Physics 601

Analytical Mechanics

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Homework #1

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1 Problem #1

We note that for a given force, $\mathbf{F}$, we can say that $\mathbf{F}$ has an associated potential energy if the force is a *conservative force*. We can test if a force is conservative if a closed path integral over $\mathbf{F}$ is *path independent* which implies that

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0. \tag{1.1}$$

Now if we apply *Stoke's Theorem* we find that equation 1.1 becomes

$$\int_A (\nabla \times \mathbf{F}) \cdot d\mathbf{a} = 0.$$

which implies that

$$\nabla \times \mathbf{F} = 0. \tag{1.2}$$

We note that equation 1.2 implies that there exists a scaler function that is given by

$$\mathbf{F} = -\nabla V(\mathbf{r}) \tag{1.3}$$

where $V(\mathbf{r})$ is called the *Potential Energy*. So, we can use equation 1.2 to test if a given force is conservative and then use equation 1.3 to find it's associated potential energy.

(a) Given that $F_x = ay$, $F_y = F_z = 0$ we can test if if equation 1.2 holds true for the force

$$\mathbf{F} = \begin{pmatrix} ay \\ 0 \\ 0 \end{pmatrix}$$

by

$$\begin{aligned} \nabla \times \mathbf{F} &= \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ ay & 0 & 0 \end{vmatrix} \\ &= 0\hat{x} + \cancelto{0}{\partial_z(ay)\hat{y}} - \partial_y(ay)\hat{z} \\ &= -a\hat{z} \neq 0 \end{aligned}$$

So, we can see that $\mathbf{F}$ is a not conservative force. Therefore there does not exist a potential energy associated with this force.

(b) Given the force

$$\mathbf{F} = a\frac{\mathbf{r}}{r^3}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and $\mathbf{r} = (x, y, z)$. We test if equation 1.2 is true by first noting that this equation is spherically symmetric so we choose to work in spherical coordinates and that the unit vector $\hat{r}$ is given by

$$\hat{r} = \frac{\mathbf{r}}{r}$$

which implies that our force can be written as

$$\mathbf{F} = a\frac{1}{r^2}\frac{\mathbf{r}}{r} = a\frac{\hat{r}}{r^2}.$$

Now we are able to solve equation 1.2 in spherical coordinates by

$$\begin{aligned}
\nabla \times \mathbf{F} &= \det \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ \partial_r & \frac{1}{r}\partial_\theta & \frac{1}{r\sin(\theta)}\partial_\phi \\ \frac{a}{r^2} & 0 & 0 \end{vmatrix} \\
&= 0\hat{r} + \cancelto{0}{\frac{1}{r\sin(\theta)}\partial_\phi\left(\frac{a}{r^2}\right)}\hat{\theta} + \cancelto{0}{\frac{1}{r}\partial_\theta\left(\frac{a}{r^2}\right)}\hat{\phi} \\
&= 0
\end{aligned}$$

So we see that equation 1.2 holds true. Therefore we can find the associated potential energy to $\mathbf{F}$ by solving equation 1.3 for $V(\mathbf{r})$.

$$\begin{gathered}
\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r}) \\
\Downarrow \\
V(\mathbf{r}) = -\int_0^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}'
\end{gathered}$$

Note in spherical coordinates our differential becomes

$$d\mathbf{r} = dr\hat{r} + rd\theta\hat{\theta} + r\sin(\theta)d\phi\hat{\phi}.$$

So we can solve for $\mathbf{F}(\mathbf{r})$ by

$$\begin{aligned}
V(\mathbf{r}) &= -\int_0^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}' \\
&= -\int_0^{\mathbf{r}} \left(a\frac{\hat{r'}}{r'^2}\right) \cdot \left(dr'\hat{r} + r'd\theta'\hat{\theta} + r'\sin(\theta')d\phi'\hat{\phi}\right) \\
&= -\int_0^{r} \frac{a}{r'^2}dr' \\
&= -a\left(\frac{1}{r'}\right|^{r} \\
&\Downarrow \\
V(r) &= a\frac{1}{r}
\end{aligned}$$

(c) Given that $F_x = a\frac{y}{r}$, $F_y = -a\frac{x}{r^2}$, and $F_z = 0$ we can test if if equation 1.2 holds true for the force

$$\mathbf{F} = \begin{pmatrix} a\frac{y}{r} \\ -a\frac{x}{r^2} \\ 0 \end{pmatrix} = \begin{pmatrix} a\frac{r\sin(\theta)\cos(\phi)}{r} \\ -a\frac{r\sin(\theta)\sin(\phi)}{r^2} \\ 0 \end{pmatrix} = \begin{pmatrix} a\sin(\theta)\cos(\phi) \\ -a\frac{\sin(\theta)\sin(\phi)}{r} \\ 0 \end{pmatrix}$$

Note that we converted x and y to spherical coordinates. Again we use equation 1.2 to see if this force is conservative.

$$
\begin{aligned}
\nabla \times \mathbf{F} &= \det \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ \partial_r & \dfrac{1}{r}\partial_\theta & \dfrac{1}{r\sin(\theta)}\partial_\phi \\ a\sin(\theta)\cos(\phi) & -a\dfrac{\sin(\theta)\sin(\phi)}{r} & 0 \end{vmatrix} \\
&= -\frac{1}{r\cancel{\sin(\theta)}}\partial_\phi\left(-a\frac{\cancel{\sin(\theta)}\sin(\phi)}{r}\right)\hat{r} \\
&\quad + \frac{1}{r\cancel{\sin(\theta)}}\partial_\phi\left(a\cancel{\sin(\theta)}\cos(\phi)\right)\hat{\theta} \\
&\quad + \left(\partial_r\left(-a\frac{\sin(\theta)\sin(\phi)}{r}\right) - \frac{1}{r}\partial_\theta\left(a\sin(\theta)\cos(\phi)\right)\right)\hat{\phi} \\
&= -\frac{a\cos(\phi)}{r^2}\hat{r} - \frac{a\sin(\phi)}{r}\hat{\theta} + \left(a\frac{\sin(\theta)\sin(\phi)}{r^2} - \frac{a\cos(\theta)\cos(\phi)}{r}\right)\hat{\phi} \neq 0
\end{aligned}
$$

Therefore the force that is given is not a conservative force. This implies that there does not exist a potential energy for said force.

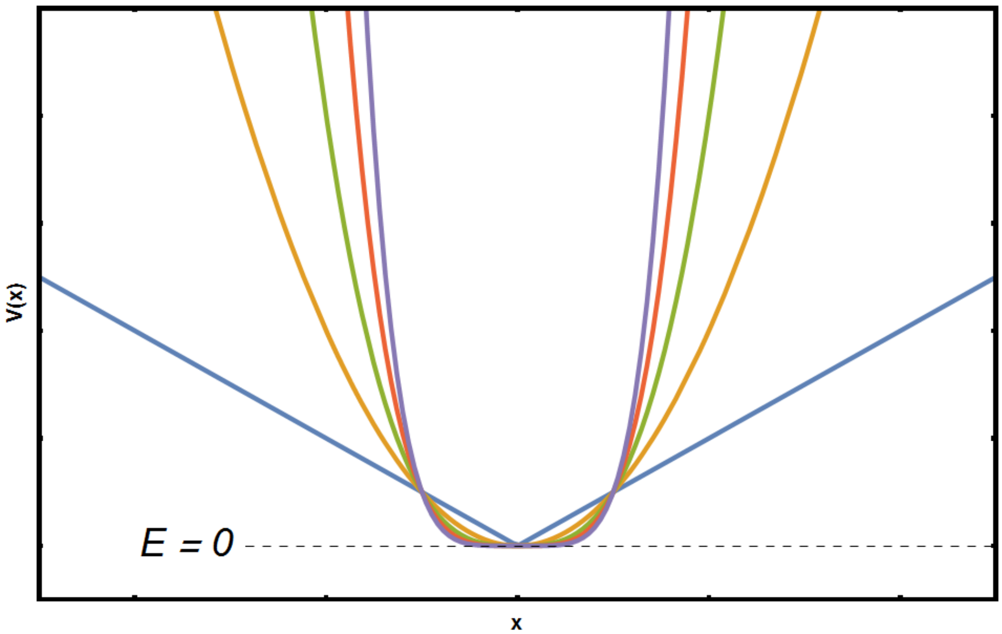

Figure 1: Plot of the potential energy $V(x) = A|x|^n$ for $n = 1, 2, 3, ..., 5$

2 Problem #2

(a) Given the potential energy $V(x) = A|x|^n$ where $A > 0$ and $n = 1, 2, 3, ...$ we can plot the potential as shown in Figure 1. Note that the dotted line designates the energies that have bounded oscillations. As we see in figure 1 that the energy range for bounded oscillations is

$$0 < E < \infty.$$

For this given potential we can calculate the period of oscillations for the case where $n = 2$ by first solving the *Integral of Motion*

$$dt = \int \frac{dx}{\sqrt{2/m(E - V(x))}}. \tag{2.1}$$

Therefore for Ax^2 we see equation 2.1 becomes

$$dt = \int \frac{dx}{\sqrt{2/m(E - Ax^2)}}$$
$$\Downarrow$$
$$dt = \int \frac{dx}{\sqrt{2A/m(E/A - x^2)}}$$

Which we need to integrate from the turning points given by

$$E = V(x) = Ax^2$$
$$\Downarrow$$
$$x_{\pm} = \pm\sqrt{\frac{E}{A}}$$

to find the total period by

$$T = 2\int_{x_-}^{x_+} \frac{dx}{\sqrt{2A/m(E/A - x^2)}}.$$

This allows us to use a substitution where

$$x = \sqrt{\frac{E}{A}}\sin(\theta)$$

and

$$dx = \sqrt{\frac{E}{A}}\cos(\theta)d\theta.$$

So, by changing from x to θ we have

$$T = 2\int_{x_-}^{x_+} \frac{dx}{\sqrt{2A/m(E/A - x^2)}}$$

$$\Downarrow$$

$$\begin{aligned}
T &= 2\sqrt{\frac{m}{2A}}\int_{x_-(\theta)}^{x_+(\theta)} \frac{\sqrt{E/A}\cos(\theta)d\theta}{\sqrt{(E/A - E/A\sin^2(\theta))}} \\
&= 2\sqrt{\frac{m}{2A}}\int_{x_-(\theta)}^{x_+(\theta)} \frac{\cancel{\sqrt{E/A}\cos(\theta)}d\theta}{\cancel{\sqrt{E/A}\cos(\theta)}} \\
&= 2\sqrt{\frac{m}{2A}}\int_{x_-(\theta)}^{x_+(\theta)} d\theta
\end{aligned}$$

$$\Downarrow$$

$$T = 2\sqrt{\frac{m}{2A}}\theta\Big|_{x_-(\theta)}^{x_+(\theta)}$$

$$\Downarrow$$

$$\begin{aligned}
T &= 2\sqrt{\frac{m}{2A}}\arcsin\left(\frac{x}{\sqrt{E/A}}\right)\Bigg|_{x_-}^{x_+} \\
&= 2\sqrt{\frac{m}{2A}}\left(\arcsin\left(\frac{\sqrt{E/A}}{\sqrt{E/A}}\right) - \arcsin\left(\frac{-\sqrt{E/A}}{\sqrt{E/A}}\right)\right) \\
&= 2\sqrt{\frac{m}{2A}}\left(\arcsin(1) - \arcsin(-1)\right) \\
&= 2\sqrt{\frac{m}{2A}}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) \\
&= 2\sqrt{\frac{m}{2A}}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) \\
T &= 2\sqrt{\frac{m}{2A}}\pi
\end{aligned}$$

Note that the period of oscillation does not depend on the energy of the system E this is expected for a harmonic potential.

(b) Given the potential energy $V(x) = -\frac{V_0}{\cosh^2(\alpha x)}$ where V_0 and α are positive constants. We plotted this potential in Figure 2. Note the two dotted lines at $E = -V_0$ and $E = 0$ these points define the range of energies that allow for bounded motion given by

$$-V_0 < E < 0.$$

For this potential we can calculate the period of oscillations by first solving equation 2.1 by

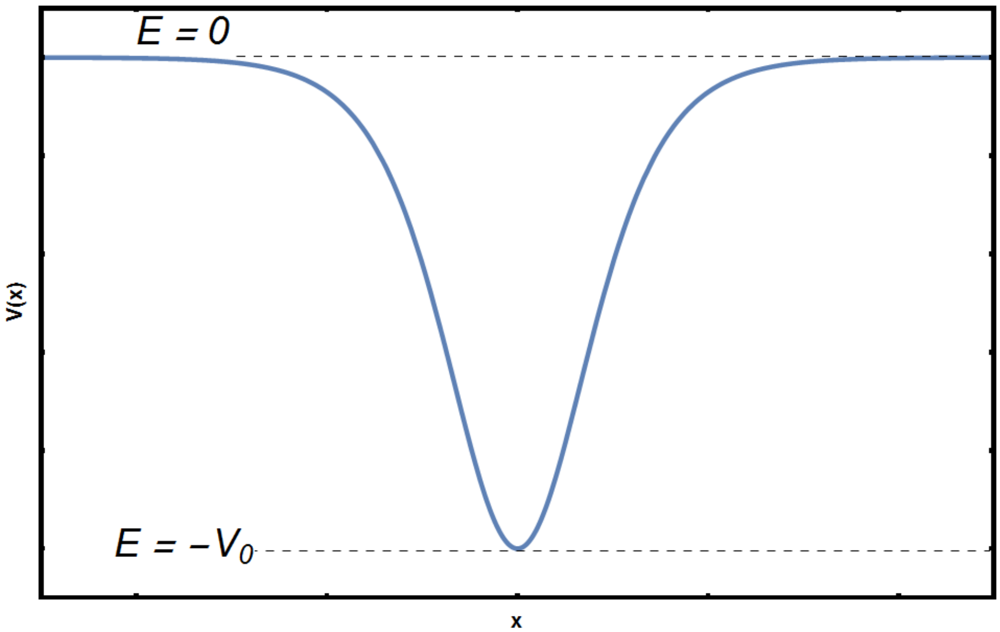

Figure 2: Plot of the potential energy $V(x) = -V_0 \cosh^2(\alpha x)$

$$dt = \int \frac{dx}{\sqrt{2/m(E - V(x))}}$$
$$\Downarrow$$
$$dt = \int \frac{dx}{\sqrt{2/m(E + V_0/\cosh^2(\alpha x))}}$$

over the bounds given by the turning points found by

$$E = V(x) = -\frac{V_0}{\cosh^2(\alpha x)}$$
$$\Downarrow$$
$$x_\pm = \pm \frac{\cosh^{-1}\left(\sqrt{\frac{V_0}{E}}\right)}{\alpha}$$

Now equation 2.1 becomes a definite integral which allows us to find the period of oscillations by

$$dt = \int \frac{dx}{\sqrt{2/m(E + V_0/\cosh^2(\alpha x))}}$$
$$\Downarrow$$
$$\begin{aligned} T &= 2\int_{x_-}^{x_+} \frac{dx}{\sqrt{2/m(E + V_0/\cosh^2(\alpha x))}} \\ &= 2\int_{x_-}^{x_+} \frac{\cosh(\alpha x)dx}{\sqrt{2/m(E\cosh^2(\alpha x) + V_0)}} \end{aligned}$$

We note that $\cosh^2(\alpha x) = 1 + \sinh^2(\alpha x)$ so that

$$\begin{aligned}
&\Rightarrow 2\int_{x_-}^{x_+} \frac{\cosh(\alpha x)dx}{\sqrt{2/m(E(1+\sinh^2(\alpha x))+V_0)}} \\
&= 2\int_{x_-}^{x_+} \frac{\cosh(\alpha x)dx}{\sqrt{2E/m(1+V_0/E+\sinh^2(\alpha x)}}
\end{aligned}$$

Now we let $a = \sqrt{1+V_0/E}$ and $t = \sinh(\alpha x)$ where

$$dt = \alpha\cosh(\alpha x)dx$$

which transforms our integral into

$$\Rightarrow \frac{2}{\alpha}\sqrt{\frac{m}{2E}}\int_{t(x_-)}^{t(x_+)} \frac{dt}{\sqrt{a^2+t^2}}$$

Next we follow another substitution such that

$$t = a\sinh(u)$$

and

$$dt = a\cosh(u)du$$

so the integral becomes

$$\begin{aligned}
&\Rightarrow \frac{2}{\alpha}\sqrt{\frac{m}{2E}}\int_{u(t(x_-))}^{u(t(x_+))} \frac{a\cosh(u)du}{\sqrt{a^2+a^2\sinh(u)^2}} \\
&= \frac{2}{\alpha}\sqrt{\frac{m}{2E}}\int_{u(t(x_-))}^{u(t(x_+))} \frac{a\cosh(u)du}{\sqrt{a^2\cosh^2(u)}} \\
&= \frac{2}{\alpha}\sqrt{\frac{m}{2E}}\int_{u(t(x_-))}^{u(t(x_+))} \frac{a\cosh(u)du}{a\cosh(u)} \\
&= \frac{2}{\alpha}\sqrt{\frac{m}{2E}}\int_{u(t(x_-))}^{u(t(x_+))} du \\
&= \frac{2}{\alpha}\sqrt{\frac{m}{2E}}u\Big|_{u(t(x_-))}^{u(t(x_+))} \\
&\Downarrow \\
&= \frac{2}{\alpha}\sqrt{\frac{m}{2E}}\sinh^{-1}(t/a)\Big|_{t(x_-)}^{t(x_+)} \\
&\Downarrow \\
&= \frac{4}{\alpha}\sqrt{\frac{m}{2E}}\sinh^{-1}\left(\frac{\sinh(\alpha x)}{\sqrt{1+V_0/E}}\right)\Big|_{0}^{x_+}
\end{aligned}$$

Note due to the symmetry of the potential we changed the bounds of integration from $x_- \to x_+$

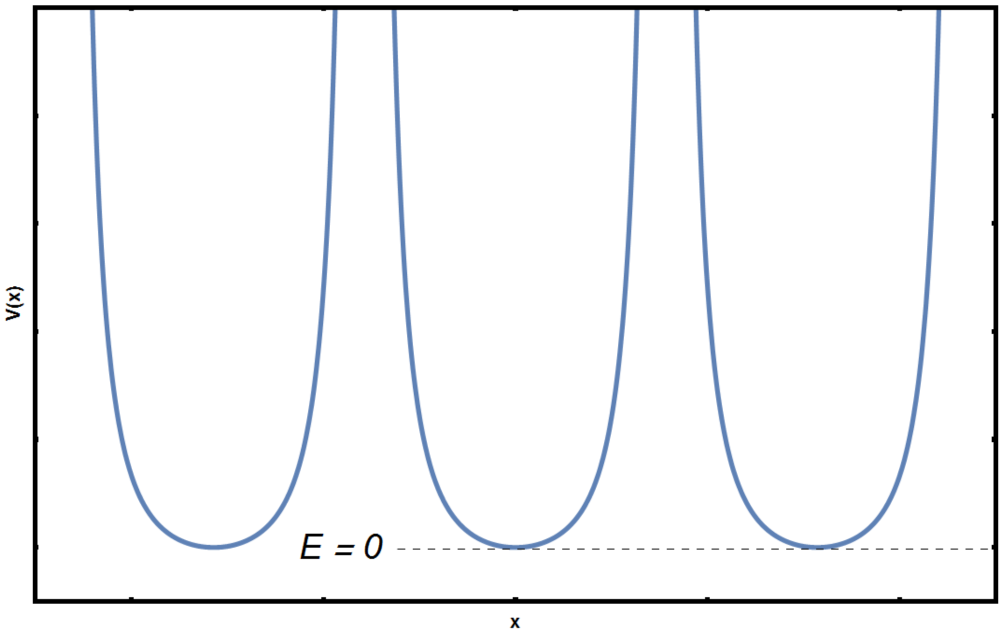

Figure 3: Plot of the potential energy $V(x) = V_0 \tan^2(\alpha x)$

to $0 \to x_+$ and doubled the integral without changing the calculation. Now we note that

$$
\begin{aligned}
\sinh(\alpha x_+) &= \sinh\left(\alpha \frac{\cosh^{-1}\left(\sqrt{\frac{V_0}{E}}\right)}{\alpha}\right) \\
&= \sinh\left(\cosh^{-1}\left(\sqrt{V_0/E}\right)\right) \\
&= \sqrt{\frac{V_0}{E} - 1}
\end{aligned}
$$

Which if we replace into our expression for the period we find

$$
\begin{aligned}
T &= \frac{4}{\alpha}\sqrt{\frac{m}{2E}} \sinh^{-1}\left(\frac{\sinh(\alpha x_+)}{\sqrt{1 + V_0/E}}\right) - \cancelto{0}{\frac{4}{\alpha}\sqrt{\frac{m}{2E}} \sinh^{-1}\left(\frac{\sinh(0)}{\sqrt{1 + V_0/E}}\right)} \\
&= \frac{4}{\alpha}\sqrt{\frac{m}{2E}} \sinh^{-1}\left(\frac{\sqrt{V_0/E - 1}}{\sqrt{1 + V_0/E}}\right) \\
&= \frac{4}{\alpha}\sqrt{\frac{m}{2E}} \sinh^{-1}\left(\sqrt{\frac{V_0/E - 1}{V_0/E + 1}}\right)
\end{aligned}
$$

(c) Given the potential energy $V(x) = V_0 \tan^2(\alpha x)$ where V_0 is a positive constant. This potential is shown in Figure 3. Note that in the range

$$0 < E < \infty$$

we have bounded oscillatory motion. Now we can use equation 2.1 with the turning points given by

$$E = V_0 \tan^2(\alpha x)$$
$$\Downarrow$$
$$x_\pm = \pm \frac{\arctan\left(\sqrt{E/V_0}\right)}{\alpha}$$

we can now integrate equation 2.1 over $x_\pm$ to find the period.

$$T = 2\int_{x_-}^{x_+} \frac{dx}{\sqrt{2/m(E - V_0 \tan^2(\alpha x))}} \qquad \Downarrow$$
$$T = 4\sqrt{\frac{m}{2E}} \int_0^{x_+} \frac{dx}{\sqrt{1 - V_0/E \tan^2(\alpha x)}}$$

Using *Mathematica* we evaluate this integral as

$$T = 4\sqrt{\frac{m}{2E}} \left. \frac{\arctan\left(\frac{\sqrt{2(1+V_0/E)}\sin(\alpha x)}{\sqrt{1-V_0/E+(1+V_0/E)\cos(2\alpha x)}}\right)\sqrt{1 - V_0/E + (1+V_0/E)\cos(2\alpha x)}\sec(\alpha x)}{\sqrt{2(1+V_0/E)(1 - V_0/E\tan^2(\alpha x))}} \right|_0^{x_+}$$
$$= 4\sqrt{\frac{m}{2E}} \frac{\arctan\left(\frac{\sqrt{2(1+V_0/E)}\sin(\alpha x_+)}{\sqrt{1-V_0/E+(1+V_0/E)\cos(2\alpha x_+)}}\right)\sqrt{1 - V_0/E + (1+V_0/E)\cos(2\alpha x_+)}\sec(\alpha x_+)}{\sqrt{2(1+V_0/E)(1 - V_0/E\tan^2(\alpha x_+))}}$$

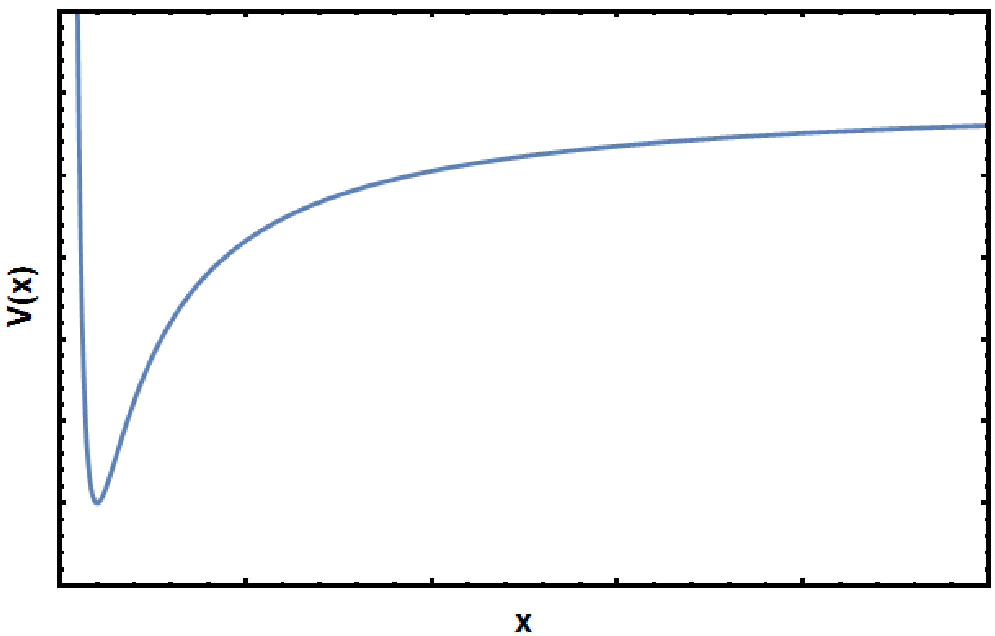

Figure 4: Plot of the effective potential for the *Yukawa-potential*.

3 Problem #3

(a) For the central force field given by the *Yukawa-potential*

$$V(r) = -k\frac{e^{-\alpha r}}{r}, \tag{3.1}$$

where k and α are positive constants. Note that this potential can be reduced to a one-dimensional potential using an effective potential given by

$$V_{eff}(r) = V(r) + \frac{L^2}{2mr^2}. \tag{3.2}$$

Note that L is the conserved quantity related to conservation of angular momentum. So we can combine equations 3.1 and 3.2 to find the effective potential for this system as

$$V_{eff}(r) = -k\frac{e^{-\alpha r}}{r} + \frac{L^2}{2mr^2}.$$

which we plot in Figure 4. We can rearrange the effective potential in such a way to illustrate the effect of angular momentum

$$V_{eff}(r) = A\left(\frac{1}{r^2} - \frac{2km}{L^2}\frac{e^{-\alpha r}}{r}\right)$$

we see that for large L the r^{-2} term dominates and we have a positive energy. This corresponds to an unbound orbit. For small L we have energies that are negative. This implies that for these L there exists bound orbits with two turning points.

(b) When we compare the potential given by equation 3.1 we see that it is like a inverse square law potential given by

$$V(r) = -k\frac{1}{r}$$

the notable difference being the factor of $e^{-\alpha r}$. This factor makes the potential become smaller faster as r increases. This implies that a $1/r$ potential would have a further reach as compared to the *Yukawa-potential*. This implies that there exists bound orbits with a greater r_{max}.

(c) We see that in Figure 4 that $V_{eff}(r)$ has a minimum. This implies that there exists a ground state or circular orbit. We note that this occurs at r_0 defined by

$$\frac{dV_{eff}(r_0)}{dr} = 0. \tag{3.3}$$

Where we first need to change variables such that $r \to 1/u$ which makes our effective potential become

$$\begin{aligned} V_{eff}(r) &= V_{eff}(1/u) \\ &= -k\frac{e^{-\alpha/u}}{1/u} + \frac{L^2}{2m(1/u)^2} \\ &= -kue^{-\alpha/u} + \frac{L^2}{2m}u^2 \end{aligned}$$

Therefore we can solve for the radius of the ground state in terms of u by

$$\begin{aligned} \left.\frac{dV_{eff}(u)}{du}\right|_{u_0} &= 0 \\ &\Downarrow \\ 0 &= \frac{d}{du}\left(-kue^{-\alpha/u} + \frac{L^2}{2m}u^2\right) \\ &= -ke^{-\alpha/u} - kue^{-\alpha/u}\left(\alpha u^{-2}\right) + \frac{L^2}{m}u \\ &= -ke^{-\alpha/u} - k\alpha e^{-\alpha/u}\frac{1}{u} + \frac{L^2}{m}u \\ &= -ke^{-\alpha/u}\left(1 + \alpha\frac{1}{u}\right) + \frac{L^2}{m}u \\ &\Downarrow \\ \frac{L^2}{km} &= r_0\left(1 + \alpha r_0\right)e^{-\alpha r_0} \end{aligned}$$

Note we can not solve this equation explicitly. So we look at the function

$$f(r_0) = r_0\left(1 + \alpha r_0\right)e^{-\alpha r_0}$$

and note that the maximum of $f(r_0)$ is found by

$$\begin{aligned} f'(r_0) = 0 &= \frac{d}{dr_0}\left(\ \left(r_0 + \alpha r_0^2\right)e^{-\alpha r_0}\right) \\ &= -\left(r_0 + \alpha r_0^2\right)\alpha e^{-\alpha r_0} + \left(1 + 2\alpha r_0\right)e^{-\alpha r_0} \\ &\Downarrow \\ \alpha(r_0 + \alpha r_0^2) &= 1 + 2\alpha r_0 \\ (\alpha r_0)^2 - \alpha r_0 &= 1 \\ &\Downarrow \\ r_0^{max} &= \frac{-1 \pm \sqrt{5}}{2\alpha} \end{aligned}$$

Therefore this constrains our angular momentum by the inequality

$$\frac{L^2}{km} < f(r_0^{max}).$$

Where we can solve

$$\begin{aligned}
f(r_0^{max}) &= r_0^{max}\left(1+\alpha r_0^{max}\right)e^{-\alpha r_0^{max}} \\
&= \left(\frac{-1+\sqrt{5}}{2\alpha}\right)\left(1+\alpha\frac{-1+\sqrt{5}}{2\alpha}\right)e^{-\alpha(-1+\sqrt{5})/2\alpha} \\
&= \frac{1}{\alpha}\left(\frac{-1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)e^{(1-\sqrt{5})/2)} \\
&= \frac{1}{\alpha}2e^{1-\sqrt{5}/2)}
\end{aligned}$$

Which implies that

$$\frac{\alpha L^2}{km} < 2e^{(1-\sqrt{5})/2}$$

Therefore, for large L we see that we have no possible circular orbits.

(d) If we add a small perturbation to the ground state orbit which we will call Δr we can find the small oscillations about the circular orbit. Doing this we can expand $V_{eff}(r)$ about r_0 to get

$$V_{eff}(r) = V_eff(r_0) + \cancelto{0}{\Delta r\frac{dV_{eff}(r)}{dr}} + \frac{1}{2}(\Delta r)^2\frac{d^2V_{eff}(r_0)}{dr^2}$$

we note that $V_eff(r_0)$ is an additive constant and due to the fact that potentials are relative it can be ignored. Also we define the constant

$$K \equiv \frac{d^2V_{eff}(r_0)}{dr^2}$$

so we can say our equation of motion for the perturbed state is

$$V_{eff}(\Delta r) = m\ddot{\Delta r} = -K\Delta r.$$

Note that this produces a solution of the form

$$\Delta r = A\cos(\omega_r t)$$

where

$$\omega_r = \sqrt{\frac{K}{m}}$$

so we can see that the period of oscillation is given by ω_r by

$$T = \frac{2\pi}{\omega_r} = 2\pi\sqrt{\frac{m}{K}}$$

so to find the period of oscillation around the circular orbit we need to find k. Note that we use the variable u as before

$$
\begin{aligned}
\left.\frac{d^2V_{eff}(u)}{du^2}\right|_{u_0} &= \frac{d}{du}\left(-ke^{-\alpha/u}\left(1+\frac{\alpha}{u}\right)+\frac{L^2}{m}u\right) \\
&= -k(\alpha u^{-2})e^{-\alpha/u}\left(1+\frac{\alpha}{u}\right) - ke^{-\alpha/u}\left(-\alpha\frac{1}{u^2}\right)+\frac{L^2}{m} \\
&= -k(\alpha u^{-2})e^{-\alpha/u}\left(1+\frac{\alpha}{u}\right) + k\alpha e^{\alpha/u}\left(\frac{1}{u^2}\right)+\frac{L^2}{m} \\
&= -k\alpha e^{-\alpha/u}\left(\frac{1}{u^2}+\frac{\alpha}{u^3}-\frac{1}{u^2}\right)+\frac{L^2}{m} \\
&= -k\alpha^2 e^{-\alpha/u}\frac{1}{u^3}+\frac{L^2}{m} \\
&\Downarrow \\
&= -k\alpha^2 e^{-\alpha r_0} r_0^3+\frac{L^2}{m}
\end{aligned}
$$

Note that from part (c) we found that for circular orbits

$$\frac{L^2}{m} = -ke^{-\alpha r}(1+\alpha r)r$$

so we can say that

$$K = -ke^{-\alpha r_0}(\alpha^2 r_0^3 + \alpha r_0^2 + r_0)$$

so the period of small oscillations is given by

$$T = 2\pi\sqrt{\frac{m}{-ke^{-\alpha r_0}(\alpha^2 r_0^3 + \alpha r_0^2 + r_0)}}$$

4 Problem #4

(a) Given a central force potential

$$V(r) = \alpha \log(r)$$

we can write the effective potential by equation 3.2 so that

$$V_{eff}(r) = \alpha \log(r) + \frac{L^2}{2mr^2}$$

Recall for circular orbits we know that

$$\left.\frac{dV_{eff}(r)}{dr}\right|_{r_0} = 0$$

$$\Downarrow$$

$$0 = \frac{d}{dr}\left(\alpha \log(r) + \frac{L^2}{2mr^2}\right)$$

$$0 = \alpha\frac{1}{r_0} - \frac{L^2}{mr_0^3}$$

$$\Downarrow$$

$$\alpha\frac{1}{r_0} = \frac{L^2}{mr_0^3}$$

$$\Downarrow$$

$$\alpha\frac{r_0^3}{r_0} = \frac{L^2}{m}$$

$$\Downarrow$$

$$r_0 = \frac{L}{\sqrt{m\alpha}}$$

(b) We can test if the circular orbit in part (a) is stable by

$$\frac{d^2V_{eff}(r_0)}{dr^2} > 0$$

So we can calculate

$$\frac{d^2V_{eff}(r)}{dr^2} = \frac{d}{dr}\left(\alpha\frac{1}{r} - \frac{L^2}{mr^3}\right)$$

$$= -\alpha\frac{1}{r^2} + \frac{3L^2}{mr^4}$$

And now we can replace

$$r = r_0 = \frac{L}{\sqrt{2m\alpha}}$$

such that

$$\frac{d^2V_{eff}(r_0)}{dr^2} = -\alpha\frac{1}{r_0^2} + \frac{3L^2}{mr_0^4}$$

$$= -\alpha\frac{m\alpha}{L^2} + \frac{3L^2}{m}\frac{(m\alpha)^2}{L^4}$$

$$= -\frac{m\alpha^2}{L^2} + \frac{3m\alpha^2}{L^2}$$

$$= \frac{2m\alpha^2}{L^2}$$

Note that by definition α, m, and L are positive constants therefore

$$\frac{d^2 V_{eff}(r_0)}{dr^2} > 0.$$

Which implies that the circular orbit at $r = r_0$ is stable.

(c) We can find the frequency of small oscillations about the circular orbit by

$$\Delta r = A\cos(\omega_r t)$$

where

$$\omega_r = \sqrt{\frac{k}{m}}$$

and

$$k \equiv \frac{d^2 V_{eff}(r_0)}{dr^2}.$$

Note see the derivation of these equations in problem 3 part (d). Given that

$$\frac{d^2 V_{eff}(r_0)}{dr^2} = \frac{2m\alpha^2}{L^2}$$

we can find ω_r by

$$\begin{aligned}
\omega_r &= \sqrt{\frac{k}{m}} \\
&= \sqrt{\frac{2m\alpha^2}{mL^2}} \\
&= \sqrt{\frac{2\alpha^2}{L^2}} \\
&= \sqrt{2}\frac{\alpha}{L}
\end{aligned}$$

(d) To test if the small oscillations about r given above forms a closed orbit we test if

$$\frac{\omega_r}{\omega_\theta} \in \mathbb{Q}$$

which states that the ratio between the frequency of small oscillations, ω_r, and the orbital frequency, ω_θ, is a rational quotient. Therefore we note that for a circular orbit there exists a *centripetal force* given by

$$-F_c(r) = \frac{dV(r)}{dr} = m\omega_\theta^2 r. \tag{4.1}$$

So, for $V(r) = \alpha\log(r)$ we see that solving equation 4.1 for ω_θ yields

$$\begin{aligned}
m\omega_\theta^2 r &= \frac{d}{dr}\left(\alpha\log(r)\right) \\
m\omega_\theta^2 r &= \frac{\alpha}{r} \\
&\Downarrow \\
\omega_\theta &= \sqrt{\frac{\alpha}{mr_0^2}} \\
&= \sqrt{\frac{\alpha}{m}\frac{m\alpha}{L^2}} \\
&= \sqrt{\frac{2\alpha^2}{L^2}} \\
&= \frac{\alpha}{L}
\end{aligned}$$

Note we evaluate equation 4.1 at the radius of circular orbit, r_0. So we can evaluate

$$\begin{aligned}\frac{\omega_r}{\omega_\theta} &= \frac{\sqrt{2}\frac{\cancel{\alpha}}{\cancel{L}}}{\frac{\cancel{\alpha}}{\cancel{L}}} \\ &= \sqrt{2} \notin \mathbb{Q}\end{aligned}$$

We see that the ratio is not a rational quotient, therefore the oscillations about the circular orbit do not form a closed orbit.