# Physics 601

## Analytical Mechanics

Professor Siu Chin

# Homework #10

Joe Becker
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# 1 Problem #1

If we preform a canonical transformation from $(q,p) \to (Q,P)$ by

$$F_2(q,P) = qP + \frac{1}{2}\epsilon H(q,P)$$

where the Hamiltonian is given as

$$H(q,p) = \frac{p^2}{2m} + V(q)$$

we can solve for the transformation variables by

$$p = \frac{\partial F_2}{\partial q} = P + \epsilon \frac{1}{2}\frac{dV}{dq}$$
$$\Downarrow$$
$$P = p - \epsilon \frac{1}{2}\frac{dV}{dq}$$

and

$$Q = \frac{\partial F_2}{\partial P} = q + \epsilon \frac{1}{2}\frac{P}{m}$$
$$\Downarrow$$
$$Q = q + \epsilon \frac{p}{2m} - \epsilon^2 \frac{1}{4m}\frac{dV}{dq}$$

This allows us to preform a new canonical transformation from $(Q,P) \to (q',p')$ by

$$F_3(P,q') = -Pq' + \frac{1}{2}\epsilon H(q',P)$$

Which yields

$$Q = -\frac{\partial F_3}{\partial P} = q' - \epsilon \frac{1}{2}\frac{P}{m}$$
$$\Downarrow$$
$$q' = Q + \epsilon \frac{1}{2}\frac{P}{m}$$

and

$$\begin{aligned} p' = -\frac{\partial F_3}{\partial q'} &= P - \epsilon \frac{1}{2}\frac{dV}{dq'} \\ &= P - \epsilon \frac{1}{2}\frac{dV}{dQ}\cancelto{1}{\frac{dQ}{dq'}} \end{aligned}$$
$$\Downarrow$$
$$p' = P - \epsilon \frac{1}{2}\frac{dV}{dQ}$$

Now we can transform $(Q,P)$ back to $(q,p)$ by the transformations we first found to get the transformation from $(q,p) \to (q',p')$. So first for $q'$ we have

$$\begin{aligned} q' &= Q + \epsilon \frac{1}{2}\frac{P}{m} \\ &= q + \epsilon \frac{p}{2m} - \epsilon^2 \frac{1}{4m}\frac{dV}{dq} + \epsilon \frac{1}{2m}\left(p - \epsilon \frac{1}{2}\frac{dV}{dq}\right) \\ &= q + \epsilon \frac{p}{m} - \epsilon^2 \frac{1}{2m}\frac{dV}{dq} \end{aligned}$$

We note the expansion of $q(t+\epsilon)$ about epsilon yields

$$q(t+\epsilon) = q + \epsilon\frac{dq}{dt} + \epsilon^2\frac{1}{2}\frac{d^2q}{dt^2}$$

we note that this is the same as the transformed result we found, due to the fact that

$$\frac{dq}{dt} = \frac{p}{m}, \qquad \frac{d^2q}{dt^2} = -\frac{1}{m}\frac{dV}{dq}$$

Which implies that $q' = q(t+\epsilon)$. Now we can do the same with $p'$ as

$$\begin{aligned}
p' &= P - \epsilon\frac{1}{2}\frac{dV(Q)}{dQ}\frac{dQ}{dq} \\
&= p - \epsilon\frac{1}{2}\frac{dV}{dq} - \epsilon\frac{1}{2}\frac{dV}{dq}\frac{d}{dq}\left(q + \epsilon\frac{p}{2m} - \epsilon^2\frac{1}{4m}\frac{dV}{dq}\right) \\
&= p - \epsilon\frac{1}{2}\frac{dV}{dq} - \epsilon\frac{1}{2}\frac{dV}{dq}\left(1 - \epsilon^2\frac{1}{4m}\frac{d^2V}{dq^2}\right) \\
&= p - \epsilon\frac{dV}{dq} + \mathcal{O}(\epsilon^3)
\end{aligned}$$

We see that this agrees to second order due to the fact that $\ddot{p} = 0$.

## 2 Problem #2

(a) For the free fall Hamiltonian

$$H = \frac{p^2}{2m} + mgq$$

where we are given the principle function

$$S(q, \alpha, t) = W(q, \alpha) - Et$$

Using this we can apply the Hamilton-Jacobi equation

$$H + \frac{dS}{dt} = 0$$

where we note that $p$ is given by

$$p = \frac{dS}{dq} = \frac{dW}{dq}$$

So we have the equation

$$0 = \frac{1}{2m}\left(\frac{dW}{dq}\right)^2 + mgq - E$$

(b) Using the equation if part (a) we can solve for $W(q, \alpha)$ by

$$\begin{aligned}
0 &= \frac{1}{2m}\left(\frac{dW}{dq}\right)^2 + mgq - E \\
&\Downarrow \\
\frac{dW}{dq} &= \Big(2m(E - mgq)\Big)^{1/2} \\
&\Downarrow \\
W &= C + \int \Big(2m(E - mgq)\Big)^{1/2} dq \\
&= C - \frac{1}{3m^2 g}\Big(2m(E - mgq)\Big)^{3/2}
\end{aligned}$$

Note we take $\alpha$ as a constant of motion which we see is $E$ so

$$W(q, \alpha) = C - \frac{1}{3m^2 g}\Big(2m(\alpha - mgq)\Big)^{3/2}$$

(c) We can solve for the equation of motion, $q(t)$, by taking the transformation

$$\begin{aligned}
\beta &= \frac{\partial S}{\partial \alpha} = \frac{\partial W}{\partial \alpha} - t \\
&\Downarrow \\
\beta + t &= -\frac{1}{3m^2 g}\frac{3}{2}\Big(2m(\alpha - mgq)\Big)^{1/2} 2m \\
&= -\frac{1}{mg}\Big(2m(\alpha - mgq)\Big)^{1/2} \\
&\Downarrow \\
(mg(\beta + t))^2 - 2m\alpha &= -2m^2 gq \\
&\Downarrow \\
q(t) &= \frac{\alpha}{mg} - \frac{1}{2}g(\beta + t)^2 = \frac{\alpha}{mg} - \frac{1}{2}g\beta^2 - g\beta t - \frac{1}{2}gt^2
\end{aligned}$$

(d) Now we can apply the initial conditions

$$p(0) = mv_0, \qquad q(0) = q_0$$

to find

$$q(0) = q_0 = \frac{\alpha}{mg} - \frac{1}{2}g\beta^2$$

and we take the momentum initial condition as

$$\begin{aligned}
p(0) = mv_0 &= \Big(2m(\alpha - mgq(0))\Big)^{1/2} \\
&= \Big(2m(\alpha - \alpha + 1/2mg^2\beta^2)\Big)^{1/2} \\
&= \pm mg\beta \\
&\Downarrow \\
\beta &= \pm\frac{v_0}{g}
\end{aligned}$$

So replacing in terms of $q_0$ and $v_0$ we find

$$\begin{aligned}
q(t) &= \frac{\alpha}{mg} - \frac{1}{2}g\beta^2 - g\beta t - \frac{1}{2}gt^2 \\
&\Downarrow \\
q(t) &= q_0 \pm v_0 t - \frac{1}{2}gt^2
\end{aligned}$$

## 3 Problem #3

(a) For the simple harmonic oscillator we have the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2$$

which gives us the Hamilton-Jacobi equation using the same principle functional form as in problem (2) as

$$0 = \frac{1}{2m}\left(\frac{dW}{dq}\right)^2 + \frac{1}{2}kq^2 - E$$

(b) We can solve for $W(q,\alpha)$ where we take $\alpha = E$ as

$$\frac{dW}{dq} = \left(2m(\alpha - 1/2kq^2)\right)^{1/2}$$
$$\Downarrow$$
$$W = C + \int \left(2m(\alpha - 1/2kq^2)\right)^{1/2} dq$$

Note we will solve the integral when solving for $q$

(c) We solve for $q(t)$ by

$$\begin{aligned}
\beta = \frac{\partial S}{\partial \alpha} &= \frac{\partial W}{\partial \alpha} - t \\
&\Downarrow \\
\beta + t &= \frac{\partial}{\partial \alpha}\int \left(2m(\alpha - 1/2kq^2)\right)^{1/2} dq \\
&= \int \frac{1}{2}\left(2m(\alpha - 1/2kq^2)\right)^{-1/2} 2m dq \\
&= \sqrt{\frac{m}{2\alpha}}\int \left(1 - \frac{kq^2}{2\alpha}\right)^{-1/2} dq \\
&= \sqrt{\frac{m}{2\alpha}} \arcsin\left(\sqrt{\frac{k}{2\alpha}}q\right)\sqrt{\frac{2\alpha}{k}} \\
&\Downarrow \\
q(t) &= \sqrt{\frac{2\alpha}{k}}\sin\left(\sqrt{\frac{k}{m}}(\beta + t)\right)
\end{aligned}$$

(d) We can apply the initial conditions $q(0) = q_0$ and $p(0) = p_0$. This implies that

$$q_0 = \sqrt{\frac{2\alpha}{k}}\sin\left(\sqrt{\frac{k}{m}}\beta\right)$$

and we can find

$$\begin{aligned}
p_0 &= \left(2m(\alpha - 1/2kq_0^2)\right)^{1/2} \\
&= \left(2m\alpha - 2m\alpha\sin^2\left(\sqrt{\frac{k}{m}}\beta\right)\right)^{1/2} \\
&= \sqrt{2m\alpha}\cos\left(\sqrt{\frac{k}{m}}\beta\right)
\end{aligned}$$

We note that we can expand $q(t)$ using trigonometric addition

$$
\begin{aligned}
q(t) &= \sqrt{\frac{2\alpha}{k}} \sin\left(\sqrt{\frac{k}{m}}(\beta + t)\right) \\
&\Downarrow \\
q(t) &= \sqrt{\frac{2\alpha}{k}} \sin\left(\sqrt{\frac{k}{m}}\beta\right) \cos\left(\sqrt{\frac{k}{m}}t\right) + \sqrt{\frac{2\alpha}{k}} \sin\left(\sqrt{\frac{k}{m}}\beta\right) \sin\left(\sqrt{\frac{k}{m}}t\right) \\
q(t) &= q_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t)
\end{aligned}
$$

Where we defined $\omega^2 = k/m$. We can use this result to get $p(t)$ by

$$
\begin{aligned}
p(t) &= \left(2m(\alpha - 1/2kq(t)^2)\right)^{1/2} \\
&\Downarrow \\
p(t) &= \left(2m\alpha - 2m\alpha k \frac{2\alpha}{k} \sin^2(\omega(\beta + t))\right)^{1/2} \\
&= \sqrt{2m\alpha} \cos(\omega\beta) \cos(\omega t) - \sqrt{2m\alpha} \sin(\omega\beta) \sin(\omega t) \\
&= p_0 \cos(\omega t) - q_0 m\omega \sin(\omega t)
\end{aligned}
$$

Note as we expect $p_0 = m\dot{q}$.

# 4 Problem #4

For the system with the following kinetic energy, $T$, and potential energy, $V$

$$T = \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2)(q_1^2 + q_2^2), \qquad V = (q_1^2 + q_2^2)^{-1}$$

We have the Lagrangian

$$\frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2)(q_1^2 + q_2^2) - (q_1^2 + q_2^2)^{-1}$$

which allows us to calculate the generalized momenta as

$$\begin{aligned} p_1 &= \frac{\partial L}{\partial \dot{q}_1} = \dot{q}_1(q_1^2 + q_2^2) \\ p_2 &= \frac{\partial L}{\partial \dot{q}_2} = \dot{q}_2(q_1^2 + q_2^2) \end{aligned}$$

Using these we can calculate the Hamiltonian by

$$\begin{aligned} H &= p_1\dot{q}_1 + p_2\dot{q}_2 - L \\ &\Downarrow \\ &= \frac{p_1^2}{q_1^2 + q_2^2} + \frac{p_2^2}{q_1^2 + q_2^2} - \frac{1}{2}\left(\frac{p_1^2}{(q_1^2 + q_2^2)^2}\frac{p_2^2}{(q_1^2 + q_2^2)^2}\right) + (q_1^2 + q_2^2)^{-1} \\ &= \frac{p_1^2 + p_2^2 + 2}{2(q_1^2 + q_2^2)} \end{aligned}$$

This yields the Hamilton-Jacobi equation

$$(q_1^2 + q_2^2)^{-1}\left(\frac{1}{2}\left(\frac{\partial S}{\partial q_1}\right)^2 + \frac{1}{2}\left(\frac{\partial S}{\partial q_2}\right)^2 + 1\right) + \frac{\partial S}{\partial t} = 0$$

where we define the principle function as

$$S(q_1, q_2, \alpha_1, \alpha_2, t) = W_1(q_1, \alpha_1, \alpha_2) + W(q_2, \alpha_1, \alpha_2) - Et$$

This makes the Hamilton-Jacobi equation become

$$\begin{aligned} 0 &= (q_1^2 + q_2^2)^{-1}\left(\frac{1}{2}\left(\frac{\partial S}{\partial q_1}\right)^2 + \frac{1}{2}\left(\frac{\partial S}{\partial q_2}\right)^2 + 1\right) + \frac{\partial S}{\partial t} \\ &\Downarrow \\ E(q_1^2 + q_2^2) &= \frac{1}{2}\left(\frac{\partial W_1}{\partial q_1}\right)^2 + \frac{1}{2}\left(\frac{\partial W_2}{\partial q_2}\right)^2 + 1 \\ &\Downarrow \\ Eq_1^2 - \frac{1}{2}\left(\frac{\partial W_1}{\partial q_1}\right)^2 &= \frac{1}{2}\left(\frac{\partial W_2}{\partial q_2}\right)^2 + 1 - Eq_2^2 \end{aligned}$$

We note that we have a function of $q_1$ on the left and a function of $q_2$ on the right. This implies that both are constant and equal. This allows us to solve for $W_1$ and $W_2$.

$$C = Eq_1^2 - \frac{1}{2}\left(\frac{\partial W_1}{\partial q_1}\right)^2$$
$$\Downarrow$$
$$\frac{\partial W_1}{\partial q_1} = \sqrt{2(Eq_1^2 - C)}$$
$$\Downarrow$$
$$W_1(q_1, \alpha_1, \alpha_2) = \int \sqrt{2(Eq_1^2 - C)}dq_1$$

By the same process we can find $W_2$ as

$$C = \frac{1}{2}\left(\frac{\partial W_2}{\partial q_2}\right)^2 + 1 - Eq_2^2$$
$$\Downarrow$$
$$W_2(q_2, \alpha_1, \alpha_2) = \int \sqrt{2(C + Eq_2^2 - 1)}dq_2$$

This gives us the principle function where we pick $\alpha_1 = E$ and $\alpha_2 = C$ as they are the constants of motion this makes our principle function become

$$S(q_1, q_2, \alpha_1, \alpha_2) = \int \sqrt{2(\alpha_1 q_1^2 - \alpha_2)}dq_1 + \int \sqrt{2(\alpha_2 + \alpha_1 q_2^2 - 1)}dq_2 - \alpha_1 t$$

Where we can solve for the dynamics by

$$\beta_1 = \frac{\partial S}{\partial \alpha_1} = \int \frac{q_1^2}{\sqrt{2(\alpha_1 q_1^2 - \alpha_2)}}dq_1 + \int \frac{q_2^2}{\sqrt{2(\alpha_2 + \alpha_1 q_2^2 - 1)}}dq_2 - t$$
$$\beta_2 = \frac{\partial S}{\partial \alpha_2} = \int \frac{1}{\sqrt{2(\alpha_1 q_1^2 - \alpha_2)}}dq_1 + \int \frac{1}{\sqrt{2(\alpha_2 + \alpha_1 q_2^2 - 1)}}dq_2$$