

Physics 606
Quantum Mechanics I
Professor Aleksei Zheltikov

Homework #3

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1 Problem #1

For the operator given by

$$\hat{A} = -i\hbar \frac{\partial}{\partial \varphi} + a \sin \varphi$$

we assume there exists an eigenfunction which is a solution to the equation

$$(\hat{A} - A)\psi(\varphi) = 0$$

where A is the eigenvalue of \hat{A} . We can see that for the given \hat{A} we have

$$\begin{aligned} \left(-i\hbar \frac{\partial}{\partial \varphi} + a \sin \varphi\right) \psi(\varphi) - A\psi(\varphi) &= 0 \\ -i\hbar \frac{\partial}{\partial \varphi} \psi(\varphi) + a \sin \varphi \psi(\varphi) - A\psi(\varphi) &= 0 \\ \Downarrow \\ -i\hbar \frac{\partial}{\partial \varphi} \psi(\varphi) &= -a \sin \varphi \psi(\varphi) + A\psi(\varphi) \\ \frac{\partial}{\partial \varphi} \psi(\varphi) &= i \frac{A - a \sin \varphi}{\hbar} \psi(\varphi) \end{aligned}$$

This equation implies that

$$\psi(\varphi) = C \exp\left(i \frac{A}{\hbar} \varphi\right) \exp\left(i \frac{a \cos \varphi}{\hbar}\right)$$

We can find the normalization constant, C , by normalizing

$$\begin{aligned} 1 &= \int_0^{2\pi} \psi^*(\varphi) \psi(\varphi) d\varphi = C^2 \int_0^{2\pi} \exp\left(-i \frac{A}{\hbar} \varphi\right) \exp\left(i \frac{A}{\hbar} \varphi\right) \exp\left(-i \frac{a \cos \varphi}{\hbar}\right) \exp\left(i \frac{a \cos \varphi}{\hbar}\right) d\varphi \\ &= C^2 \int_0^{2\pi} d\varphi = C^2 2\pi \\ \Downarrow \\ C &= \frac{1}{\sqrt{2\pi}} \end{aligned}$$

Therefore our normalized eigenfunction is

$$\psi(\varphi) = \frac{1}{\sqrt{2\pi}} \exp\left(i \frac{A}{\hbar} \varphi\right) \exp\left(i \frac{a \cos \varphi}{\hbar}\right)$$

where we apply the periodic boundary condition that $\psi(\varphi + 2\pi) = \psi(\varphi)$ we note that the second term satisfies this requirement and the first term is periodic for

$$A = n\hbar \quad n = 1, 2, 3, \dots$$

so we finally have the eigenfunction

$$\psi(\varphi) = \frac{1}{\sqrt{2\pi}} \exp(in\varphi) \exp\left(i \frac{a \cos \varphi}{\hbar}\right)$$

with the eigenvalue

$$A = n\hbar \quad n = 1, 2, 3, \dots$$

2 Problem #2

- (a) For a general potential $U(x)$ the Hamiltonian operator in three-dimensions, \hat{H} , becomes

$$\hat{H} = \frac{\mathbf{p}^2}{2m} + U(x)$$

where in three-dimensions

$$\mathbf{p}^2 = \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2$$

For this Hamiltonian we can calculate the commutator $[\hat{H}, y]$ by

$$\begin{aligned} [\hat{H}, y] &= \frac{1}{2m} \left(\cancel{[\hat{p}_x^2, y]}^0 + [\hat{p}_y^2, y] + \cancel{[\hat{p}_z^2, y]}^0 \right) + \cancel{[U(x), y]}^0 \\ &= \frac{1}{2m} [\hat{p}_y^2, y] \\ &= \frac{1}{2m} (\hat{p}_y [\hat{p}_y, y] + [\hat{p}_y, y] \hat{p}_y) \\ &= \frac{1}{2m} (\hat{p}_y (-i\hbar) + (-i\hbar) \hat{p}_y) \\ &= \frac{-i\hbar}{m} \hat{p}_y \end{aligned}$$

- (b) For the same Hamiltonian we calculate

$$\begin{aligned} [\hat{H}, \hat{p}_x] &= \frac{1}{2m} \left(\cancel{[\hat{p}_x^2, \hat{p}_x]}^0 + \cancel{[\hat{p}_y^2, \hat{p}_x]}^0 + \cancel{[\hat{p}_z^2, \hat{p}_x]}^0 \right) + [U(x), \hat{p}_x] \\ &= [U(x), \hat{p}_x] \end{aligned}$$

We can calculate this commutation by multiplying by a function f

$$\begin{aligned} [U(x), \hat{p}_x] f(x) &= U(x) \hat{p}_x f(x) - \hat{p}_x (U(x) f(x)) \\ &= U(x) (-i\hbar) \frac{\partial}{\partial x} f(x) - (-i\hbar) \frac{\partial}{\partial x} (U(x) f(x)) \\ &= -i\hbar U(x) \frac{\partial f(x)}{\partial x} + i\hbar \frac{\partial f(x)}{\partial x} U(x) + i\hbar \frac{\partial U(x)}{\partial x} f(x) \\ &\Downarrow \\ [U(x), \hat{p}_x] f(x) &= i\hbar \frac{\partial U(x)}{\partial x} f(x) \\ [U(x), \hat{p}_x] &= i\hbar \frac{\partial U(x)}{\partial x} \end{aligned}$$

So we have

$$[\hat{H}, \hat{p}_x] = i\hbar \frac{\partial U(x)}{\partial x}$$

- (c) Next we calculate how \hat{p}_x^2 commutes with the Hamiltonian

$$\begin{aligned} [\hat{H}, \hat{p}_x^2] &= \frac{1}{2m} (\cancel{[\hat{p}_x^2, \hat{p}_x^2]} + \cancel{[\hat{p}_y^2, \hat{p}_x^2]} + \cancel{[\hat{p}_z^2, \hat{p}_x^2]}) + [U(x), \hat{p}_x^2] \\ &= [U(x), \hat{p}_x^2] = [U(x), \hat{p}_x] \hat{p}_x + \hat{p}_x [U(x), \hat{p}_x] \\ &= i\hbar \frac{\partial U(x)}{\partial x} \hat{p}_x + \hat{p}_x i\hbar \frac{\partial U(x)}{\partial x} \\ &= i\hbar \frac{\partial U(x)}{\partial x} \hat{p}_x + i\hbar (-i\hbar) \frac{\partial}{\partial x} \frac{\partial U(x)}{\partial x} \\ &= i\hbar \frac{\partial U(x)}{\partial x} \hat{p}_x + \hbar^2 \frac{\partial^2 U(x)}{\partial x^2} \end{aligned}$$

3 Problem #3

In order to express the translation operator $\hat{T}_{\mathbf{a}}$ which is defined as a spacial translation \mathbf{a} given by

$$\hat{T}_{\mathbf{a}}\psi(\mathbf{r}) = \psi(\mathbf{r} + \mathbf{a})$$

in terms of the momentum operator $\hat{\mathbf{p}}$ by noting that the expansion

$$\psi(\mathbf{r} + \mathbf{a}) = \sum_{n=0}^{\infty} \frac{1}{n!} \nabla^n(\psi(\mathbf{r})) \cdot \mathbf{a}^n$$

We can place this into terms of \mathbf{P} by noting that

$$\mathbf{p} = -i\hbar\nabla$$

which implies that

$$\begin{aligned} \psi(\mathbf{r} + \mathbf{a}) &= \sum_{n=0}^{\infty} \frac{1}{n!} \nabla^n(\psi(\mathbf{r})) \cdot \mathbf{a}^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{a}^n \cdot \left(\frac{i}{\hbar} \mathbf{p} \right)^n \psi(\mathbf{r}) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{\hbar} \mathbf{a} \cdot \mathbf{p} \right)^n \psi(\mathbf{r}) \end{aligned}$$

We note that we have a Taylor expansion of the exponential

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{\hbar} \mathbf{a} \cdot \mathbf{p} \right)^n = \exp \left(i \frac{\mathbf{a} \cdot \mathbf{p}}{\hbar} \right)$$

So we see that we have found that

$$\psi(\mathbf{r} + \mathbf{a}) = \exp \left(i \frac{\mathbf{a} \cdot \mathbf{p}}{\hbar} \right) \psi(\mathbf{r})$$

So the translation operator in terms of the momentum operator is given by

$$\hat{T}_{\mathbf{a}} = \exp \left(i \frac{\mathbf{a} \cdot \mathbf{p}}{\hbar} \right)$$

4 Problem #4

For a free particle in one-dimension in the presence of an impenetrable wall (or barrier) which is given by the potential

$$U(x) = \begin{cases} 0 & x > 0 \\ \infty & x < 0 \end{cases}$$

We note that in the region $x < 0$ there is no wave-function because we have an infinite potential therefore we have $\psi(x) = 0$ for $x \leq 0$. And away from the barrier we have a free particle given by

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

We note that in general the term kx is found by the integral

$$k(x) = - \int_{x_0}^x \frac{1}{\hbar} \sqrt{2\mu(E - U(x'))} dx'$$

but for $U(x) = 0$ we have a constant integral where

$$k = \frac{\sqrt{2\mu E}}{\hbar}$$

end Next we apply the continuity condition where $\psi(0) = 0$. We note that due to the infinite change in potential we cannot have a continuous derivative. So we see that

$$\begin{aligned} \psi(0) &= 0 = Ae^{ik0} + Be^{-ik0} \\ \psi(0) &= 0 = A + B \\ &\Downarrow \\ A &= -B \end{aligned}$$

So our solution becomes

$$\psi(x) = A \left(\exp\left(\frac{\sqrt{2\mu E}}{\hbar} x\right) - \exp\left(-\frac{\sqrt{2\mu E}}{\hbar} x\right) \right)$$

Now we can normalize our $\psi(x)$ to a delta function in E . By calculating

$$\begin{aligned} \delta(E - E') &= A^2 \int_0^E \left(\exp\left(-\frac{\sqrt{2\mu E'}}{\hbar} x\right) - \exp\left(\frac{\sqrt{2\mu E'}}{\hbar} x\right) \right) \left(\exp\left(\frac{\sqrt{2\mu E'}}{\hbar} x\right) - \exp\left(-\frac{\sqrt{2\mu E'}}{\hbar} x\right) \right) dE' \\ &= A^2 \int_0^E 2 \exp\left(\frac{\sqrt{2\mu E'}}{\hbar} x\right) \exp\left(-\frac{\sqrt{2\mu E'}}{\hbar} x\right) dE' \\ &= A^2 \int_0^E 2 dE' \\ &= 2A^2 E \\ &\Downarrow \\ A &= \frac{1}{2E\delta(E - E')} \end{aligned}$$

So we have

$$\psi(x) = \frac{1}{2E\delta(E)} \left(\exp\left(\frac{\sqrt{2\mu E}}{\hbar} x\right) - \exp\left(-\frac{\sqrt{2\mu E}}{\hbar} x\right) \right)$$