Physics 611

Electromagnetic Theory II

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Homework #2

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1 Problem #1

(a) Given that $\phi = V^\mu U_\mu$ is a Lorentz scalar, where V^μ is an arbitrary 4-vector this allows us to determine if U^μ is a 4-vector. Note that we know V^μ transforms as 4-vector which implies that

$$V'^\mu = \Lambda^\mu{}_\nu V^\nu.$$

We also know that ϕ is a Lorentz scalar which implies that it is invariant under transformation $\phi' = \phi$. If we take an arbitrary transformation, $T_\mu{}^\rho$, of U_μ we have

$$\begin{aligned} \phi' &= V'^\mu U'_\mu \\ &= \Lambda^\mu{}_\nu V^\nu T_\mu{}^\rho U_\rho \\ &= \Lambda^\mu{}_\nu T_\mu{}^\rho V^\nu U_\rho \end{aligned}$$

Therefore for ϕ to remain Lorentz invariant we have the condition

$$\Lambda^\mu{}_\nu T_\mu{}^\rho = \delta^\rho_\nu$$

this implies that $T_\mu{}^\rho = \Lambda_\mu{}^\rho$. Therefore U_μ must be a 4-vector if V^μ is a 4-vector.

(b) For the tensor defined as

$$S_{\mu\nu} \equiv W_{\mu\rho} W_\nu{}^\rho$$

we can see that for any 4-tensor $W_{\mu\rho}$ we can calculate $S_{\nu\mu}$ noting that we can raise and lower the indices of the W 4-tensor by

$$W_{\nu\rho} = \eta_{\sigma\rho} W_\nu{}^\sigma \qquad W_\mu{}^\rho = \eta^{\rho\lambda} W_{\mu\lambda}$$

$$\begin{aligned} S_{\nu\mu} &= W_{\nu\rho} W_\mu{}^\rho \\ &= \eta_{\sigma\rho} W_\nu{}^\sigma \eta^{\rho\lambda} W_{\mu\lambda} \\ &= \eta_{\sigma\rho} \eta^{\rho\lambda} W_\nu{}^\sigma W_{\mu\lambda} \\ &= \delta^\lambda_\sigma W_\nu{}^\sigma W_{\mu\lambda} \\ &= W_\nu{}^\sigma W_{\mu\sigma} \\ &= W_{\mu\rho} W_\nu{}^\rho = S_{\mu\nu} \end{aligned}$$

Note that we changed the dummy index $\sigma \to \rho$. So we see that $S_{\mu\nu} = S_{\nu\mu}$ this implies that $S_{\mu\nu}$ is symmetric for any 4-tensor $W_{\mu\rho}$.

(c) Given that k^μ is a *lightlike* vector, that is $k^\mu k_\mu = 0$, and a non-spacelike 4-vector, V^μ, that is orthogonal to k^μ we write each vector as

$$k^\mu = (k^0, \mathbf{k}) \qquad V^\mu = (V^0, \mathbf{V})$$

Which allows us to write the condition on the components of k^μ

$$\begin{gathered} k^\mu k_\mu = 0 \Rightarrow (k^0)^2 = |\mathbf{k}|^2 \Rightarrow k^0 = |\mathbf{k}| \\ V^\mu V_\mu \leq 0 \Rightarrow |\mathbf{V}|^2 \leq (V^0)^2 \Rightarrow |\mathbf{V}| \leq V^0 \end{gathered}$$

where $\mathbf{k}$ and $\mathbf{V}$ are Euclidean space 3-vectors. Note by orthogonality we see that

$$\begin{gathered} -k^0 V^0 + \mathbf{k} \cdot \mathbf{V} = 0 \\ \Downarrow \\ k^0 V_0 = \mathbf{k} \cdot \mathbf{V} \leq |\mathbf{k}||\mathbf{V}| \\ \Downarrow \\ 0 \leq -k^0 V_0 + |\mathbf{k}||\mathbf{V}| \end{gathered}$$

But if we take the conditions we first take for k^μ and V^μ we have

$$
\begin{aligned}
|\mathbf{V}| &\leq V^0 \\
&\Downarrow \\
|\mathbf{k}||\mathbf{V}| &\leq k^0 V^0 \\
&\Downarrow \\
-k^0 V^0 + |\mathbf{k}||\mathbf{V}| &\leq 0
\end{aligned}
$$

This result implies that the only result that does not contradict the orthogonality condition is if $V^0 = |\mathbf{V}|$. This means that V^μ is also a timelike vector and must be a multiple of k^μ.

2 Problem #2

We can derive the *Lorentz transformation* that gives $\mathbf{B}'$ in terms of $\mathbf{E}$ and $\mathbf{B}$ for an arbitrary Lorentz boost with velocity, $\mathbf{v}$. First, we note that we can write the magnetic field in terms of the *Field Tensor*, $F^{\mu\nu}$, by

$$B_i = -\frac{1}{2}\epsilon_{ijk}F^{jk}.$$

So, we can find B_i' by the transformation

$$\begin{aligned}
B_i' = \frac{1}{2}\epsilon_{ijk}F'^{jk} &= \frac{1}{2}\epsilon_{ijk}\Lambda^j{}_\rho\Lambda^k{}_\sigma F^{\rho\sigma} \\
&= \frac{1}{2}\epsilon_{ijk}\Lambda^j{}_0\Lambda^k{}_l F^{0l} + \frac{1}{2}\epsilon_{ijk}\Lambda^j{}_l\Lambda^k{}_0 F^{l0} + \frac{1}{2}\epsilon_{ijk}\Lambda^j{}_l\Lambda^k{}_m F^{lm} \\
&= \frac{1}{2}\epsilon_{ijk}\Lambda^j{}_0\Lambda^k{}_l F^{0l} - \frac{1}{2}\epsilon_{ijk}\Lambda^j{}_l\Lambda^k{}_0 F^{0l} + \frac{1}{2}\epsilon_{ijk}\Lambda^j{}_l\Lambda^k{}_m F^{lm} \\
&= \frac{1}{2}\epsilon_{ijk}\Lambda^j{}_0\Lambda^k{}_l F^{0l} - \frac{1}{2}\epsilon_{ikj}\Lambda^k{}_l\Lambda^j{}_0 F^{0l} + \frac{1}{2}\epsilon_{ijk}\Lambda^j{}_l\Lambda^k{}_m F^{lm} \\
&= \epsilon_{ijk}\Lambda^j{}_0\Lambda^k{}_l F^{0l} + \frac{1}{2}\epsilon_{ijk}\Lambda^j{}_l\Lambda^k{}_m F^{lm} \\
&= \epsilon_{ijk}(-\gamma v_j)\left(\delta_{kl} + \frac{\gamma-1}{v^2}v_k v_l\right)E_l + \frac{1}{2}\epsilon_{ijk}\left(\delta_{jl} + \frac{\gamma-1}{v^2}v_j v_l\right)\left(\delta_{km} + \frac{\gamma-1}{v^2}v_k v_m\right)\epsilon_{lmn}B_n \\
&= -\gamma\left(\epsilon_{ijk}\delta_{kl}v_j + \frac{\gamma-1}{v^2}\cancelto{0}{\epsilon_{ijk}v_j v_k v_l}\right)E_l + \frac{1}{2}\epsilon_{ijk}\left(\delta_{jl} + \frac{\gamma-1}{v^2}v_j v_l\right)\left(\delta_{km} + \frac{\gamma-1}{v^2}v_k v_m\right)\epsilon_{lmn}B_n \\
&= -\gamma\epsilon_{ijk}v_j E_k + \frac{1}{2}\epsilon_{ijk}\epsilon_{lmn}B_n\left(\delta_{jl}\delta_{km} + \delta_{km}\frac{\gamma-1}{v^2}v_j v_l + \delta_{jl}\frac{\gamma-1}{v^2}v_k v_m + \cancel{\frac{\gamma-1}{v^2}v_j v_k v_l v_m}\right) \\
&= -\gamma\epsilon_{ijk}v_j E_k + \frac{1}{2}\epsilon_{ijk}\epsilon_{njk}B_n + \frac{\gamma-1}{2v^2}\epsilon_{ijk}\epsilon_{lkn}B_n v_j v_l + \frac{\gamma-1}{2v^2}\epsilon_{ijk}\epsilon_{jmn}B_n v_k v_m \\
&= -\gamma\epsilon_{ijk}v_j E_k + \frac{1}{2}2\delta_{in}B_n + \frac{\gamma-1}{2v^2}\epsilon_{ijk}\epsilon_{lkn}B_n v_j v_l + \frac{\gamma-1}{2v^2}\epsilon_{ikj}\epsilon_{kln}B_n v_j v_l \\
&= -\gamma\epsilon_{ijk}v_j E_k + B_i + \frac{\gamma-1}{2v^2}\epsilon_{ijk}\epsilon_{lkn}B_n v_j v_l + \frac{\gamma-1}{2v^2}\epsilon_{ijk}\epsilon_{lkn}B_n v_j v_l \\
&= -\gamma\epsilon_{ijk}v_j E_k + B_i + \frac{\gamma-1}{v^2}\epsilon_{kji}\epsilon_{kln}B_n v_j v_l \\
&= -\gamma\epsilon_{ijk}v_j E_k + B_i + \frac{\gamma-1}{v^2}(\delta_{jl}\delta_{in} - \delta_{jn}\delta_{il})B_n v_j v_l \\
&= -\gamma\epsilon_{ijk}v_j E_k + B_i + \frac{\gamma-1}{v^2}(B_i v_j v_j - v_i B_j v_j) \\
&= -\gamma\epsilon_{ijk}v_j E_k + \gamma B_i - \frac{\gamma-1}{v^2}v_i B_j v_j) \\
&= \gamma(B_i - \epsilon_{ijk}v_j E_k) - \frac{\gamma-1}{v^2}v_i B_j v_j)
\end{aligned}$$

This gives us the transformation result we we can write in vector notation as

$$\mathbf{B}' = \gamma(\mathbf{B} - \mathbf{v}\times\mathbf{E}) - \frac{\gamma-1}{v^2}(\mathbf{v}\cdot\mathbf{B})\mathbf{v}$$

3 Problem #3

(a) Given the scalar quantity, R, defined by

$$R^2 \equiv \eta_{\mu\nu} x^\mu x^\nu$$

we can see if we take the derivative ∂_μ we have

$$\begin{aligned}
\partial_\mu R &= \partial_\mu (\eta_{\mu\nu} x^\mu x^\nu)^{1/2} \\
&= \frac{1}{2} (\eta_{\mu\nu} x^\mu x^\nu)^{-1/2} \eta_{\mu\nu} x^\nu (\partial_\mu x^\mu) \\
&= \frac{1}{2} (\eta_{\mu\nu} x^\mu x^\nu)^{-1/2} \eta_{\mu\nu} x^\nu (-1+3) \\
&= (\eta_{\mu\nu} x^\mu x^\nu)^{-1/2} \eta_{\mu\nu} x^\nu \\
&= \frac{\eta_{\mu\nu} x^\nu}{R}
\end{aligned}$$

(b) Using the result from part (a) we can see that

$$\begin{aligned}
\Box \frac{1}{R^2} &= \partial^\mu \partial_\mu \frac{1}{R^2} \\
&= \eta^{\mu\nu} \partial_\nu \left(\frac{-2}{R^3} \frac{\eta_{\mu\nu} x^\nu}{R} \right) \\
&= -2 \eta^{\mu\nu} \eta_{\mu\nu} \partial_\nu \left(\frac{1}{R^4} x^\nu \right) \\
&= -8 \left(\frac{-4}{R^5} \frac{\eta_{\mu\nu} x^\mu x^\nu}{R} + \frac{1}{R^4} \partial_\nu x^\nu \right) \\
&= -8 \left(\frac{-4}{R^6} R^2 + \frac{4}{R^4} \partial_\nu x^\nu \right) \\
&= -8 \left(\frac{-4}{R^4} + \frac{4}{R^4} \right) = 0
\end{aligned}$$

4 Problem #4

Given the constant 4-vector k_μ such that

$$\phi \equiv e^{ik_\mu x^\mu} \tag{4.1}$$

we can find the condition on k_μ that solves the wave equation

$$\Box\phi = 0$$

where $\Box$ is the *d'Alembertian operator* defined as

$$\Box \equiv \partial_\mu \partial^\mu = -\partial_0^2 + \partial_i^2 \tag{4.2}$$

So if we apply equation 4.2 to equation 4.1 we find that

$$\begin{aligned}
\Box\phi = 0 &= (-\partial_0^2 + \partial_i^2)e^{ik_\mu x^\mu} \\
&= -(ik_0)^2 e^{ik_\mu} + (ik_1)^2 e^{ik_\mu} + (ik_2)^2 e^{ik_\mu} + (ik_3)^2 e^{ik_\mu} \\
&= (k_0^2 - k_1^2 - k_2^2 - k_3^2)e^{ik_\mu} \\
&\Downarrow \\
0 &= -k_0^2 + k_1^2 + k_2^2 + k_3^2 \\
&\Downarrow \\
k_\mu k^\mu &= 0
\end{aligned}$$

Therefore the magnitude of k_μ must be zero (lightlike) in order for equation 4.1 to satisfy the wave equation.