

Physics 611  
Electromagnetic Theory II  
Professor Christopher Pope

Homework #4

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October 19th, 2016

# 1 Problem #1

(a) Given the energy-momentum tensor for the electromagnetic field

$$T_{\mu\nu} = \frac{1}{8\pi} (F_{\mu\rho} F_{\nu}{}^{\rho} + {}^*F_{\mu\rho} {}^*F_{\nu}{}^{\rho}) \quad (1.1)$$

where  ${}^*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ . We can see that equation 1.1 can be reduced by noting that

$$\begin{aligned} {}^*F_{\mu\rho} {}^*F_{\nu}{}^{\rho} &= \left( \frac{1}{2}\epsilon_{\mu\rho\pi\sigma}F^{\pi\sigma} \right) \eta_{\nu\lambda} {}^*F^{\lambda\rho} \\ &= \left( \frac{1}{2}\epsilon_{\mu\rho\pi\sigma}F^{\pi\sigma} \right) \eta_{\nu\lambda} \frac{1}{2}\epsilon^{\lambda\rho\phi\omega}F_{\phi\omega} \\ &= \frac{1}{4}\eta_{\nu\lambda}F^{\pi\sigma}F_{\phi\omega} \left( -\delta_{\mu}^{\lambda}\delta_{\pi}^{\phi}\delta_{\sigma}^{\omega} - \delta_{\mu}^{\phi}\delta_{\pi}^{\omega}\delta_{\sigma}^{\lambda} - \delta_{\mu}^{\omega}\delta_{\pi}^{\lambda}\delta_{\sigma}^{\phi} + \delta_{\mu}^{\phi}\delta_{\pi}^{\lambda}\delta_{\sigma}^{\omega} + \delta_{\mu}^{\lambda}\delta_{\pi}^{\omega}\delta_{\sigma}^{\phi} + \delta_{\mu}^{\omega}\delta_{\pi}^{\phi}\delta_{\sigma}^{\lambda} \right) \\ &= \frac{1}{4}(-\eta_{\nu\mu}F^{\pi\sigma}F_{\pi\sigma} - \eta_{\nu\sigma}F^{\pi\sigma}F_{\mu\pi} - \eta_{\nu\pi}F^{\sigma\pi}F_{\mu\sigma} + \eta_{\nu\pi}F^{\pi\sigma}F_{\mu\sigma} + \eta_{\nu\mu}F^{\pi\sigma}F_{\sigma\pi} + \eta_{\nu\sigma}F^{\pi\sigma}F_{\pi\mu}) \\ &= \frac{1}{4}(-2\eta_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} + 4\eta_{\nu\sigma}F^{\sigma\rho}F_{\mu\rho}) \\ &= -\frac{1}{2}(\eta_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} - 2F_{\nu}{}^{\rho}F_{\mu\rho}) \end{aligned}$$

We see that equation 1.1 becomes

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{8\pi} (F_{\mu\rho} F_{\nu}{}^{\rho} + {}^*F_{\mu\rho} {}^*F_{\nu}{}^{\rho}) \\ &\Downarrow \\ &= \frac{1}{8\pi} \left( F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{2}\eta_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} + F_{\nu}{}^{\rho}F_{\mu\rho} \right) \\ &= \frac{1}{4\pi} \left( F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{4}\eta_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} \right) \end{aligned}$$

Note this is the typical form of the electromagnetic field energy momentum tensor.

(b) We can find the expression for  $T_{\mu\rho}T^{\nu\rho}$  by noting that

$$\begin{aligned} T^{\nu\rho} &= \eta^{\nu\sigma}\eta^{\rho\lambda}T_{\sigma\lambda} \\ &= \frac{1}{4\pi} \left( \eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}{}^{\phi} - \frac{1}{4}\eta^{\nu\sigma}\eta^{\rho\lambda}\eta_{\sigma\lambda}F^{\phi\omega}F_{\phi\omega} \right) \\ &= \frac{1}{4\pi} \left( \eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}{}^{\phi} - \frac{1}{4}\eta^{\nu\rho}F^{\sigma\lambda}F_{\sigma\lambda} \right) \end{aligned}$$

So we can calculate defining  $F^2 \equiv F^{\mu\nu}F_{\mu\nu}$

$$\begin{aligned} T_{\mu\rho}T^{\nu\rho} &= \frac{1}{(4\pi)^2} \left( F_{\mu\sigma}F_{\rho}{}^{\sigma} - \frac{1}{4}\eta_{\mu\rho}F^{\sigma\lambda}F_{\sigma\lambda} \right) \left( \eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}{}^{\phi} - \frac{1}{4}\eta^{\nu\rho}F^{\sigma\lambda}F_{\sigma\lambda} \right) \\ &= \frac{1}{(4\pi)^2} \left( \eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}{}^{\phi}F_{\mu\omega}F_{\rho}{}^{\omega} + \frac{1}{16}\eta_{\mu\rho}\eta^{\nu\rho}(F^2)^2 - \frac{1}{4}F^2\eta^{\nu\rho}F_{\mu\sigma}F_{\rho}{}^{\sigma} - \frac{1}{4}F^2\eta_{\mu\rho}\eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}{}^{\phi} \right) \\ &= \frac{1}{(4\pi)^2} \left( \eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}{}^{\phi}F_{\mu\omega}F_{\rho}{}^{\omega} + \frac{1}{16}\delta_{\mu}^{\nu}(F^2)^2 - \frac{1}{4}F^2\eta^{\nu\rho}F_{\mu\sigma}F_{\rho}{}^{\sigma} - \frac{1}{4}F^2\eta^{\nu\rho}F_{\rho\sigma}F_{\mu}{}^{\sigma} \right) \end{aligned}$$

Next we consider the term

$$\eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}^{\phi}F_{\mu\omega}F_{\rho}^{\omega}=\eta^{\nu\sigma}\eta^{\rho\lambda}F_{\sigma\phi}F_{\lambda}^{\phi}F_{\mu\omega}F_{\rho}^{\omega}$$

## 2 Problem #2

(a) Given the Lagrangian density

$$\mathcal{L} = -\frac{1}{16}F^{\mu\nu}F_{\mu\nu} - \frac{m^2}{8\pi}A^\mu A_\mu + J^\mu A_\mu \quad (2.1)$$

we can derive the equations of motion from the *Euler-Lagrange equations*

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \right) = 0 \quad (2.2)$$

Noting that  $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$  we have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial A_\mu} &= -\frac{m^2}{8\pi}A^\mu + J^\mu \\ \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} &= \frac{\partial}{\partial(\partial_\nu A_\mu)} \left( -\frac{1}{16}F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) \right) = \frac{1}{4}F^{\mu\nu} \end{aligned}$$

So equation 2.2 yields the equation of motion

$$-\partial_\nu F^{\mu\nu} + m^2 A^\mu = 4\pi J^\mu$$

(b) Using the result from part (a) we can find the solution for the scalar potential  $\phi \equiv A^0$  for a point charge  $q$  located at the origin. This implies that we have  $J^0 = q\delta^3(\mathbf{r})$  which allows us to solve for  $\mu = 0$

$$\begin{aligned} -\partial_\nu F^{0\nu} + m^2 A^0 &= 4\pi q\delta^3(\mathbf{r}) \\ \Downarrow \\ -\nabla^2 \phi + m^2 \phi &= 4\pi q\delta^3(\mathbf{r}) \\ \Downarrow \\ -\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + m^2 \phi &= 4\pi q\delta(r) \\ -\frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + r^2 m^2 \phi &= 4\pi r^2 q\delta(r) \\ \Downarrow \\ r\phi(r) &= Ce^{-mr} \end{aligned}$$

Where we can solve for the constant  $C$  by noting that for *Gauss' law* to hold  $C = q$  therefore

$$\phi(r) = \frac{qe^{-mr}}{r}$$

### 3 Problem #3

- (a) Given the potentials describing an electric charge,  $e$ , moving with constant velocity,  $\mathbf{v}$

$$\phi(\mathbf{r}, t) = \frac{e\gamma}{r'}, \quad \mathbf{A}(\mathbf{r}, t) = \mathbf{v}\phi$$

we note that the magnetic field is given as

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} \\ &= \nabla \times (\mathbf{v}\phi) \\ &= \nabla\phi \times \mathbf{v} + \phi(\nabla \times \mathbf{v})^0 \\ &= -\mathbf{E} \times \mathbf{v} \\ &= \mathbf{v} \times \mathbf{E} \end{aligned}$$

- (b) Using the expressions we were given in part (a) we can find the electric field by

$$\begin{aligned} \mathbf{E} &= -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \\ &= -\nabla \frac{e\gamma}{r'} - \mathbf{v} \frac{\partial}{\partial t} \frac{e\gamma}{r'} \end{aligned}$$

we note that we can assume that  $\mathbf{v}$  is pointing along the  $\hat{x}$  direction this gives us

$$r' = \gamma \sqrt{(x - vt)^2 + y^2 + z^2}$$

which allows us to calculate

$$\begin{aligned} \frac{\partial r'}{\partial t} &= \gamma \frac{\partial}{\partial t} \left( \sqrt{(x - vt)^2 + y^2 + z^2} \right) = \gamma \left( (x - vt)^2 + y^2 + z^2 \right)^{-1/2} 2(x - vt)(-v) \\ &= -\frac{2\gamma^2(x - vt)v}{r'} \\ \frac{\partial r'}{\partial x} &= \frac{2\gamma^2(x - vt)}{r'} \\ \frac{\partial r'}{\partial y} &= \frac{2\gamma^2 y}{r'} \\ \frac{\partial r'}{\partial z} &= \frac{2\gamma^2 z}{r'} \end{aligned}$$

So this gives the resulting electric field

$$\begin{aligned} \mathbf{E} &= -\nabla \frac{e\gamma}{r'} - \mathbf{v} \frac{\partial}{\partial t} \frac{e\gamma}{r'} \\ &= \frac{e\gamma}{r'^2} \nabla r' + v \hat{x} \frac{e\gamma}{r'^2} \end{aligned}$$