Physics 601

Analytical Mechanics

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Homework #3

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1 Problem #1

Given the *Laplace-Runge-Lenz vector* (LRL-vector) defined as

$$\mathbf{A} = \frac{1}{mk}\mathbf{p} \times \mathbf{L} - \hat{\mathbf{r}}. \tag{1.1}$$

Note, for the central potential $V(r) = -k/r$ the LRL-vector is conserved.

(a) We can show that the momentum vector, $\mathbf{p}$, traces out a circle in momentum space by first calculating $\mathbf{L} \times \mathbf{A}$, which yields

$$\mathbf{L} \times \mathbf{A} = \frac{1}{mk}\mathbf{L} \times (\mathbf{p} \times \mathbf{L}) - \mathbf{L} \times \hat{\mathbf{r}}$$

Where we calculate

$$\begin{aligned}
\mathbf{L} \times (\mathbf{p} \times \mathbf{L}) &= \mathbf{p}\,(\mathbf{L} \cdot \mathbf{L}) - \mathbf{L}\,(\mathbf{p} \cdot \mathbf{L}) \\
&= L^2\mathbf{p} - \mathbf{L}\,(\mathbf{p} \cdot (\mathbf{r} \times \mathbf{p})) \\
&= L^2\mathbf{p} - \mathbf{L}\left(\mathbf{r} \cdot \left(\cancelto{0}{\mathbf{p} \times \mathbf{p}}\right)\right) \\
&= L^2\mathbf{p}
\end{aligned}$$

Next, we assume that $\mathbf{A}$ points in the $\hat{x}$ direction. This coupled with the knowledge that our motion is constrained to xy-plane implies that $\mathbf{L}$ points in the $\hat{z}$ direction. Therefore

$$\mathbf{L} \times \mathbf{A} = LA(\hat{\mathbf{z}} \times \hat{\mathbf{x}}) = LA\hat{\mathbf{y}}$$

Using this same assumption that $\mathbf{L} = L\hat{\mathbf{z}}$ we find

$$\mathbf{L} \times \hat{\mathbf{r}} = L(\hat{\mathbf{z}} \times \hat{\mathbf{r}}) = -L\hat{\theta}$$

This allows us to rearrange our equation into the form

$$\begin{aligned}
\mathbf{L} \times \mathbf{A} &= \frac{1}{mk}\mathbf{L} \times (\mathbf{p} \times \mathbf{L}) - \mathbf{L} \times \hat{\mathbf{r}} \\
\mathbf{L} \times \mathbf{A} - \frac{1}{mk}\mathbf{L} \times (\mathbf{p} \times \mathbf{L}) &= -\mathbf{L} \times \hat{\mathbf{r}} \\
&\Downarrow \\
L\hat{\theta} &= LA\hat{\mathbf{y}} - \frac{L^2}{mk}\mathbf{p} \\
L\hat{\theta} &= LA\hat{\mathbf{y}} - \frac{L^2}{mk}(p_x\hat{\mathbf{x}} + p_y\hat{\mathbf{y}}) \\
L\hat{\theta} &= \left(LA - \frac{L^2}{mk}p_y\right)\hat{\mathbf{y}} - \frac{L^2}{mk}p_x\hat{\mathbf{x}}
\end{aligned}$$

Now we calculate the magnitude of both sides to get

$$\begin{aligned}
L^2 &= \left(LA - \frac{L^2}{mk}p_y\right)^2 - \left(\frac{L^2}{mk}p_x\right)^2 \\
&= (LA)^2 + \left(\frac{L^2}{mk}\right)^2 p_y^2 - 2\frac{AL^3}{mk}p_y + \left(\frac{L^2}{mk}\right)^2 p_x^2
\end{aligned}$$

Which we can rearrange by grouping together the p_x and p_y terms

$$\begin{aligned}
\Rightarrow \left(\frac{mk}{L}\right)^2 &= p_x^2 + p_y^2 - 2\frac{m^2k^2}{L^4}\frac{AL^3}{mk}p_y + \frac{m^2k^2}{L^4}(AL)^2 \\
\left(\frac{mk}{L}\right)^2 &= p_x^2 + p_y^2 - 2\frac{Amk}{L}p_y + \left(\frac{mkA}{L}\right)^2 \\
&\Downarrow \\
\left(\frac{mk}{L}\right)^2 &= p_x^2 + \left(p_y - \frac{Amk}{L}\right)^2
\end{aligned}$$

We recall that the magnitude of the LRL-vector is the eccentricity, e, which we replace as

$$p_x^2 + \left(p_y - \frac{mke}{L}\right)^2 = \left(\frac{mk}{L}\right)^2 \tag{1.2}$$

which is the equation for a circle centered at $(0, mke/L)$ with a radius of mk/L in momentum space.

(b) We can take the result from part (a) given by equation 1.2 and find where it crosses the p_x axis by setting $p_y = 0$. This yields

$$\begin{aligned}
p_x^2 + \left(0 - \frac{mke}{L}\right)^2 &= \left(\frac{mk}{L}\right)^2 \\
&\Downarrow \\
p_x^2 &= \left(\frac{mke}{L}\right)^2 - \left(\frac{mke}{L}\right)^2 \\
&= \left(\frac{mk}{L}\right)^2 (1 - e^2) \\
&\Downarrow \\
p_0 &\equiv \pm\frac{mk}{L}\sqrt{1 - e^2}
\end{aligned}$$

Where we defined the momentum at this point as p_0. We note that for the condition of bound orbits, $|e| < 1$, this gives a real result. Note the result that eccentricity is given by

$$e = \sqrt{\frac{2L^2E}{mk^2} + 1} \tag{1.3}$$

for elliptical orbits. We can replace this into the result from above to get

$$\begin{aligned}
p_0 = \pm\frac{mk}{L}\sqrt{1 - e^2} &= \pm\frac{mk}{L}\sqrt{1 - \frac{2L^2E}{mk^2} + 1} \\
&= \pm\sqrt{\frac{m^2k^2}{L^2}\frac{2L^2E}{mk^2}} \\
&= \pm\sqrt{2mE}
\end{aligned}$$

This result can be written in the familiar form as

$$E = \frac{p_0^2}{2m}$$

which implies when we are at p_0 the energy is entirely kinetic. This corresponds to the two turning points of our system.

(c) Given the p_0 we found in part (b) we can rescale the momentum to a dimensionless quantity $\mathbf{R} = \mathbf{p}/p_0$. Which we can transform equation 1.2 by dividing both sides by p_0^2

$$\left(\frac{p_x}{p_0}\right)^2 + \left(\frac{p_y}{p_0} - \frac{mke}{Lp_0}\right)^2 = \left(\frac{mk}{Lp_0}\right)^2$$
$$\Downarrow$$
$$R_x^2 + \left(R_y - \frac{e}{\sqrt{1-e^2}}\right)^2 = \frac{1}{1-e^2}$$

Note that for we have the intersections ± 1 on the X axis. Which is consistent with the result from part (b). Now, we can find the intersections of the Y axis by

$$\cancelto{0}{R_x^2} + \left(R_y - \frac{e}{\sqrt{1-e^2}}\right)^2 = \frac{1}{1-e^2}$$
$$\Downarrow$$
$$\begin{aligned} R_y - \frac{e}{\sqrt{1-e^2}} &= \pm\frac{1}{\sqrt{1-e^2}} \\ R_y &= \pm\frac{1}{\sqrt{1-e^2}} + \frac{e}{\sqrt{1-e^2}} \\ &= \frac{e \pm 1}{\sqrt{1-e^2}} \end{aligned}$$

So our two intersection points on Y are $e - 1/\sqrt{1-e^2}$ and $e + 1/\sqrt{1-e^2}$.

(d) In part (c) we created the $X - Y$ plane. We can take a unit sphere and intersect this plane at the equator of the unit sphere. We can calculate the points that pierce the unit sphere by a line drawn from the north pole of the unit sphere at location $(0, 0, 1)$ to the point on the $X - Y$ plane by

$$x = \frac{2X}{1 + X^2 + Y^2}, \qquad y = \frac{2Y}{1 + X^2 + Y^2}, \qquad z = \frac{-1 + X^2 + Y^2}{1 + X^2 + Y^2}$$

For the points that intersect the X axis we have the points $(1, 0)$ and $(-1, 0)$ and it is easy to see that for the point $(1, 0)$

$$\begin{aligned} x &= \frac{2(1)}{1+1} = 1 \\ y &= \frac{0}{1+1} = 0 \\ z &= \frac{-1+1+0}{1+1} = 0 \end{aligned}$$

and for $(-1, 0)$ the piercing point is

$$\begin{aligned} x &= \frac{2(-1)}{1+1} = -1 \\ y &= \frac{0}{1+1} = 0 \\ z &= \frac{-1+1+0}{1+1} = 0 \end{aligned}$$

Now, we can calculate the piercing coordinates for the position $(0, (e+1)/\sqrt{1-e^2})$ by finding y and z with $x = 0$.

$$
\begin{aligned}
y &= \frac{2\frac{e+1}{\sqrt{1-e^2}}}{1+\frac{(e+1)^2}{1-e^2}} \\
&= \frac{2(e+1)}{\sqrt{1-e^2}} \frac{1-e^2}{1-e^2+(e+1)^2} \\
&= \frac{2(e+1)}{\sqrt{1-e^2}} \frac{1-e^2}{1-e^2+e^2+1+2e} \\
&= \frac{2(e+1)}{\sqrt{1-e^2}} \frac{1-e^2}{2(e+1)} \\
&= \sqrt{1-e^2}
\end{aligned}
$$

and

$$
\begin{aligned}
z &= \frac{-1+\frac{(e+1)^2}{1-e^2}}{1+\frac{(e+1)^2}{1-e^2}} \\
&= \frac{(e+1)^2-1+e^2}{1-e^2} \frac{1-e^2}{(e+1)^2+1-e^2} \\
&= \frac{e^2+2e+1-1+e^2}{e^2+2e+1+1-e^2} \\
&= \frac{2e(e+1)}{2(e+1)} \\
&= e
\end{aligned}
$$

And we can repeat this calculation for $(0, (e-1)/\sqrt{1-e^2})$ by

$$
\begin{aligned}
y &= \frac{2\frac{e-1}{\sqrt{1-e^2}}}{1+\frac{(e-1)^2}{1-e^2}} \\
&= \frac{2(e-1)}{\sqrt{1-e^2}} \frac{1-e^2}{1-e^2+(e-1)^2} \\
&= \frac{2(e-1)}{\sqrt{1-e^2}} \frac{1-e^2}{1-e^2+e^2+1-2e} \\
&= \frac{2(e-1)}{\sqrt{1-e^2}} \frac{1-e^2}{2(1-e)} \\
&= -\sqrt{1-e^2}
\end{aligned}
$$

and

$$
\begin{aligned}
z &= \frac{-1+\frac{(e-1)^2}{1-e^2}}{1+\frac{(e-1)^2}{1-e^2}} \\
&= \frac{(e-1)^2-1+e^2}{1-e^2} \frac{1-e^2}{(e-1)^2+1-e^2} \\
&= \frac{e^2-2e+1-1+e^2}{e^2-2e+1+1-e^2} \\
&= \frac{2e(e-1)}{-2(e-1)} \\
&= -e
\end{aligned}
$$

So we can see the four points found in part (c) correspond to the pierced points

$$
\begin{aligned}
(1,0) &\Rightarrow (1,0,0)\\
(-1,0) &\Rightarrow (-1,0,0)\\
\left(0,\frac{e+1}{\sqrt{1-e^2}}\right) &\Rightarrow (0,\sqrt{1-e^2},e)\\
\left(0,\frac{e-1}{\sqrt{1-e^2}}\right) &\Rightarrow (0,-\sqrt{1-e^2},-e)
\end{aligned}
$$

We can calculate the volume made by three of these points by

$$
\begin{aligned}
(-1,0,0)\cdot((1,0,0)\times(0,\sqrt{1-e^2},e) &= 0\\
&\text{or}\\
(-1,0,0)\cdot((0,-\sqrt{1-e^2},-e)\times(0,\sqrt{1-e^2},e) &= 0\\
(-1,0,0)\cdot((0,-\sqrt{1-e^2},-e)\times(0,\sqrt{1-e^2},e) &= 0
\end{aligned}
$$

that these points lie on the same plane, and we know they intersect the unit sphere. This implies that these points must make a greater circle on the unit sphere. We can test to see that this holds true for a general point on the normalized Y circle by seeing that for $X = R_x$ and $Y = R_y - e/\sqrt{1-e^2}$ where the equation of our circle becomes

$$X^2 + Y^2 = \frac{1}{1-e^2}$$

which implies that

$$1 + X^2 + Y^2 = \frac{2-e^2}{1-e^2}$$

so our piecing points become

$$
\begin{aligned}
x &= 2\frac{1-e^2}{2-e^2}R_x\\
y &= 2\frac{1-e^2}{2-e^2}R_y - 2\frac{e\sqrt{1-e^2}}{2-e^2}\\
z &= -\frac{e^2}{1-e^2}
\end{aligned}
$$

we note that the above z does not depend on R_x or R_y so we can infer that all R_x and R_y lie on a plane. Therefore for all R_x and R_y we project onto a great circle on the unit sphere.

(e) We see that the hidden symmetry of the Kepler problem arises from the forcing of the momentum space to fit onto a circle in the X-Y plane.

2 Problem #2

Given Kepler's equation

$$\psi - e\sin\psi = \frac{2\pi t}{T} \equiv \omega t \tag{2.1}$$

we can expand over a small e using the recursion relation

$$\psi_{n+1} = \omega t + e\sin\psi_n \tag{2.2}$$

with $\psi_0 = \omega t$. Calculating to first order in e is easy to see

$$\psi_1 = \omega t + e\sin\psi_0 = \omega t + e\sin\omega t$$

Now to calculate to second order in e we need to use the fact that e is small.

$$\begin{aligned}
\psi_2 &= \omega t + e\sin\psi_1 \\
&= \omega t + e\sin(\omega t + e\sin\omega t) \\
&= \omega t + e(\sin\omega t\cos(e\sin\omega t) + \sin(e\sin\omega t)\cos\omega t) \\
&= \omega t + e\sin\omega t + e^2\sin\omega t\cos\omega t
\end{aligned}$$

And finally to third order we get

$$\begin{aligned}
\psi_3 &= \omega t + e\sin\psi_2 \\
&= \omega t + e\sin(\omega t + e\sin\omega t + e^2\sin\omega t\cos\omega t) \\
&= \omega t + e\sin\Big(\omega t + e\sin\omega t(1 + e\cos\omega t)\Big) \\
&= \omega t + e\left(\sin\omega t\cancelto{1}{\cos(e\sin\omega t(1 + e\cos\omega t))} + \sin(e\sin\omega t(1 + e\cos\omega t))\cos\omega t\right) \\
&= \omega t + e\Big(\sin\omega t + e\sin\omega t(1 + e\cos\omega t))\cos\omega t\Big) \\
&= \omega t + e\sin\omega t + e^2\sin\omega t\cos\omega t + e^3\sin\omega t\cos^2\omega t)
\end{aligned}$$

3 Problem #3

For a gravitational potential with a perturbation given as

$$V(r) = -\frac{k}{r} + \frac{\beta}{r^n}$$

we can calculate the resulting precession by the integral

$$\Delta\theta = \int_0^T \Omega(t)dt$$

where $\Omega(t)t$ is the angle of rotation of the LRL-vector about a vector $\mathbf{\Omega}$. Where we know that Ω is given by

$$\Omega = \frac{-f(r)}{mk}\frac{\cos\theta}{e}L$$

which allows us to change the integral over time into an integral over angle θ.

$$\begin{aligned}
\Delta\theta &= \int_0^T \Omega(t)dt \\
&= \frac{1}{mke}\int_0^T (-f(r))\cos\theta L dt \\
&= \frac{1}{mke}\int_0^{2\pi} (-f(r))\cos\theta \left(mr^2\frac{d\theta}{dt}\right) dt \\
&= \frac{1}{ke}\int_0^{2\pi} (-f(r)r^2)\cos\theta d\theta
\end{aligned}$$

Where $f(r)$ is the perturbative force which for our case we can calculate as

$$\begin{aligned}
f(r) = -V_p'(r) &= -\frac{d}{dr}\frac{\beta}{r^n} \\
&= \frac{n\beta}{r^{n+1}}
\end{aligned}$$

Which gives the equation for $\Delta\theta$

$$\Delta\theta = \frac{n\beta}{ke}\int_0^{2\pi} r^{1-n}(\theta)\cos\theta d\theta$$

We note that r is a function of θ and taking the perturbation to be small we can assume that r is the unperturbed result given by

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

which replacing into the integral yields

$$\Delta\theta_n = \frac{n\beta(a(1-e^2))^{1-n}}{ke}\int_0^{2\pi}\frac{\cos\theta}{(1+e\cos\theta)^{1-n}}d\theta$$

Now we can calculate the period for $n = 2, 3, 4$ by taking the integrals. First we have

$$\begin{aligned}
\Delta\theta_2 &= \frac{2\beta(a(1-e^2))^{-1}}{ke}\int_0^{2\pi}\frac{\cos\theta}{(1+e\cos\theta)^{-1}}d\theta \\
&= \frac{2\beta}{kae(1-e^2)}\int_0^{2\pi}(1+e\cos\theta)\cos\theta d\theta \\
&= \frac{2\beta\pi}{kae(1-e^2)}e\pi = \frac{2\beta\pi}{ka(1-e^2)}
\end{aligned}$$

Now for $n = 3$ we can calculate the integral for $\Delta\theta_3$.

$$\begin{aligned}
\Delta\theta_3 &= \frac{3\beta a(1-e^2)^{-2}}{ke}\int_0^{2\pi}\frac{\cos\theta}{(1+e\cos\theta)^{-2}}d\theta \\
&= \frac{3\beta}{ke(a(1-e^2))^2}\int_0^{2\pi}(1+e\cos\theta)^2\cos\theta d\theta \\
&= \frac{3\beta}{ke(a(1-e^2))^2}\int_0^{2\pi}(1+2e\cos\theta+e^2\cos^2\theta)\cos\theta d\theta \\
&= \frac{3\beta}{ke(a(1-e^2))^2}\int_0^{2\pi}\cos\theta+2e\cos^2\theta+e^2\cos^3\theta d\theta \\
&= \frac{3\beta}{ke(a(1-e^2))^2}2e\pi = \frac{6\beta a\pi}{k(a(1-e^2))^2}
\end{aligned}$$

And finally for $\Delta\theta_4$ we have

$$\begin{aligned}
\Delta\theta_4 &= \frac{4\beta}{ke(a(1-e^2))^3}\int_0^{2\pi}(1+e\cos\theta)^3\cos\theta d\theta \\
&= \frac{4\beta a}{ke(a(1-e^2))^3}\int_0^{2\pi}(1+3e\cos\theta+3e^2\cos^2\theta+e^3\cos^3\theta)\cos\theta d\theta \\
&= \frac{4\beta a}{ke(a(1-e^2))^3}\int_0^{2\pi}\cos\theta+3e\cos^2\theta+3e^2\cos^3\theta+e^3\cos^4\theta d\theta \\
&= \frac{4\beta a}{ke(a(1-e^2))^3}\left(3e\pi+\frac{3}{4}e^3\pi\right) = \frac{3\beta\pi(4+e^2)}{k(a(1-e^2))^3}
\end{aligned}$$

4 Problem #4

For the precession of the perihelion of the orbit of Mercury we take a perturbation on the potential as

$$V(r) = \frac{-mMG}{r}\left(1 + \alpha\frac{GM}{rc^2}\right)$$

we see that the perturbation is on the order of $1/r^2$. So we can use $\Delta\theta_2$ given by

$$\Delta\theta_2 = \frac{2\beta\pi}{ka(1-e^2)}$$

where

$$k = mMG, \qquad \beta = k\alpha\frac{GM}{c^2}$$

So we can solve for α by taking $\Delta\theta_2$ for Mercury.

$$\begin{aligned}
\Delta\theta_2 &= \frac{2\beta\pi}{ka(1-e^2)} \\
&= \frac{2k\pi}{ka(1-e^2)}\frac{GM}{c^2}\alpha \\
&= \frac{2\pi}{a(1-e^2)}\frac{GM}{c^2}\alpha \\
&\Downarrow \\
\alpha &= \Delta\theta_2\frac{c^2}{GM}\frac{a(1-e^2)}{2\pi}
\end{aligned}$$

So in order calculate α we can calculate $\Delta\theta_2$ by using the fact that the period of the orbit is $T = 0.241\,\text{yr}$ and the rate of the perihelion's advancement is $\Omega = 43\,\text{arcsec/century}$ which we can convert to radians per year

$$\Omega = \frac{43\,\text{arcsec}}{1\,\text{century}}\frac{1\,\text{century}}{100\,\text{yr}}\frac{4.85\times10^{-6}\,\text{radians}}{1\,\text{arcsec}} = 2.08\times10^{-6}\,\text{yr}^{-1}$$

So we can calculate $\Delta\theta_2$ by

$$\Delta\theta_2 = \Omega T = 2.08\times10^{-6}\,\text{yr}^{-1}(0.241\,\text{yr}) = 5.03\times10^{-7}$$

and we use the given values

$$\begin{aligned}
a &= 57900000\,\text{km} \\
e &= 0.206 \\
GM/c^2 &= 1.475\,\text{km}
\end{aligned}$$

to calculate

$$\alpha = 3.01$$

We note that this perturbation on the potential does not act as an addition to the barrier from the angular momentum. This is because the motion of the orbit is completely constrained in the plane. This means any extra terms results in precession.