Physics 615

Methods of Theoretical Physics I

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Homework #10

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1 Problem #1

For the integral

$$I(x) = \int_0^\infty e^{xt - e^{2t}} dt$$

we can find the leading order behavior for $x \to \infty$. To do this we change variables by $u = e^{2t}/x$ which implies that

$$du = 2\frac{e^{2t}}{x} dt = 2u dt$$

So we can change variables to

$$\begin{aligned} I(x) &= \int_0^\infty e^{xt - e^{2t}} dt \\ &\Downarrow \\ I(x) &= \int_{1/x}^\infty \exp\left(\frac{1}{2} x \log(xu) - xu\right) \frac{1}{2u} du \\ &= e^{x/2 \log(x)} \int_{1/x}^\infty \exp\left(\frac{1}{2} x \log(u) - xu\right) \frac{1}{2u} du \\ &= x^{x/2} \int_{1/x}^\infty \exp\left(-x(u - \log(u^{1/2})\right) \frac{1}{2u} du \end{aligned}$$

We note that this is a Laplace type integral where $\phi(u) = u - \log(u^{1/2})$. We note that $\phi(u)$ has an extrema, c, which we find at

$$\begin{gathered} \phi'(u) = 0 = 1 - \frac{1}{2u} \\ \Downarrow \\ c = \frac{1}{2} \end{gathered}$$

We can test to see that the extrema at c is a minimum by noting

$$\phi''(u) = \frac{1}{2u^2} \qquad \Rightarrow \qquad \phi''(c) = 2$$

So we use that asymptotic solution given by

$$I(x) \approx f(c) e^{-x\phi(c)} \sqrt{\frac{2\pi}{x\phi''(c)}}$$

where $f(c) = x^{x/2}$ and $\phi(c) = 1/2(1 + \log(2))$ so

$$I(x) \approx x^{x/2} e^{-x/2(1+\log(2))} \sqrt{\frac{\pi}{x}} = \sqrt{\frac{\pi}{x}} \left(\frac{x}{2e}\right)^{x/2}$$

2 Problem #2

We can find the leading behavior of the integral

$$I(x) = \int_0^\infty \cos\left(xt^2 - t\right) dt$$

by complexifying this integral and writing it in the form $e^{x\phi(z)}$ by

$$I(x) = \frac{1}{2}\int_0^\infty \left(e^{x(iz^2 - iz/x)} + e^{-x(iz^2 - iz/x)}\right) dz$$

which allows us to approximate this integral by the method of steepest descent which yields

$$I(x) \approx f(z_0) e^{i\theta} e^{x\phi(z_0)} \sqrt{\frac{2\pi}{xa}}$$

Where for the first term we have

$$\phi(z) = iz^2 - \frac{i}{x} z$$

which has a critical point

$$\phi'(z) = 0 = 2iz - \frac{i}{x} \qquad \Rightarrow \qquad z_0 = \frac{1}{2x}$$

Using this critical point we can calculate α setting

$$\phi''(z_0) = ae^{i\alpha}$$

so we have $\phi''(z) = 2i$ which implies that $a = 2$ and $\alpha = \pi/2$. Using α we can determine θ by

$$\theta = -\frac{\alpha}{2} \pm \frac{\pi}{2}$$

so $\theta = \pi/4, -3\pi/4$ where we can pick either solution as still be able to deform the contour back to the real axis. So this first integral has the solution

$$\begin{aligned} I_1(x) &\approx e^{i\pi/4} e^{x(i/4x^2 - i/2x^2)} \sqrt{\frac{2\pi}{2x}} \\ &\approx e^{i\pi/4} e^{-i/4x} \sqrt{\frac{\pi}{x}} \end{aligned}$$

Now we can solve the negative integral by noting that

$$\phi(z) = -iz^2 + \frac{i}{x} z$$

which has a critical point

$$\phi'(z) = 0 = -2iz + \frac{i}{x} \qquad \Rightarrow \qquad z_0 = \frac{1}{2x}$$

Now we can find α by seeing $\phi''(z) = -2i$ which implies that $a = 2$ and $\alpha = -\pi/2$. So we can find $\theta = -\pi/4, 3\pi/2$ So we have an approximate solution by picking $\theta = -\pi/4$

$$\begin{aligned} I_2(x) &\approx e^{-i\pi/4} e^{x(-i/4x^2 + i/2x^2)} \sqrt{\frac{2\pi}{2x}} \\ &\approx e^{-i\pi/4} e^{+i/4x} \sqrt{\frac{\pi}{x}} \end{aligned}$$

So we can find the total integral by

$$
\begin{aligned}
I(x) &\approx \frac{1}{2}\left(e^{i\pi/4}e^{-i/4x}\sqrt{\frac{\pi}{x}} + e^{-i\pi/4}e^{+i/4x}\sqrt{\frac{\pi}{x}}\right) \\
&\approx \frac{\sqrt{\pi}}{2\sqrt{x}}\left(e^{-i(1/4x-\pi/4)} + e^{i(1/4x-\pi/4)}\right) \\
&\approx \sqrt{\frac{\pi}{x}}\cos\left(\frac{1}{4x} - \frac{\pi}{4}\right)
\end{aligned}
$$

3 Problem #3

We can take the Fourier transform, $F(\omega)$, of the Gaussian function given by

$$
f(t) = e^{-a^2t^2}, \qquad a \in \mathbb{R}
$$

by calculating the integral

$$
\begin{aligned}
F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t} \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2t^2}e^{i\omega t} \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2t^2+i\omega t} \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2t^2+i\omega t+\omega^2/4a^2-\omega^2/4a^2} \\
&= \frac{1}{\sqrt{2\pi}}e^{-\omega^2/4a^2}\int_{-\infty}^{\infty} e^{-(at+i\omega/2a)^2} \\
&= \frac{1}{\sqrt{2\pi}}e^{-\omega^2/4a^2}\frac{\sqrt{\pi}}{a} \\
&= \frac{1}{\sqrt{2}a}e^{-\omega^2/(2a)^2}
\end{aligned}
$$