Physics 611

Electromagnetic Theory II

Professor Christopher Pope

Homework #6

Joe Becker
UID: 125-00-4128
November 2nd, 2016

1 Problem #1

For a small test particle with mass, m, and positive charge, q, which has a *circular* orbit in the $x-y$ plane around a fixed positive charge Q at the origin. This is due to the presence of a uniform magnetic field B oriented along the z direction. Given that the orbit is circular we can say the that the velocity of the particle is given by

$$\mathbf{v} = \omega R\hat{\theta}$$

where ω is the angular frequency and R is the radius of the orbit. Note that we are in plane so we can work within *cylindrical coordinates.* This implies that the *fully relativistic Lorentz force equation*

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \tag{1.1}$$

can be written in the form

$$\begin{aligned}
\frac{d\mathbf{p}}{dt} &= q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \\
&\Downarrow \\
\frac{d}{dt}(m\gamma\mathbf{v}) &= q\left(\frac{Q}{R^2}\hat{r} + \omega RB(\hat{\theta} \times \hat{z})\right) \\
m\gamma\frac{d\mathbf{v}}{dt} &= q\left(\frac{Q}{R^2} - \omega RB\right)\hat{r}
\end{aligned}$$

Note that for circular motion to exist we need a radial acceleration of the form $d\mathbf{v}/dt = -\omega^2 R\hat{r}$ this implies that

$$\begin{aligned}
-m\gamma\omega^2 R &= q\left(\frac{Q}{R^2} - \omega RB\right) \\
&\Downarrow \\
\frac{q}{m} &= \frac{-\gamma\omega^2 R}{\left(\frac{Q}{R^2} - \omega RB\right)} \\
&= \frac{-\omega^2 R}{\sqrt{1-(\omega R)^2}}\frac{1}{\frac{Q}{R^2} - \omega RB} \\
&= \frac{\omega^2}{\sqrt{1-(\omega R)^2}}\frac{1}{\omega B - \frac{Q}{R^3}}
\end{aligned}$$

Note that $\gamma = (1-v^2)^{-1/2} = (1-(\omega R)^2)^{-1/2}$.

2 Problem #2

(a) For an electromagnetic wave for which the electric field is given as

$$\mathbf{E} = E_0 \sin\omega t \,(\sin\omega z, \cos\omega z, 0) \tag{2.1}$$

where E_0 and ω are constant we can solve for $\mathbf{B}$ by noting the maxwell equation

$$\begin{aligned}
-\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times \mathbf{E} \\
&= \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_0 \sin\omega t \sin\omega z & E_0 \sin\omega t \cos\omega z & 0 \end{vmatrix} \\
&= (E_0\omega \sin\omega t \sin\omega z, E_0\omega \sin\omega t \cos\omega z, 0) \\
&\Downarrow \\
\mathbf{B} &= E_0 \cos\omega t \,(\sin\omega z, \cos\omega z, 0)
\end{aligned}$$

Note that this satisfies one of the four source-free Maxwell equation. The remaining three are

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}$$

We can quickly verify the first two equations by calculating

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= E_0 \sin\omega t \left(\cancelto{0}{\frac{\partial}{\partial x}(\sin\omega z)} + \cancelto{0}{\frac{\partial}{\partial y}(\cos\omega z)} + \frac{\partial}{\partial z}(0) \right) = 0 \\
\nabla \cdot \mathbf{B} &= E_0 \cos\omega t \left(\cancelto{0}{\frac{\partial}{\partial x}(\sin\omega z)} + \cancelto{0}{\frac{\partial}{\partial y}(\cos\omega z)} + \frac{\partial}{\partial z}(0) \right) = 0
\end{aligned}$$

and for the last we equation we calculate

$$\begin{aligned}
\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_0 \cos\omega t \sin\omega z & E_0 \cos\omega t \cos\omega z & 0 \end{vmatrix} - \frac{\partial}{\partial t}(E_0 \sin\omega t)\,(\sin\omega z, \cos\omega z, 0) \\
&= E_0\omega \cos\omega t(\sin\omega z, \cos\omega z, 0) - E_0\omega \cos\omega t\,(\sin\omega z, \cos\omega z, 0) = 0
\end{aligned}$$

Therefore all the source-free Maxwell equations hold.

(b) For the electromagnetic wave in part (a) we can calculate the energy density, W, by

$$\begin{aligned}
W = \frac{1}{8\pi}\left(E^2 + B^2\right) &= \frac{1}{8\pi}\left(E_0^2 \sin^2\omega t\cancelto{1}{(\sin^2\omega z + \cos^2\omega z)} + E_0^2 \cos^2\omega t\cancelto{1}{(\cos^2\omega z + \sin^2\omega z)} \right) \\
&= \frac{1}{8\pi}\left(E_0^2\cancelto{1}{(\sin^2\omega t + \cos^2\omega t)} \right) \\
&= \frac{E_0^2}{8\pi}
\end{aligned}$$

We can also calculate the *Poynting vector*, $\mathbf{S}$, as

$$\begin{aligned}
\mathbf{S} = \frac{1}{4\pi}\mathbf{E}\times\mathbf{B} = \frac{1}{4\pi}\det&\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ E_0\sin\omega t\sin\omega z & E_0\sin\omega t\cos\omega z & 0 \\ E_0\cos\omega t\sin\omega z & E_0\cos\omega t\cos\omega z & 0 \end{vmatrix} \\
&= 0\hat{x} + 0\hat{y} + \frac{E_0^2}{4\pi}(\sin\omega t\cos\omega t\sin\omega z\cos\omega z - \sin\omega t\cos\omega t\sin\omega z\cos\omega z)\hat{z} \\
&= 0
\end{aligned}$$

(c) We can repeat the process from parts (a) and (b) for the electric field

$$\mathbf{E} = E_0\cos\omega z\,(\cos\omega t, -\sin\omega t, 0) \tag{2.2}$$

which gives us

$$\begin{aligned}
-\frac{\partial\mathbf{B}}{\partial t} &= \nabla\times\mathbf{E} \\
&= \det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_0\cos\omega t\cos\omega z & -E_0\sin\omega t\cos\omega z & 0 \end{vmatrix} \\
&= (-E_0\omega\sin\omega t\sin\omega z, -E_0\omega\cos\omega t\sin\omega z, 0) \\
&\Downarrow \\
\mathbf{B} &= E_0\sin\omega z\,(\cos\omega t, -\sin\omega t, 0)
\end{aligned}$$

which allows us to calculate

$$\nabla\cdot\mathbf{E} = E_0\left(\cos\omega t\,\cancelto{0}{\frac{\partial}{\partial x}(\cos\omega z)} - \sin\omega t\,\cancelto{0}{\frac{\partial}{\partial y}(\cos\omega z)} + \frac{\partial}{\partial z}(0)\right) = 0$$

$$\nabla\cdot\mathbf{B} = E_0\left(\cos\omega t\,\cancelto{0}{\frac{\partial}{\partial x}(\sin\omega z)} - \sin\omega t\,\cancelto{0}{\frac{\partial}{\partial y}(\sin\omega z)} + \frac{\partial}{\partial z}(0)\right) = 0$$

and

$$\begin{aligned}
\nabla\times\mathbf{B} - \frac{\partial\mathbf{E}}{\partial t} = \det&\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_0\cos\omega t\sin\omega z & -E_0\sin\omega t\sin\omega z & 0 \end{vmatrix} - \frac{\partial}{\partial t}(E_0\cos\omega z)\,(\cos\omega t, -\sin\omega t, 0) \\
&= E_0\omega\cos\omega z(-\sin\omega t, -\cos\omega t, 0) - E_0\omega\cos\omega z\,(-\sin\omega t, -\cos\omega t, 0) = 0
\end{aligned}$$

We also calculate

$$\begin{aligned}
W = \frac{1}{8\pi}\left(E^2 + B^2\right) &= \frac{1}{8\pi}\left(E_0^2\cos^2\omega z\cancelto{1}{(\sin^2\omega t + \cos^2\omega t)} + E_0^2\cos^2\omega z\cancelto{1}{(\cos^2\omega t + \sin^2\omega t)}\right) \\
&= \frac{1}{8\pi}\left(E_0^2\cancelto{1}{(\sin^2\omega z + \cos^2\omega z)}\right) \\
&= \frac{E_0^2}{8\pi}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{S} = \frac{1}{4\pi}\mathbf{E}\times\mathbf{B} = \frac{1}{4\pi}\det &\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ E_0\cos\omega t\cos\omega z & -E_0\sin\omega t\cos\omega z & 0 \\ E_0\cos\omega t\sin\omega z & -E_0\sin\omega t\sin\omega z & 0 \end{vmatrix} \\
&= 0\hat{x} + 0\hat{y} + \frac{E_0^2}{4\pi}(\sin\omega t\cos\omega t\sin\omega z\cos\omega z - \sin\omega t\cos\omega t\sin\omega z\cos\omega z)\hat{z} \\
&= 0
\end{aligned}$$

3 Problem #3

(a) For constant $\mathbf{E}$ and $\mathbf{B}$ fields we can show that in general it is possible to find a new *Lorentz frame* related to the original by a boost velocity of the form

$$\mathbf{v} = \lambda\mathbf{E}\times\mathbf{B}$$

such that in the boosted frame $\mathbf{E}'$ and $\mathbf{B}'$ are parallel. We can do this by taking the boosted field equations

$$\begin{aligned}
\mathbf{E}' &= \gamma(\mathbf{E} + \mathbf{v}\times\mathbf{B}) - \frac{\gamma-1}{v^2}(\mathbf{v}\cdot\mathbf{E})\mathbf{v} \\
\mathbf{B}' &= \gamma(\mathbf{B} - \mathbf{v}\times\mathbf{E}) - \frac{\gamma-1}{v^2}(\mathbf{v}\cdot\mathbf{B})\mathbf{v}
\end{aligned}$$

Now if we replace $\mathbf{v}$ with the velocity we were given we find

$$\begin{aligned}
\mathbf{E}' &= \gamma(\mathbf{E} + (\lambda\mathbf{E}\times\mathbf{B})\times\mathbf{B}) - \frac{\gamma-1}{v^2}\cancelto{0}{((\lambda\mathbf{E}\times\mathbf{B})\cdot\mathbf{E})}(\lambda\mathbf{E}\times\mathbf{B}) \\
&= \gamma\left(\mathbf{E} + \lambda\left((\mathbf{E}\cdot\mathbf{B})\mathbf{B} - (\mathbf{B}\cdot\mathbf{B})\mathbf{E}\right)\right) \\
&= \gamma\left((1-\lambda B^2)\mathbf{E} + \lambda(\mathbf{E}\cdot\mathbf{B})\mathbf{B}\right)
\end{aligned}$$

And similarly we find

$$\begin{aligned}
\mathbf{B}' &= \gamma(\mathbf{B} - (\lambda\mathbf{E}\times\mathbf{B})\times\mathbf{E}) - \frac{\gamma-1}{v^2}\cancelto{0}{((\lambda\mathbf{E}\times\mathbf{B})\cdot\mathbf{B})}(\lambda\mathbf{E}\times\mathbf{B}) \\
&= \gamma\left(\mathbf{B} - \lambda\left((\mathbf{E}\cdot\mathbf{E})\mathbf{B} - (\mathbf{B}\cdot\mathbf{E})\mathbf{E}\right)\right) \\
&= \gamma\left((1-\lambda E^2)\mathbf{B} + \lambda(\mathbf{E}\cdot\mathbf{B})\mathbf{E}\right)
\end{aligned}$$

This allows us to calculate

$$\begin{aligned}
\mathbf{E}'\times\mathbf{B} &= \gamma^2\left((1-\lambda B^2)(1-\lambda E^2)\mathbf{E}\times\mathbf{B} + \lambda^2(\mathbf{E}\cdot\mathbf{B})^2\mathbf{B}\times\mathbf{E}\right) \\
&= \gamma^2\left((1-\lambda B^2)(1-\lambda E^2) - \lambda^2(\mathbf{E}\cdot\mathbf{B})^2\right)\mathbf{B}\times\mathbf{E}
\end{aligned}$$

Therefore we see that $\mathbf{E}'$ and $\mathbf{B}'$ are parallel if

$$(1-\lambda B^2)(1-\lambda E^2) - \lambda^2(\mathbf{E}\cdot\mathbf{B})^2 = 0$$
$$\Downarrow$$
$$\lambda = \frac{(E^2+B^2) \pm \sqrt{E^4 + B^4 + 4(\mathbf{E}\cdot\mathbf{B})^2 - 2E^2B^2}}{-2(\mathbf{E}\cdot\mathbf{B})^2 - B^2E^2}$$

Note this exists for any $\mathbf{E}$ and $\mathbf{B}$.

(b) If $\mathbf{E}\cdot\mathbf{B} = 0$ and $|\mathbf{E}| = |\mathbf{B}|$ are true then we can see that $\mathbf{E}$ is perpendicular to $\mathbf{B}$ in all reference frames due to the fact that $\mathbf{E}\cdot\mathbf{B}$ is *Lorentz invariant.* Therefore it is not possible to construct a frame in with $\mathbf{E}$ and $\mathbf{B}$ are parallel. Note this follows from the result from part (a) as λ goes to infinity in this case.