

Physics 606
Quantum Mechanics I
Professor Aleksei Zheltikov

Homework #8

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1 Problem #1

For the addition of the angular momenta for two particles with the quantum numbers $l_1 = l_2 = 1$, $m_1 = 0, \pm 1$, and $m_2 = 0, \pm 1$. For the net states $|l = 2, m = 0\rangle$ and $|l = 2, m = -1\rangle$ we can write it as a linear combination of $|m_1, m_2\rangle$ by using the ladder operator given by

$$L_{\pm}|lm\rangle = \sqrt{(l \mp m)(l \pm m + 1)}|lm \pm 1\rangle$$

where we take the total angular momentum ladder operator is

$$L_{\pm} = L_{1\pm} + L_{2\pm}$$

This allows us to act on the state $|l = 2, m = -2\rangle = |m_1 = -1, m_2 = -1\rangle$ which yields

$$L_+|l = 2, m = -2\rangle = (L_{1+} + L_{2+})|m_1 = -1, m_2 = -1\rangle$$

$$\Downarrow$$

$$\sqrt{(2+2)(2-2+1)}|l = 2, m = -1\rangle = \sqrt{(1+1)(1-1+1)}|m_1 = 0, m_2 = -1\rangle + \sqrt{(1+1)(1-1+1)}|m_1 = -1, m_2 = 0\rangle$$

$$2|l = 2, m = -1\rangle = \sqrt{2}|m_1 = 0, m_2 = -1\rangle + \sqrt{2}|m_1 = -1, m_2 = 0\rangle$$

$$|l = 2, m = -1\rangle = \frac{\sqrt{2}}{2} \left(|m_1 = 0, m_2 = -1\rangle + |m_1 = -1, m_2 = 0\rangle \right)$$

Which allows us to act L_+ again as

$$L_+|l = 2, m = -1\rangle = (L_{1+} + L_{2+})\frac{\sqrt{2}}{2} (|m_1 = 0, m_2 = -1\rangle + |m_1 = -1, m_2 = 0\rangle)$$

$$\Downarrow$$

$$\sqrt{(2+1)(2-1+1)}|l = 2, m = 0\rangle = \frac{\sqrt{2}}{2}L_{1+} (|m_1 = 0, m_2 = -1\rangle + |m_1 = -1, m_2 = 0\rangle)$$

$$+ \frac{\sqrt{2}}{2}L_{2+} (|m_1 = 0, m_2 = -1\rangle + |m_1 = -1, m_2 = 0\rangle)$$

$$\sqrt{6}|l = 2, m = 0\rangle = \frac{\sqrt{2}}{2} \left(\sqrt{2}|m_1 = 1, m_2 = -1\rangle + \sqrt{2}|m_1 = 0, m_2 = 0\rangle \right)$$

$$+ \frac{\sqrt{2}}{2} \left(|m_1 = 0, m_2 = -1\rangle + \sqrt{2}|m_1 = -1, m_2 = 0\rangle \right)$$

$$|l = 2, m = 0\rangle = \frac{\sqrt{6}}{6} \left(|m_1 = 1, m_2 = -1\rangle + |m_1 = -1, m_2 = 1\rangle + 2|m_1 = 0, m_2 = 0\rangle \right)$$

$$= \frac{\sqrt{6}}{6}|m_1 = 1, m_2 = -1\rangle + \frac{\sqrt{6}}{6}|m_1 = -1, m_2 = 1\rangle + \frac{\sqrt{6}}{3}|m_1 = 0, m_2 = 0\rangle$$

2 Problem #2

- (a) To calculate the matrix elements for a $l = 1$ state of \hat{l}_x , \hat{l}_y , and \hat{l}_\pm which are the ladder operators defined as

$$\hat{l}_\pm \equiv \hat{l}_x \pm i\hat{l}_y$$

which allows us to say that

$$\begin{aligned}\hat{l}_x &= \frac{1}{2} (\hat{l}_+ + \hat{l}_-) \\ \hat{l}_y &= \frac{1}{2i} (\hat{l}_+ - \hat{l}_-)\end{aligned}$$

So the matrix elements for $l = 1$ we have

$$\begin{aligned}\langle 1m' | \hat{l}_x | 1m \rangle &= \sum_{m'=-1}^1 \sum_{m=-1}^1 \frac{1}{2} \langle 1m' | (\hat{l}_+ + \hat{l}_-) | 1m \rangle \\ &= \sum_{m'=-1}^1 \sum_{m=-1}^1 \frac{\hbar}{2} \langle 1m' | \left(\sqrt{(1-m)(1+m+1)} | 1m+1 \rangle + \sqrt{(1+m)(1-m+1)} | 1m-1 \rangle \right) \\ &= \sum_{m'=-1}^1 \sum_{m=-1}^1 \frac{\hbar}{2} \left(\sqrt{(1-m)(2+m)} \langle 1m' | 1m+1 \rangle + \sqrt{(1+m)(2-m)} \langle 1m' | 1m-1 \rangle \right)\end{aligned}$$

Which if we manually calculate each entry we have

$$\hat{l}_x = \hbar \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Now we repeat the process with \hat{l}_y

$$\begin{aligned}\langle 1m' | \hat{l}_y | 1m \rangle &= \sum_{m'=-1}^1 \sum_{m=-1}^1 \frac{1}{2i} \langle 1m' | (\hat{l}_+ - \hat{l}_-) | 1m \rangle \\ &= \sum_{m'=-1}^1 \sum_{m=-1}^1 \frac{\hbar}{2i} \left(\sqrt{(1-m)(2+m)} \langle 1m' | 1m+1 \rangle - \sqrt{(1+m)(2-m)} \langle 1m' | 1m-1 \rangle \right)\end{aligned}$$

Which gives

$$\hat{l}_y = \hbar \frac{\sqrt{2}}{2i} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Which allows us to calculate the ladder operators by definition

$$\begin{aligned}\hat{l}_+ &= \hbar \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ \hat{l}_- &= \hbar \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\end{aligned}$$

- (b) We can use the above result to find the state, ψ_{l_x} with $l_x = 0$ which implies that

$$\hat{l}_x \psi_{l_x} = 0$$

which we can calculate as

$$\begin{aligned}\hat{l}_x \psi_{lx} &= \hbar \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\ &\Downarrow \\ 0 &= \hbar \frac{\sqrt{2}}{2} \begin{pmatrix} b \\ a+c \\ b \end{pmatrix}\end{aligned}$$

This implies that $b = 0$ and $c = -a$ so our state is

$$\psi_{lx} = a \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

where we can find a by normalizing

$$\begin{aligned}|\psi_{lx}|^2 &= 1 = a^2(1+1) \\ &\Downarrow \\ a &= \frac{\sqrt{2}}{2}\end{aligned}$$

So we have the state

$$\psi_{lx} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

3 Problem #3

- (a) For an eigenfunction, $|m\rangle$, of the orbital angular momentum operator \hat{l}_z . We can find expectation values of the operators \hat{l}_x and \hat{l}_y by using the which allows us to calculate

$$\begin{aligned}\langle \hat{l}_y \rangle &= \langle m | \frac{1}{2i} (\hat{l}_+ - \hat{l}_-) | m \rangle \\ &= \frac{1}{2i} (\langle m | \hat{l}_+ | m \rangle - \langle m | \hat{l}_- | m \rangle) \\ &= \frac{1}{2i} (\hbar A \langle m | m+1 \rangle - \hbar B \langle m | m-1 \rangle) \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\langle \hat{l}_x \rangle &= \langle m | \frac{1}{2} (\hat{l}_+ + \hat{l}_-) | m \rangle \\ &= \frac{1}{2} (\langle m | \hat{l}_+ | m \rangle + \langle m | \hat{l}_- | m \rangle) \\ &= \frac{1}{2} (\hbar A \langle m | m+1 \rangle + \hbar B \langle m | m-1 \rangle) \\ &= 0\end{aligned}$$

- (b) To calculate the expectation value of the operator $\hat{l}_x \hat{l}_y + \hat{l}_y \hat{l}_x$ in the state $|m\rangle$ we use ladder operators to find

$$\begin{aligned}\hat{l}_x \hat{l}_y + \hat{l}_y \hat{l}_x &= \frac{1}{4i} \left((\hat{l}_+ + \hat{l}_-) (\hat{l}_+ - \hat{l}_-) + (\hat{l}_+ - \hat{l}_-) (\hat{l}_+ + \hat{l}_-) \right) \\ &= \frac{1}{4i} (\hat{l}_+ \hat{l}_+ - \hat{l}_+ \hat{l}_- + \hat{l}_- \hat{l}_+ - \hat{l}_- \hat{l}_- + \hat{l}_+ \hat{l}_+ + \hat{l}_+ \hat{l}_- - \hat{l}_- \hat{l}_+ - \hat{l}_- \hat{l}_-) \\ &= \frac{1}{2i} (\hat{l}_+ \hat{l}_+ - \hat{l}_- \hat{l}_-)\end{aligned}$$

Note the cross terms cancel so neither operator will bring the state back to $|m\rangle$ which implies that

$$\langle \hat{l}_x \hat{l}_y + \hat{l}_y \hat{l}_x \rangle = 0$$

- (c) Note for the operators \hat{l}_x^2 and \hat{l}_y^2 we have

$$\begin{aligned}\hat{l}_x^2 &= \frac{1}{4} (\hat{l}_+ + \hat{l}_-)^2 \\ &= \frac{1}{4} (\hat{l}_+ \hat{l}_+ + \hat{l}_- \hat{l}_- + \hat{l}_+ \hat{l}_- + \hat{l}_- \hat{l}_+)\end{aligned}$$

and

$$\begin{aligned}\hat{l}_y^2 &= -\frac{1}{4} (\hat{l}_+ - \hat{l}_-)^2 \\ &= -\frac{1}{4} (\hat{l}_+ \hat{l}_+ + \hat{l}_- \hat{l}_- - \hat{l}_+ \hat{l}_- - \hat{l}_- \hat{l}_+)\end{aligned}$$

+ We note that only the cross terms have nonzero contributions to the expectation values of

these operators. So for a state in $|lm\rangle$ we have

$$\begin{aligned}
\langle \hat{l}_x^2 \rangle &= \frac{1}{4} \langle lm | \left(\hat{l}_+ \hat{l}_+ + \hat{l}_- \hat{l}_- + \hat{l}_+ \hat{l}_- + \hat{l}_- \hat{l}_+ \right) | lm \rangle \\
&= \frac{1}{4} \left(\langle lm | \hat{l}_+ \hat{l}_- | lm \rangle + \langle lm | \hat{l}_- \hat{l}_+ | lm \rangle \right) \\
&= \frac{\hbar}{4} \left(\sqrt{(l+m)(l-m+1)} \langle lm | \hat{l}_+ | lm-1 \rangle + \sqrt{(l-m)(l+m+1)} \langle lm | \hat{l}_- | lm+1 \rangle \right) \\
&= \frac{\hbar^2}{4} \left(\sqrt{(l+m)(l-m+1)} \sqrt{(l-m+1)(l+m+1-1)} \langle lm || lm \rangle \right. \\
&\quad \left. + \sqrt{(l-m)(l+m+1)} \sqrt{(l+m+1)(l-m-1+1)} \langle lm || lm \rangle \right) \\
&= \frac{\hbar^2}{2} (l(l+1) - m^2)
\end{aligned}$$

and

$$\begin{aligned}
\langle \hat{l}_y^2 \rangle &= -\frac{1}{4} \langle lm | \left(\hat{l}_+ \hat{l}_+ + \hat{l}_- \hat{l}_- - \hat{l}_+ \hat{l}_- - \hat{l}_- \hat{l}_+ \right) | lm \rangle \\
&= \frac{1}{4} \left(\langle lm | \hat{l}_+ \hat{l}_- | lm \rangle + \langle lm | \hat{l}_- \hat{l}_+ | lm \rangle \right) \\
&= \frac{\hbar^2}{2} (l(l+1) - m^2)
\end{aligned}$$

4 Problem #4

For a wave function that describes a planer rotor

$$\psi(\varphi) = A \sin^2 \varphi$$

we can write it as eigenfunctions of the \hat{l}_z operator which are of the form $e^{im\varphi}$ by

$$\begin{aligned}\psi(\varphi) &= A \sin^2 \varphi = A \left(e^{i\phi} - e^{-i\phi} \right)^2 \\ &= A \left(e^{i2\phi} + e^{-i2\phi} - 1 \right) \\ &= A \left(|m=2\rangle + |m=-2\rangle - |m=0\rangle \right)\end{aligned}$$

Where we can find A by normalizing

$$\begin{aligned}|\psi(\varphi)|^2 &= 1 = A^2 \left(\langle m=2|m=2\rangle + \langle m=-2|m=-2\rangle - \langle m=0|m=0\rangle \right) \\ &= A^2(1 + 1 - 1) \\ &\Downarrow \\ A &= 1\end{aligned}$$

So we have the expectation values

$$\begin{aligned}\langle \hat{L}_z \rangle &= \langle m=2|\hat{L}_z|m=2\rangle + \langle m=-2|\hat{L}_z|m=-2\rangle - \langle m=0|\hat{L}_z|m=0\rangle \\ &= 2\hbar - 2\hbar - 1\hbar = -\hbar\end{aligned}$$

and

$$\begin{aligned}\langle \hat{L}_z^2 \rangle &= \langle m=2|\hat{L}_z^2|m=2\rangle + \langle m=-2|\hat{L}_z^2|m=-2\rangle - \langle m=0|\hat{L}_z^2|m=0\rangle \\ &= 4\hbar^2 + 4\hbar^2 + 1\hbar^2 = 9\hbar^2\end{aligned}$$