Physics 615

Methods of Theoretical Physics I

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Homework #3

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1 Problem #1

For the differential equation given by

$$(1-x^3)y'' - 6x^2y' - 6xy = 0 \tag{1.1}$$

with the boundary conditions

$$y(0) = 1, \qquad y'(0) = 0$$

can be solved by finding a series for y in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Where we note that

$$\begin{aligned}
y'(x) &= \sum_{n=0}^{\infty} a_n n x^{n-1} \\
y''(x) &= \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2}
\end{aligned}$$

and we replace these sums into equation 1.1 to get

$$\begin{aligned}
(1-x^3)\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - 6x^2\sum_{n=0}^{\infty} a_n n x^{n-1} - 6x\sum_{n=0}^{\infty} a_n x^n &= 0 \\
\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - x^3\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - 6x^2\sum_{n=0}^{\infty} a_n n x^{n-1} - 6x\sum_{n=0}^{\infty} a_n x^n &= 0 \\
\sum_{n=-3}^{\infty} a_{n+3}(n+2)(n+3)x^{n+1} - \sum_{n=0}^{\infty} a_n n(n-1)x^{n+1} - \sum_{n=0}^{\infty} 6a_n n x^{n+1} - \sum_{n=0}^{\infty} 6a_n x^{n+1} &= 0 \\
a_2 + \sum_{n=0}^{\infty}\Big(a_{n+3}(n+2)(n+3) - a_n n(n-1) - 6a_n n - 6a_n\Big)x^{n+1} &= 0
\end{aligned}$$

Note that we changed the dummy index over the first summation to $n \to n+3$. We note that the $n = -3, -2$ terms are zero and the $n = -1$ term is a_2 which is the pre-factor. We now see that for this equation to be true for all x the following needs to be true. This implies that $a_2 = 0$.

$$\begin{aligned}
0 &= a_{n+3}(n+2)(n+3) - a_n n(n-1) - 6a_n n - 6a_n \\
0 &= a_{n+3}(n+2)(n+3) - a_n(n(n-1) + 6n + 6) \\
0 &= a_{n+3}(n+2)(n+3) - a_n(n^2 + 5n + 6) \\
0 &= a_{n+3}(n+2)(n+3) - a_n((n+2)(n+3)) \\
&\Downarrow \\
a_{n+3} &= \frac{(n+2)(n+3)}{(n+2)(n+3)}a_n \\
a_{n+3} &= a_n
\end{aligned}$$

Now that we have a recursion relation for the coefficients a_n we can apply the boundary conditions to determine the values for a_n. Applying $y(0) = 1$ gives

$$\begin{aligned} y(0) = 1 &= \sum_{n=0}^{\infty} a_n (0)^n \\ &= a_0 + \cancelto{0}{\sum_{n=1}^{\infty} a_n (0)^n} \\ &\Downarrow \\ 1 &= a_0 \end{aligned}$$

and

$$\begin{aligned} y'(0) = 0 &= \sum_{n=0}^{\infty} a_n n (0)^{n-1} \\ &\Downarrow \\ 0 &= a_1 \end{aligned}$$

Therefore we see only the $a_0, a_3, a_6, ...$ terms are non-zero and are equal to 1. This makes the series for y become

$$y(x) = 1 + x^3 + x^6 + x^9 + ... = \sum_{n=0}^{\infty} x^{3n} = \frac{1}{1 - x^3}$$

we can verify this is a solution by first noting that

$$\begin{aligned} y(x) &= \frac{1}{1 - x^3} \\ y'(x) &= \frac{3x^2}{(1 - x^3)^2} \\ y''(x) &= 3\left(\frac{2x}{(1 - x^3)^2} + \frac{6x^4}{(1 - x^3)^3}\right) = \frac{6x(2x^3 + 1)}{(1 - x^3)^3} \end{aligned}$$

and plugging back into equation 1.1 which yields

$$\begin{aligned} (1 - x^3)y'' - 6x^2 y' - 6xy &= (1 - x^3)\frac{6x(2x^3 + 1)}{(1 - x^3)^3} - 6x^2 \frac{3x^2}{(1 - x^3)^2} - 6x\frac{1}{1 - x^3} \\ &= \frac{6x(2x^3 + 1)}{(1 - x^3)^2} - \frac{18x^4}{(1 - x^3)^2} - \frac{6x(1 - x^3)}{(1 - x^3)^2} \\ &= \frac{12x^4 + 6x - 18x^4 - 6x + 6x^4)}{(1 - x^3)^2} \\ &= 0 \end{aligned}$$

Therefore,

$$y(x) = \frac{1}{1 - x^3}$$

is a solution for equation 1.1.

2 Problem #2

For the differential equation

$$y'' - 2xy' - 2y = 0 \tag{2.1}$$

we can find a solution by a power series like in Problem 1. With

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y'(x) &= \sum_{n=0}^{\infty} a_n n x^{n-1} \\
y''(x) &= \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2}
\end{aligned}$$

which makes equation 2.1 become

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - 2x\sum_{n=0}^{\infty} a_n n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=-2}^{\infty} a_{n+2}(n+1)(n+2)x^n - 2\sum_{n=0}^{\infty} a_n n x^n - 2\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} \Big(a_{n+2}(n+1)(n+2) - 2a_n n - 2a_n \Big) x^n
\end{aligned}$$

This leads us to the condition that for all x

$$\begin{aligned}
0 &= a_{n+2}(n+1)(n+2) - 2a_n n - 2a_n \\
&\Downarrow \\
a_{n+2} &= \frac{2n+2}{(n+1)(n+2)} a_n \\
&= \frac{2}{(n+2)} a_n
\end{aligned}$$

We see that even n a_n are proportional to a_0 and odd n, a_n are proportional to a_1. This gives us two solutions to equation 2.1 given by

$$\begin{aligned}
y_1(x) &= a_0 + a_2x^2 + a_4x^4 + a_6x^6 ... \\
&= a_0 + a_0x^2 + \frac{1}{2}a_2x^4 + \frac{1}{3}a_4x^6 ... \\
&= a_0 + a_0x^2 + \frac{1}{2}a_0x^4 + \frac{1}{6}a_0x^6 ... \\
&= a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \\
&= a_0 e^{x^2}
\end{aligned}$$

and the odd terms yield

$$\begin{aligned}
y_2(x) &= a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 + ... \\
&= a_1 x + \frac{2}{3} a_1 x^3 + \frac{2}{5} a_3 x^5 + \frac{2}{7} a_5 x^7 + ... \\
&= a_1 \left(x + \frac{2}{3} x^3 + \frac{4}{5 \times 3} x^5 + \frac{8}{7 \times 5 \times 3} x^7 + ... \right) \\
&= a_1 \sum_{n=0}^{\infty} \frac{2^n x^{2n+1}}{(2n+1)!!} \\
&= a_1 \frac{\sqrt{\pi}}{2} e^{x^2} \operatorname{erf}(x)
\end{aligned}$$

We can test to see if these two solutions are linearly independent by calculating the *Wronskian* and if the result is nonzero then we can say the two solutions are linearly independent. We note that for a second order ODE the *Wronskian* is given by

$$W = y_1 y_2' - y_2 y_1'$$

Which we can calculate $y_1'(x)$ as

$$\begin{aligned}
y_1'(x) &= \frac{d}{dx} \left(a_0 e^{x^2} \right) \\
&= a_0 (2x) e^{x^2}
\end{aligned}$$

and we calculate $y_2'(x)$ as

$$\begin{aligned}
y_2'(x) &= \frac{d}{dx} \left(a_1 \frac{\sqrt{\pi}}{2} e^{x^2} \operatorname{erf}(x) \right) \\
&= a_1 \frac{\sqrt{\pi}}{2} \left(2x e^{x^2} \operatorname{erf}(x) + e^{x^2} \frac{2e^{-x^2}}{\pi} \right) \\
&= a_1 \frac{\sqrt{\pi}}{2} \left(2x e^{x^2} \operatorname{erf}(x) + \frac{2}{\pi} \right)
\end{aligned}$$

Now we can calculate W

$$\begin{aligned}
W &= y_1 y_2' - y_2 y_1' \\
&= a_0 e^{x^2} a_1 \frac{\sqrt{\pi}}{2} \left(2x e^{x^2} \operatorname{erf}(x) + \frac{2}{\pi} \right) - a_1 \frac{\sqrt{\pi}}{2} e^{x^2} \operatorname{erf}(x) a_0 (2x) e^{x^2} \\
&= a_0 a_1 e^{x^2} \frac{\sqrt{\pi}}{2} \left(2x e^{x^2} \operatorname{erf}(x) + \frac{2}{\pi} - 2x e^{x^2} \operatorname{erf}(x) \right) \\
&= a_0 a_1 e^{x^2} \frac{1}{\sqrt{\pi}}
\end{aligned}$$

as nonzero for any value of x and nonzero values of a_0 and a_1. Therefore, the solutions we found are linearly independent.

3 Problem #3

For the differential equation

$$xy'' + \frac{3}{x}y = 1 + x^2 \tag{3.1}$$

We first solve the homogeneous equation

$$xy'' + \frac{3}{x}y = 0$$

with a solution of the form

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n$$

due to the regular singularity at $x = 0$. We note that

$$\begin{aligned} y'(x) &= \sum_{n=0}^{\infty} a_n (n+s) x^{n+s-1} \\ y''(x) &= \sum_{n=0}^{\infty} a_n (n+s)(n+s-1) x^{n+s-2} \end{aligned}$$

which we plug into equation 3.1 to yield

$$\begin{aligned} 0 &= x \sum_{n=0}^{\infty} a_n (n+s)(n+s-1) x^{n+s-2} + \frac{3}{x} \sum_{n=0}^{\infty} a_n x^{n+s} \\ &= \sum_{n=0}^{\infty} a_n (n+s)(n+s-1) x^{n+s-1} + 3 \sum_{n=0}^{\infty} a_n x^{n+s-1} \\ &= \sum_{n=0}^{\infty} a_n \Big((n+s)(n+s-1) + 3 \Big) x^{n+s-1} \\ &= a_0 \Big(s(s-1) + 3 \Big) x^{s-1} + \sum_{n=1}^{\infty} a_n \Big((n+s)(n+s-1) + 3 \Big) x^{n+s-1} \end{aligned}$$

Note that we pulled the $n = 0$ term out because we want the x^{s-1} term to vanish for all x. This implies

$$\begin{aligned} 0 &= a_0 \Big(s(s-1) + 3 \Big) \\ &\Downarrow \\ 0 &= s^2 - s + 3 \end{aligned}$$

We see that this has a complex solution where the value of s becomes $s = 1/2(1 \pm i\sqrt{11})$. So we can pick the positive solution to find an indicial equation by

$$\begin{aligned} 0 &= a_n \Big((n+s)(n+s-1) + 3 \Big) \\ &\Downarrow \\ 0 &= (n + 1/2 + i/2\sqrt{11}))(n + 1/2 + i/2\sqrt{11} - 1) + 3 \\ &= (n + 1/2(1 + i\sqrt{11}))(n + 1/2(i\sqrt{11} - 1)) + 3 \\ &= n^2 + \frac{1}{4}(1 + i\sqrt{11})(i\sqrt{11} - 1) + n\frac{1}{2}(i\sqrt{11} - 1) + n\frac{1}{2}(1 + i\sqrt{11}) + 3 \\ &= n^2 - \frac{12}{4} + in\sqrt{11} + 3 \\ &= n^2 + in\sqrt{11} = n(n + i\sqrt{11}) \end{aligned}$$

We have two solutions for n where $n = 0, -i\sqrt{11}$. We note that there is no recursion relation so in general our solution is of the form $x^(s+n)$ for both values of n. Therefore our series becomes

$$\begin{aligned}
y_0(x) = a_0 x^{1/2+i/2\sqrt{11}+0} + a_1 x^{1/2+i/2\sqrt{11}-i\sqrt{11}} &= a_0\sqrt{x}x^{i/2\sqrt{11}} + a_1\sqrt{x}x^{-i/2\sqrt{11}} \\
&= a_0\sqrt{x}\exp\left(\log(x^{i/2\sqrt{11}})\right) + a_1\sqrt{x}\exp\left(\log(x^{-i/2\sqrt{11}})\right) \\
&= a_0\sqrt{x}\exp\left(\frac{i\sqrt{11}}{2}\log x\right) + a_1\sqrt{x}\exp\left(\frac{-i\sqrt{11}}{2}\log x\right) \\
&= a_0\sqrt{x}\cos\left(\frac{\sqrt{11}}{2}\log x\right) + a_1\sqrt{x}\sin\left(\frac{\sqrt{11}}{2}\log x\right)
\end{aligned}$$

Now we can find the particular solution of equation 3.1 by first calculating the *Wronskian* by

$$\begin{aligned}
W &= y_1 y_2' - y_1' y_2 \\
&= a_0\sqrt{x}\cos\left(\frac{\sqrt{11}}{2}\log x\right)\frac{a_1}{2\sqrt{x}}\left(\sin\left(\frac{\sqrt{11}}{2}\log x\right) + \sqrt{11}\cos\left(\frac{\sqrt{11}}{2}\log x\right)\right) \\
&\qquad - a_1\sqrt{x}\sin\left(\frac{\sqrt{11}}{2}\log x\right)\frac{a_0}{2\sqrt{x}}\left(\cos\left(\frac{\sqrt{11}}{2}\log x\right) - \sqrt{11}\sin\left(\frac{\sqrt{11}}{2}\log x\right)\right) \\
&= \frac{a_0 a_1\sqrt{11}}{2}\left(\sin^2\left(\frac{\sqrt{11}}{2}\log x\right) + \cos^2\left(\frac{\sqrt{11}}{2}\log x\right)\right) \\
&= \frac{a_0 a_1\sqrt{11}}{2}
\end{aligned}$$

We note that this value is nonzero for all x which confirms that y_1 and y_2 are linearly independent. So we can find y_p by

$$y_p = y_2\int\frac{y_1 f}{W} - y_1\int\frac{y_2 f}{W} \tag{3.2}$$

Because W is shown to be a constant we can just remove it from the integral in equation 3.2. So for $f = (1+x^2)/x$ we calculate

$$\begin{aligned}
\int y_1\frac{(1+x^2)}{x}dx &= \frac{a_0}{18}x^{1/2}\left((3+5x^2)\cos(\sqrt{11}/2\log x) + \sqrt{11}(3+x^2)\sin(\sqrt{11}/2\log x)\right) \\
\int y_2\frac{(1+x^2)}{x}dx &= \frac{a_1}{18}x^{1/2}\left((3+5x^2)\sin(\sqrt{11}/2\log x) - \sqrt{11}(3+x^2)\cos(\sqrt{11}/2\log x)\right)
\end{aligned}$$

using Mathematica. Which gives the particular solution by equation 3.2

$$\begin{aligned}
y_p(x) &= \frac{x}{9\sqrt{11}}\left(\cancel{(3+5x^2)\cos(\sqrt{11}/2\log x)\sin(\sqrt{11}/\log x)} + \sqrt{11}(3+x^2)\sin^2(\sqrt{11}/2\log x)\right) \\
&\quad + \frac{x}{9\sqrt{11}}\left(\cancel{-(3+5x^2)\cos(\sqrt{11}/2\log x)\sin(\sqrt{11}/\log x)} + \sqrt{11}(3+x^2)\cos^2(\sqrt{11}/2\log x)\right) \\
&= \frac{x}{9}(3+x^2)
\end{aligned}$$

Which gives the general solution

$$y(x) = a_0\sqrt{x}\cos(\sqrt{11}/2\log x) + a_1\sqrt{x}\sin(\sqrt{11}/2\log x) + \frac{x}{9}(3+x^2)$$

4 Problem #4

For the differential equation

$$2xy'' + y' + xy = 0 \tag{4.1}$$

which can be rewritten as

$$y'' + \frac{1}{2x}y' + \frac{1}{2}y = 0$$

we note that we have a *regular singular point* at $x = 0$ this implies that we try a solution in the form of a power series of the form

$$\begin{aligned}
y(x) &= x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+s} \\
y'(x) &= \sum_{n=0}^{\infty} a_n (n+s) x^{n+s-1} \\
y''(x) &= \sum_{n=0}^{\infty} a_n (n+s)(n+s-1) x^{n+s-2}
\end{aligned}$$

which

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} a_n (n+s)(n+s-1) x^{n+s-2} + \frac{1}{2x} \sum_{n=0}^{\infty} a_n (n+s) x^{n+s-1} + \frac{1}{2} \sum_{n=0}^{\infty} a_n x^{n+s} \\
&= \sum_{n=-2}^{\infty} a_{n+2}(n+s+2)(n+s+1) x^{n+s} + \frac{1}{2} \sum_{n=0}^{\infty} a_n (n+s) x^{n+s-2} + \frac{1}{2} \sum_{n=0}^{\infty} a_n x^{n+s} \\
&= \sum_{n=-2}^{\infty} a_{n+2}(n+s+2)(n+s+1) x^{n+s} + \frac{1}{2} \sum_{n=-2}^{\infty} a_{n+2}(n+s+2) x^{n+s} + \frac{1}{2} \sum_{n=0}^{\infty} a_n x^{n+s} \\
&= \left(a_0(s)(s-1) + \frac{1}{2} a_0 s \right) x^{s-2} + \left(a_1 s(s+1) + \frac{1}{2} a_1 (s+1) \right) x^{s-1} \\
&\quad + \sum_{n=-0}^{\infty} \left(a_{n+2}(n+s+2)(n+s+1) + \frac{1}{2} a_{n+2}(n+s+2) + \frac{1}{2} a_n \right) x^{n+s}
\end{aligned}$$

Now we take the coefficients the lowest powers of x to vanish, namely x^{s-2} which is given by the first term

$$\begin{aligned}
a_0(s)(s-1) + \frac{1}{2} a_0 s &= 0 \\
a_0 \left(s^2 - s + \frac{1}{2} s \right) &= 0 \\
&\Downarrow \\
s \left(s - \frac{1}{2} \right) &= 0
\end{aligned}$$

assuming a_0 is nonzero then we have

$$s = 0, \frac{1}{2}$$

next we take the coefficients with the term x^{s-1} which are from the second term this yields

$$\begin{aligned}
\left(a_1 s(s+1) + \frac{1}{2}a_1(s+1)\right) &= 0 \\
a_1\left(s^2 + s + \frac{1}{2}s + \frac{1}{2}\right) &= 0 \\
a_1\left(s^2 + \frac{3}{2}s + \frac{1}{2}\right) &= 0 \\
a_1(s+1)\left(s + \frac{1}{2}\right) &= 0
\end{aligned}$$

We note for both solutions $s = 0, 1/2$ the only way for the above to hold true is if $a_1 = 0$. This is okay as the values of s are separated by a non-integer value, which implies that the solutions for each s are linearly independent. Next we take the x^{n+s} coefficients which includes all term to get

$$\begin{aligned}
0 &= a_{n+2}(n+s+2)(n+s+1) + \frac{1}{2}a_{n+2}(n+s+2) + \frac{1}{2}a_n \\
0 &= a_{n+2}\left((n+s+2)(n+s+1) + \frac{1}{2}(n+s+2)\right) + \frac{1}{2}a_n \\
&\Downarrow \\
a_{n+2} &= -\frac{a_n}{2(n+s+2)(n+s+1) + (n+s+2)}
\end{aligned}$$

We note that due to this recursion relation we have $a_1 = a_{\text{odd}} = 0$. We can choose $s = 0$ so that the relation becomes

$$a_{n+2} = -\frac{a_n}{2(n+1)(n+2) + (n+2)}$$

which allows us to calculate coefficients in terms of a_0 by

$$a_2 = -\frac{1}{2(1)(2)+2}a_0 = -\frac{1}{6}a_0 a_4 \quad = -\frac{1}{2(3)(4)+2}a_0 = \frac{1}{168}a_0 a_6 = -\frac{1}{2(5)(6)+2}a_0 = -\frac{1}{10416}a_0 \quad \vdots$$

So we can say that our first solution is

$$y_1(x) = a_0\left(-\frac{1}{6} + \frac{1}{168}x^2 - \frac{1}{10416}x^4 + ...\right)$$

Now we can find the other solution by choosing $s = 1/2$ while keeping $a_1 = 0$ to give the recursion relation

$$a_{n+2} = -\frac{a_n}{2(n+5/2)(n+3/2) + (n+5/2)} = -\frac{2a_n}{(2n+5)(2n+3) + (2n+5)}$$

Calculating our coefficients yields

$$\begin{aligned}
a_2 &= -\frac{1}{10}a_0 \\
a_4 &= -\frac{1}{36}a_2 = \frac{1}{360}a_0 \\
a_6 &= -\frac{1}{55}a_4 = -\frac{1}{19800}a_0 \qquad \vdots
\end{aligned}$$

which yields the second solution

$$y_2(x) = a_0\left(-\frac{1}{10}x^{1/2} + \frac{1}{360}x^{5/2} - \frac{1}{19800}x^{7/2} + ...\right)$$