Physics 606

Quantum Mechanics I

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Homework #3

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1 Problem #1

For the operator given by

$$\hat{A} = -i\hbar\frac{\partial}{\partial\varphi} + a\sin\varphi$$

we assume there exists an eigenfunction which is a solution to the equation

$$(\hat{A} - A)\psi(\varphi) = 0$$

where A is the eigenvalue of $\hat{A}$. We can see that for the given $\hat{A}$ we have

$$\left(-i\hbar\frac{\partial}{\partial\varphi} + a\sin\varphi\right)\psi(\varphi) - A\psi(\varphi) = 0$$
$$-i\hbar\frac{\partial}{\partial\varphi}\psi(\varphi) + a\sin\varphi\psi(\varphi) - A\psi(\varphi) = 0$$
$$\Downarrow$$
$$-i\hbar\frac{\partial}{\partial\varphi}\psi(\varphi) = -a\sin\varphi\psi(\varphi) + A\psi(\varphi)$$
$$\frac{\partial}{\partial\varphi}\psi(\varphi) = i\frac{A - a\sin\varphi}{\hbar}\psi(\varphi)$$

This equation implies that

$$\psi(\varphi) = C\exp\left(i\frac{A}{\hbar}\varphi\right)\exp\left(i\frac{a\cos\varphi}{\hbar}\right)$$

We can find the normalization constant, C, by normalizing

$$1 = \int_0^{2\pi}\psi^*(\varphi)\psi(\varphi)d\phi = C^2\int_0^{2\pi}\exp\left(-i\frac{A}{\hbar}\varphi\right)\exp\left(i\frac{A}{\hbar}\varphi\right)\exp\left(-i\frac{a\cos\varphi}{\hbar}\right)\exp\left(i\frac{a\cos\varphi}{\hbar}\right)d\varphi$$
$$= C^2\int_0^{2\pi}d\varphi = C^2 2\pi$$
$$\Downarrow$$
$$C = \frac{1}{\sqrt{2\pi}}$$

Therefore our normalized eigenfunction is

$$\psi(\varphi) = \frac{1}{\sqrt{2\pi}}\exp\left(i\frac{A}{\hbar}\varphi\right)\exp\left(i\frac{a\cos\varphi}{\hbar}\right)$$

where we apply the periodic boundary condition that $\psi(\varphi + 2\pi) = \psi(\varphi)$ we note that the second term satisfies this requirement and the first term is periodic for

$$A = n\hbar \qquad n = 1, 2, 3, ...$$

so we finally have the eigenfunction

$$\psi(\varphi) = \frac{1}{\sqrt{2\pi}}\exp(in\varphi)\exp\left(i\frac{a\cos\varphi}{\hbar}\right)$$

with the eigenvalue

$$A = n\hbar \qquad n = 1, 2, 3...$$

2 Problem #2

(a) For a general potential $U(x)$ the Hamiltonian operator in three-dimensions, $\hat{H}$, becomes

$$\hat{H} = \frac{\mathbf{p}^2}{2m} + U(x)$$

where in three-dimensions

$$\mathbf{p}^2 = \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2$$

For this Hamiltonian we can calculate the commutator $[\hat{H}, y]$ by

$$\begin{aligned}
\left[\hat{H}, y\right] &= \frac{1}{2m}\left(\cancelto{0}{[\hat{p}_x^2, y]} + [\hat{p}_y^2, y] + \cancelto{0}{[\hat{p}_z^2, y]}\right) + \cancelto{0}{[U(x), y]} \\
&= \frac{1}{2m}[\hat{p}_y^2, y] \\
&= \frac{1}{2m}\left(\hat{p}_y[\hat{p}_y, y] + [\hat{p}_y, y]\hat{p}_y\right) \\
&= \frac{1}{2m}\left(\hat{p}_y(-i\hbar) + (-i\hbar)\hat{p}_y\right) \\
&= \frac{-i\hbar}{m}\hat{p}_y
\end{aligned}$$

(b) For the same Hamiltonian we calculate

$$\begin{aligned}
[\hat{H}, \hat{p}_x] &= \frac{1}{2m}\left(\cancelto{0}{[\hat{p}_x^2, \hat{p}_x]} + \cancelto{0}{[\hat{p}_y^2, \hat{p}_x]} + \cancelto{0}{[\hat{p}_z^2, \hat{p}_x]}\right) + [U(x), \hat{p}_x] \\
&= [U(x), \hat{p}_x]
\end{aligned}$$

We can calculate this commutation by multiplying by a function f

$$\begin{aligned}
[U(x), \hat{p}_x]f(x) &= U(x)\hat{p}_x f(x) - \hat{p}_x(U(x)f(x)) \\
&= U(x)(-i\hbar)\frac{\partial}{\partial d}f(x) - (-i\hbar)\frac{\partial}{\partial x}(U(x)f(x)) \\
&= -i\hbar U(x)\frac{\partial f(x)}{\partial d} + i\hbar\frac{\partial f(x)}{\partial x}U(x) + i\hbar\frac{\partial U(x)}{\partial x}f(x) \\
&\Downarrow \\
[U(x), \hat{p}_x]\cancel{f(x)} &= i\hbar\frac{\partial U(x)}{\partial x}\cancel{f(x)} \\
[U(x), \hat{p}_x] &= i\hbar\frac{\partial U(x)}{\partial x}
\end{aligned}$$

So we have

$$[\hat{H}, \hat{p}_x] = i\hbar\frac{\partial U(x)}{\partial x}$$

(c) Next we calculate how $\hat{p}_x^2$ commutes with the Hamiltonian

$$\begin{aligned}
[\hat{H}, \hat{p}_x^2] &= \cancel{\frac{1}{2m}([\hat{p}_x^2, \hat{p}_x^2] + [\hat{p}_y^2, \hat{p}_x^2] + [\hat{p}_z^2, \hat{p}_x^2])} + [U(x), \hat{p}_x^2] \\
&= [U(x), \hat{p}_x^2] = [U(x), \hat{p}_x]\hat{p}_x + \hat{p}_x[U(x), \hat{p}_x] \\
&= i\hbar\frac{\partial U(x)}{\partial x}\hat{p}_x + \hat{p}_x i\hbar\frac{\partial U(x)}{\partial x} \\
&= i\hbar\frac{\partial U(x)}{\partial x}\hat{p}_x + i\hbar(-i\hbar)\frac{\partial}{\partial x}\frac{\partial U(x)}{\partial x} \\
&= i\hbar\frac{\partial U(x)}{\partial x}\hat{p}_x + \hbar^2\frac{\partial^2 U(x)}{\partial x^2}
\end{aligned}$$

3 Problem #3

In order to express the translation operator $\hat{T}_{\mathbf{a}}$ which is defined as a spacial translation $\mathbf{a}$ given by

$$\hat{T}_{\mathbf{a}}\psi(\mathbf{r}) = \psi(\mathbf{r}+\mathbf{a})$$

in terms of the momentum operator $\hat{\mathbf{p}}$ by noting that the expansion

$$\psi(\mathbf{r}+\mathbf{a}) = \sum_{n=0}^{\infty} \frac{1}{n!} \nabla^n(\psi(\mathbf{r})) \cdot \mathbf{a}^n$$

We can place this into terms of $\mathbf{P}$ by noting that

$$\mathbf{p} = -i\hbar\nabla$$

which implies that

$$\begin{aligned}
\psi(\mathbf{r}+\mathbf{a}) &= \sum_{n=0}^{\infty} \frac{1}{n!} \nabla^n(\psi(\mathbf{r})) \cdot \mathbf{a}^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{a}^n \cdot \left(\frac{i}{\hbar}\mathbf{p}\right)^n \psi(\mathbf{r}) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{\hbar}\mathbf{a}\cdot\mathbf{p}\right)^n \psi(\mathbf{r})
\end{aligned}$$

We note that we have a Taylor expansion of the exponential

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{\hbar}\mathbf{a}\cdot\mathbf{p}\right)^n = \exp\left(i\frac{\mathbf{a}\cdot\mathbf{p}}{\hbar}\right)$$

So we see that we have found that

$$\psi(\mathbf{r}+\mathbf{a}) = \exp\left(i\frac{\mathbf{a}\cdot\mathbf{p}}{\hbar}\right)\psi(\mathbf{r})$$

So the translation operator in terms of the momentum operator is given by

$$\hat{T}_{\mathbf{a}} = \exp\left(i\frac{\mathbf{a}\cdot\mathbf{p}}{\hbar}\right)$$

4 Problem #4

For a free particle in one-dimension in the presence of an impenetrable wall (or barrier) which is given by the potential

$$U(x) = \begin{cases} 0 & x > 0 \\ \infty & x < 0 \end{cases}$$

We note that in the region $x < 0$ there is no wave-function because we have an infinite potential therefore we have $\psi(x) = 0$ for $x \leq 0$. And away from the barrier we have a free particle given by

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

We note that in general the term kx is found by the integral

$$k(x) = -\int_{x_0}^{x} \frac{1}{\hbar}\sqrt{2\mu(E - U(x'))}dx'$$

but for $U(x) = 0$ we have a constant integral where

$$k = \frac{\sqrt{2\mu E}}{\hbar}$$

end Next we apply the continuity condition where $\psi(0) = 0$. We note that due to the infinte change in potential we cannot have a continuous derivative. So we see that

$$\begin{gathered} \psi(0) = 0 = Ae^{ik0} + Be^{-ik0} \\ \psi(0) = 0 = A + B \\ \Downarrow \\ A = -B \end{gathered}$$

So our solution becomes

$$\psi(x) = A\left(\exp\left(\frac{\sqrt{2\mu E}}{\hbar}\right) - \exp\left(-\frac{\sqrt{2\mu E}}{\hbar}\right)\right)$$

Now we can normalize our $\psi(x)$ to a delta function in E. By calculating

$$\begin{aligned} \delta(E - E') &= A^2 \int_0^E \left(\exp\left(-\frac{\sqrt{2\mu E'}}{\hbar}\right) - \exp\left(\frac{\sqrt{2\mu E'}}{\hbar}\right)\right)\left(\exp\left(\frac{\sqrt{2\mu E'}}{\hbar}\right) - \exp\left(-\frac{\sqrt{2\mu E'}}{\hbar}\right)\right) dE' \\ &= A^2 \int_0^E 2\exp\left(\frac{\sqrt{2\mu E'}}{\hbar}\right)\exp\left(-\frac{\sqrt{2\mu E'}}{\hbar}\right) dE \\ &= A^2 \int_0^E 2dE \\ &= 2A^2E \\ &\Downarrow \\ A &= \frac{1}{2E\delta(E - E')} \end{aligned}$$

So we have

$$\psi(x) = \frac{1}{2E\delta(E)}\left(\exp\left(\frac{\sqrt{2\mu E}}{\hbar}\right) - \exp\left(-\frac{\sqrt{2\mu E}}{\hbar}\right)\right)$$