Which is in agreement with equation 1.1.

2 Problem #2

Given the differential equation

$$xy^2y' - \frac{1}{3}(x^3 + y^3) = 0 \tag{2.1}$$

we can rearrange equation 2.1 into the form

$$A(x,y)dx + B(x,y)dy = 0$$

which we get as

$$y^2dy - \frac{1}{3}\left(x^2 + \frac{y^3}{x}\right)dx = 0$$

where $A(x,y) = \frac{1}{3}\left(x^2 + \frac{y^3}{x}\right)$ and $B(x,y) = y^2$. We verify that the conditions

$$\begin{aligned} A(ax,ay) &= a^r A(x,y) \\ B(ax,ay) &= a^r B(x,y) \end{aligned}$$

by

$$\begin{aligned} A(ax,ay) &= \frac{1}{3}\left((ax)^2 + \frac{(ay)^3}{ax}\right) \\ &= \frac{1}{3}\left((ax)^2 + \frac{(ay)^3}{ax}\right) \\ &= \frac{1}{3}\left(a^2x^2 + \frac{a^3y^3}{ax}\right) \\ &= \frac{1}{3}\left(a^2x^2 + a^2\frac{y^3}{x}\right) \\ &= a^2\frac{1}{3}\left(2x^2 + \frac{y^3}{x}\right) \\ &= a^2A(x,y) \end{aligned}$$

and

$$\begin{aligned} B(ax,ay) &= (ay)^2 \\ &= a^2y^2 \\ &= a^2B(x,y). \end{aligned}$$

This implies that we can set a change of variables

$$x, y \to x, v = \frac{y}{x}$$

which transforms equation 2.1 into

$$\begin{aligned} &0 = y^2dy - \frac{1}{3}\left(x^2 + \frac{y^3}{x}\right)dx \\ &\Downarrow \\ &0 = (vx)^2(vdx + xdv) - \frac{1}{3}\left(x^2 + \frac{(vx)^3}{x}\right)dx \end{aligned}$$

Which allows us to use separation of variables by

$$\begin{aligned}
0 &= v^2x^3dv + \left(v^3x^2 - \frac{1}{3}x^2 - \frac{1}{3}v^3x^2\right)dx \\
0 &= v^2x^3dv + \left(-\frac{1}{3}x^2 + \frac{2}{3}v^3x^2\right)dx \\
0 &= v^2x^3dv - \frac{1}{3}x^2\left(1-2v^3\right)dx \\
&\Downarrow \\
\frac{3v^2}{1-2v^3}dv &= \frac{1}{x}dx
\end{aligned}$$

Now we can solve by integrating both sides

$$\begin{aligned}
\int \frac{1}{x}dx &= \int \frac{3v^2}{1-2v^3}dv \\
&\Downarrow \\
\log(x) &= \int \frac{3v^2}{1-2v^3}dv
\end{aligned}$$

where we use a substitution $u = 1-2v^3$ and $du = -6v^2dv$ to get

$$\begin{aligned}
\log(x) &= \int \frac{3v^2}{1-2v^3}dv \\
&\Downarrow \\
\log(x) &= \frac{1}{2}\int \frac{du}{u} \\
\log(x) &= -\frac{1}{2}\log(1-2v^3) + c \\
&\Downarrow \\
x &= C\exp\left[\log((1-2v^3)^{-1/2})\right] \\
x &= C(1-2v^3)^{-1/2} \\
&\Downarrow \\
x &= C\left(1-2\left(\frac{y}{x}\right)^3\right)^{-1/2} \\
&\Downarrow \\
-\frac{Cx^{-2}-1}{2} &= \frac{y}{x} \\
&\Downarrow \\
y^3 &= \frac{Cx+x^3}{2} \\
&\Downarrow \\
y(x) &= \left(\frac{x(x^2+C)}{2}\right)^{1/3}
\end{aligned}$$

3 Problem #3

Given the second order differential equation

$$y'' + 3xy' - y = 0 \tag{3.1}$$

we can solve for the general solution by using a power series given by

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} c_n x^n \\ y'(x) &= \sum_{n=0}^{\infty} c_n n x^{n-1} \\ y''(x) &= \sum_{n=0}^{\infty} c_n n(n-1) x^{n-2} \end{aligned}$$

Note we can shift the indices of $y''(x)$ such that

$$y''(x) = \sum_{n=0}^{\infty} c_{n+2}(n+1)(n+2)x^n.$$

So, we replace the power series into equation 3.1 and get

$$\begin{aligned} 0 &= y'' + 3xy' - y \\ &\Downarrow \\ 0 &= \sum_{n=0}^{\infty} \left(c_{n+2}(n+1)(n+2)x^n + 3c_n n x^n - c_n x^n\right) \\ 0 &= \sum_{n=0}^{\infty} \left(c_{n+2}(n+1)(n+2) + 3c_n n - c_n\right) x^n \end{aligned}$$

Which implies that for all x

$$0 = c_{n+2}(n+1)(n+2) + c_n(3n-1)$$

must hold true. This leads to a recursion relation

$$c_{n+2} = -\frac{3n-1}{(n+1)(n+2)} c_n \tag{3.2}$$

We expect to have two free constants due to the fact that this is a second order equation. We note that two constants define the power series where c_0 defines the even indices and c_1 defines the odd indices by equation 3.2. These are our two free constants. So, we can write the general solution by

$$y(x) = c_0 \left(1 + \frac{1}{2}x^2 - \frac{5}{24}x^4 + \frac{55}{720}x^6 + ...\right) + c_1 \left(x - \frac{1}{3}x^3 + \frac{2}{15}x^5 + ...\right)$$

4 Problem #4

Given the differential equation

$$y'' - 4y' + 3y = e^{2x} + 3x^2 \tag{4.1}$$

we can solve for $y(x)$ by first solving the homogeneous version of equation 4.1 with the ansatz $y_0(x) = e^{ax}$. This yields the equation

$$\begin{aligned} a^2 - 4a + 3 &= 0 \\ (a-3)(a-1) &= 0 \\ &\Downarrow \\ a &= 3, 1 \end{aligned}$$

So we can say that

$$y_0(x) = c_1 e^{3x} + c_2 e^x.$$

Now we need to find the particular solution by using the ansatz

$$y_p(x) = ae^{2x} + bx^2 + cx + d$$

which we can calculate

$$\begin{aligned} y_p'(x) &= 2ae^{2x} + 2bx + c \\ y_p''(x) &= 4ae^{2x} + 2b. \end{aligned}$$

Then we plug $y_p(x)$, $y_p'(x)$, and $y_p''(x)$ in equation 4.1

$$\begin{aligned} y_p'' - 4y_p' + 3y_p &= e^{2x} + 3x^2 \\ &\Downarrow \\ 4ae^{2x} + 2b - 4(2ae^{2x} + 2bx + c) + 3(ae^{2x} + bx^2 + cx + d) &= e^{2x} + 3x^2 \\ 4ae^{2x} - 8ae^{2x} + 3ae^{2x} + 3bx^2 - 8bx + 3cx + 2b - 4c + 3d &= e^{2x} + 3x^2 \\ -ae^{2x} + 3bx^2 + (3c - 8b)x + 2b - 4c + 3d &= e^{2x} + 3x^2 \end{aligned}$$

and solve for the coefficients by the system of equations

$$\begin{aligned} -a &= 1 \\ 3b &= 3 \\ 3c - 8c &= 0 \\ 2b - 4c + 3d &= 0 \end{aligned}$$

which gives the solution

$$\begin{aligned} a &= -1 \\ b &= 1 \\ c &= \frac{8}{3} \\ d &= \frac{26}{9} \end{aligned}$$

Which results in the particular solution

$$y_p(x) = -e^{2x} + x^2 + \frac{8}{3}x + \frac{26}{9}$$

and the total solution

$$y(x) = c_1 e^{3x} + c_2 e^x - e^{2x} + x^2 + \frac{8}{3}x + \frac{26}{9}$$