Physics 615

Methods of Theoretical Physics I

Professor Katrin Becker

Homework #4

Joe Becker
UID: 125-00-4128
September 30th, 2015

1 Problem #1

For the integral

$$\int_0^{(1+i)} \bar{z} dz$$

for $z \in \mathbb{C}$ we can test if the solution to this integral is path dependent by integrating over two different paths. For the first path we integrate over C_1 and C_2 where C_1 is along the real axes for $y = 0, x$ and C_2 is along the line where $y, x = 1$. So,

$$\begin{aligned}
\int_0^{(1+i)} \bar{z} dz &= \int_{C_1} \bar{z} dz + \int_{C_2} \bar{z} dz \\
&= \int_0^1 dx(x - i0) + \int_0^1 idy(1 - iy) \\
&= \left.\frac{x^2}{2}\right|_0^1 + \int_0^1 dy(i + y) \\
&= \frac{1}{2} - 0 + iy + \left.\frac{y^2}{2}\right|_0^1 \\
&= \frac{1}{2} - 0 + (i + \frac{1}{2} - 0) \\
&= 1 + i
\end{aligned}$$

Now we integrate over C_3 and C_4 where C_3 is the path along the imaginary axes for $y, x = 0$ and C_4 is the path where $y = 1, x$. So we calculate

$$\begin{aligned}
\int_0^{(1+i)} \bar{z} dz &= \int_{C_3} \bar{z} dz + \int_{C_4} \bar{z} dz \\
&= \int_0^1 idy(0 - iy) + \int_0^1 dx(x - i) \\
&= \int_0^1 ydy + \int_0^1 (x - i)dx \\
&= \left.\frac{y^2}{2}\right|_0^1 + \left.\frac{x^2}{2} - ix\right|_0^1 \\
&= \frac{1}{2} - 0 + \frac{1^2}{2} - i - 0 \\
&= 1 - i
\end{aligned}$$

We see that the integral of $\bar{z}$ is path dependent.

2 Problem #2

For $u(x,y) = xy$ we can test if it is a harmonic function by calculating

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = y\frac{\partial^2}{\partial x^2}x + x\frac{\partial^2}{\partial y^2}y = 0$$

So we confirm that $u(x,y)$ is a harmonic function. To find the harmonic conjugate we construct a complex function $f = u + iv$ where $v(x,y)$ is a harmonic function that makes f analytic. We can find v by the *Cauchy-Riemann* equations

$$\frac{\partial}{\partial x}u = \frac{\partial}{\partial y}v \tag{2.1}$$

$$\frac{\partial}{\partial x}v = -\frac{\partial}{\partial y}u \tag{2.2}$$

Which yields

$$\begin{gathered}\frac{\partial}{\partial y}v = \frac{\partial}{\partial x}(xy)\\ \frac{\partial}{\partial y}v = y\\ \Downarrow\\ v(x,y) = \frac{y^2}{2} + C(x)\end{gathered}$$

We find $C(x)$ by the other equation

$$\begin{gathered}\frac{\partial}{\partial x}\left(\frac{y^2}{2} + C(x)\right) = -\frac{\partial}{\partial y}(xy)\\ C'(x) = -x\\ \Downarrow\\ C(x) = -\frac{x^2}{2}\end{gathered}$$

So the harmonic conjugate of $u(x,y)$ is

$$v(x,y) = \frac{y^2}{2} - \frac{x^2}{2}$$

Note that we test

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2}{\partial x^2}\frac{-x^2}{2} + \frac{\partial^2}{\partial y^2}\frac{y^2}{2} = -1 + 1 = 0$$

so we confirm that $v(x,y)$ is harmonic. We can repeat this process for $u(x,y) = \cosh x \sin y$. So we apply equations 2.1 and 2.2 to find $v(x,y)$

$$\begin{gathered}\frac{\partial v}{\partial y} = \sin y\frac{\partial}{\partial x}\cosh x\\ \frac{\partial v}{\partial y} = \sin y \sinh x\\ \Downarrow\\ v(x,y) = -\cos y \sinh x + C(x)\end{gathered}$$

and

$$\begin{aligned}
\frac{\partial v}{\partial x} &= -\cosh x \frac{\partial}{\partial y} \sin y \\
\frac{\partial v}{\partial x} &= -\cosh x \cos y \\
-\cos y \cosh x + C'(x) &= -\cosh x \cos y
\end{aligned}$$

So $C'(x) = 0$ which implies that $C(x)$ is a constant. So we can write

$$v(x, y) = -\cos y \sinh x$$

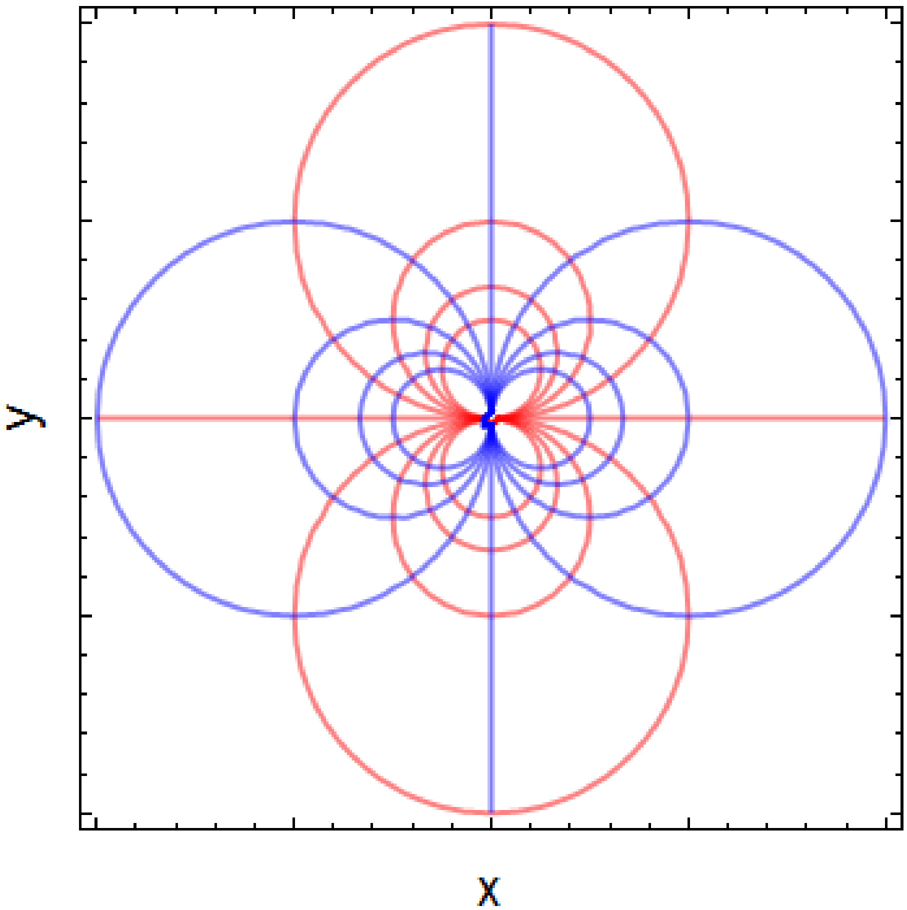

Figure 1: Plot of level curves of $u(x,y)$ (in blue) and $v(x,y)$ (in red) for $f = 1/z$.

3 Problem #3

(a) For the function $f = 1/z$ where $z \in \mathbb{C}$. We want f to be in the form of $f = u + iv$. We rationalize the denominator to do this

$$f = \frac{1}{z} = \frac{1}{x+iy}\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$$

So we can see that

$$u(x,y) = \frac{x}{x^2+y^2}$$
$$v(x,y) = \frac{-y}{x^2+y^2}$$

We can plot the level curves given by these functions. The result is shown in Figure 1. Note that the curves of $u(x,y)$ are orthogonal to the curves of $v(x,y)$ which implies that f is analytic.

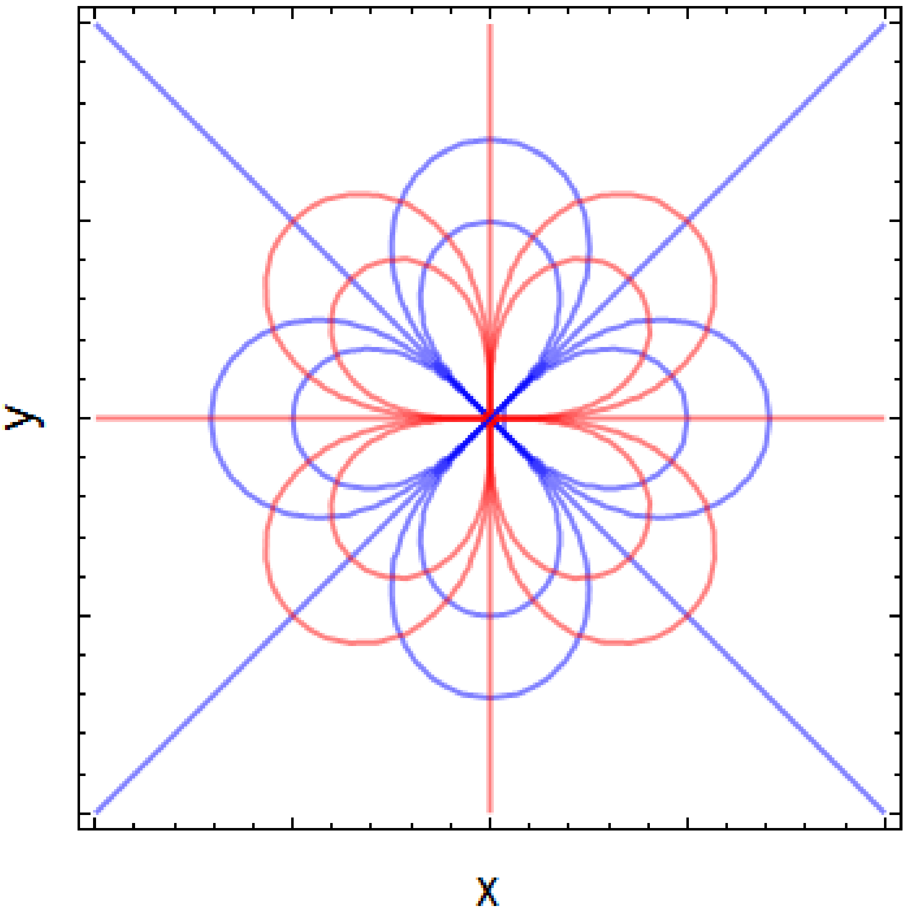

Figure 2: Plot of level curves of $u'(x,y)$ (in blue) and $v'(x,y)$ (in red) for $f = 1/z^2$.

(b) For $f = 1/z^2$ where $z \in \mathbb{C}$ we want to rearrange into the form $f = u + iv$ by

$$\begin{aligned}
\frac{1}{z^2} &= \frac{1}{(x+iy)^2} \\
&= \frac{1}{x^2 - y^2 + 2ixy} \frac{x^2 - y^2 - 2ixy}{x^2 - y^2 - 2ixy} \\
&= \frac{x^2 - y^2 - 2ixy}{x^4 - x^2y^2 + 2ix^3y - x^2y^2 + y^4 - 2ixy^3 - 2ix^3y + 2ixy^3 + 4x^2y^2} \\
&= \frac{x^2 - y^2 - 2ixy}{x^4 - y^4 + 2x^2y^2} \\
&= \frac{x^2 - y^2}{x^4 - y^4 + 2x^2y^2} + i\frac{-2xy}{x^4 - y^4 + 2x^2y^2}
\end{aligned}$$

So we have the functions u and v

$$\begin{aligned}
u'(x,y) &= \frac{x^2 - y^2}{x^4 - y^4 + 2x^2y^2} \\
v'(x,y) &= \frac{-2xy}{x^4 - y^4 + 2x^2y^2}
\end{aligned}$$

We plot the functions u' and v' in figure 2. Again we see that the level curves are orthogonal therefore we infer that $1/z^2$ is analytic.

4 Problem #4

To confirm the given identity

$$\left(\frac{ia-1}{ia+1}\right)^{ib} = \exp[-2b\text{arccot}(a)] \tag{4.1}$$

where $a, b \in \mathbf{C}$, we want to write the fraction inside the parentheses in radial form . Where for $ia-1$ we have

$$r = \sqrt{a^2+1}, \qquad \theta = \text{arccot}(-a)$$

which means that we can write

$$ia - 1 = \sqrt{a^2+1}\exp(i\ \text{arccot}(-a))$$

and for $ia+1$ we have

$$r = \sqrt{a^2+1}, \qquad \theta = \text{arccot}(a)$$

which implies that

$$ia + 1 = \sqrt{a^2+1}\exp(i\text{arccot}(a))$$

So for the identity in equation 4.1 we have

$$\begin{aligned}
\left(\frac{ia-1}{ia+1}\right)^{ib} &= \left(\frac{\sqrt{a^2+1}\exp(i\text{arccot}(a))}{\sqrt{a^2+1}\exp(i\text{arccot}(-a))}\right)^{ib} \\
&= \left(\frac{\exp(i\text{arccot}(a))}{\exp(i\text{arccot}(-a))}\right)^{ib} \\
&= (\exp(i\text{arccot}(a))\exp(-i\ \text{arccot}(-a)))^{ib} \\
&= (\exp(i(\text{arccot}(a) - \ \text{arccot}(-a))))^{ib} \\
&= (\exp(i(\text{arccot}(a) + \ \text{arccot}(a))))^{ib} \\
&= (\exp(2i\text{arccot}(a)))^{ib} \\
&= \exp\Big[(ib)2i\text{arccot}(a)\Big] \\
&= \exp\Big[-2b\text{arccot}(a)\Big]
\end{aligned}$$

So we confirmed the identity.