Physics 606

Quantum Mechanics I

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Homework #5

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1 Problem #1

(a) Given the translation operator defined as

$$\hat{T}_a \Psi(x) = \Psi(x+a)$$

which using coordinate representation we can write it as

$$\hat{T}_a \langle x|a\rangle = \langle x|b\rangle$$

We can find the momentum representation of $\hat{T}_a$ by first finding the matrix elements by

$$\langle p'|\hat{T}_a|p\rangle = \int dx \langle p'|x\rangle \hat{T}_a \langle x|p\rangle$$

We note that the eigenfunctions of the momentum operator in coordinate representation is given by

$$\langle x|p\rangle = (2\pi\hbar)^{-1/2} \exp\left(ip\frac{x}{\hbar}\right)$$

Acting $\hat{T}_a$ on $\langle x|p\rangle$ yields

$$\begin{aligned}
\hat{T}_a \langle x|p\rangle &= \hat{T}_a (2\pi\hbar)^{-1/2} \exp\left(ip\frac{x}{\hbar}\right) \\
&= (2\pi\hbar)^{-1/2} \exp\left(ip\frac{x+a}{\hbar}\right) \\
&= \exp\left(ip\frac{a}{\hbar}\right) (2\pi\hbar)^{-1/2} \exp\left(ip\frac{x}{\hbar}\right) \\
&= \exp\left(ip\frac{a}{\hbar}\right) \langle x|p\rangle
\end{aligned}$$

Therefore we can find the matrix elements by

$$\begin{aligned}
\langle p'|\hat{T}_a|p\rangle &= \int dx \langle p'|x\rangle \hat{T}_a \langle x|p\rangle \\
&= \exp\left(ip\frac{a}{\hbar}\right) \int dx \langle p'|x\rangle \langle x|p\rangle \\
&= \exp\left(ip\frac{a}{\hbar}\right) \delta(p'-p)
\end{aligned}$$

Note this forms a matrix of continuous indices. Using the matrix elements we can find how this operator acts on a wave function in momentum space $\langle p|a\rangle$ as

$$\begin{aligned}
\langle p'|b\rangle &= \int dp \langle p'|\hat{T}_a|p\rangle \langle p|a\rangle \\
&= \exp\left(ip\frac{a}{\hbar}\right) \int dp\, \delta(p'-p) \langle p|a\rangle \\
&= \exp\left(ip\frac{a}{\hbar}\right) \langle p'|a\rangle
\end{aligned}$$

So in momentum space

$$\hat{T}_a = \exp\left(ip\frac{a}{\hbar}\right)$$

(b) For the inversion operator, $\hat{I}$, defined as

$$\hat{I}\psi(x) = \psi(-x)$$

which in coordinate representation is

$$\hat{I}\langle x|a\rangle = \langle x|b\rangle$$

We calculate the momentum representation of $\hat{I}$ by finding the matrix elements like in part (a) noting that

$$\begin{aligned}\hat{I}\langle x|p\rangle &= \hat{I}(2\pi\hbar)^{-1/2}\exp\left(ip\frac{x}{\hbar}\right)\\ &= (2\pi\hbar)^{-1/2}\exp\left(ip\frac{-x}{\hbar}\right)\\ &= \langle x|-p\rangle\end{aligned}$$

So this allows us to find the matrix elements of $\hat{I}$ by

$$\begin{aligned}\langle p'|\hat{I}|p\rangle &= \int dx\langle p'|x\rangle\hat{I}\langle x|p\rangle\\ &= \int dx\langle p'|x\rangle\langle x|-p\rangle\\ &= \delta(p'+p)\end{aligned}$$

Note that this is a matrix that represents a momentum in the opposite direction as we would expect. Using this we can act the $\hat{I}$ operator on a wave function in momentum space.

$$\begin{aligned}\langle p'|b\rangle &= \int dp\langle p'|\hat{I}|p\rangle\langle p|a\rangle\\ &= \int dp\delta(p'+p)\langle p|a\rangle\\ &= \langle -p'|a\rangle\end{aligned}$$

So we can say the inversion operator in momentum space acts like

$$\hat{I}\psi(p) = \psi(-p)$$

2 Problem #2

(a) For the translation operator $\hat{T}_a$ we note that the inverse of $\hat{T}_a$ is $\hat{T}_{-a}$ which implies that these represent a unitary transformation by

$$\hat{T}_a \hat{T}_{-a} = 1$$

This allows us to do a unitary transformation by the general transformation of an operator $\hat{F}$

$$\hat{F}' = S\hat{F}S^{-1}$$

So the transformation of the position operator is given by

$$\begin{aligned} \hat{x'} &= \hat{T}_a \hat{x} \hat{T}_{-a} \\ &= (x+a)\hat{T}_{-a} \end{aligned}$$

Then we can find the transformation of the operator $\hat{p}$ by using $\hat{T}_a$ in momentum space which we found in problem one.

$$\begin{aligned} \hat{p}' &= \hat{T}_a \hat{p} \hat{T}_{-a} \\ &= \exp\left(ip\frac{a}{\hbar}\right)\hat{p}\exp\left(ip\frac{-a}{\hbar}\right) \\ &= \hat{p}\exp\left(ip\frac{a}{\hbar}\right)\exp\left(ip\frac{-a}{\hbar}\right) \\ &= \hat{p} \end{aligned}$$

So the operator $\hat{p}$ is invariant under the unitary transformation $\hat{T}_a$.

(b) We note that the inversion operator's inverse operator is itself $\hat{I}^{-1} = \hat{I}$. Which we note $\hat{I}$ represents an unitary transformation. Therefore we can calculate the transformation of $\hat{x}$ as

$$\begin{aligned} \hat{x}' &= \hat{I}\hat{x}\hat{I}^{-1} \\ &= -\hat{x}\hat{I} \end{aligned}$$

And for the momentum operator we have a analogous result

$$\begin{aligned} \hat{p}' &= \hat{I}\hat{p}\hat{I}^{-1} \\ &= -\hat{p}\hat{I} \end{aligned}$$

3 Problem #3

For the wave function in coordinate representation

$$\psi(x) = \langle x|\psi\rangle = \begin{cases} a^{-1/2}\exp\left(\frac{i}{\hbar}xp_0\right) & -a/2 \leq x \leq a/2 \\ 0 & |x| > a/2 \end{cases}$$

This allows us the find the momentum representation, $\langle p|\psi\rangle$, by

$$\begin{aligned}
\langle p|\psi\rangle &= \int dx \langle p|x\rangle\langle x|\psi\rangle \\
&= \int_{-a/2}^{a/2} dx (2\pi\hbar)^{-1/2}\exp\left(-\frac{i}{\hbar}xp\right) a^{-1/2}\exp\left(\frac{i}{\hbar}xp_0\right) \\
&= \sqrt{\frac{1}{2a\pi\hbar}}\int_{-a/2}^{a/2} dx \exp\left(\frac{i}{\hbar}x(p_0-p)\right) \\
&= -\sqrt{\frac{1}{2a\pi\hbar}}\frac{i\hbar}{p_0-p}\left(\exp\left(\frac{i}{\hbar}x(p_0-p)\right)\right|_{-a/2}^{a/2} \\
&= -\sqrt{\frac{1}{2a\pi\hbar}}\frac{i\hbar}{p_0-p}\left(\exp\left(i\frac{a}{2\hbar}(p_0-p)\right) - \exp\left(-i\frac{a}{2\hbar}(p_0-p)\right)\right) \\
&= -\sqrt{\frac{1}{2a\pi\hbar}}\frac{i\hbar}{p_0-p}2i\sin\left(\frac{a}{2\hbar}(p_0-p)\right) \\
&= \sqrt{\frac{1}{2a\pi\hbar}}\frac{2\hbar}{p_0-p}\sin\left(\frac{a}{2\hbar}(p_0-p)\right)
\end{aligned}$$

4 Problem #4

For a particle in a potential $U(x) = \alpha x$ we can find the eigenvalues and eigenfunctions of the energy operator, $\hat{H}$, in the momentum representation. Where in coordinate representation we have

$$\hat{H}_x = -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2} + \alpha x$$

by writing the Hamiltonian in momentum representation as

$$\hat{H}_p = \frac{p^2}{2\mu} + i\alpha\hbar\frac{\partial}{\partial p}$$

So by solving the eigenvalue problem we can find the eigenvalues, E_p, and eigenfunctions, $\phi(p)$.

$$\begin{aligned}
\hat{H}_p\phi(p) &= E_p\phi(p) \\
&\Downarrow \\
\left[\frac{p^2}{2\mu} + i\alpha\hbar\frac{\partial}{\partial p}\right]\phi(p) &= E_p\phi(p) \\
&\Downarrow \\
i\alpha\hbar\frac{\partial\phi}{\partial p} &= E_p\phi(p) - \frac{p^2}{2\mu}\phi(p) \\
&\Downarrow \\
\int\frac{\partial(\phi(p))}{\phi(p)} &= -\int\frac{i}{\alpha\hbar}\left(E_p - \frac{p^2}{2\mu}\right)\partial p \\
\log(\phi(p)) &= -\frac{i}{\alpha\hbar}\left(E_p p - \frac{p^3}{6\mu}\right) + C \\
\phi(p) &= A\exp\left(-\frac{iE_p}{\alpha\hbar}p + \frac{i}{6\alpha\hbar\mu}p^3\right)
\end{aligned}$$

We note that this is a free particle so E_p represents a continuous spectrum of eigenvalues.