Physics 606

Quantum Mechanics I

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Homework #8

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1 Problem #1

For the addition of the angular momenta for two particles with the quantum numbers $l_1 = l_2 = 1$, $m_1 = 0, \pm 1$, and $m_2 = 0, \pm 1$. For the net states $|l = 2, m = 0\rangle$ and $|l = 2, m = -1\rangle$ we can write it as a linear combination of $|m_1, m_2\rangle$ by using the ladder operator given by

$$L_{\pm}|lm\rangle = \sqrt{(l \mp m)(l \pm m + 1)}|lm - 1\rangle$$

where we take the total angular momentum ladder operator is

$$L_{\pm} = L_{1\pm} + L_{2\pm}$$

This allows us to act on the state $|l = 2, m = -2\rangle = |m_1 = -1, m_2 = -1\rangle$ which yields

$$\begin{aligned}
L_{+}|l = 2, m = -2\rangle &= (L_{1+} + L_{2+})|m_1 = -1, m_2 = -1\rangle \\
&\Downarrow \\
\sqrt{(2+2)(2-2+1)}|l = 2, m = -1\rangle &= \sqrt{(1+1)(1-1+1)}|m_1 = 0, m_2 = -1\rangle + \sqrt{(1+1)(1-1+1)}|m_1 = -1, m_2 = 0\rangle \\
2|l = 2, m = -1\rangle &= \sqrt{2}|m_1 = 0, m_2 = -1\rangle + \sqrt{2}|m_1 = -1, m_2 = 0\rangle \\
|l = 2, m = -1\rangle &= \frac{\sqrt{2}}{2}\Big(|m_1 = 0, m_2 = -1\rangle + |m_1 = -1, m_2 = 0\rangle\Big)
\end{aligned}$$

Which allows us to act L_+ again as

$$\begin{aligned}
L_{+}|l = 2, m = -1\rangle &= (L_{1+} + L_{2+})\frac{\sqrt{2}}{2}(|m_1 = 0, m_2 = -1\rangle + |m_1 = -1, m_2 = 0\rangle) \\
&\Downarrow \\
\sqrt{(2+1)(2-1+1)}|l = 2, m = 0\rangle &= \frac{\sqrt{2}}{2}L_{1+}(|m_1 = 0, m_2 = -1\rangle + |m_1 = -1, m_2 = 0\rangle) \\
&\quad + \frac{\sqrt{2}}{2}L_{2+}(|m_1 = 0, m_2 = -1\rangle + |m_1 = -1, m_2 = 0\rangle) \\
\sqrt{6}|l = 2, m = 0\rangle &= \frac{\sqrt{2}}{2}\Big(\sqrt{2}|m_1 = 1, m_2 = -1\rangle + \sqrt{2}|m_1 = 0, m_2 = 0\rangle\Big) \\
&\quad + \frac{\sqrt{2}}{2}\Big(|m_1 = 0, m_2 = -1\rangle + \sqrt{2}|m_1 = -1, m_2 = 0\rangle\Big) \\
|l = 2, m = 0\rangle &= \frac{\sqrt{6}}{6}\Big(|m_1 = 1, m_2 = -1\rangle + |m_1 = -1, m_2 = 1\rangle + 2|m_1 = 0, m_2 = 0\rangle\Big) \\
&= \frac{\sqrt{6}}{6}|m_1 = 1, m_2 = -1\rangle + \frac{\sqrt{6}}{6}|m_1 = -1, m_2 = 1\rangle + \frac{\sqrt{6}}{3}|m_1 = 0, m_2 = 0\rangle
\end{aligned}$$

2 Problem #2

(a) To calculate the matrix elements for a $l = 1$ state of $\hat{l}_x$, $\hat{l}_y$, and $\hat{l}_\pm$ which are the ladder operators defined as

$$\hat{l}_\pm \equiv \hat{l}_x \pm i\hat{l}_y$$

which allows us to say that

$$\hat{l}_x = \frac{1}{2}\left(\hat{l}_+ + \hat{l}_-\right)$$
$$\hat{l}_y = \frac{1}{2i}\left(\hat{l}_+ - \hat{l}_-\right)$$

So the matrix elements for $l = 1$ we have

$$\begin{aligned}\langle 1m'|\hat{l}_x|1m\rangle &= \sum_{m'=-1}^{1}\sum_{m=-1}^{1}\frac{1}{2}\langle 1m'|\left(\hat{l}_+ + \hat{l}_-\right)|1m\rangle \\ &= \sum_{m'=-1}^{1}\sum_{m=-1}^{1}\frac{\hbar}{2}\langle 1m'|\left(\sqrt{(1-m)(1+m+1)}|1m+1\rangle + \sqrt{(1+m)(1-m+1)}|1m-1\rangle\right) \\ &= \sum_{m'=-1}^{1}\sum_{m=-1}^{1}\frac{\hbar}{2}\left(\sqrt{(1-m)(2+m)}\langle 1m'|1m+1\rangle + \sqrt{(1+m)(2-m)}\langle 1m'|1m-1\rangle\right)\end{aligned}$$

Which if we manually calculate each entry we have

$$\hat{l}_x = \hbar\frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Now we repeat the process with $\hat{l}_y$

$$\begin{aligned}\langle 1m'|\hat{l}_y|1m\rangle &= \sum_{m'=-1}^{1}\sum_{m=-1}^{1}\frac{1}{2i}\langle 1m'|\left(\hat{l}_+ - \hat{l}_-\right)|1m\rangle \\ &= \sum_{m'=-1}^{1}\sum_{m=-1}^{1}\frac{\hbar}{2i}\left(\sqrt{(1-m)(2+m)}\langle 1m'|1m+1\rangle - \sqrt{(1+m)(2-m)}\langle 1m'|1m-1\rangle\right)\end{aligned}$$

Which gives

$$\hat{l}_y = \hbar\frac{\sqrt{2}}{2i}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Which allows us to calculate the ladder operators by definition

$$\hat{l}_+ = \hbar\sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\hat{l}_- = \hbar\sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(b) We can use the above result to find the state, ψ_{lx} with $l_x = 0$ which implies that

$$\hat{l}_x\psi_{lx} = 0$$

which we can calculate as

$$\hat{l}_x \psi_{lx} = \hbar \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Downarrow$$

$$0 = \hbar \frac{\sqrt{2}}{2} \begin{pmatrix} b \\ a+c \\ b \end{pmatrix}$$

This implies that $b = 0$ and $c = -a$ so our state is

$$\psi_{lx} = a \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

where we can find a by normalizing

$$|\psi_{lx}|^2 = 1 = a^2(1+1)$$

$$\Downarrow$$

$$a = \frac{\sqrt{2}}{2}$$

So we have the state

$$\psi_{lx} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

3 Problem #3

(a) For an eigenfunction, $|m\rangle$, of the orbital angular momentum operator $\hat{l}_z$. We can find expectation values of the operators $\hat{l}_x$ and $\hat{l}_y$ by using the which allows us to calculate

$$\begin{aligned}
\left\langle \hat{l}_y \right\rangle &= \langle m|\frac{1}{2i}\left(\hat{l}_+ - \hat{l}_-\right)|m\rangle \\
&= \frac{1}{2i}\left(\langle m|\hat{l}_+|m\rangle - \langle m|\hat{l}_-|m\rangle\right) \\
&= \frac{1}{2i}\left(\hbar A\langle m|m+1\rangle - \hbar B\langle m|m-1\rangle\right) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\left\langle \hat{l}_x \right\rangle &= \langle m|\frac{1}{2}\left(\hat{l}_+ + \hat{l}_-\right)|m\rangle \\
&= \frac{1}{2}\left(\langle m|\hat{l}_+|m\rangle + \langle m|\hat{l}_-|m\rangle\right) \\
&= \frac{1}{2}\left(\hbar A\langle m|m+1\rangle + \hbar B\langle m|m-1\rangle\right) \\
&= 0
\end{aligned}$$

(b) To calculate the expectation value of the operator $\hat{l}_x\hat{l}_y + \hat{l}_y\hat{l}_x$ in the state $|m\rangle$ we use ladder operators to find

$$\begin{aligned}
\hat{l}_x\hat{l}_y + \hat{l}_y\hat{l}_x &= \frac{1}{4i}\left(\left(\hat{l}_+ + \hat{l}_-\right)\left(\hat{l}_+ - \hat{l}_-\right) + \left(\hat{l}_+ - \hat{l}_-\right)\left(\hat{l}_+ + \hat{l}_-\right)\right) \\
&= \frac{1}{4i}\left(\hat{l}_+\hat{l}_+ - \hat{l}_+\hat{l}_- + \hat{l}_-\hat{l}_+ - \hat{l}_-\hat{l}_- + \hat{l}_+\hat{l}_+ + \hat{l}_+\hat{l}_- - \hat{l}_-\hat{l}_+ - \hat{l}_-\hat{l}_-\right) \\
&= \frac{1}{2i}\left(\hat{l}_+\hat{l}_+ - \hat{l}_-\hat{l}_-\right)
\end{aligned}$$

Note the cross terms cancel so neither operator will bring the state back to $|m\rangle$ which implies that

$$\left\langle \hat{l}_x\hat{l}_y + \hat{l}_y\hat{l}_x \right\rangle = 0$$

(c) Note for the operators $\hat{l}_x^2$ and $\hat{l}_y^2$ we have

$$\begin{aligned}
\hat{l}_x^2 &= \frac{1}{4}\left(\hat{l}_+ + \hat{l}_-\right)^2 \\
&= \frac{1}{4}\left(\hat{l}_+\hat{l}_+ + \hat{l}_-\hat{l}_- + \hat{l}_+\hat{l}_- + \hat{l}_-\hat{l}_+\right)
\end{aligned}$$

and

$$\begin{aligned}
\hat{l}_y^2 &= -\frac{1}{4}\left(\hat{l}_+ - \hat{l}_-\right)^2 \\
&= -\frac{1}{4}\left(\hat{l}_+\hat{l}_+ + \hat{l}_-\hat{l}_- - \hat{l}_+\hat{l}_- - \hat{l}_-\hat{l}_+\right)
\end{aligned}$$

+ We note that only the cross terms have nonzero contributions to the expectation values of these operators. So for a state in $|lm\rangle$ we have

$$
\begin{aligned}
\left\langle \hat{l}_x^2 \right\rangle &= \frac{1}{4}\langle lm| \left(\hat{l}_+\hat{l}_+ + \hat{l}_-\hat{l}_- + \hat{l}_+\hat{l}_- + \hat{l}_-\hat{l}_+\right)|lm\rangle \\
&= \frac{1}{4}\left(\langle lm|\hat{l}_+\hat{l}_-|lm\rangle + \langle lm|\hat{l}_-\hat{l}_+|lm\rangle\right) \\
&= \frac{\hbar}{4}\left(\sqrt{(l+m)(l-m+1)}\langle lm|\hat{l}_+|lm-1\rangle + \sqrt{(l-m)(l+m+1)}\langle lm|\hat{l}_-|lm+1\rangle\right) \\
&= \frac{\hbar^2}{4}\left(\sqrt{(l+m)(l-m+1)}\sqrt{(l-m+1)(l+m+1-1)}\langle lm||lm\rangle\right. \\
&\quad \left. +\sqrt{(l-m)(l+m+1)}\sqrt{(l+m+1)(l-m-1+1)}\langle lm||lm\rangle\right) \\
&= \frac{\hbar^2}{2}\left(l(l+1)-m^2\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\left\langle \hat{l}_y^2 \right\rangle &= -\frac{1}{4}\langle lm| \left(\hat{l}_+\hat{l}_+ + \hat{l}_-\hat{l}_- - \hat{l}_+\hat{l}_- - \hat{l}_-\hat{l}_+\right)|lm\rangle \\
&= \frac{1}{4}\left(\langle lm|\hat{l}_+\hat{l}_-|lm\rangle + \langle lm|\hat{l}_-\hat{l}_+|lm\rangle\right) \\
&= \frac{\hbar^2}{2}\left(l(l+1)-m^2\right)
\end{aligned}
$$

4 Problem #4

For a wave function that describes a planer rotor

$$\psi(\varphi) = A \sin^2 \varphi$$

we can write it as eigenfunctions of the $\hat{l}_z$ operator which are of the form $e^{im\varphi}$ by

$$\begin{aligned}
\psi(\varphi) = A \sin^2 \varphi &= A \Big(e^{i\phi} - e^{i\phi} \Big)^2 \\
&= A \Big(e^{i2\phi} + e^{i2\phi} - 1 \Big) \\
&= A \Big(|m=2\rangle + |m=-2\rangle - |m=0\rangle \Big)
\end{aligned}$$

Where we can find A by normalizing

$$\begin{aligned}
|\psi(\varphi)|^2 = 1 &= A^2 \Big(\langle m=2|m=2\rangle + \langle m=-2|m=-2\rangle - \langle m=0|m=0\rangle \Big) \\
&= A^2(1+1-1) \\
&\Downarrow \\
A &= 1
\end{aligned}$$

So we have the expectation values

$$\begin{aligned}
\left\langle \hat{L}_z \right\rangle &= \langle m=2|\hat{L}_z|m=2\rangle + \langle m=-2|\hat{L}_z|m=-2\rangle - \langle m=0|\hat{L}_z|m=0\rangle \\
&= 2\hbar - 2\hbar - 1\hbar = -\hbar
\end{aligned}$$

and

$$\begin{aligned}
\left\langle \hat{L}_z^2 \right\rangle &= \langle m=2|\hat{L}_z^2|m=2\rangle + \langle m=-2|\hat{L}_z^2|m=-2\rangle - \langle m=0|\hat{L}_z^2|m=0\rangle \\
&= 4\hbar^2 + 4\hbar^2 + 1\hbar^2 = 9\hbar
\end{aligned}$$