

Physics 601
Analytical Mechanics
Professor Siu Chin

Homework #10

Joe Becker
UID: 125-00-4128
November 18th, 2015

1 Problem #1

If we perform a canonical transformation from $(q, p) \rightarrow (Q, P)$ by

$$F_2(q, P) = qP + \frac{1}{2}\epsilon H(q, P)$$

where the Hamiltonian is given as

$$H(q, p) = \frac{p^2}{2m} + V(q)$$

we can solve for the transformation variables by

$$p = \frac{\partial F_2}{\partial q} = P + \epsilon \frac{1}{2} \frac{dV}{dq}$$

\Downarrow

$$P = p - \epsilon \frac{1}{2} \frac{dV}{dq}$$

and

$$Q = \frac{\partial F_2}{\partial P} = q + \epsilon \frac{1}{2} \frac{P}{m}$$

\Downarrow

$$Q = q + \epsilon \frac{p}{2m} - \epsilon^2 \frac{1}{4m} \frac{dV}{dq}$$

This allows us to perform a new canonical transformation from $(Q, P) \rightarrow (q', p')$ by

$$F_3(P, q') = -Pq' + \frac{1}{2}\epsilon H(q', P)$$

Which yields

$$Q = -\frac{\partial F_3}{\partial P} = q' - \epsilon \frac{1}{2} \frac{P}{m}$$

\Downarrow

$$q' = Q + \epsilon \frac{1}{2} \frac{P}{m}$$

and

$$p' = -\frac{\partial F_3}{\partial q'} = P - \epsilon \frac{1}{2} \frac{dV}{dq'}$$

$$= P - \epsilon \frac{1}{2} \frac{dV}{dQ} \frac{dQ}{dq'} \overset{1}{\nearrow}$$

\Downarrow

$$p' = P - \epsilon \frac{1}{2} \frac{dV}{dQ}$$

Now we can transform (Q, P) back to (q, p) by the transformations we first found to get the transformation from $(q, p) \rightarrow (q', p')$. So first for q' we have

$$\begin{aligned} q' &= Q + \epsilon \frac{1}{2} \frac{P}{m} \\ &= q + \epsilon \frac{p}{2m} - \epsilon^2 \frac{1}{4m} \frac{dV}{dq} + \epsilon \frac{1}{2m} \left(p - \epsilon \frac{1}{2} \frac{dV}{dq} \right) \\ &= q + \epsilon \frac{p}{m} - \epsilon^2 \frac{1}{2m} \frac{dV}{dq} \end{aligned}$$

We note the expansion of $q(t + \epsilon)$ about epsilon yields

$$q(t + \epsilon) = q + \epsilon \frac{dq}{dt} + \epsilon^2 \frac{1}{2} \frac{d^2 q}{dt^2}$$

we note that this is the same as the transformed result we found, due to the fact that

$$\frac{dq}{dt} = \frac{p}{m}, \quad \frac{d^2 q}{dt^2} = -\frac{1}{m} \frac{dV}{dq}$$

Which implies that $q' = q(t + \epsilon)$. Now we can do the same with p' as

$$\begin{aligned} p' &= P - \epsilon \frac{1}{2} \frac{dV(Q)}{dQ} \frac{dQ}{dq} \\ &= p - \epsilon \frac{1}{2} \frac{dV}{dq} - \epsilon \frac{1}{2} \frac{dV}{dq} \frac{d}{dq} \left(q + \epsilon \frac{p}{2m} - \epsilon^2 \frac{1}{4m} \frac{dV}{dq} \right) \\ &= p - \epsilon \frac{1}{2} \frac{dV}{dq} - \epsilon \frac{1}{2} \frac{dV}{dq} \left(1 - \epsilon^2 \frac{1}{4m} \frac{d^2 V}{dq^2} \right) \\ &= p - \epsilon \frac{dV}{dq} + \mathcal{O}(\epsilon^3) \end{aligned}$$

We see that this agrees to second order due to the fact that $\ddot{p} = 0$.

2 Problem #2

(a) For the free fall Hamiltonian

$$H = \frac{p^2}{2m} + mgq$$

where we are given the principle function

$$S(q, \alpha, t) = W(q, \alpha) - Et$$

Using this we can apply the Hamilton-Jacobi equation

$$H + \frac{dS}{dt} = 0$$

where we note that p is given by

$$p = \frac{dS}{dq} = \frac{dW}{dq}$$

So we have the equation

$$0 = \frac{1}{2m} \left(\frac{dW}{dq} \right)^2 + mgq - E$$

(b) Using the equation if part (a) we can solve for $W(q, \alpha)$ by

$$\begin{aligned} 0 &= \frac{1}{2m} \left(\frac{dW}{dq} \right)^2 + mgq - E \\ \Downarrow \\ \frac{dW}{dq} &= \left(2m(E - mgq) \right)^{1/2} \\ \Downarrow \\ W &= C + \int \left(2m(E - mgq) \right)^{1/2} dq \\ &= C - \frac{1}{3m^2g} \left(2m(E - mgq) \right)^{3/2} \end{aligned}$$

Note we take α as a constant of motion which we see is E so

$$W(q, \alpha) = C - \frac{1}{3m^2g} \left(2m(\alpha - mgq) \right)^{3/2}$$

(c) We can solve for the equation of motion, $q(t)$, by taking the transformation

$$\begin{aligned} \beta &= \frac{\partial S}{\partial \alpha} = \frac{\partial W}{\partial \alpha} - t \\ \Downarrow \\ \beta + t &= -\frac{1}{3m^2g} \frac{3}{2} \left(2m(\alpha - mgq) \right)^{1/2} 2m \\ &= -\frac{1}{mg} \left(2m(\alpha - mgq) \right)^{1/2} \\ \Downarrow \\ (mg(\beta + t))^2 - 2m\alpha &= -2m^2gq \\ \Downarrow \\ q(t) &= \frac{\alpha}{mg} - \frac{1}{2}g(\beta + t)^2 = \frac{\alpha}{mg} - \frac{1}{2}g\beta^2 - g\beta t - \frac{1}{2}gt^2 \end{aligned}$$

(d) Now we can apply the initial conditions

$$p(0) = mv_0, \quad q(0) = q_0$$

to find

$$q(0) = q_0 = \frac{\alpha}{mg} - \frac{1}{2}g\beta^2$$

and we take the momentum initial condition as

$$\begin{aligned} p(0) &= mv_0 = \left(2m(\alpha - mgq(0)) \right)^{1/2} \\ &= \left(2m(\alpha - \alpha + 1/2mg^2\beta^2) \right)^{1/2} \\ &= \pm mg\beta \\ &\Downarrow \\ \beta &= \pm \frac{v_0}{g} \end{aligned}$$

So replacing in terms of q_0 and v_0 we find

$$\begin{aligned} q(t) &= \frac{\alpha}{mg} - \frac{1}{2}g\beta^2 - g\beta t - \frac{1}{2}gt^2 \\ &\Downarrow \\ q(t) &= q_0 \pm v_0 t - \frac{1}{2}gt^2 \end{aligned}$$

3 Problem #3

(a) For the simple harmonic oscillator we have the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2$$

which gives us the Hamilton-Jacobi equation using the same principle functional form as in problem (2) as

$$0 = \frac{1}{2m} \left(\frac{dW}{dq} \right)^2 + \frac{1}{2}kq^2 - E$$

(b) We can solve for $W(q, \alpha)$ where we take $\alpha = E$ as

$$\begin{aligned} \frac{dW}{dq} &= (2m(\alpha - 1/2kq^2))^{1/2} \\ \Downarrow \\ W &= C + \int (2m(\alpha - 1/2kq^2))^{1/2} dq \end{aligned}$$

Note we will solve the integral when solving for q

(c) We solve for $q(t)$ by

$$\begin{aligned} \beta &= \frac{\partial S}{\partial \alpha} = \frac{\partial W}{\partial \alpha} - t \\ \Downarrow \\ \beta - t &= \frac{\partial}{\partial \alpha} \int (2m(\alpha - 1/2kq^2))^{1/2} dq \\ &= \int \frac{1}{2} (2m(\alpha - 1/2kq^2))^{-1/2} 2mdq \\ &= \sqrt{\frac{m}{2\alpha}} \int \left(1 - \frac{kq^2}{2\alpha} \right)^{-1/2} dq \\ &= \sqrt{\frac{m}{2\alpha}} \arcsin \left(\sqrt{\frac{k}{2\alpha}} q \right) \sqrt{\frac{2\alpha}{k}} \\ \Downarrow \\ q(t) &= \sqrt{\frac{2\alpha}{k}} \sin \left(\sqrt{\frac{k}{m}} (\beta - t) \right) \end{aligned}$$

(d) We can apply the initial conditions $q(0) = q_0$ and $p(0) = p_0$. This implies that

$$q_0 = \sqrt{\frac{2\alpha}{k}} \sin \left(\sqrt{\frac{k}{m}} \beta \right)$$

and we can find

$$\begin{aligned} p_0 &= (2m(\alpha - 1/2kq_0^2))^{1/2} \\ &= \left(2m\alpha - 2m\alpha \sin^2 \left(\sqrt{\frac{k}{m}} \beta \right) \right)^{1/2} \\ &= \sqrt{2m\alpha} \cos \left(\sqrt{\frac{k}{m}} \beta \right) \end{aligned}$$

We note that we can expand $q(t)$ using trigonometric addition

$$\begin{aligned}
q(t) &= \sqrt{\frac{2\alpha}{k}} \sin \left(\sqrt{\frac{k}{m}}(\beta - t) \right) \\
&\Downarrow \\
q(t) &= \sqrt{\frac{2\alpha}{k}} \sin \left(\sqrt{\frac{k}{m}}\beta \right) \cos \left(\sqrt{\frac{k}{m}}t \right) - \sqrt{\frac{2\alpha}{k}} \sin \left(\sqrt{\frac{k}{m}}\beta \right) \sin \left(\sqrt{\frac{k}{m}}t \right) \\
q(t) &= q_0 \cos(\omega t) - \frac{p_0}{m\omega} \sin(\omega t)
\end{aligned}$$

Where we defined $\omega^2 = k/m$. We can use this result to get $p(t)$ by

$$\begin{aligned}
p(t) &= \left(2m(\alpha - 1/2kq(t)^2) \right)^{1/2} \\
&\Downarrow \\
p(t) &= \left(2m\alpha - 2m\alpha k \frac{2\alpha}{k} \sin^2(\omega(\beta - t)) \right)^{1/2} \\
&= \sqrt{2m\alpha} \cos(\omega\beta) \cos(\omega t) + \sqrt{2m\alpha} \sin(\omega\beta) \sin(\omega t) \\
&= p_0 \cos(\omega t) + q_0 m\omega \sin(\omega t)
\end{aligned}$$

Note as we expect $p_0 = m\dot{q}$.

4 Problem #4

For the system with the following kinetic energy, T , and potential energy, V

$$T = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2)(q_1^2 + q_2^2), \quad V = (q_1^2 + q_2^2)^{-1}$$

We have the Lagrangian

$$\frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2)(q_1^2 + q_2^2) - (q_1^2 + q_2^2)^{-1}$$

which allows us to calculate the generalized momenta as

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = \dot{q}_1(q_1^2 + q_2^2)$$

$$p_2 = \frac{\partial L}{\partial \dot{q}_2} = \dot{q}_2(q_1^2 + q_2^2)$$

Using these we can calculate the Hamiltonian by

$$H = p_1 \dot{q}_1 + p_2 \dot{q}_2 - L$$

$$\Downarrow$$

$$= \frac{p_1^2}{q_1^2 + q_2^2} + \frac{p_2^2}{q_1^2 + q_2^2} - \frac{1}{2} \left(\frac{p_1^2}{(q_1^2 + q_2^2)^2} \frac{p_2^2}{(q_1^2 + q_2^2)^2} \right) + (q_1^2 + q_2^2)^{-1}$$

$$= \frac{p_1^2 + p_2^2 + 2}{2(q_1^2 + q_2^2)}$$

This yields the Hamilton-Jacobi equation

$$(q_1^2 + q_2^2)^{-1} \left(\frac{1}{2} \left(\frac{\partial S}{\partial q_1} \right)^2 + \frac{1}{2} \left(\frac{\partial S}{\partial q_2} \right)^2 + 1 \right) + \frac{\partial S}{\partial t} = 0$$

where we define the principle function as

$$S(q_1, q_2, \alpha_1, \alpha_2, t) = W_1(q_1, \alpha_1, \alpha_2) + W_2(q_2, \alpha_1, \alpha_2) - Et$$

This makes the Hamilton-Jacobi equation become

$$0 = (q_1^2 + q_2^2)^{-1} \left(\frac{1}{2} \left(\frac{\partial S}{\partial q_1} \right)^2 + \frac{1}{2} \left(\frac{\partial S}{\partial q_2} \right)^2 + 1 \right) + \frac{\partial S}{\partial t}$$

$$\Downarrow$$

$$E(q_1^2 + q_2^2) = \frac{1}{2} \left(\frac{\partial W_1}{\partial q_1} \right)^2 + \frac{1}{2} \left(\frac{\partial W_2}{\partial q_2} \right)^2 + 1$$

$$\Downarrow$$

$$Eq_1^2 - \frac{1}{2} \left(\frac{\partial W_1}{\partial q_1} \right)^2 = \frac{1}{2} \left(\frac{\partial W_2}{\partial q_2} \right)^2 + 1 - Eq_2^2$$

We note that we have a function of q_1 on the left and a function of q_2 on the right. This implies that both are constant and equal. This allows us to solve for W_1 and W_2 .

$$\begin{aligned}
C &= Eq_1^2 - \frac{1}{2} \left(\frac{\partial W_1}{\partial q_1} \right)^2 \\
&\Downarrow \\
\frac{\partial W_1}{\partial q_1} &= \sqrt{2(Eq_1^2 - C)} \\
&\Downarrow \\
W_1(q_1, \alpha_1, \alpha_2) &= \int \sqrt{2(Eq_1^2 - C)} dq_1
\end{aligned}$$

By the same process we can find W_2 as

$$\begin{aligned}
C &= \frac{1}{2} \left(\frac{\partial W_2}{\partial q_2} \right)^2 + 1 - Eq_2^2 \\
&\Downarrow \\
W_2(q_2, \alpha_1, \alpha_2) &= \int \sqrt{2(C + Eq_2^2 - 1)} dq_2
\end{aligned}$$

This gives us the principle function where we pick $\alpha_1 = E$ and $\alpha_2 = C$ as they are the constants of motion this makes our principle function become

$$S(q_1, q_2, \alpha_1, \alpha_2) = \int \sqrt{2(\alpha_1 q_1^2 - \alpha_2)} dq_1 + \int \sqrt{2(\alpha_2 + \alpha_1 q_2^2 - 1)} dq_2 - \alpha_1 t$$

Where we can solve for the dynamics by

$$\begin{aligned}
\beta_1 &= \frac{\partial S}{\partial \alpha_1} = \int \frac{q_1^2}{\sqrt{2(\alpha_1 q_1^2 - \alpha_2)}} dq_1 + \int \frac{q_2^2}{\sqrt{2(\alpha_2 + \alpha_1 q_2^2 - 1)}} dq_2 - t \\
\beta_2 &= \frac{\partial S}{\partial \alpha_2} = \int \frac{1}{\sqrt{2(\alpha_1 q_1^2 - \alpha_2)}} dq_1 + \int \frac{1}{\sqrt{2(\alpha_2 + \alpha_1 q_2^2 - 1)}} dq_2
\end{aligned}$$