Physics 607

Statistical Physics and Thermodynamics

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Homework #1

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1 Problem #1

(a) Given a gas with a large number, N, of non-interacting particles in a box with a volume, V. We can find the probability that we will find fixed number, $N_1 < N$, of particles in a sub-volume, $V_1 < V$, by noting that this probability follows a binomial distribution where the probability of finding a single particle in the volume, V_1, is given by

$$p = \frac{V_1}{V}$$

therefore if we expand to N_1 particles found we have $N - N_1$ particles not in V_1. So it follows that the total probability of finding N_1 particles within V_1 is

$$P(N_1) = \binom{N}{N_1}\left(1 - \frac{V_1}{V}\right)^{N-N_1}\left(\frac{V_1}{V}\right)^{N_1} \tag{1.1}$$

where we note the combinatoric is defined as

$$\binom{N}{N_1} \equiv \frac{N!}{(N-N_1)!N_1!}.$$

We can find the value N_1 that maximizes equation 1.1 by noting that the N_1 that gives the mean value which is found by

$$\overline{N_1} = \sum_{N_1=0}^{N} N_1 P(N_1)$$

has the maximum probability. For any binomial distribution this value is given by the total number of trials multiplied by the probability. So equation 1.1 is maximized for the value

$$\overline{N_1} = N\frac{V_1}{V}$$

which makes intuitive sense as the assumption that the particle has equal probability of being found in any location implies that there is a uniform distribution of particles. We note that the average square fluctuations, $\overline{(\Delta N_1)^2}$, is given as

$$\overline{(\Delta N_1)^2} = N\frac{V_1}{V}\left(1 - \frac{V_1}{V}\right)$$

which follows from the standard variance of a binomial distribution. Now if we take the number of particles to be large, $N = 10^{20}$, we note that we can approximate the binomial distribution as a Gaussian distribution where

$$P(N_1) \approx \frac{1}{\sqrt{2\pi N p(1-p)}}\exp\left(\frac{(N_1 - \overline{N_1})^2}{2\overline{(\Delta N_1)^2}}\right).$$

Now if we take the size of the sub-volume to be $V_1 = V/2$ we see that $p = 1/2$ and $\overline{N_1} = N/2$ and $\overline{(\Delta N_1)^2} = N/4$. So we have the probability distribution

$$P(N_1) \approx \sqrt{\frac{2}{\pi N}}\exp\left(\frac{(N_1 - N/2)^2}{N/2}\right).$$

So using this distribution we can calculate the probability that the number, N_1 deviates from $N/2$ by $(10^{-4})\%$ by solving the integral

$$\int_{N/2(1-10^{-6})}^{N/2(1+10^{-6})} P(N_1)dN_1$$

But we note that the variance goes by the $\sqrt{N/4}$ which is equal to $1/2 \times 10^{10}$ which implies that the distribution does not vary out side of $10^{-8}\%$ with any real probability. Therefore we approximate the variance as zero.

(b) Now, if we extend the problem to the case where the total volume is divided into an arbitrary number, n, boxes with volumes $V_1, V_2, ..., V_n$. We can find the probability and assuming that the probabilities are independent we can calculate the probability that there are N_1 particles in V_1, N_2 in V_2 and so forth by calculating the product

$$P(N_1, N_2, ..., N_n) = \prod_{k=1}^{n} \binom{N}{N_k} \left(1 - \frac{V_k}{V}\right)^{N-N_k} \left(\frac{V_k}{V}\right)^{N_i}$$

(c) The probability found in part (b) is at a maximum when each probability is at it's mean given by

$$\overline{N_k} = N\frac{V_k}{V}$$

2 Problem #2

(a) For a large number, N, of non-interacting identical quantum particles with a mass, m, confined in a cubic box with the side of length, L, we can estimate the density of states for a fixed total energy, E. We first assume that $N >> 1$ and $kL >> 1$ where

$$k = \sqrt{2m\epsilon}/\hbar \qquad \text{with} \qquad \epsilon = E/N.$$

We first consider a single particle in the box with energy ϵ this implies that for a wave function of the form

$$\psi_{\mathbf{p}} = e^{i\mathbf{p}\cdot\mathbf{r}}$$

with the periodic boundary condition that $r_i = r_i + L$ for $i = x, y, z$. Which implies that

$$\frac{p_i L}{\hbar} = 2\pi n_i$$

which quasiclassically yields the number of states for three degrees of freedom as

$$\frac{dpdq}{(2\pi\hbar)^3}$$

where dq becomes defined by the allowed position space given as L^3 and dp is defined by the allowed energy as the energy is fully kinetic. So

$$dp = \frac{d\epsilon}{\sqrt{8\pi\epsilon}}$$

So the density of states for N particles with $\epsilon = E/N$ we have

$$D(N) = \left(\frac{L}{2\pi\hbar}\right)^{3N}\sqrt{\frac{N}{8\pi E}}$$