

Physics 611
Electromagnetic Theory II
Professor Christopher Pope

Homework #4

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October 19th, 2016

1 Problem #1

(a) Given the energy-momentum tensor for the electromagnetic field

$$T_{\mu\nu} = \frac{1}{8\pi} (F_{\mu\rho} F_{\nu}{}^{\rho} + {}^*F_{\mu\rho} {}^*F_{\nu}{}^{\rho}) \quad (1.1)$$

where ${}^*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. We can see that equation 1.1 can be reduced by noting that

$$\begin{aligned} {}^*F_{\mu\rho} {}^*F_{\nu}{}^{\rho} &= \left(\frac{1}{2}\epsilon_{\mu\rho\pi\sigma}F^{\pi\sigma} \right) \eta_{\nu\lambda} {}^*F^{\lambda\rho} \\ &= \left(\frac{1}{2}\epsilon_{\mu\rho\pi\sigma}F^{\pi\sigma} \right) \eta_{\nu\lambda} \frac{1}{2}\epsilon^{\lambda\rho\phi\omega}F_{\phi\omega} \\ &= \frac{1}{4}\eta_{\nu\lambda}F^{\pi\sigma}F_{\phi\omega} \left(-\delta_{\mu}^{\lambda}\delta_{\pi}^{\phi}\delta_{\sigma}^{\omega} - \delta_{\mu}^{\phi}\delta_{\pi}^{\omega}\delta_{\sigma}^{\lambda} - \delta_{\mu}^{\omega}\delta_{\pi}^{\lambda}\delta_{\sigma}^{\phi} + \delta_{\mu}^{\phi}\delta_{\pi}^{\lambda}\delta_{\sigma}^{\omega} + \delta_{\mu}^{\lambda}\delta_{\pi}^{\omega}\delta_{\sigma}^{\phi} + \delta_{\mu}^{\omega}\delta_{\pi}^{\phi}\delta_{\sigma}^{\lambda} \right) \\ &= \frac{1}{4}(-\eta_{\nu\mu}F^{\pi\sigma}F_{\pi\sigma} - \eta_{\nu\sigma}F^{\pi\sigma}F_{\mu\pi} - \eta_{\nu\pi}F^{\sigma\pi}F_{\mu\sigma} + \eta_{\nu\pi}F^{\pi\sigma}F_{\mu\sigma} + \eta_{\nu\mu}F^{\pi\sigma}F_{\sigma\pi} + \eta_{\nu\sigma}F^{\pi\sigma}F_{\pi\mu}) \\ &= \frac{1}{4}(-2\eta_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} + 4\eta_{\nu\sigma}F^{\sigma\rho}F_{\mu\rho}) \\ &= -\frac{1}{2}(\eta_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} - 2F_{\nu}{}^{\rho}F_{\mu\rho}) \end{aligned}$$

We see that equation 1.1 becomes

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{8\pi} (F_{\mu\rho} F_{\nu}{}^{\rho} + {}^*F_{\mu\rho} {}^*F_{\nu}{}^{\rho}) \\ &\Downarrow \\ &= \frac{1}{8\pi} \left(F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{2}\eta_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} + F_{\nu}{}^{\rho}F_{\mu\rho} \right) \\ &= \frac{1}{4\pi} \left(F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{4}\eta_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} \right) \end{aligned}$$

Note this is the typical form of the electromagnetic field energy momentum tensor.

(b) We can find the expression for $T_{\mu\rho}T^{\nu\rho}$ by noting that

$$\begin{aligned} T^{\nu\rho} &= \eta^{\nu\sigma}\eta^{\rho\lambda}T_{\sigma\lambda} \\ &= \frac{1}{4\pi} \left(\eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}{}^{\phi} - \frac{1}{4}\eta^{\nu\sigma}\eta^{\rho\lambda}\eta_{\sigma\lambda}F^{\phi\omega}F_{\phi\omega} \right) \\ &= \frac{1}{4\pi} \left(\eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}{}^{\phi} - \frac{1}{4}\eta^{\nu\rho}F^{\sigma\lambda}F_{\sigma\lambda} \right) \end{aligned}$$

So we can calculate defining $F^2 \equiv F^{\mu\nu}F_{\mu\nu}$

$$\begin{aligned} T_{\mu\rho}T^{\nu\rho} &= \frac{1}{(4\pi)^2} \left(F_{\mu\sigma}F_{\rho}{}^{\sigma} - \frac{1}{4}\eta_{\mu\rho}F^{\sigma\lambda}F_{\sigma\lambda} \right) \left(\eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}{}^{\phi} - \frac{1}{4}\eta^{\nu\rho}F^{\sigma\lambda}F_{\sigma\lambda} \right) \\ &= \frac{1}{(4\pi)^2} \left(\eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}{}^{\phi}F_{\mu\omega}F_{\rho}{}^{\omega} + \frac{1}{16}\eta_{\mu\rho}\eta^{\nu\rho}(F^2)^2 - \frac{1}{4}F^2\eta^{\nu\rho}F_{\mu\sigma}F_{\rho}{}^{\sigma} - \frac{1}{4}F^2\eta_{\mu\rho}\eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}{}^{\phi} \right) \\ &= \frac{1}{(4\pi)^2} \left(\eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}{}^{\phi}F_{\mu\omega}F_{\rho}{}^{\omega} + \frac{1}{16}\delta_{\mu}^{\nu}(F^2)^2 - \frac{1}{4}F^2\eta^{\nu\rho}F_{\mu\sigma}F_{\rho}{}^{\sigma} - \frac{1}{4}F^2\eta^{\nu\rho}F_{\rho\sigma}F_{\mu}{}^{\sigma} \right) \end{aligned}$$

Next we consider the term

$$\eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}^{\phi}F_{\mu\omega}F_{\rho}^{\omega}=\eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}^{\phi}F_{\mu\omega}F_{\rho}^{\omega}$$

2 Problem #2

(a) Given the Lagrangian density

$$\mathcal{L} = -\frac{1}{16}F^{\mu\nu}F_{\mu\nu} - \frac{m^2}{8\pi}A^\mu A_\mu + J^\mu A_\mu \quad (2.1)$$

we can derive the equations of motion from the *Euler-Lagrange equations*

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\nu A_\mu)} \right) = 0 \quad (2.2)$$

Noting that $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ we have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial A_\mu} &= -\frac{m^2}{8\pi}A^\mu + J^\mu \\ \frac{\partial \mathcal{L}}{\partial (\partial_\nu A_\mu)} &= \frac{\partial}{\partial (\partial_\nu A_\mu)} \left(-\frac{1}{16}F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) \right) = \frac{1}{4}F^{\mu\nu} \end{aligned}$$

So equation 2.2 yields the equation of motion

$$-\partial_\nu F^{\mu\nu} + m^2 A^\mu = 4\pi J^\mu$$

(b) Using the result from part (a) we can find the solution for the scalar potential $\phi \equiv A^0$ for a point charge q located at the origin. This implies that we have $J^0 = q\delta^3(\mathbf{r})$ which allows us to solve for $\mu = 0$

$$\begin{aligned} -\partial_\nu F^{0\nu} + m^2 A^0 &= 4\pi q\delta^3(\mathbf{r}) \\ \Downarrow \\ -\nabla^2 \phi + m^2 \phi &= 4\pi q\delta^3(\mathbf{r}) \\ \Downarrow \\ -\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + m^2 \phi &= 4\pi q\delta(r) \\ -\frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + r^2 m^2 \phi &= 4\pi r^2 q\delta(r) \\ \Downarrow \\ r\phi(r) &= Ce^{-mr} \end{aligned}$$

Where we can solve for the constant C by noting that for *Gauss' law* to hold $C = q$ therefore

$$\phi(r) = \frac{qe^{-mr}}{r}$$

3 Problem #3

- (a) Given the potentials describing an electric charge, e , moving with constant velocity, \mathbf{v}

$$\phi(\mathbf{r}, t) = \frac{e\gamma}{r'}, \quad \mathbf{A}(\mathbf{r}, t) = \mathbf{v}\phi$$

we note that the magnetic field is given as

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} \\ &= \nabla \times (\mathbf{v}\phi) \\ &= \nabla\phi \times \mathbf{v} + \phi(\nabla \times \mathbf{v})^0 \\ &= -\mathbf{E} \times \mathbf{v} \\ &= \mathbf{v} \times \mathbf{E} \end{aligned}$$

- (b) Using the expressions we were given in part (a) we can find the electric field by

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$