Physics 648

Quantum Optics and Laser Phyiscs

Professor Muhammad Suhail Zubairy

Homework #1

Joe Becker
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1 Problem #1

For two noncommuting operators $\hat{A}$ and $\hat{B}$ which satisfy the conditions

$$[[\hat{A},\hat{B}],\hat{A}] = [[\hat{A},\hat{B}],\hat{B}] = 0 \tag{1.1}$$

we can find an expression for $e^{\hat{A}+\hat{B}}$ we define a function

$$f(x) \equiv e^{\hat{A}x}e^{\hat{B}x}$$

and take the derivate with respect to x to find that

$$\begin{aligned}
\frac{df(x)}{dx} &= \hat{A}e^{\hat{A}x}e^{\hat{B}x} + e^{\hat{A}x}e^{\hat{B}x}\hat{B} \\
&= e^{\hat{A}x}e^{\hat{B}x}\left(e^{-\hat{B}x}\hat{A}e^{\hat{B}x} + \hat{B}\right) \\
&= f(x)\left(\left(\hat{A} + [\hat{A},\hat{B}]x + \cancelto{0}{\frac{1}{2!}[[\hat{A},\hat{B}],\hat{B}] + ...}\right) + \hat{B}\right) \\
&\Downarrow \\
\frac{df}{f} &= \left(\hat{A} + [\hat{A},\hat{B}]x + \hat{B}\right)dx \\
\ln(f) &= \hat{A}x + \frac{1}{2}[\hat{A},\hat{B}]x^2 + \hat{B}x \\
&\Downarrow \\
f(x) &= \exp\left(\hat{A}x + \frac{1}{2}[\hat{A},\hat{B}]x^2 + \hat{B}x\right)
\end{aligned}$$

Note that we used the *Baker-Hausdorff formula*

$$e^{-\hat{T}}\hat{A}e^{\hat{T}} = \hat{A} + [\hat{A},\hat{B}]x + \frac{1}{2!}[[\hat{A},\hat{B}],\hat{B}] + \frac{1}{3!}[[[\hat{A},\hat{T}],\hat{T},\hat{T}]] + ... \tag{1.2}$$

which due to equation 1.1 is zero for all terms except the first two. Now for $x = 1$ we find that

$$\begin{aligned}
e^{\hat{A}}e^{\hat{B}} &= e^{\hat{A}+\hat{B}}e^{\frac{1}{2}[\hat{A},\hat{B}]} \\
&\Downarrow \\
e^{\hat{A}+\hat{B}} &= e^{-\frac{1}{2}[\hat{A},\hat{B}]}e^{\hat{A}}e^{\hat{B}}
\end{aligned}$$

Note that we can factor $f(x)$ differently which yields

$$\frac{df(x)}{dx} = e^{\hat{B}x}e^{\hat{A}x}\left(\hat{A} + e^{-\hat{A}x}\hat{B}e^{\hat{A}x}\right)$$

which results in

$$e^{\hat{A}+\hat{B}} = e^{\frac{1}{2}[\hat{A},\hat{B}]}e^{\hat{B}}e^{\hat{A}}$$

by following the same steps as above.

2 Problem #2

For two noncommuting operators $\hat{A}$ and $\hat{B}$ and the parameter α we can see find a generalized result for equation 1.2 by taking $e^{-\alpha\hat{A}}\hat{B}e^{\alpha\hat{A}}$ and expanding each exponential to yield

$$\begin{aligned} e^{-\alpha\hat{A}}\hat{B}e^{\alpha\hat{A}} &= \left(1-\alpha\hat{A}+\frac{(\alpha\hat{A})^2}{2!}+...\right)\hat{B}\left(1+\alpha\hat{A}+\frac{(\alpha\hat{A})^2}{2}+...\right) \\ &= \hat{B}-\alpha[\hat{A},\hat{B}]+\frac{\alpha^2}{2!}[\hat{A},[\hat{A},\hat{B}]]+... \end{aligned}$$

Note that the result follows from simply expanding the products and grouping powers of α.

3 Problem #3

Given a function, $f(a,a^\dagger)$ which can be expanded in a power series of a and $a^\dagger$

$$f(a,a^\dagger)=1+\frac{\partial f}{\partial a}a+\frac{\partial f}{\partial a^\dagger}a^\dagger+\frac{\partial f}{\partial aa^\dagger}aa^\dagger+\frac{\partial f}{\partial a^\dagger a}a^\dagger a+... \tag{3.1}$$

while noting that $[a,a^\dagger]=1$ we can find the following relations

(a)

$$\begin{aligned} [a,f(a,a^\dagger)] &= \frac{\partial f}{\partial a}\cancel{[a,a]}+\frac{\partial f}{\partial a^\dagger}[a,a^\dagger]+\frac{\partial f}{\partial aa^\dagger}\cancel{[}a,aa^\dagger]+\frac{\partial f}{\partial a^\dagger a}\cancel{[a,a^\dagger a]}+... \\ &= \frac{\partial f}{\partial a^\dagger} \end{aligned}$$

(b)

$$\begin{aligned} [a^\dagger,f(a,a^\dagger)] &= \frac{\partial f}{\partial a}[a^\dagger,a]+\frac{\partial f}{\partial a^\dagger}\cancel{[a^\dagger,a^\dagger]}+\frac{\partial f}{\partial aa^\dagger}\cancel{[}a^\dagger,aa^\dagger]+\frac{\partial f}{\partial a^\dagger a}\cancel{[a^\dagger,a^\dagger a]}+... \\ &= -\frac{\partial f}{\partial a} \end{aligned}$$

(c)

$$\begin{aligned} e^{-\alpha aa^\dagger}f(a,a^\dagger)e^{\alpha aa^\dagger} &= f(a,a^\dagger)-\alpha[aa^\dagger,f(a,a^\dagger)]+... \\ &= f(a,a^\dagger)-\alpha a^\dagger[a,f(a,a^\dagger]-\alpha[a^\dagger,f(a,a^\dagger]a+... \quad = f(a,a^\dagger)-\alpha a^\dagger\frac{\partial f}{\partial a^\dagger}+\alpha a\frac{\partial f}{\partial a}+... \\ &= f(ae^\alpha,a^\dagger e^{-\alpha}) \end{aligned}$$

4 Problem #4

Expanding the exponential we can show that

$$\begin{aligned}
[a, e^{-\alpha a^\dagger a}] &= [a, 1 - \alpha a^\dagger a + (\alpha a^\dagger a)^2 + ...] \\
&= -\alpha[a, a^\dagger a] + \alpha^2[a, (a^\dagger a)^2] + ... \\
&= -\alpha a + \alpha^2(a - 2a^\dagger a) + ... \\
&= \left(-\alpha + \frac{1}{2}\alpha^2 + ...\right)\left(1 - \alpha a^\dagger a + ...\right) a \\
&= \left(e^{-\alpha} - 1\right) e^{-\alpha a^\dagger a} a
\end{aligned}$$

and

$$\begin{aligned}
[a^\dagger, e^{-\alpha a^\dagger a}] &= [a^\dagger, 1 - \alpha a^\dagger a + (\alpha a^\dagger a)^2 + ...] \\
&= -\alpha[a^\dagger, a^\dagger a] + \alpha^2[a^\dagger, (a^\dagger a)^2] + ... \\
&= \alpha a^\dagger + \alpha^2(a - 2a^\dagger a^\dagger) + ... \\
&= \left(\alpha + \frac{1}{2}\alpha^2 + ...\right)\left(1 - \alpha a^\dagger a + ...\right) a^\dagger \\
&= \left(e^{\alpha} - 1\right) e^{-\alpha a^\dagger a} a^\dagger
\end{aligned}$$

5 Problem #5

We can verify that $\sum_i \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_i = \mathbf{1}$ by taking the dot product with any vector $\mathbf{v}$ which we define as

$$\mathbf{v} = \sum_i v_i \hat{\mathbf{e}}_i \tag{5.1}$$

we note that by equation 5.1 we find that

$$\mathbf{v} \cdot \mathbf{v} = \sum_i v_i^2 \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_i = \sum_i v_i^2 \sum_i \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_i = \sum_i v_i^2$$

which only holds true if

$$\sum_i \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_i = \mathbf{1}$$

So, if we define the direction of our basis to be along the wavevetor $\mathbf{k}$ we have

$$\hat{\mathbf{e}}_1 = \hat{\epsilon}_{\mathbf{k}}^{(1)}, \qquad \hat{\mathbf{e}}_2 = \hat{\epsilon}_{\mathbf{k}}^{(2)}, \qquad \hat{\mathbf{e}}_3 = \mathbf{k}/k$$

which if we take to be in polar coordinates we can find that

$$\begin{aligned}
\hat{\epsilon}_{\mathbf{k}}^{(1)} &\equiv (\sin\phi, -\cos\phi, 0) \\
\hat{\epsilon}_{\mathbf{k}}^{(2)} &\equiv (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)
\end{aligned}$$

Which if we take the dot product we find that

$$\begin{aligned}
\mathbf{1} &= \hat{\epsilon}_{\mathbf{k}}^{(1)} \cdot \hat{\epsilon}_{\mathbf{k}}^{(1)} + \hat{\epsilon}_{\mathbf{k}}^{(2)} \cdot \hat{\epsilon}_{\mathbf{k}}^{(2)} + k^2/k^2 \\
&= \sin^2\phi + \cos^2\phi + \cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta + 1 \\
&= 3
\end{aligned}$$

So it follows that

$$\epsilon_{\mathbf{k}i}^{(1)}\epsilon_{\mathbf{k}j}^{(1)} + \epsilon_{\mathbf{k}i}^{(2)}\epsilon_{\mathbf{k}j}^{(2)} = \delta_{ij} - \frac{k_i k_j}{k^2}$$

6 Problem #6

For a one dimensional system where two conducting reflecting mirrors are placed at a distance L apart we see that

$$H_0^{box} = \sum_l \frac{1}{2}\hbar\nu_l = \sum_l \frac{1}{2}\hbar c \frac{l\pi}{L}$$

and for outside the box we have the vacuum energy

$$H_0^{vac} = \frac{c\hbar}{2}\int_0^\infty \frac{l\pi}{L'}dl$$

this allows us to calculate the difference in energy as

$$\begin{aligned} H_0^{box} - H_0^{vac} &= \sum_l \frac{1}{2}\hbar c \frac{l\pi}{L} - \frac{c\hbar}{2}\int_0^\infty \frac{l\pi}{L'}dl \\ &= -\frac{c\hbar}{2L}\frac{1}{180} \end{aligned}$$