Physics 615

Methods of Theoretical Physics I

Professor Katrin Becker

Homework #1

Joe Becker
UID: 125-00-4128
September 9th, 2015

1 Problem #1

We can solve the ordinary differential equation

$$\frac{y'}{y} = x\log(y) + 2x \tag{1.1}$$

through the method of separation of variables where we place equation 1.1 in the form

$$A(y)dy = B(x)dx.$$

We can do this by

$$\begin{aligned}
\frac{y'}{y} &= x\log(y) + 2x \\
&\Downarrow \\
\frac{dy}{dx}\frac{1}{y} &= x(\log(y) + 2) \\
&\Downarrow \\
\frac{dy}{y} &= xdx(\log(y) + 2) \\
\frac{dy}{y(\log(y)+2)} &= xdx \\
&\Downarrow \\
\int \frac{dy}{y(\log(y)+2)} &= \int xdx = \frac{1}{2}x^2 + C
\end{aligned}$$

Note we can solve the integral with by substituting

$$\begin{aligned}
u &= \log(y) + 2 \\
du &= \frac{1}{y}dy.
\end{aligned}$$

Therefore we can solve the integral by

$$\begin{aligned}
\int \frac{dy}{y(\log(y)+2)} &\Rightarrow \int \frac{du}{u} \\
&= \log(u) \Rightarrow \log(\log(y) + 2)
\end{aligned}$$

Now we can solve for $y(x)$ as

$$\begin{aligned}
\log(\log(y) + 2) &= \frac{1}{2}x^2 + C \\
&\Downarrow \\
\log(y) + 2 &= Ce^{x^2/2} \\
\log(y) &= Ce^{x^2/2} - 2 \\
&\Downarrow \\
y(x) &= \exp(C\exp(x^2/2) - 2) \\
&= \exp(\exp(x^2/2) - 2/C) \\
&= C\exp(\exp(x^2/2))
\end{aligned}$$

Note that we grouped the constants $\exp(-C/2)$ into another constant C without loss of generality.

2 Problem #2

For the differential equation

$$y' = \sin(x+y) - 1, \tag{2.1}$$

we change variables using the equality

$$z = \alpha x + \beta y + \gamma$$

where we choose $\alpha = \beta = 1$ and $\gamma = 0$ so that we can have

$$z = x + y$$

with

$$\frac{dz}{dx} = 1 + \frac{dy}{dx}$$

solving for

$$y'$$

we see that we have

$$\frac{dy}{dx} = \frac{dz}{dx} - 1$$

Which implies that equation 2.1 becomes

$$\frac{dy}{dx} = \sin(x+y) - 1 \Rightarrow \sin(z) - 1 = \frac{dz}{dx} - 1.$$

Therefore by changing to the variable z we have a separable differential equation

$$\frac{dz}{dx} = \sin(z).$$

Which we separate as

$$\begin{aligned}
\frac{dz}{dx} &= \sin(z) \\
\frac{dz}{\sin(z)} &= dx \\
&\Downarrow \\
\int dx &= \int \frac{dz}{\sin(z)} \\
x + C &= \int \csc(z) dz \\
&= -\ln(\cos(z/2)) + \ln(\sin(z/2)) \\
&= -\ln\left(\frac{\sin(z/2)}{\cos(z/2)}\right) \\
&= -\ln(\tan(z/2))
\end{aligned}$$

Next we solve for z by

$$\begin{aligned}
x + C &= -\ln(\tan(z/2)) + \\
&\Downarrow \\
Ae^{-x} &= \tan(z/2) \\
&\Downarrow \\
z &= 2\arctan\left(Ae^{-x}\right)
\end{aligned}$$

Now we replace z in the solution for y that is given by $y = z - x$. Therefore,

$$y = 2\arctan\left(Ae^{-x}\right) - x$$

3 Problem #3

To solve the ordinary differential equation

$$3x^2y^2 + 2x^3yy' + 10y^4y' = 0 \tag{3.1}$$

we note that equation 3.1 can be rewritten as

$$\begin{aligned}
3x^2y^2 + 2x^3yy' + 10y^4y' &= 0 \\
&\Downarrow \\
3x^2y^2 &= -(2x^3y + 10y^4)y' \\
&\Downarrow \\
(3x^2y^2)dx + (2x^3y + 10y^4)dy &= 0
\end{aligned}$$

which is in the form

$$A(x,y)dx + B(x,y)dy = 0$$

where

$$\begin{aligned}
A(x,y) &= 3x^2y^2 \\
B(x,y) &= 2x^3y + 10y^4
\end{aligned}$$

We test to see if this equation is *exact* by the condition

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}. \tag{3.2}$$

We calculate

$$\begin{aligned}
\frac{\partial A}{\partial y} &= \frac{\partial}{\partial y}(3x^2y^2) \\
&= 6x^2y
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial B}{\partial x} &= \frac{\partial}{\partial x}(2x^3y + 10y^4) \\
&= 6x^2y
\end{aligned}$$

So, we see that equation 3.2 holds true which implies that equation 3.1 is exact. Therefore, there exists a function u such that

$$\begin{aligned}
\frac{\partial u}{\partial x} &= A(x,y) \\
\frac{\partial u}{\partial y} &= B(x,y).
\end{aligned}$$

We note that we can solve for u using the integral

$$\begin{aligned}
u(x,y) &= \int A(x,y)dx + \int B(x,y)dy = C = \text{const.} \\
&\Downarrow \\
u(x,y) &= \int 3x^2y^2dx + \int 2x^3y + 10y^4dy \\
&= x^3y^2 + x^3y^2 + 2y^5dy + C \\
&= 2x^3y^2 + 2y^5 + A
\end{aligned}$$

So we can see that the solution of equation 3.1 is

$$2x^3y^2 + 2y^5 = C$$

Note we combined constants A and C without loss of generality.

4 Problem #4

For the given second order ODE

$$y'' - 3y' + 2y = e^{3x}(x + x^2) \tag{4.1}$$

we note that this is a nonhomogeneous linear ODE with constant coefficients. Therefore we first need to solve for the homogeneous solution,

$$0 = y_0'' - 3y_0' + 2y_0,$$

by the ansatz

$$y_0 = e^{mx}.$$

We see that our homogeneous version of equation 4.1 becomes

$$\begin{aligned} 0 &= y_0'' - 3y_0' + 2y_0 \\ &= m^2\cancel{e^{mx}} - 3m\cancel{e^{mx}} + 2\cancel{e^{mx}} \\ &= m^2 - 3m + 2 \\ &= (m-1)(m-2). \end{aligned}$$

Solving for m gives us $m = 1, 2$ so y_0 is given by a linear combination

$$y_0(x) = C_1 e^x + C_2 e^{2x}$$

Next we need to construct the particular solution, y_p, by the ansatz

$$y_p = e^{3x}(ax^2 + bx + c)$$

where

$$\begin{aligned} y_p' &= \frac{d}{dx}\left(e^{3x}(ax^2 + bx + c)\right) \\ &= e^{3x}(3)(ax^2 + bx + c) + e^{3x}(2ax + b) \\ &= e^{3x}(3ax^2 + 3bx + 3c + 2ax + b) \\ &= e^{3x}(3ax^2 + (3b + 2a)x + 3c + b) \end{aligned}$$

and

$$\begin{aligned} y_p'' &= \frac{d}{dx}\left(e^{3x}(3ax^2 + (3b + 2a)x + 3c + b)\right) \\ &= e^{3x}(3)(3ax^2 + (3b + 2a)x + 3c + b) + e^{3x}(6ax + (3b + 2a)) \\ &= e^{3x}(9ax^2 + (9b + 6a)x + 9c + 3b + 6ax + (3b + 2a)) \\ &= e^{3x}(9ax^2 + (12a + 9b)x + 9c + 6b + 2a)) \end{aligned}$$

Now we can replace y_p into equation 4.1 by first noting the common e^{3x} factor that we can cancel yielding the resulting equation

$$9ax^2 + (12a + 9b)x + 9c + 6b + 2a - 3(3ax^2 + (2a + 3b)x + 3c + b) + 2(ax^2 + bx + c) = x + x^2$$

Grouping like terms gives us

$$\begin{aligned} \cancel{9ax^2} + (12a + 9b)x + 9c + 6b + 2a - \cancel{9ax^2} - (9b + 6a)x - 9c - 3b + 2ax^2 + 2bx + 2c &= x + x^2 \\ 2ax^2 + (12a + 9b - 9b - 6a + 2b)x + 9c + 6b + 2a - 9c - 3b + 2c &= x + x^2 \\ 2ax^2 + (6a + 2b)x + 2a + 3b + 2c &= x + x^2 \end{aligned}$$

Which yields the system of equations

$$\begin{aligned} 2a &= 1 \\ 6a + 2b &= 1 \\ 2a + 3b + 2c &= 0 \end{aligned}$$

Which implies that

$$\begin{aligned} 2 &= \frac{1}{2} \\ b &= -1 \\ c &= 1 \end{aligned}$$

$$\Rightarrow y_p(x) = e^{3x} \left(\frac{1}{2}x^2 - x + 1 \right)$$

Therefore our solution is a linear combination of both y_0 and y_p given by

$$y(x) = e^{3x} \left(\frac{1}{2}x^2 - x + 1 \right) + y_0(x) = C_1 e^x + C_2 e^{2x}$$