Physics 624

Quantum Mechanics II

Professor Aleksei Zheltikov

Homework #7

Joe Becker
UID: 125-00-4128
April 27th, 2016

1 Problem #1

Given the momentum operator

$$\hat{\mathbf{P}} = -i\hbar \int \hat{\Psi}^\dagger(\xi') \frac{\partial}{\partial \mathbf{r}'} \hat{\Psi}(\xi') d\xi'$$

defined on the field $\hat{\Psi}(\xi)$ we can find the commutator $[\hat{\mathbf{P}}, \hat{\Psi}(\xi)]$.

(a) For a system of identical fermions we have the anti-commutation relations

$$\begin{aligned}
\left\{\hat{\Psi}(\xi), \hat{\Psi}(\xi')\right\} &= 0 \\
\left\{\hat{\Psi}^\dagger(\xi), \hat{\Psi}^\dagger(\xi')\right\} &= 0 \\
\left\{\hat{\Psi}(\xi), \hat{\Psi}^\dagger(\xi')\right\} &= \delta(\xi - \xi')
\end{aligned}$$

which we can use to calculate

$$\begin{aligned}
\left[\hat{\mathbf{P}}, \hat{\Psi}(\xi)\right] &= \hat{\mathbf{P}}\hat{\Psi}(\xi) - \hat{\Psi}(\xi)\hat{\mathbf{P}} \\
&= -i\hbar \int \hat{\Psi}^\dagger(\xi') \frac{\partial}{\partial \mathbf{r}'} \hat{\Psi}(\xi') d\xi \hat{\Psi}(\xi) + i\hbar \hat{\Psi}(\xi) \int \hat{\Psi}^\dagger(\xi') \frac{\partial}{\partial \mathbf{r}'} \hat{\Psi}(\xi') d\xi' \\
&= i\hbar \int \hat{\Psi}^\dagger(\xi') \hat{\Psi}(\xi) \frac{\partial}{\partial \mathbf{r}'} \hat{\Psi}(\xi') d\xi + i\hbar \int \hat{\Psi}(\xi) \hat{\Psi}^\dagger(\xi') \frac{\partial}{\partial \mathbf{r}'} \hat{\Psi}(\xi') d\xi' \\
&= i\hbar \int \left(\hat{\Psi}^\dagger(\xi')\hat{\Psi}(\xi) + \hat{\Psi}(\xi)\hat{\Psi}^\dagger(\xi')\right) \frac{\partial}{\partial \mathbf{r}'} \hat{\Psi}(\xi') d\xi' \\
&= i\hbar \int \left\{\hat{\Psi}(\xi), \hat{\Psi}^\dagger(\xi')\right\} \frac{\partial}{\partial \mathbf{r}'} \hat{\Psi}(\xi') d\xi' \\
&= i\hbar \int \delta(\xi - \xi') \frac{\partial}{\partial \mathbf{r}'} \hat{\Psi}(\xi') d\xi' \\
&= i\hbar \frac{\partial}{\partial \mathbf{r}} \hat{\Psi}(\xi)
\end{aligned}$$

(b) For a system of identical bosons we use the commutation relations

$$\begin{aligned}
\left[\hat{\Psi}(\xi), \hat{\Psi}(\xi')\right] &= 0 \\
\left[\hat{\Psi}^\dagger(\xi), \hat{\Psi}^\dagger(\xi')\right] &= 0 \\
\left[\hat{\Psi}(\xi), \hat{\Psi}^\dagger(\xi')\right] &= \delta(\xi - \xi')
\end{aligned}$$

which yields

$$\begin{aligned}
\left[\hat{\mathbf{P}}, \hat{\Psi}(\xi)\right] = \hat{\mathbf{P}}\hat{\Psi}(\xi) - \hat{\Psi}(\xi)\hat{\mathbf{P}} &= -i\hbar \int \hat{\Psi}^\dagger(\xi') \frac{\partial}{\partial \mathbf{r}'} \hat{\Psi}(\xi') d\xi \hat{\Psi}(\xi) + i\hbar \hat{\Psi}(\xi) \int \hat{\Psi}^\dagger(\xi') \frac{\partial}{\partial \mathbf{r}'} \hat{\Psi}(\xi') d\xi' \\
&= -i\hbar \int \hat{\Psi}^\dagger(\xi') \hat{\Psi}(\xi) \frac{\partial}{\partial \mathbf{r}'} \hat{\Psi}(\xi') d\xi + i\hbar \int \hat{\Psi}(\xi) \hat{\Psi}^\dagger(\xi') \frac{\partial}{\partial \mathbf{r}'} \hat{\Psi}(\xi') d\xi' \\
&= i\hbar \int \left[\hat{\Psi}(\xi), \hat{\Psi}^\dagger(\xi')\right] \frac{\partial}{\partial \mathbf{r}'} \hat{\Psi}(\xi') d\xi' \\
&= i\hbar \int \delta(\xi - \xi') \frac{\partial}{\partial \mathbf{r}'} \hat{\Psi}(\xi') d\xi' \\
&= i\hbar \frac{\partial}{\partial \mathbf{r}} \hat{\Psi}(\xi)
\end{aligned}$$

2 Problem #2

Given the Bosonic creation and annihilation operators defined as

$$\hat{a} = \alpha\hat{x} + \beta\hat{p}$$
$$\hat{a}^\dagger = \alpha^*\hat{x} + \beta^*\hat{p}$$

where $\hat{x}$ and $\hat{p}$ are the coordinate and momentum operator. Let

$$\alpha = C\frac{\hbar}{L}, \qquad \beta = iCL$$

we can find the normalization constant, C, that satisfies the commutation rule for bosonic creation and annihilation operators by

$$\begin{aligned}
[\hat{a}, \hat{a}^\dagger] = 1 &= (\alpha\hat{x} + \beta\hat{p})(\alpha^*\hat{x} + \beta^*\hat{p}) - (\alpha^*\hat{x} + \beta^*\hat{p})(\alpha\hat{x} + \beta\hat{p}) \\
&= \cancel{|\alpha|^2\hat{x}^2 + |\beta|^2\hat{p}^2} + \alpha\beta^*\hat{x}\hat{p} + \beta\alpha^*\hat{p}\hat{x} - \cancel{|\alpha|^2\hat{x}^2 - |\beta|^2\hat{p}^2} - \alpha\beta^*\hat{p}\hat{x} - \beta\alpha^*\hat{x}\hat{p} \\
&= \alpha\beta^*[\hat{x}, \hat{p}] + \beta\alpha^*[\hat{p}, \hat{x}] \\
&= (\alpha\beta^* - \alpha^*\beta)[\hat{x}, \hat{p}] \\
&\Downarrow \\
&= i\hbar\left(-i\hbar C^2 - i\hbar C^2\right) \\
&= 2\hbar^2 C^2 \\
&\Downarrow \\
C &= \frac{1}{\sqrt{2}\hbar}
\end{aligned}$$

Using this we can find the wave function of the vacuum state, $|0\rangle$, by noting that $\hat{a}|0\rangle = 0$ which if we calculate in coordinate representation we find

$$\begin{aligned}
\left(\frac{1}{\sqrt{2}L}\hat{x} + \frac{iL}{\sqrt{2}\hbar}\hat{p}\right)|0\rangle \Rightarrow \left(\frac{1}{\sqrt{2}L}x - \frac{iL}{\sqrt{2}\hbar}i\hbar\frac{\partial}{\partial x}\right)\psi_0(x) &= 0 \\
\left(\frac{1}{L}x + L\frac{\partial}{\partial x}\right)\psi_0(x) &= 0 \\
&\Downarrow \\
\frac{1}{\psi_0(x)}\frac{\partial\psi_0(x)}{\partial x} &= -\frac{x}{L^2} \\
&\Downarrow \\
\psi_0(x) &= Ae^{-x^2/2L^2}
\end{aligned}$$

We find A by the normalization condition as

$$\begin{aligned}
\langle 0|0\rangle = 1 &= |A|^2\int e^{-x^2/L^2}dx \\
&= |A|^2\frac{\sqrt{\pi}L}{2} \\
&\Downarrow \\
A &= \sqrt{\frac{2}{\sqrt{\pi}L}}
\end{aligned}$$

So we have the vacuum wave function as

$$\psi_0(x) = \left(\frac{4}{\pi L^2}\right)^{1/4} e^{-x^2/2L^2}$$

3 Problem #3

Given a two-particle state of a system of identical particles described by the eigenfunction

$$|2\rangle = C\hat{a}^\dagger_{f_1}\hat{a}^\dagger_{f_2}|0\rangle.$$

(a) For a system of bosons we normalize using the commutation relations

$$[\hat{a}_i, \hat{a}^\dagger_j] = \delta_{ij}, \qquad [\hat{a}_i, \hat{a}_j] = [\hat{a}^\dagger_i, \hat{a}^\dagger_j] = 0$$

(i) This allows us to calculate the normalization for $f_1 \neq f_2$ which yields

$$\begin{aligned}
\langle 2|2\rangle = 1 &= |C|^2\langle 0|\hat{a}_{f_2}\hat{a}_{f_1}\hat{a}^\dagger_{f_1}\hat{a}^\dagger_{f_2}|0\rangle \\
&= |C|^2\langle 0|1 + \hat{a}^\dagger_{f_2}\hat{a}_{f_2} + \hat{a}_{f_2}\hat{a}^\dagger_{f_1}\hat{a}_{f_1}\hat{a}^\dagger_{f_2}|0\rangle \\
&= |C|^2\langle 0|1 - \cancel{\hat{a}^\dagger_{f_2}\hat{a}_{f_2}} - \cancel{\hat{a}_{f_2}\hat{a}^\dagger_{f_1}\hat{a}^\dagger_{f_2}\hat{a}_{f_1}}|0\rangle \\
&= |C|^2\langle 0|0\rangle \\
&\Downarrow \\
C &= 1
\end{aligned}$$

(ii) For $f_1 = f_2$ we can repeat the normalization calculation

$$\begin{aligned}
\langle 2|2\rangle = 1 &= |C|^2\langle 0|\hat{a}_{f_1}\hat{a}_{f_1}\hat{a}^\dagger_{f_1}\hat{a}^\dagger_{f_1}|0\rangle \\
&= |C|^2\langle 0|1 + \hat{a}^\dagger_{f_1}\hat{a}_{f_1} + \hat{a}_{f_1}\hat{a}^\dagger_{f_1}\hat{a}_{f_1}\hat{a}^\dagger_{f_1}|0\rangle \\
&= |C|^2\langle 0|0\rangle + |C|^2\langle 0|\hat{a}_{f_1}\hat{a}^\dagger_{f_1}\hat{a}_{f_1}\hat{a}^\dagger_{f_1}|0\rangle \\
&= |C|^2\langle 0|0\rangle + |C|^2\langle 0|0\rangle \\
&= 2|C|^2 \\
&\Downarrow \\
C &= \frac{1}{\sqrt{2}}
\end{aligned}$$

(b) For a system of fermions we normalize using the anti-commutation relations

$$\{\hat{a}_i, \hat{a}^\dagger_j\} = \delta_{ij}, \qquad \{\hat{a}_i, \hat{a}_j\} = \{\hat{a}^\dagger_i, \hat{a}^\dagger_j\} = 0$$

(i) This allows us to calculate the normalization for $f_1 \neq f_2$ which yields

$$\begin{aligned}
\langle 2|2\rangle = 1 &= |C|^2\langle 0|\hat{a}_{f_2}\hat{a}_{f_1}\hat{a}^\dagger_{f_1}\hat{a}^\dagger_{f_2}|0\rangle \\
&= |C|^2\langle 0|1 - \hat{a}^\dagger_{f_2}\hat{a}_{f_2} - \hat{a}_{f_2}\hat{a}^\dagger_{f_1}\hat{a}_{f_1}\hat{a}^\dagger_{f_2}|0\rangle \\
&= |C|^2\langle 0|0\rangle \\
&\Downarrow \\
C &= 1
\end{aligned}$$

(ii) For $f_1 = f_2$ we can repeat the normalization calculation

$$\begin{aligned}
\langle 2|2\rangle = 1 &= |C|^2\langle 0|\hat{a}_{f_1}\hat{a}_{f_1}\hat{a}^\dagger_{f_1}\hat{a}^\dagger_{f_1}|0\rangle \\
&= |C|^2\langle 0|0\rangle - |C|^2\langle 0|\hat{a}_{f_1}\hat{a}^\dagger_{f_1}\hat{a}_{f_1}\hat{a}^\dagger_{f_1}|0\rangle \\
&= |C|^2\langle 0|0\rangle - |C|^2\langle 0|0\rangle \neq 1
\end{aligned}$$

Note for fermions the state where two particles occupy the same state is no allowed. Therefore the normalization condition in this case is not allowed. This follows from the anti-symmetry of fermions.