Physics 624

Quantum Mechanics II

Professor Aleksei Zheltikov

Homework #4

Joe Becker
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1 Problem #1

(a) For a Dirac particle we can find the commutator $[\gamma_5, H]$ where

$$\gamma_5 = -\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

where I is the 2×2 identity matrix and the Hamiltonian is

$$H = c\boldsymbol{\alpha}\cdot\mathbf{p} + mc^2\beta$$

where β is the matrix

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

and each component of $\boldsymbol{\alpha}$ is a matrix given as

$$\alpha^k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}$$

So we note that we can represent the Hamiltonian as the matrix

$$H = c\begin{pmatrix} mcI & \sigma_k p_k \\ \sigma_k p_k & -mcI \end{pmatrix}$$

note we are summing over the index k over the three dimensions. So

$$\begin{aligned}
[\gamma_5, H] &= -c\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} mcI & \sigma_k p_k \\ \sigma_k p_k & -mcI \end{pmatrix} - c\begin{pmatrix} mcI & \sigma_k p_k \\ \sigma_k p_k & -mcI \end{pmatrix}\begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \\
&= -c\begin{pmatrix} \sigma_k p_k & -mcI \\ mcI & \sigma_k p_k \end{pmatrix} + c\begin{pmatrix} \sigma_k p_k & mcI \\ -mcI & \sigma_k p_k \end{pmatrix} \\
&= 2mc^2\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}
\end{aligned}$$

(b) Next we calculate the commutator $[\mathbf{l}^2, H]$ where $\mathbf{l} = [\mathbf{r}\times\mathbf{p}]$. So we calculate

$$\begin{aligned}
[\mathbf{l}^2, H] &= [\mathbf{l}^2, c\boldsymbol{\alpha}\cdot\mathbf{p}] + \cancelto{0}{[\mathbf{l}^2, mc^2\beta]} \\
&= [l_i l_i, c\alpha_k p_k] \\
&= c\alpha_k(l_i[l_i, p_k] + [l_i, p_k]l_i) \\
&= c\alpha_k(l_i i\hbar\epsilon_{ikl}p_l + i\hbar\epsilon_{ikl}p_l l_i) \\
&= i\hbar c\epsilon_{ikl}\alpha_k(l_i p_l + p_l l_i)
\end{aligned}$$

(c) And for the inversion operator $J\psi(\mathbf{r}) = \psi(-\mathbf{r})$ we calculate the commutation relation

$$\begin{aligned}
[J, H] &= [J, c\boldsymbol{\alpha}\cdot\mathbf{p}] + \cancelto{0}{[J, mc^2\beta]} \\
&= c\boldsymbol{\alpha}\cdot(J\mathbf{p}) - c\boldsymbol{\alpha}\cdot(\mathbf{p}J) \\
&= c\boldsymbol{\alpha}\cdot(-\mathbf{p}J) - c\boldsymbol{\alpha}\cdot(\mathbf{p}J) \\
&= -2c(\boldsymbol{\alpha}\cdot\mathbf{p})J
\end{aligned}$$

2 Problem #2

(a) We can find the non-relativistic limit for the expression for the charge density

$$\rho = e\psi^*\psi$$

by taking the lower spinor, χ, of the wave function

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

as

$$\chi = \frac{1}{2mc}\boldsymbol{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)\varphi$$

Note this follows from the solution to the Hamiltonian for a Dirac particle

$$i\hbar\frac{\partial\psi}{\partial t} = (c\boldsymbol{\alpha} \cdot \mathbf{p} + mc^2\beta)\psi$$

which yields the system of equations

$$(mc^2 - \varepsilon)\varphi + c\boldsymbol{\sigma} \cdot \mathbf{p}\chi = 0$$
$$c\boldsymbol{\sigma} \cdot \mathbf{p}\varphi - (mc^2 + \varepsilon)\chi = 0$$

Which if we solve for χ and taking the non-relativistic limit $\varepsilon << mc^2$ we find

$$\chi = \frac{c\boldsymbol{\sigma} \cdot \mathbf{p}}{2mc^2 - \varepsilon}\varphi \approx \frac{1}{2mc}\boldsymbol{\sigma} \cdot \left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)\varphi$$

note we take the canonical momentum $\mathbf{p} = \mathbf{p} - e/c\mathbf{A}$. So we calculate ρ as

$$\begin{aligned}\rho = e\psi^*\psi &= e\begin{pmatrix} \varphi^*, & \frac{1}{2mc}\left(\boldsymbol{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)\varphi\right)^* \end{pmatrix}\begin{pmatrix} \varphi \\ \frac{1}{2mc}\boldsymbol{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)\varphi \end{pmatrix} \\ &= e\varphi^*\varphi + \mathcal{O}\left(\frac{1}{c^2}\right)\end{aligned}$$

(b) Using the same wave function we found in part (a) we can calculate the current density

$$\begin{aligned}
\mathbf{j} &= ec\psi^*\boldsymbol{\alpha}\psi \\
&= ec\psi^*\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}\begin{pmatrix} \varphi \\ \frac{1}{2mc}\boldsymbol{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)\varphi \end{pmatrix} \\
&= ec\begin{pmatrix} \varphi^*, & \frac{1}{2mc}\left(\boldsymbol{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)\varphi\right)^* \end{pmatrix}\begin{pmatrix} \frac{\boldsymbol{\sigma}}{2mc}\boldsymbol{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)\varphi \\ \boldsymbol{\sigma}\varphi \end{pmatrix} \\
&= \frac{ec}{2mc}\left[\varphi^*\boldsymbol{\sigma}\left(\boldsymbol{\sigma} \cdot \left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)\right)\varphi - \left(\left(\hat{\mathbf{p}} + \frac{e}{c}\mathbf{A}\right) \cdot \varphi^*\boldsymbol{\sigma}\right)\boldsymbol{\sigma}\varphi\right] \\
&\Downarrow \\
j_i &= \frac{e}{2m}\left[\varphi^*\sigma_i\left(\sigma_k\left(\hat{p}_k - \frac{e}{c}A_k\right)\right)\varphi - \left(\left(\hat{p}_k + \frac{e}{c}A_k\right)\varphi^*\sigma_k\right)\sigma_i\varphi\right] \\
&= \frac{e}{2m}\left[\varphi^*(\delta_{ik} + i\epsilon_{ikl}\sigma_l)\left(\hat{p}_k - \frac{e}{c}A_k\right)\varphi - \left(\hat{p}_k + \frac{e}{c}A_k\right)\varphi^*(\delta_{ik} + i\epsilon_{ikl}\sigma_l)\varphi\right] \\
&= \frac{e}{2m}\left[\varphi^*\hat{p}_i\varphi - \hat{p}_i\varphi^*\varphi - \frac{2e}{c}A_i\varphi^*\varphi + i\epsilon_{ikl}\left(\varphi^*\sigma_l\hat{p}_k\varphi + \hat{p}_k\varphi^*\sigma_l\varphi\right)\right] \\
&\Downarrow \\
\mathbf{j} &= -\frac{i\hbar e}{2m}\left[\varphi^*\boldsymbol{\nabla}\varphi - (\boldsymbol{\nabla}\varphi^*)\varphi\right] - \frac{e^2}{mc}\mathbf{A}\varphi^*\varphi + \frac{e\hbar}{2m}\boldsymbol{\nabla} \times (\varphi^*\boldsymbol{\sigma}\varphi)
\end{aligned}$$

3 Problem #3

Using the Born approximation and the stationary Klein-Gordon equation

$$\left\{-\frac{\hbar^2}{2m}\boldsymbol{\nabla} + U(r)\right\}\psi = \frac{p_0^2}{2m}\psi$$

where $c^2p_0^2 = E^2 - m^2c^4$ we can find the dependence of the scattering cross section, $\sigma(E)$, on the energy, E, for a spinless relativistic particle in an external scalar field $U(r)$ in the ultra-relativistic limiting case ($E \to \infty$). In order to use the Born approximation we need to write the stationary Klein-Gordon equation in the form of the Schrödinger equation as

$$\left\{-\frac{\hbar^2}{2m}\boldsymbol{\nabla} + U(r) - \frac{U(r)^2}{2mc^2} + \frac{EU(r)}{mc^2} - \frac{E^2}{2mc^2}\right\}\psi = E\psi$$

Neglecting the constant term this yields an effective potential of the form

$$U_{eff}(r) = \frac{(E+mc^2)U(r)}{mc^2} - \frac{U(r)^2}{2mc^2}$$

using this potential we can find the scattering amplitude by the Born approximation

$$\begin{aligned} f_B &= -\frac{m}{2\pi\hbar^2}\int U_{eff}e^{-i\mathbf{q}\cdot\mathbf{r}}d^3r \\ B &= -\frac{1}{2\pi\hbar^2c^2}\int\left((E+mc^2)U(r) - \frac{1}{2}U(r)^2\right)e^{-i\mathbf{q}\cdot\mathbf{r}}d^3r \end{aligned}$$

In the ultra-relativistic case we note that the first term is dominant and we take $E \to cp_0$. This results in the integral becoming a Fourier transformation of $U(r)$ by

$$\begin{aligned} f_B &= -\frac{p_0}{2\pi\hbar^2c}\int U(r)e^{-i\mathbf{q}\cdot\mathbf{r}}d^3r \\ &= -\frac{p_0}{2\pi\hbar^2c}U(q) \end{aligned}$$

Using this result we can take the integral over $d\Omega$ where

$$d\Omega = \frac{\pi\hbar^2}{p_0^2}dq^2$$

which yields

$$\begin{aligned} \sigma &= \int |f_B|^2 d\Omega \\ &= \frac{p_0^2}{4\pi^2\hbar^4c^2}\frac{\pi\hbar^2}{p_0^2}\int |U(q)|^2dq^2 \\ &= \frac{1}{4\pi\hbar^2c^2}\int_0^\infty |U(q)|^2dq^2 \end{aligned}$$

Note that as $E \to \infty$ we take $p \to \infty$ which in turn implies that $q \to \infty$. Therefore the integral in σ goes to infinity. Note that if $U(r)$ goes to zero as $r \to \infty$ we have σ as a constant.