Physics 611

Electromagnetic Theory II

Professor Christopher Pope

Homework #1

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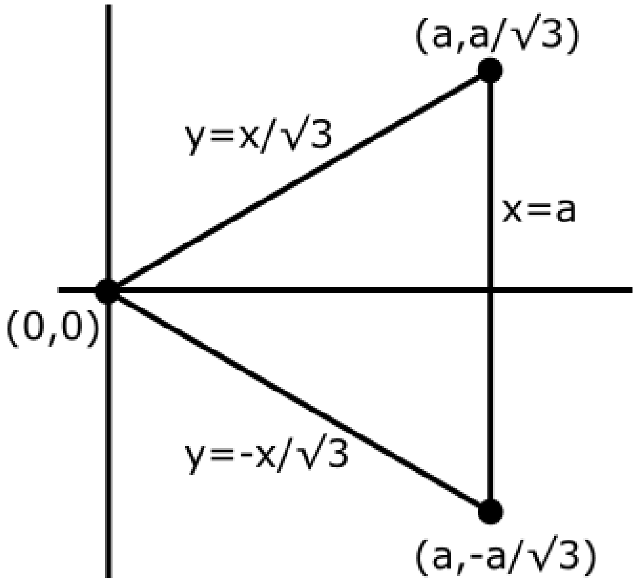

Figure 1: Equilateral triangle waveguide geometry.

1 Problem #1

(a) For a waveguide whose cross-section is an equilateral triangle whose vertices are at

$$(x, y) = \left\{(0,0), (a, a/\sqrt{3}), (a, -a/\sqrt{3})\right\}$$

we can verify that the function

$$\psi_{mn} = \sin\frac{l\pi x}{a}\sin\frac{(m-n)\pi y}{a\sqrt{3}} + \sin\frac{m\pi x}{a}\sin\frac{(n-l)\pi y}{a\sqrt{3}} + \sin\frac{n\pi x}{a}\sin\frac{(l-m)\pi y}{a\sqrt{3}}$$

where $l \equiv -m-n$ satisfy the TM boundary conditions as shown in figure 1 If we recall the result from homework 7 we see that

$$\psi_{nm}(a, y) = \cancelto{0}{\sin l\pi}\sin\frac{(m-n)\pi y}{a\sqrt{3}} + \cancelto{0}{\sin m\pi}\sin\frac{(n-l)\pi y}{a\sqrt{3}} + \cancelto{0}{\sin n\pi}\sin\frac{(l-m)\pi y}{a\sqrt{3}} = 0$$

and

$$\begin{aligned}
\psi_{nm}(x, x/\sqrt{3}) &= \sin\frac{l\pi x}{a}\sin\frac{(m-n)\pi x}{3a} + \sin\frac{m\pi x}{a}\sin\frac{(n-l)\pi x}{3a} + \sin\frac{n\pi x}{a}\sin\frac{(l-m)\pi x}{3a} \\
&= \frac{1}{2}\left[\cos\left(\frac{\pi x}{3a}(4m+2n)\right) - \cos\left(\frac{\pi x}{3a}(4m+2n)\right)\right. \\
&\quad + \cos\left(\frac{\pi x}{3a}(2m-2n)\right) - \cos\left(\frac{\pi x}{3a}(2m-2n)\right) \\
&\quad \left. + \cos\left(\frac{\pi x}{3a}(4n+2m)\right) - \cos\left(\frac{\pi x}{3a}(4n+2m)\right)\right] = 0
\end{aligned}$$

Note that for the final boundary condition we can use the result from above and the fact that sine is a odd function to see

$$\begin{aligned}
\psi_{nm}(x, -x/\sqrt{3}) &= \sin\frac{l\pi x}{a}\sin\frac{-(m-n)\pi x}{3a} + \sin\frac{m\pi x}{a}\sin\frac{-(n-l)\pi x}{3a} + \sin\frac{n\pi x}{a}\sin\frac{-(l-m)\pi x}{3a} \\
&= -\left(\sin\frac{l\pi x}{a}\sin\frac{(m-n)\pi x}{3a} + \sin\frac{m\pi x}{a}\sin\frac{(n-l)\pi x}{3a} + \sin\frac{n\pi x}{a}\sin\frac{(l-m)\pi x}{3a}\right) \\
&= -\psi_{nm}(x, x/\sqrt{3}) = 0
\end{aligned}$$

So we see that all TM boundary conditions are met.

(b) There exists a further set of TM modes, but not the modes described by the bisected equilateral triangular waveguide described in part (a). Note that we already have met the boundary condition for the full equilateral triangle with an additional boundary condition that $\psi_{mn}(x,0)=0$. This implies that we should exchange the sine functions that depend on y to be cosine functions that depend on y, such that

$$\psi_{mn} = \sin\frac{l\pi x}{a}\cos\frac{(m-n)\pi y}{a\sqrt{3}} + \sin\frac{m\pi x}{a}\cos\frac{(n-l)\pi y}{a\sqrt{3}} + \sin\frac{n\pi x}{a}\cos\frac{(l-m)\pi y}{a\sqrt{3}}$$

Note that the boundary condition $\psi_{mn}(a,y)=0$ still holds true as each sine function goes to zero. Next we check that

$$\begin{aligned}
\psi_{mn}(x,x/\sqrt{3}) &= \sin\frac{l\pi x}{a}\cos\frac{(m-n)\pi x}{3a} + \sin\frac{m\pi x}{a}\cos\frac{(n-l)\pi x}{3a} + \sin\frac{n\pi x}{a}\cos\frac{(l-m)\pi x}{3a} \\
&= \frac{1}{2}\Big[\sin\Big(\frac{\pi x}{3a}(3l-m+n)\Big) + \sin\Big(\frac{\pi x}{3a}(3l+m-n)\Big) \\
&\quad + \sin\Big(\frac{\pi x}{3a}(3m-n+l)\Big) + \sin\Big(\frac{\pi x}{3a}(3m+n-l)\Big) \\
&\quad + \sin\Big(\frac{\pi x}{3a}(3n-l+m)\Big) + \sin\Big(\frac{\pi x}{3a}(3n+l-m)\Big)\Big] \\
&= \frac{1}{2}\Big[\sin\Big(\frac{\pi x}{3a}(-4m-2n)\Big) + \sin\Big(\frac{\pi x}{3a}(-2m-4n)\Big) \\
&\quad + \sin\Big(\frac{\pi x}{3a}(2m-2n)\Big) + \sin\Big(\frac{\pi x}{3a}(4m+2n)\Big) \\
&\quad + \sin\Big(\frac{\pi x}{3a}(4n+2m)\Big) + \sin\Big(\frac{\pi x}{3a}(2n-2m)\Big)\Big] \\
&= \frac{1}{2}\Big[\sin\Big(\frac{\pi x}{3a}(4m+2n)\Big) - \sin\Big(\frac{\pi x}{3a}(4m+2n)\Big) \\
&\quad + \sin\Big(\frac{\pi x}{3a}(2m-2n)\Big) - \sin\Big(\frac{\pi x}{3a}(2m-2n)\Big) \\
&\quad + \sin\Big(\frac{\pi x}{3a}(4n+2m)\Big) - \sin\Big(\frac{\pi x}{3a}(4n+2m)\Big)\Big] \\
&= 0
\end{aligned}$$

Now for the final boundary condition we see that

$$\begin{aligned}
\psi_{nm}(x,-x/\sqrt{3}) &= \sin\frac{l\pi x}{a}\cos\frac{-(m-n)\pi x}{3a} + \sin\frac{m\pi x}{a}\cos\frac{-(n-l)\pi x}{3a} + \sin\frac{n\pi x}{a}\cos\frac{-(l-m)\pi x}{3a} \\
&= \left(\sin\frac{l\pi x}{a}\cos\frac{(m-n)\pi x}{3a} + \sin\frac{m\pi x}{a}\cos\frac{(n-l)\pi x}{3a} + \sin\frac{n\pi x}{a}\cos\frac{(l-m)\pi x}{3a}\right) \\
&= \psi_{mn}(x,x/\sqrt{3}) = 0
\end{aligned}$$

Note we can verify that this equation solves the *Helmholtz Equation* by

$$\begin{aligned}
\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} &= -\Omega^2\psi \\
&\Downarrow \\
-\Omega^2\psi = -\frac{\pi^2}{a^2}\bigg(\bigg(l^2 + \frac{(m-n)^2}{3}\bigg)&\sin\frac{l\pi x}{a}\cos\frac{(m-n)\pi y}{a\sqrt{3}} \\
+ \bigg(m^2 + \frac{(n-l)^2}{3}\bigg)&\sin\frac{m\pi x}{a}\cos\frac{(n-l)\pi y}{a\sqrt{3}} \\
+ \bigg(n^2 + \frac{(l-m)^2}{3}\bigg)&\sin\frac{n\pi x}{a}\cos\frac{(l-m)\pi y}{a\sqrt{3}}\bigg) \\
= -\frac{4\pi^2}{3a^2}(m^2+n^2+mn)&\psi_{mn}(x,y)
\end{aligned}$$

Where we found that the eigenvalue is

$$\Omega_{mn}^2 = \frac{4\pi^2}{3a^2}(m^2 + n^2 + mn)$$

(c) To find the lowest frequency that can propagate through the equilateral triangular waveguide we note that we need to find the lowest mode that results in a oscillating field. This is when $m = 1, n = 1$ note that $m = 0, n = 0$ or $m = 1, n = 0$ results in a null field. We see that for these modes we have

$$\Omega = \frac{2\pi}{a} = \omega_{\text{min}}$$

Note that for the bisected triangular waveguide described in part (a) we have the lowest possible mode as $m = 2, n = 1$ or $m = 1, n = 2$ due to the double sines. Note that the $m = 1, n = 1$ mode results in a null field, this implies that

$$\Omega^{\text{bi}} = \sqrt{\frac{28}{3}}\frac{\pi}{a} = \omega_{\text{min}}^{\text{bi}}$$

2 Problem #2

(a) Given the *Lienard-Wiechert potentials*

$$\phi(\mathbf{r},t) = \frac{e}{R - \mathbf{v}\cdot\mathbf{R}} \qquad \mathbf{A}(\mathbf{r},t) = \frac{e\mathbf{v}}{R - \mathbf{v}\cdot\mathbf{R}} \tag{2.1}$$

we can calculate the magnetic field which follows from an accelerating point charge by

$$\begin{aligned}
\mathbf{B} = \nabla\times\mathbf{A} &= \nabla\times\frac{e\mathbf{v}}{R-\mathbf{v}\cdot\mathbf{R}} \\
&= e\epsilon_{ijk}\partial_j\frac{v_k}{R-\mathbf{v}\cdot\mathbf{R}} \\
&= e\epsilon_{ijk}\frac{\partial_j v_k(R-\mathbf{v}\cdot\mathbf{R}) - \partial_j(R-\mathbf{v}\cdot\mathbf{R})v_k}{(R-\mathbf{v}\cdot\mathbf{R})^2}
\end{aligned}$$

Note the following results from the class notes

$$\begin{aligned}
\partial_i R &= \frac{R_i}{R-\mathbf{v}\cdot\mathbf{R}} \\
\partial_i R_j &= \delta_{ij} + \frac{v_j R_i}{R-\mathbf{v}\cdot\mathbf{R}} \\
\partial_i v_j &= -\frac{\dot{v}_j R_i}{R-\mathbf{v}\cdot\mathbf{R}}
\end{aligned}$$

we can calculate

$$\begin{aligned}
\nabla\times\mathbf{A} &= \frac{e\epsilon_{ijk}}{(R-\mathbf{v}\cdot\mathbf{R})^2}\left(-\frac{\dot{v}_k R_j\cancel{(R-\mathbf{v}\cdot\mathbf{R})}}{\cancel{R-\mathbf{v}\cdot\mathbf{R}}} - \frac{R_j v_k}{R-\mathbf{v}\cdot\mathbf{R}} + \partial_j(v_l R_l)v_k\right) \\
&= \frac{e\epsilon_{ijk}}{(R-\mathbf{v}\cdot\mathbf{R})^2}\left(-\dot{v}_k R_j - \frac{R_j v_k}{R-\mathbf{v}\cdot\mathbf{R}} + (\partial_j v_l)R_l v_k + (\partial_j R_l)v_l v_k\right) \\
&= \frac{e\epsilon_{ijk}}{(R-\mathbf{v}\cdot\mathbf{R})^2}\left(-\dot{v}_k R_j - \frac{R_j v_k}{R-\mathbf{v}\cdot\mathbf{R}} - R_l v_k\frac{\dot{v}_l R_j}{R-\mathbf{v}\cdot\mathbf{R}} + v_l v_k\delta_{jl} + v_l v_k\frac{v_l R_j}{R-\mathbf{v}\cdot\mathbf{R}}\right) \\
&= \frac{e}{(R-\mathbf{v}\cdot\mathbf{R})^3}\left(\mathbf{a}\times\mathbf{R}(R-\mathbf{v}\cdot\mathbf{R}) + \mathbf{v}\times\mathbf{R} + (\mathbf{a}\cdot\mathbf{R})(\mathbf{v}\times\mathbf{R}) - v^2(\mathbf{v}\times\mathbf{R})\right) \\
&= \frac{e}{(R-\mathbf{v}\cdot\mathbf{R})^3}\left(\mathbf{a}\times\mathbf{R}(R-\mathbf{v}\cdot\mathbf{R}) + \mathbf{v}\times\mathbf{R}(1-v^2+\mathbf{a}\cdot\mathbf{R})\right)
\end{aligned}$$

(b) Given that the electric field from a accelerating charge is given by

$$\mathbf{E} = \frac{e}{(R-\mathbf{v}\cdot\mathbf{R})^3}\Big((1-v^2)(\mathbf{R}-\mathbf{v}R) + \mathbf{R}\times[(\mathbf{R}-\mathbf{v}R)\times\mathbf{a}]\Big) \tag{2.2}$$

we can calculate the magnetic field by

$$\begin{aligned}
\mathbf{B} &= \frac{\mathbf{R}\times\mathbf{E}}{R} \\
&= \frac{e}{R(R-\mathbf{v}\cdot\mathbf{R})^3}\left((1-v^2)\epsilon_{ijk}R_j(R_k - v_k R) + \epsilon_{ijk}R_j\epsilon_{klm}R_l\epsilon_{mno}(R_n - v_n R)a_o\right) \\
&= \frac{e}{R(R-\mathbf{v}\cdot\mathbf{R})^3}\left(-R(1-v^2)\epsilon_{ijk}R_j v_k + \epsilon_{ijk}R_j(\delta_{kn}\delta_{lo} - \delta_{ko}\delta_{ln})R_l(R_n - v_n R)a_o\right) \\
&= \frac{e}{R(R-\mathbf{v}\cdot\mathbf{R})^3}\left(-R(1-v^2)\epsilon_{ijk}R_j v_k + \epsilon_{ijk}R_j(R_l a_l(R_k - v_k R) - R_l(R_l - v_l R)a_k)\right) \\
&= \frac{e}{R(R-\mathbf{v}\cdot\mathbf{R})^3}\left(R(1-v^2)\mathbf{v}\times\mathbf{R} + R(\mathbf{a}\cdot\mathbf{R})(\mathbf{v}\times\mathbf{R}) + (R^2 - R\mathbf{v}\cdot\mathbf{R})\mathbf{a}\times\mathbf{R}\right) \\
&= \frac{e}{(R-\mathbf{v}\cdot\mathbf{R})^3}\left((R-\mathbf{v}\cdot\mathbf{R})\mathbf{a}\times\mathbf{R} + (1-v^2+\mathbf{a}\cdot\mathbf{R})\mathbf{v}\times\mathbf{R}\right)
\end{aligned}$$

Recovering the result from part (a).