Physics 615

Methods of Theoretical Physics I

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Homework #8

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1 Problem #1

To evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx$$

We can complexify the integral and integrate over a contour from $-R$ to R along the real axis and then over a semicircle of radius R. We then take R to infinity. We note the integral along the semicircle z becomes $Re^{i\phi}$ and we integrate from 0 to π so as we take the limit as $R \to \infty$ we see

$$\lim_{R\to\infty} \int_{C_R} \frac{z^2}{z^4+1} dz = \lim_{R\to\infty} \int_0^{\pi} d\phi \frac{R^2 e^{i2\phi}}{R^4 e^{i4\phi}+1} = 0$$

Due to the fact that our integrand goes by $1/R^2$. So we can say that the integral over the reals is also

$$I = \oint_{C_\infty} \frac{z^2}{z^4+1} dz = \oint_{C_\infty} \frac{z^2}{(z-e^{i\pi/4})(z-e^{i3\pi/4})(z-e^{i5\pi/4})(z-e^{i7\pi/4})} dz$$

where the contour is over the positive complex plane. We can solve this integral by noting that there are two residues within this contour $z_0 = e^{i\pi/4}$ and $z_0 = e^{i3\pi/4}$ which are both simple poles. This means we can calculate the residues by noting that we are in the form of

$$f(z) = \frac{g(z)}{h(z)}$$

$$\begin{aligned}
\text{Res}\left[f(z), e^{i\pi/4}\right] &= \frac{g(e^{i\pi/4})}{h'(e^{i\pi/4})} \\
&= \frac{(e^{i\pi/4})^2}{4(e^{i\pi/4})^3} \\
&= \frac{1}{4} e^{-i\pi/4}
\end{aligned}$$

and

$$\begin{aligned}
\text{Res}\left[f(z), e^{i3\pi/4}\right] &= \frac{g(e^{i3\pi/4})}{h'(e^{i3\pi/4})} \\
&= \frac{(e^{i3\pi/4})^2}{4(e^{i3\pi/4})^3} \\
&= \frac{1}{4} e^{-i3\pi/4}
\end{aligned}$$

So the integral is the sum of the residues

$$\begin{aligned}
I &= 2\pi i \frac{1}{4}\left(e^{-i\pi/4} + e^{-i3\pi/4}\right) \\
&= \pi i \frac{1}{2}\left(\cos(\pi/4) - i\sin(\pi/4) + \cos(3\pi/4) - i\sin(3\pi/4)\right) \\
&= \pi i \frac{1}{2}\left(-i\sqrt{2}\right) \\
&= \frac{\sqrt{2}}{2}\pi
\end{aligned}$$

2 Problem #2

For the integral

$$I = \int_{-\infty}^{\infty} \frac{1}{x^6+1} dx$$

we see that we can use the same contour as before and know that the semicircle contribution is zero again. So we note that we have poles in the contour of $z_0 = e^{i\pi/6}$, $z_0 = e^{i3\pi/6} = i$, and $z_0 = e^{i5\pi/6}$. So like before we can calculate

$$\begin{aligned}
\text{Res}\left[f(z), e^{i\pi/6}\right] &= \frac{1}{h'(e^{i\pi/6})} \\
&= \frac{1}{6(e^{i\pi/6})^5} \\
&= \frac{1}{6} e^{-i5\pi/6}
\end{aligned}$$

and for $z_0 = e^{i5\pi/6}$

$$\begin{aligned}
\text{Res}\left[f(z), e^{i5\pi/6}\right] &= \frac{1}{h'(e^{i\pi/6})} \\
&= \frac{1}{6(e^{i5\pi/6})^5} \\
&= \frac{1}{6} e^{-i25\pi/6}
\end{aligned}$$

and finally for $z_0 = i$ we have

$$\begin{aligned}
\text{Res}\left[f(z), i\right] &= \frac{1}{h'(i)} \\
&= \frac{1}{6(i)^5} \\
&= -\frac{1}{6} i
\end{aligned}$$

So we can solve for I by summing the residues

$$\begin{aligned}
I &= 2\pi i \frac{1}{6}\left(e^{-i5\pi/6} + e^{-i25\pi/6} - i\right) \\
&= \frac{\pi i}{3}\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2} + \frac{\sqrt{3}}{2} - i\frac{1}{2} - i\right) \\
&= \frac{2\pi}{3}
\end{aligned}$$

3 Problem #3

For the given integral

$$I = \int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2 + 2x + 5} dx$$

we can complexify the integral by noting that

$$\sin(\pi z) = \Im[e^{i\pi z}]$$

we note that along semicircle contour we used in problems one and two the integrand only goes by $1/R$. Therefore we need to see that the integrand is of the form $g(z)e^{iaz}$ which allows us to apply *Jordan's lemma* to say that the semicircle contribution goes to zero as the radius is taken to infinity. Therefore

$$I = \Im\left[\int_C \frac{z(e^{i\pi z})}{z^2 + 2z + 5} dz\right]$$

which implies that we can calculate I by residue theorem where we only need the residue of the simple pole at $z_0 = -1 + 2i$ which allows us to calculate

$$\begin{aligned}
\operatorname{Res}\left[f(z), -1+2i\right] &= \lim_{z \to -1+2i} (z - (-1+2i)) \frac{z(e^{i\pi z})}{z^2 + 2z + 5} \\
&= \lim_{z \to -1+2i} \frac{z(e^{i\pi z}}{(z + (1+2i))} \\
&= \frac{(-1+2i)(e^{i\pi(-1+2i)})}{-1+2i+1+2i} \\
&= \frac{(-1+2i)(-e^{-2\pi})}{4i} \\
&= \frac{-2-i}{4} e^{-2\pi}
\end{aligned}$$

So we can calculate I by

$$\begin{aligned}
I &= \Im[2\pi i \frac{-2-i}{4} e^{-2\pi}] \\
&= 2\pi e^{-2\pi} \Im[\frac{-2i+1}{4}] \\
&= \pi e^{-2\pi}
\end{aligned}$$

4 Problem #4

We can evaluate the integral

$$I = \int_0^\infty \frac{\log x}{(x+a)(x+b)} dx, \qquad a, b > 0, a \neq b$$

by considering the integral of the complex function

$$f(z) = \frac{(\log z)^2}{(z+a)(z+b)}$$

over a keyhole contour, C, around the branch cut of $\log z$. Where we define the branch cut of the $\log(z)$ function to be the positive real axis. We note that as we take the circle of radius ϵ to zero the integral becomes zero. The same holds true when we take R to infinity

$$\begin{aligned}
\oint_C f(z)dz &= \lim_{\epsilon \to 0} \lim_{R \to \infty} \cancelto{0}{\oint_{C_R} f(z)dz} + \cancelto{0}{\oint_{C_\epsilon} f(z)dz} \int_\epsilon^R f(z)dz + \int_R^\epsilon f(ze^{2\pi i})dz \\
&= \int_0^\infty f(z)dz + \int_\infty^0 f(ze^{2\pi i})dz
\end{aligned}$$

We note that due to the branch cut this integral does not equal zero. So we can see that

$$\begin{aligned}
\oint_C f(z)dz = \int_0^\infty f(z)dz + \int_\infty^0 f(ze^{2\pi i})dz &= \int_0^\infty \frac{(\log x)^2}{(x+a)(x+b)} dx + \int_\infty^0 \frac{(\log xe^{2\pi i})^2}{(x+a)(x+b)} dx \\
&= \int_0^\infty \frac{(\log x)^2}{(x+a)(x+b)} dx - \int_0^\infty \frac{(\log x + 2\pi i)^2}{(x+a)(x+b)} dx \\
&= \int_0^\infty \frac{(\log x)^2}{(x+a)(x+b)} dx - \int_0^\infty \frac{(\log x)^2 - 4\pi + 4\pi i \log x}{(x+a)(x+b)} dx \\
&= 4\pi \int_0^\infty \frac{1}{(x+a)(x+b)} dx - 4\pi i \int_0^\infty \frac{\log x}{(x+a)(x+b)} dx \\
&\Downarrow
\end{aligned}$$

$$\int_0^\infty \frac{\log x}{(x+a)(x+b)} dx = -i \int_0^\infty \frac{1}{(x+a)(x+b)} dx - \frac{1}{4\pi i} \oint_C f(z)dz$$

We note that the integral solution to the first integral is

$$\int_0^\infty \frac{1}{(x+a)(x+b)} dx = \frac{\log(a/b)}{b-a}$$

which implies that we just need to find the solution to the complex integral which we do using residue theorem. Where we can say that

$$\begin{aligned}
\text{Res}\left[f(z), -a\right] &= \frac{(\log -a)^2}{b-a} = -\frac{(\log a)^2 - \pi + 2\pi i \log a}{a-b} \\
\text{Res}\left[f(z), -b\right] &= \frac{(\log -b)^2}{a-b} = \frac{(\log b)^2 - \pi + 2\pi i \log b}{a-b}
\end{aligned}$$

So we have the integral as

$$\begin{aligned}
I &= -i\frac{\log(a/b)}{b-a} - \frac{2\pi i}{4\pi i}\left(\frac{(\log b)^2 - \pi + 2\pi i \log b}{a-b} - \frac{(\log a)^2 - \pi + 2\pi i \log a}{a-b}\right) \\
&= i\frac{\log(a/b)}{a-b} - \frac{(\log b)^2 - (\log a)^2}{2(a-b)} - i\frac{\log b - \log a}{a-b} \\
&= \frac{(\log a)^2 - (\log b)^2}{2(a-b)}
\end{aligned}$$