# Physics 615

## Methods of Theoretical Physics I

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# Homework #6

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# 1 Problem #1

For the function

$$f(z) = \frac{e^z}{z(z^2 - 16)}$$

we can see that $f(z)$ is analytic everywhere except $z = 0$ and $|z| = 4$. So we can calculate the integral

$$I = \oint_C \frac{e^z}{z(z^2 - 16)}$$

where the contour $C$ is defined as the boundary of the annulus between the circles $|z| = 1$ and $|z| = 3$. We note that $f(z)$ is analytic everywhere along $C$ which implies that

$$I = \oint_C \frac{e^z}{z(z^2 - 16)} = 0$$

due to the fact that $C$ is a closed contour.

# 2 Problem #2

1) For the integral

$$\oint_C e^{x^2} \left( \frac{1}{z^2} - \frac{1}{z^3} \right)$$

where $C$ is the unit circle centered at the origin. We first expand $e^{z^2}$ by

$$e^{z^2} = 1 + z^2 + \frac{z^4}{2} + ...$$

Which makes our integral become

$$\begin{aligned} \oint_C e^{x^2} \left( \frac{1}{z^2} - \frac{1}{z^3} \right) &= \oint_C (1 + z^2 + \frac{z^4}{2} + ...) \left( \frac{1}{z^2} - \frac{1}{z^3} \right) \\ &= \oint_C \frac{1}{z^2} - \frac{1}{z^3} + 1 - \frac{1}{z} + ... \end{aligned}$$

We note that the only integral in this sum that is nonzero is the $z^{-1}$. We can use the fact that $z$ on the unit circle is given by $z = e^{i\phi}$ which allows us to integrate over the circle by

$$\begin{aligned} -\oint_C \frac{1}{z} dz &= -\oint_C \frac{1}{e^{i\phi}} d(e^{i\phi}) \\ &= -\oint_0^{2\pi} \frac{1}{e^{i\phi}} i e^{i\phi} d\phi \\ &= -\oint_0^{2\pi} i d\phi \\ &= -2\pi i \end{aligned}$$

So the solution to the integral is

$$\oint_C e^{x^2} \left( \frac{1}{z^2} - \frac{1}{z^3} \right) = -2\pi i$$

2) For the integral

$$\oint_C \frac{(\sin z)^6}{(z-\frac{\pi}{6})^3}$$

we note the derivative formula

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(w)}{(w-z)^{n+1}}$$

where we can rearrange

$$\begin{aligned}
f^{(n)}(z) &= \frac{n!}{2\pi i}\oint_C \frac{f(w)}{(w-z)^{n+1}} \\
&\Downarrow \\
\oint_C \frac{(\sin z)^6}{(z-\pi/6)^3} &= \frac{n!}{2\pi i} f^{(2)}(z) \\
&= \frac{2\pi i}{2!}\frac{d^2}{dz^2}(\sin z)^6\bigg|_{\pi/6} \\
&= \frac{2\pi i}{2}\frac{d}{dz}6(\sin z)^5\cos z\bigg|_{\pi/6} \\
&= i\pi\left(\ 30(\sin z)^4(\cos z)^2 - 6(\sin z)^6\right|_{\pi/6} \\
&= i\pi\left(30(\sin\pi/6)^4(\cos\pi/6)^2 - 6(\sin\pi/6)^6\right) \\
&= i\pi\left(30\frac{1}{2^4}\frac{3}{4} - 6\frac{1}{2^6}\right) \\
&= i\pi\left(\frac{90}{64} - \frac{6}{64}\right) \\
&= i\pi\left(\frac{84}{64}\right) \\
&= \frac{21}{16}\pi i
\end{aligned}$$

## 3 Problem #3

Given *Rodrigues' formula*

$$P_n(t) = \frac{1}{2^n n!} \left(\frac{d}{dx}\right)^n [(x^2-1)^n] \tag{3.1}$$

we can write the derivative as a complex integral by the formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}}$$

where we note that

$$f^{(n)}(z) = \left(\frac{d}{dz}\right)^n [(z^2-1)^n]$$

which implies that $f(z) = (z^2-1)^n$. We can take the nth derivative of $f(z)$ by the integral

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{(z^2-1)^n}{(z-t)^{n+1}}$$

And by replacing into equation 3.1 we get

$$\begin{aligned} P_n(t) &= \frac{1}{2^n n!} \left(\frac{d}{dx}\right)^n [(x^2-1)^n] \\ &\Downarrow \\ P_n(t) &= \frac{1}{2^n n!} \frac{n!}{2\pi i} \oint_C \frac{(z^2-1)^n}{(z-t)^{n+1}} \\ &= \frac{1}{2\pi i \frac{1}{2^n}} \oint_C \frac{(z^2-1)^n}{(z-t)^{n+1}} \end{aligned}$$

## 4 Problem #4

For the function

$$f(z) = \frac{1}{e^z - 1}$$

we can calculate the first three non-zero terms in the *Laurent expansion* where we expand about the point $z = 0$. We expand

$$e^z = 1 + z + z^2 + z^3 + ...$$

which makes $f(z)$ become

$$f(z) = \frac{1}{z + z^2/2 + ...} = \frac{1}{z} \frac{1}{1 + z/2 + ...}$$

Now we can expand

$$\frac{1}{1+z/2} = 1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8}$$

so the first four terms of the Laurent expansion are

$$f(z) = \frac{1}{e^z - 1} = \frac{1}{z} \frac{1}{1+z/2} = z^{-1} - \frac{1}{2} z + \frac{1}{4} z^2 - \frac{1}{8} z^3$$