Physics 601

Analytical Mechanics

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Homework #2

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1 Problem #1

(a) We note that for bound orbits a finite energy, E, must cross the effective potential for both $r \to \infty$ and $r \to 0$. This implies there exists two turning points r_{min} and r_{max}. Therefore if E is any finite value the conditions

$$\lim_{r\to\infty} V_{eff}(r) = \infty$$
$$\lim_{r\to 0} V_{eff}(r) = \infty$$

must be true for all orbits to be bound. Where V_{eff} is given by

$$V_{eff}(r) = \frac{L^2}{2mr^2} + V(r) \tag{1.1}$$

So, for the central harmonic oscillator potential

$$V(r) = \frac{1}{2}kr^2$$

which makes equation 1.1 become

$$V_{eff}(r) = \frac{L^2}{2mr^2} + \frac{1}{2}kr^2.$$

so we can see that

$$\lim_{r\to\infty} V_{eff}(r) = \cancelto{0}{\frac{L^2}{2mr^2}} + \cancelto{\infty}{\frac{1}{2}kr^2} = \infty$$

and

$$\lim_{r\to 0} V_{eff}(r) = \cancelto{\infty}{\frac{L^2}{2mr^2}} + \cancelto{0}{\frac{1}{2}kr^2} = \infty.$$

Therefore we can say that for the potential $V(r) = 1/2kr^2$ all orbits are bound. We note that there is a minimum required energy for these orbits. This occurs when $V_{eff}(r)$ is at a minimum. Which corresponds to a circular orbit. We find this minimum as

$$\frac{dV_{eff}}{dr} = 0$$
$$\Downarrow$$
$$0 = -\frac{L^2}{mr^3} + kr$$
$$\Downarrow$$
$$\frac{L^2}{mr^3} = kr$$
$$\Downarrow$$
$$r^4 = \frac{L^2}{mk}$$
$$\Downarrow$$
$$r_0 = \left(\frac{L^2}{mk}\right)^{1/4}$$

Now we plug r_0 into $V_{eff}(r)$ to find the minimum energy by

$$\begin{aligned}
V_{eff}(r_0) &= \frac{L^2}{2mr_0^2} + \frac{1}{2}kr_0^2 \\
&= \frac{L^2}{2m}\left(\frac{mk}{L^2}\right)^{2/4} + \frac{1}{2}k\left(\frac{L^2}{mk}\right)^{2/4} \\
&= \frac{1}{2}\left(\frac{L^4mk}{m^2L^2}\right)^{1/2} + \frac{1}{2}\left(\frac{k^2L^2}{mk}\right)^{1/2} \\
&= \frac{1}{2}\left(\frac{L^2k}{m}\right)^{1/2} + \frac{1}{2}\left(\frac{kL^2}{m}\right)^{1/2} \\
E_{min} &= \left(\frac{L^2k}{m}\right)^{1/2}
\end{aligned}$$

(b) To solve for the orbital motion of the given potential we recall the integral of motion

$$\frac{dr}{dt} = \pm\frac{2}{m}\sqrt{E - V_{eff}}$$

which we can convert into an equation with respect to θ instead of t by saying

$$\begin{aligned}
\frac{dr}{d\theta} &= \frac{dr}{dt}\frac{dt}{d\theta} \\
&= \frac{dr}{dt}\frac{mr^2}{L} \\
&\Downarrow \\
\frac{dr}{dt} &= \frac{L}{mr^2}\frac{dr}{d\theta}
\end{aligned}$$

Which coverts our integral into

$$\frac{1}{r^2}\frac{dr}{d\theta} = \pm\frac{2}{L}\sqrt{E - V_{eff}}.$$

Next we can we change to the variable u where $u = 1/r$ with

$$du = -\frac{1}{r^2}dr.$$

This gives us the integral

$$\frac{du}{d\theta} = \mp\frac{2}{L}\sqrt{E - V_{eff}}.$$

where

$$V_{eff}(r) = \frac{L^2}{2mr^2} + \frac{1}{2}kr^2 = \frac{L^2}{2m}u^2 + \frac{k}{2u^2}$$

So we can separate the variables to get the integral

$$\int d\theta = \int \frac{du}{2/L\sqrt{E - \frac{L^2}{2m}u^2 - \frac{k}{2u^2}}}$$

where we integrate this function using Mathematica with the command:

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Integrate[1/Sqrt[\[Alpha]-\[Beta]*u^2-\[Gamma]*u^-2],u]
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which yields the

$$\frac{iu\sqrt{\alpha - \frac{\gamma+\beta u^4}{u^2}}\log\left(2\sqrt{-\gamma-\beta u^4+\alpha u^2}+\frac{i\left(\alpha-2\beta u^2\right)}{\sqrt{\beta}}\right)}{2\sqrt{\beta}\sqrt{-\gamma-\beta u^4+\alpha u^2}}$$

We see when we replace with the coefficients of our problem the expression reduces to

$$\frac{i\log\left(2\sqrt{Eu^2-\frac{L^2}{2m}u^4-\frac{k}{2}}+\frac{i\left(E-2\frac{L^2}{2m}u^2\right)}{\sqrt{L^2/2m}}\right)}{2\sqrt{L^2/2m}}$$

which leads to the solution of the integral

$$(2/L)\theta + C = \frac{i\log\left(2\sqrt{Eu^2-\frac{L^2}{2m}u^4-\frac{k}{2}}+\frac{i\left(E-2\frac{L^2}{2m}u^2\right)}{\sqrt{L^2/2m}}\right)}{2\sqrt{L^2/2m}}$$

$$\Downarrow$$

$$2\sqrt{\frac{L^2}{2m}\frac{4}{L^2}}\theta + C = i\log\left(2\sqrt{Eu^2-\frac{L^2}{2m}u^4-\frac{k}{2}}+\frac{i\left(E-2\frac{L^2}{2m}u^2\right)}{\sqrt{L^2/2m}}\right)$$

Which leads to the result

$$\frac{1}{r^2} = \frac{2Em}{L^2} + \sqrt{\left(\frac{2Em}{L^2}\right)^2 - \frac{2k}{mL^2}}\sin(2\theta)$$

note we can write this in terms of E_{min} from part (a) to get

$$\frac{1}{r^2} = \frac{2Em}{L^2} + \sqrt{\frac{2km}{L^2}\left(\left(2\frac{E}{E_{min}k}\right)^2 - 1\right)}\sin(2\theta) \tag{1.2}$$

note the eccentricity, ϵ, is given by

$$\epsilon = \sqrt{\frac{2km}{L^2}\left(\left(2\frac{E}{E_{min}k}\right)^2 - 1\right)}$$

(c) Now that we have the solution for the elliptical orbit given by equation 1.2 we can easily see the period of the orbit of an ellipse is

$$T = 2\pi\sqrt{\frac{m}{k}}$$

independent of E or L.

2 Problem #2

Note the two orbital equations for a central force

$$\frac{d\theta}{dt} = \frac{L}{mr^2} \tag{2.1}$$

and

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{m}\Big(E - V_{eff}(r)\Big)} \tag{2.2}$$

where $V_{eff}(r)$ is given by equation 1.1. Given circular motion at radius $r = r_0$ we can say a small perturbation on this orbit oscillates harmonically by

$$r(t) = r_0 + A\cos(\omega_r t) \tag{2.3}$$

where

$$\omega_r = \sqrt{\frac{V''_{eff}(r_0)}{m}}. \tag{2.4}$$

(a) Given the potential

$$V(r) = \frac{k}{\alpha}r^\alpha$$

we can calculate ω_r by using equation 1.1 and taking a time derivatives of $V_eff(r)$ by

$$\begin{aligned} V'(r) &= \frac{d}{dt}\left(\frac{L^2}{2mr^2} + \frac{k}{\alpha}r^\alpha\right) \\ &= -\frac{L^2}{mr^3} + kr^{\alpha-1} \end{aligned}$$

Recall when $V'(r) = 0$ our radius becomes $r = r_0$. So we can solve for r_0 by

$$\begin{aligned} V'(r_0) = 0 &= -\frac{L^2}{mr_0^3} + kr_0^{\alpha-1} \\ &\Downarrow \\ \frac{L^2}{mr_0^3} &= kr_0^{\alpha-1} \\ &\Downarrow \\ r_0^{\alpha-1}r_0^3 &= \frac{L^2}{mk} \\ r_0^{\alpha+2} &= \frac{L^2}{mk} \\ &\Downarrow \\ r_0^\alpha &= \frac{L^2}{mkr_0^2} \end{aligned}$$

Next we calculate $V''_{eff}(r)$ by

$$\begin{aligned} V''_{eff}(r) &= \frac{d}{dt}\left(-\frac{L^2}{mr^3} + kr^{\alpha-1}\right) \\ &= 3\frac{L^2}{mr^4} + k(\alpha-1)r^{\alpha-2} \end{aligned}$$

We note that when $r = r_0$ we have a constant angular velocity given by ω which implies that equation 2.1 becomes

$$\frac{d\theta}{dt} = \omega = \frac{L}{mr_0^2} \tag{2.5}$$

So we can calculate $V''_{eff}(r_0)$ by

$$\begin{aligned}
V''_{eff}(r_0) &= 3\frac{L^2}{mr_0^4} + k(\alpha-1)r_0^{\alpha-2} \\
&= \frac{3L}{r_0^2}\frac{L}{mr_0^2} + \frac{k}{r_0^2}(\alpha-1)r_0^{\alpha} \\
&= \frac{3L}{r_0^2}\omega + \frac{L}{r_0^2}(\alpha-1)\frac{L}{mr_0^2} \\
&= \frac{3L}{r_0^2}\omega + \frac{L}{r_0^2}(\alpha-1)\omega \\
&= \frac{L}{r_0^2}\omega\,(3+\alpha-1) \\
&= \frac{L}{r_0^2}\omega\,(\alpha+2) \\
&= (m\omega)\omega\,(\alpha+2) \\
&= m\omega^2\,(\alpha+2)
\end{aligned}$$

Now we can calculate ω_r by equation 2.4

$$\begin{aligned}
\omega_r &= \sqrt{\frac{V''_{eff}(r_0)}{m}} \\
&= \sqrt{\frac{m\omega^2(\alpha+2}{m}} \\
&= \omega\sqrt{\alpha+2}
\end{aligned}$$

(b) We can calculate the *apsidal angle*, θ_A, which is defined by the angle between r_{min} and r_{max}. We note that by equation 2.4 we are at a maximum r when $\omega_r t = 0$ given by

$$r_{max} = r_0 + A\cos(0) = r_0 + A$$

and we are at a minimum r when $\omega_r t = \pi$ given by

$$r_{min} = r_0 + A\cos(\pi) = r_0 - A$$

noting that $\omega t = \theta$ it follows that the *apsidal angle* is the angle that

$$\begin{aligned}
\pi &= \omega_r t \\
&= \omega t\sqrt{\alpha+2} \\
&= \theta_A\sqrt{\alpha+2} \\
&\Downarrow \\
\theta_A &= \frac{\pi}{\sqrt{\alpha+2}}
\end{aligned}$$

(c) We can find the limit of the given potential as $\alpha \to 0$ by

$$\begin{aligned}
\lim_{\alpha \to 0} V(r) &= \lim_{\alpha \to 0} \frac{k}{\alpha} r^{\alpha} \\
&= \lim_{\alpha \to 0} k r^{\alpha} \ln(r) \\
&= k \ln(r)
\end{aligned}$$

We can also see that as $\alpha \to 0$ we have

$$\omega_r = \sqrt{2}\omega$$

therefore the ratio of ω_r/ω is

$$\frac{\omega_r}{\omega} = \frac{\sqrt{2}\cancel{\omega}}{\cancel{\omega}} = \sqrt{2}$$

which agrees with the result from homework #1.

3 Problem #3

(a) For $\alpha < 0$ we let $\alpha = -s$ with $0 < s < 2$ which makes our our potential from problem # 2 become

$$\frac{k}{\alpha}r^{\alpha} \to -\frac{k}{s}r^{-s}.$$

We note that for these potentials all bounded orbits have $E < 0$ with the orbital equation

$$E = \frac{1}{2}m^*\left(\frac{du}{d\theta}\right)^2 + \frac{1}{2}m^*u^2 - \frac{k}{s}u^s \tag{3.1}$$

Where we change variables such that

$$u = \frac{1}{r}$$
$$m^* = \frac{L^2}{m}.$$

Using equation 3.1 we can find the apsidal angle in the limit of $E \to 0$. This implies that

$$\frac{1}{2}m^*\left(\frac{du}{d\theta}\right)^2 + \frac{1}{2}m^*u^2 - \frac{k}{s}u^s = 0$$
$$\Downarrow$$
$$\frac{k}{s}u^s = \frac{1}{2}m^*\left(\frac{du}{d\theta}\right)^2 + \frac{1}{2}m^*u^2$$
$$\frac{k}{s} = \frac{1}{2}m^*u^{-s}\left(\frac{du}{d\theta}\right)^2 + \frac{1}{2}m^*u^{2-s}$$

We desire to change this orbital equation into the form of a harmonic oscillator so that we can solve the equation. We see that we want $u^{2-s} = x^2$ to get into this form. We note that we need to change the variables of the derivative by

$$\frac{du}{d\theta} = \frac{du}{dx}\frac{dx}{d\theta}$$

where we can find $\frac{du}{dx}$ by

$$2xdx = (2-s)u^{2-u-1}du$$
$$\Downarrow$$
$$2xdx = (2-s)u^{2-u}u^{-1}du$$
$$2xdx = (2-s)x^2u^{-1}du$$
$$\Downarrow$$
$$\frac{2}{x}dx = \frac{(2-s)}{u}du$$
$$\Downarrow$$
$$\frac{du}{dx} = \frac{2u}{(2-s)x}$$

So we can convert to our new variable x by

$$
\begin{aligned}
\frac{k}{s} &= \frac{1}{2}m^* u^{-s}\left(\frac{du}{d\theta}\right)^2 + \frac{1}{2}m^* u^{2-s} \\
&\Downarrow \\
\frac{k}{s} &= \frac{1}{2}m^* u^{-s}\left(\frac{du}{dx}\frac{dx}{d\theta}\right)^2 + \frac{1}{2}m^* x^2 \\
\frac{k}{s} &= \frac{1}{2}m^* u^{-s}\left(\frac{2u}{(2-s)x}\frac{dx}{d\theta}\right)^2 + \frac{1}{2}m^* x^2 \\
\frac{k}{s} &= \frac{1}{2}m^*\left(\frac{2}{2-s}\right)^2 \frac{u^2u^{-s}}{x^2}\left(\frac{dx}{d\theta}\right)^2 + \frac{1}{2}m^* x^2 \\
\frac{k}{s} &= \frac{1}{2}m^*\left(\frac{2}{2-s}\right)^2 \cancelto{1}{\frac{u^{2-s}}{x^2}}\left(\frac{dx}{d\theta}\right)^2 + \frac{1}{2}m^* x^2 \\
\frac{k}{s} &= \frac{1}{2}m^*\left(\frac{2}{2-s}\right)^2 \left(\frac{dx}{d\theta}\right)^2 + \frac{1}{2}m^* x^2
\end{aligned}
$$

As we see we have successfully converted our orbital equation into the form of a harmonic oscillator in the variable x under the limit $E \to 0$. Given that a harmonic oscillator has the solution of the form

$$x(\theta) = x_0 + A\cos(\omega_x \theta) \tag{3.2}$$

where

$$\omega_x = \sqrt{\frac{k}{m}}.$$

Therefore for our orbital equation we see that

$$k = m^*$$

and

$$m = m^*\left(\frac{2}{2-s}\right)^2$$

so we can see that the frequency of oscillation ω is given by

$$\omega_x = \sqrt{\frac{m^*}{m^*}\left(\frac{2}{2-s}\right)^2} = \frac{2}{2-s}.$$

Now, we can use ω_x to calculate the apsidal angle, θ_A, by noting that we go from a minimum to maximum in equation 3.2 from $\omega_x\theta = 0$ to $\omega_x\theta_A = \pi/2$. By solving for θ_A we get

$$
\begin{aligned}
\omega_x\theta_A = \frac{2}{2-s}\theta_A &= \frac{\pi}{2} \\
&\Downarrow \\
\theta_A &= \frac{\pi}{2-s}
\end{aligned}
$$

Which in terms of α we have

$$\theta_A = \frac{\pi}{2+\alpha}$$

(b) If we recall that in problem #2 for near circular orbits we found that

$$\theta_A = \frac{\pi}{\sqrt{2+\alpha}}$$

for a general energy, E, and as we see in part (a) that in the limit of $E \to 0$ the apsidal angle becomes

$$\theta_A = \frac{\pi}{2+\alpha}.$$

So, we can find the α that keeps θ_A constant for all energies we solve

$$\frac{\pi}{\sqrt{2+\alpha}} = \frac{\pi}{2+\alpha}$$
$$\Downarrow$$
$$\sqrt{2+\alpha} = 2+\alpha$$
$$\Downarrow$$
$$\alpha = -1$$

Note that we neglected the solution $\alpha = -2$ as it makes θ_A ill defined, and is outside our defined range for α.