Physics 601

Analytical Mechanics

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Homework #9

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1 Problem #1

For a one-dimensional harmonic oscillator of mass, m, and spring constant k we can construct a Hamiltonian by finding the Lagrangian as

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2$$

where $\omega = \sqrt{k/m}$. We find the canonical momentum of the system as

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}$$

which allows us to construct the Hamiltonian by

$$\begin{aligned} H = p\dot{q} - L &= \frac{p^2}{m} - \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \\ &= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \\ &= \frac{1}{2}m\left(\left(\frac{p}{m}\right)^2 + (\omega q)^2\right) \end{aligned}$$

For the *Canonical Transformation* of a one-dimensional harmonic oscillator given by

$$Q = C(p + im\omega q), \qquad P = C(p - im\omega q)$$

We can find the constant C by using the fact that the determinant of the Jacobian is unity. This implies that

$$\begin{aligned} 1 &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \\ &= (Cim\omega)C - C(-Cim\omega) \\ &2C^2 im\omega \\ &\Downarrow \\ C &= \frac{1}{\sqrt{2im\omega}} \end{aligned}$$

So our transformations become

$$Q = \frac{1}{\sqrt{2im\omega}}(p + im\omega q), \qquad P = \frac{1}{\sqrt{2im\omega}}(p - im\omega q)$$

Now we can find the generating function $F_2(q, P)$ for this transformation by noting that

$$\frac{\partial F_2}{\partial q} = p, \qquad \frac{\partial F_2}{\partial P} = Q$$

Where we can solve the transformations such that

$$p(q, P) = \sqrt{2im\omega}P + im\omega q$$

and

$$Q(q, P) = \frac{1}{\sqrt{2im\omega}}(p(q, P) + im\omega q) = P + \sqrt{2im\omega}q$$

Which allows us to find

$$\frac{\partial F_2}{\partial P} = Q = P + \sqrt{2im\omega}q$$
$$\Downarrow$$
$$F_2(q,P) = \frac{1}{2}P^2 + \sqrt{2im\omega}qP + f(q)$$

Now we can find the function of q by

$$\frac{\partial F_2}{\partial q} = P = \sqrt{2im\omega}P + f'(q)$$
$$\Downarrow$$
$$\sqrt{2im\omega}P + im\omega q = \sqrt{2im\omega}P + f'(q)$$
$$\Downarrow$$
$$f(q) = \frac{1}{2}im\omega q^2$$

So our generator function is

$$F_2(q,P) = \frac{1}{2}P^2 + \sqrt{2im\omega}qP + \frac{1}{2}im\omega q^2$$

which allows us see that

$$\begin{aligned} QP &= \frac{1}{2im\omega}(p + im\omega q)(p - im\omega q) \\ &= -\frac{i}{2m\omega}(p^2 + (m\omega q)^2) \\ &= -\frac{i}{2}m\left(\frac{1}{\omega}\left(\frac{p}{m}\right)^2 + \omega q^2\right) \end{aligned}$$

Which implies that the Hamiltonian is

$$H = i\omega QP$$

so the equations of motion are given by

$$\dot{P} = -\frac{\partial H}{\partial Q} = -i\omega P, \qquad \dot{Q} = \frac{\partial H}{\partial P} = i\omega Q$$

Which easily allows us to solve the equations of motion as

$$P(t) = Ae^{-i\omega t}, \qquad Q(t) = Be^{i\omega t}$$

2 Problem #2

We note that the generator $F(q,Q) = F_1(q,Q)$ follows from the Legendre transform that states

$$\delta \int_{t_1}^{t_2} (p\dot{q} - H) - (P\dot{Q} - K)dt = 0$$

which implies that the difference in the integrand can only differ by a total time derivative

$$(p\dot{q} - H) - (P\dot{Q} - K) = \frac{dF}{dt}$$

which implies that

$$dF = (K - H)dt + pdq - PdQ$$

We note that F is a function of q and Q. We can use this to find $F_3(p,Q)$ by noting that

$$d(pq) = pdq + qdp$$

which allows us to write

$$\begin{aligned} dF &= (K - H)dt + pdq - PdQ \\ &\Downarrow \\ dF &= (K - H)dt + d(pq) - qdp - PdQ \\ &\Downarrow \\ d(F - pq) &= (K - H)dt - qdp - PdQ \end{aligned}$$

So $F_3(p,Q) = F - pq$. Next we repeat the process to find $F_4(p,P)$ by noting that

$$d(PQ) = PdQ + QdP$$

which implies

$$\begin{aligned} dF &= (K - H)dt + pdq - PdQ \\ &\Downarrow \\ dF &= (K - H)dt + pdq - (d(PQ) - QdP) \\ &\Downarrow \\ d(F + PQ) &= (K - H)dt + pdq + QdP \end{aligned}$$

which implies that $F_4(p,P) = F + PQ$.

3 Problem #3

(a) For the infinitesimal transformation with the generator given by

$$F_2(q,P) = qP + \epsilon H(q,P)$$

where H is the Hamiltonian given by

$$H(q,p) = \frac{p^2}{2m} + V(q)$$

we can find the canonical transformations associated with $F_2(q,P)$ by

$$\begin{aligned} Q = \frac{\partial F_2}{\partial P} &= q + \epsilon \frac{\partial}{\partial P}\left(\frac{P^2}{2m} + V(q)\right) \\ &= q + \epsilon \frac{P}{m} \end{aligned}$$

Then we can find p and invert by

$$\begin{gathered} p = \frac{\partial F_2}{\partial q} = P + \epsilon \frac{dV}{dq} \\ \Downarrow \\ P = p - \epsilon \frac{dV}{dq} \end{gathered}$$

So replacing P in the Q term we get the canonical transformations

$$\begin{aligned} P &= p - \epsilon \frac{dV}{dq} \\ Q &= q + \epsilon \frac{p}{m} - \frac{\epsilon^2}{m}\frac{dV}{dq} \end{aligned}$$

(b) If we take ϵ to be a small time step we can see that for $q = q(t)$ and $p = p(t)$ we have the transformations

$$\begin{aligned} q(t+\epsilon) = Q = q(t) + \epsilon \frac{p(t)}{m} \\ p(t+\epsilon) = P = p(t) - \epsilon \frac{dV}{dq} \end{aligned}$$

If we Taylor expand $q(t+\epsilon)$ about ϵ to first order we see that

$$\begin{aligned} q(t+\epsilon) = q(t) + \dot{q}\epsilon \\ p(t+\epsilon) = p(t) + \dot{p}\epsilon \end{aligned}$$

We note that the velocity $\dot{q}$ is related to momentum p by $\dot{q} = p/m$ which implies that The q expansion is the same as the canonical transformation. We also note that $\dot{p} = F = -dV/dq$ so the expansion of p also is the same to first order.

(c) Using the results from above we can find the transformed Hamiltonian $K(P, Q)$ by noting

$$
\begin{aligned}
H(q,p) &= \frac{p^2}{2m} + V(q) \\
&\Downarrow \\
K(Q,P) &= \frac{1}{2m}\left(P + \epsilon\frac{dV}{dQ}\right)^2 + V\left(Q - \epsilon\frac{P}{m}\right) \\
&= \frac{1}{2m}\left(P^2 + 2P\epsilon\frac{dV}{dQ} + \epsilon^2\left(\frac{dV}{dQ}\right)^2\right) + V(Q) - \epsilon\frac{P}{m}\frac{dV}{dQ} + \frac{1}{2}\epsilon^2\frac{p^2}{m^2}\frac{d^2V}{dQ^2} \\
&= \frac{P^2}{2m} + V(Q) + \epsilon\left(\frac{P}{m}\frac{dV}{dQ} - \frac{P}{m}\frac{dV}{dQ}\right) + \frac{\epsilon^2}{2m}\left(\left(\frac{dV}{dQ}\right)^2 + \frac{p^2}{m}\frac{d^2V}{dQ^2}\right) \\
&= \frac{P^2}{2m} + V(Q) + \frac{\epsilon^2}{2m}\left(\left(\frac{dV}{dQ}\right)^2 + \frac{p^2}{m}\frac{d^2V}{dQ^2}\right)
\end{aligned}
$$

4 Problem #4

(a) For the generator

$$F_3(p, Q) = -pQ + \epsilon H(Q, p)$$

where H is the same Hamiltonian from problem 3 this allows up to find the transformations by

$$q = -\frac{\partial F_3}{\partial p} = Q - \epsilon \frac{p}{m}$$

and

$$P = -\frac{\partial F_3}{\partial Q} = p - \epsilon \frac{dV}{dQ}$$

So solving for Q we have the canonical transformations by noting that

$$\frac{\partial V}{\partial q} = \frac{\partial V}{\partial Q}\frac{\partial V}{\partial q}$$

but as we can see $dQ/dq = 1$ so

$$Q = q + \epsilon \frac{p}{m}$$
$$P = p - \epsilon \frac{dV}{dq}$$

which are the same canonical transformations as problem 3.

(b) For ϵ as a time step we can find the same result that

$$q(t+\epsilon) = Q = q(t) + \epsilon \frac{p(t)}{m}$$
$$p(t+\epsilon) = P = p(t) - \epsilon \frac{dV}{dq}$$

This again is the same as the Taylor expansion about ϵ to first order. Where $\dot{p} = -dV/dq$ and $\dot{q} = p/m$.

(c) We can find the transformed Hamiltonian by noting that

$$p = P + \epsilon \frac{dV}{dQ}$$
$$q = Q - \epsilon \frac{P}{m} - \frac{\epsilon^2}{m}\frac{dV}{dQ}$$

$$\begin{aligned}
H(q,p) &= \frac{p^2}{2m} + V(q) \\
&\Downarrow \\
K(Q,P) &= \frac{1}{2m}\left(P + \epsilon \frac{dV}{dQ}\right)^2 + V\left(Q - \epsilon\frac{P}{m} - \frac{\epsilon^2}{m}\frac{dV}{dQ}\right) \\
&= \frac{1}{2m}\left(P^2 + 2P\epsilon\frac{dV}{dQ} + \epsilon^2\left(\frac{dV}{dQ}\right)^2\right) + V(Q) - \epsilon\frac{P}{m}\frac{dV}{dQ} - \frac{\epsilon^2}{m}\left(\frac{dV}{dQ}\right)^2 + \frac{1}{2}\epsilon^2\frac{P^2}{m^2}\frac{d^2V}{dQ^2} \\
&= \frac{P^2}{2m} + V(Q) + \frac{\epsilon^2}{2m}\left(\frac{p^2}{m}\frac{d^2V}{dQ^2} - \left(\frac{dV}{dQ}\right)^2\right)
\end{aligned}$$

(d) We see that the change from F_2 and F_3 we have to take a second order term in Q because q is generated from the generating function not Q.

5 Problem #5

For the Kepler problem with a three dimensional Hamiltonian given by

$$H = \frac{\mathbf{p}^2}{2m} - \frac{k}{|\mathbf{r}|}.$$

Using this we can show that

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

is a constant of motion by taking the Poisson bracket with H and noting that the l component of $\mathbf{L}$ is

$$\Big(\mathbf{r} \times \mathbf{p}\Big)_l = \epsilon_{lmn} r_m p_n$$

$$\begin{aligned}
\frac{d\mathbf{L}}{dt} = \{\mathbf{L}, H\} &= \frac{\partial \mathbf{L}}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial \mathbf{L}}{\partial p_i}\frac{\partial H}{\partial q_i} \\
&= (\epsilon_{lin} p_n)\frac{p_i}{m} - (\epsilon_{lmi} r_m)\left(-\frac{r_i}{|\mathbf{r}|^3}\right) \\
&= 0
\end{aligned}$$

Note the above equation is equal to zero due to the permutations about the Levi-Civita symbol. Therefore the angular momentum is a conserved quantity as we expected. We can repeat this process for

$$\mathbf{A} = \frac{\mathbf{p} \times \mathbf{L}}{km} - \frac{\mathbf{r}}{|\mathbf{r}|}$$

where we note that the cross product is given by

$$\begin{aligned}
(\mathbf{p} \times \mathbf{L})_m = \epsilon_{mni} p_n L_i &= \epsilon_{mni}\epsilon_{ijk} p_n r_j p_k \\
&= (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}) p_n r_j p_k \\
&= r_m p_k p_k - p_m r_j p_j
\end{aligned}$$

This allows us to calculate the Poisson bracket by

$$\begin{aligned}
\frac{d\mathbf{A}}{dt} = \{\mathbf{A}, H\} &= \frac{\partial \mathbf{A}}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial \mathbf{A}}{\partial p_i}\frac{\partial H}{\partial q_i} \\
&= \left(\frac{p_k^2 - p_m p_i}{km} - \frac{1}{|\mathbf{r}|} + \frac{r_i^2}{|\mathbf{r}|^3}\right)\frac{p_i}{m} - \left(\frac{2r_m p_i - p_i r_j - p_m r_i}{km}\right)\left(-\frac{r_i}{|\mathbf{r}|^3}\right) \\
&= \left(\frac{p_k^2 - p_m p_i}{km} - \frac{1}{|\mathbf{r}|} + \frac{|\mathbf{r}|^2}{|\mathbf{r}|^3}\right)\frac{p_i}{m} - \left(\frac{2r_m p_i - 2p_i r_j}{km}\right)\left(-\frac{r_i}{|\mathbf{r}|^3}\right) \\
&= \frac{p_i p_k^2 - p_m p_i^2}{km^2} \qquad = 0
\end{aligned}$$

So under a central potential the LRL-vector is conserved as we expect.