Physics 624

Quantum Mechanics II

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Homework #3

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1 Problem #1

For a given potential we can use the *First Born Approximation* to approximate the scattering amplitude, A_{ba}, by the equation

$$A_{ba}(\mathbf{q}) = -\frac{\mu}{2\pi\hbar^2}\int e^{-i\mathbf{q}\cdot\mathbf{r}}V(\mathbf{r})d^3r \tag{1.1}$$

where $\hbar\mathbf{q} = \hbar(\mathbf{k}_a - \mathbf{k}_b)$. Note that $|A_{ba}|^2$ yields the differential cross section.

(a) For the given central potential

$$V(r) = V_0 \exp\left(-\frac{r}{R}\right)$$

we note that the potential only depends on the coordinate r which allows us to simplify equation 1.1 as

$$\begin{aligned}
A_{ba}(\mathbf{q}) &= -\frac{\mu}{2\pi\hbar^2}\int e^{i\mathbf{q}\cdot\mathbf{r}}V(\mathbf{r})d^3r \\
&\Downarrow \\
A_{ba}(q) &= -\frac{\mu}{2\pi\hbar^2}\int_0^\infty\int_0^\pi\int_0^{2\pi} e^{-iqr\cos\theta}V(r)\sin\theta r^2 d\theta d\phi dr \\
&= -\frac{\mu}{\hbar^2}\int_0^\infty \frac{e^{iqr}-e^{-iqr}}{iqr}V(r)r^2dr \\
&= -\frac{2\mu}{q\hbar^2}\int_0^\infty \sin(qr)V(r)rdr
\end{aligned}$$

So using the simplified form of equation 1.1 we have

$$\begin{aligned}
A_{ba}(q) &= -\frac{2\mu V_0}{q\hbar^2}\int_0^\infty \sin(qr)e^{-r/R}rdr \\
&= -\frac{2\mu V_0}{q\hbar^2}\frac{2qR^3}{(1+(qR)^2)^2} \\
&= -\frac{4\mu V_0 R^3}{\hbar^2(1+(qR)^2)^2}
\end{aligned}$$

Using this result and noting that $q = 2k\sin(\theta/2)$ for a spherically symmetric potential we can calculate the total cross section, $\sigma(E)$, as

$$\begin{aligned}
\sigma(k) = \int |A_{ba}|^2 d\Omega &= 2\pi\int_0^\pi |A_{ba}|^2\sin\theta d\theta \\
&= 2\pi\int_0^\pi\left(\frac{4\mu V_0R^3}{\hbar^2(1+(qR)^2)^2}\right)^2\sin\theta d\theta \\
&= \frac{32\pi\mu^2V_0^2R^6}{\hbar^4}\int_0^\pi\frac{\sin\theta}{(1+(2k\sin(\theta/2)R)^2)^4}d\theta \\
&= \frac{16\pi\mu^2V_0^2R^4}{3\hbar^4k^2}\left(1-\frac{1}{(1+4k^2R^2)^2}\right) \\
&\Downarrow \\
\sigma(E) &= \frac{8\pi\mu V_0^2R^4}{3\hbar^2E}\left(1-\frac{1}{(1+8\mu ER^2/\hbar^2)^2}\right)
\end{aligned}$$

Note we changed to a dependence on energy by $k^2 = 2\mu E/\hbar^2$.

(b) Now for the potential

$$V(r) = V_0 \exp\left(-\frac{r^2}{R^2}\right)$$

which is also a central potential we repeat the process from part (a) to find A_{ba} as

$$\begin{aligned}
A_{ba}(q) &= -\frac{2\mu V_0}{q\hbar^2}\int_0^\infty \sin(qr)e^{-r^2/R^2}rdr \\
&= -\frac{2\mu V_0}{q\hbar^2}\frac{\sqrt{\pi}qR^3e^{-q^2R^2/4}}{4} \\
&= -\frac{\sqrt{\pi}\mu V_0 R^3}{2\hbar^2}e^{-q^2R^2/4}
\end{aligned}$$

and σ as

$$\begin{aligned}
\sigma(k) &= 2\pi\int_0^\pi \left(\frac{\sqrt{\pi}\mu V_0 R^3}{2\hbar^2}e^{-q^2R^2/4}\right)^2 \sin\theta d\theta \\
&= \frac{\pi^2\mu^2V_0^2R^6}{2\hbar^4}\int_0^\pi e^{-2k^2\sin^2(\theta/2)R^2}\sin\theta d\theta \\
&= \frac{\pi^2\mu^2V_0^2R^4}{2\hbar^4k^2}\left(1-e^{-2k^2R^2}\right) \\
&\Downarrow \\
\sigma(E) &= \frac{\pi^2\mu V_0^2R^4}{4\hbar^2E}\left(1-e^{-4\mu ER^2/\hbar^2}\right)
\end{aligned}$$

Again using the free particle energy $k^2 = 2\mu E/\hbar^2$ to get the total cross section as a function of the incident energy, E.

2 Problem #2

Given a particle with mass, M, and incident wave function $\exp(ikx)$ we can calculate the wave function after scattering by a potential $V(x)$ is given by

$$\psi(x) = \exp(ikx) + \frac{2M}{\hbar^2}\int_{-\infty}^{\infty} G(x,x')V(x')\psi(x')dx' \tag{2.1}$$

where the Green's function for the one dimensional Schrödinger's equation is given as

$$G(x,x') = \begin{cases} (2ik)^{-1}\exp(ik(x-x')) & x \geq x' \\ (2ik)^{-1}\exp(-ik(x-x')) & x \leq x' \end{cases}$$

Using $G(x,x')$ we can find the explicit form of $\psi(x)$ for an attractive potential

$$V(x) = -\frac{\gamma\hbar^2}{2M}\delta x$$

where γ is a positive constant. Therefore equation 2.1 becomes

$$\begin{aligned}\psi(x) &= \exp(ikx) + \frac{i\gamma}{2k}\int_{-\infty}^{\infty} e^{ik(x-x')}\delta(x')\psi(x')dx' \\ &= \exp(ikx) + \begin{cases} \dfrac{i\gamma}{2k}\psi(0)e^{ikx} & x \geq 0 \\ \\ \dfrac{i\gamma}{2k}\psi(0)e^{-ikx} & x \leq 0 \end{cases}\end{aligned}$$

Note for $x < 0$ the exponential is negative is but so is x this implies that the wave function is

$$\psi(x) = e^{ikx} + \frac{i\gamma}{2k}\psi(0)e^{ik|x|}$$

This allows us to solve for $\psi(0)$ by

$$\begin{aligned}\psi(0) &= e^{ik(0)} + \frac{i\gamma}{2k}\psi(0)e^{ik(0)} \\ &= 1 + \frac{i\gamma}{2k}\psi(0) \\ &\Downarrow \\ 1 &= \psi(0)\left(1 - \frac{i\gamma}{2k}\right) \\ &\Downarrow \\ \psi(0) &= \frac{2k}{2k - i\gamma}\end{aligned}$$

So we have the wave-function

$$\psi(x) = e^{ikx} + \frac{i\gamma}{2k - i\gamma}e^{ik|x|}$$

which tells us that in the region left of the potential ($x < 0$) we have a scattered wave-function moving to the left with an amplitude

$$|R|^2 = \frac{\gamma^2}{4k^2 + \gamma^2}$$

and to the right of the potential ($x > 0$) we have a scattered potential moving to the right with the amplitude

$$|T|^2 = \frac{4k^2}{4k^2 + \gamma^2}$$

3 Problem #3

We can use the *First Born Approximation*, given by equation 1.1, to express the scattering amplitude by N identical scattering centers located along a straight line where b is the distance between any two neighboring centers. Note we are given the scattering amplitude, $f_0(q)$, of a single scattering center $V_0(r)$. The total potential, $V(r)$ is given by the sum

$$V(r) = \sum_{n=0}^{N} V_0(r + nb)$$

where there we apply a periodic condition that

$$V(r) = V(r + b)$$

We note that the scattering potentials only depend on r so we can calculate the total scattering amplitude, $f_N(q)$, as

$$\begin{aligned}
f_N(q) &= -\frac{\mu}{2\pi\hbar^2} \int e^{-i\mathbf{q}\cdot\mathbf{r}} V(r) d^3r \\
&= -\frac{\mu}{2\pi\hbar^2} \int e^{-i\mathbf{q}\cdot\mathbf{r}} \sum_{n=0}^{N} V(r + nb) d^3r \\
&= \sum_{n=0}^{N} -\frac{\mu}{2\pi\hbar^2} \int e^{-i(r-nb)\mathbf{q}\cdot\hat{r}} V(r) d^3r \\
&= \left(-\frac{\mu}{2\pi\hbar^2} \int e^{-i\mathbf{q}\cdot\mathbf{r}} V(r) d^3r \right) \sum_{n=0}^{N} e^{inb\mathbf{q}\cdot\hat{r}} \\
&= f_0(q) \sum_{n=0}^{N} e^{inbq\cos\theta} \\
&= f_0(q) \left(\frac{1 - e^{iNbq\cos\theta}}{1 - e^{ibq\cos\theta}} \right)
\end{aligned}$$

where we take θ as the scattering angle.