

Physics 611
Electromagnetic Theory II
Professor Christopher Pope

Homework #6

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1 Problem #1

For a small test particle with mass, m , and positive charge, q , which has a *circular* orbit in the $x - y$ plane around a fixed positive charge Q at the origin. This is due to the presence of a uniform magnetic field B oriented along the z direction. Given that the orbit is circular we can say that the velocity of the particle is given by

$$\mathbf{v} = \omega R \hat{\theta}$$

where ω is the angular frequency and R is the radius of the orbit. Note that we are in plane so we can work within *cylindrical coordinates*. This implies that the *fully relativistic Lorentz force equation*

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (1.1)$$

can be written in the form

$$\begin{aligned} \frac{d\mathbf{p}}{dt} &= q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\ \Downarrow \\ \frac{d}{dt}(m\gamma\mathbf{v}) &= q\left(\frac{Q}{R^2}\hat{r} + \omega RB(\hat{\theta} \times \hat{z})\right) \\ m\gamma\frac{d\mathbf{v}}{dt} &= q\left(\frac{Q}{R^2} - \omega RB\right)\hat{r} \end{aligned}$$

Note that for circular motion to exist we need a radial acceleration of the form $d\mathbf{v}/dt = -\omega^2 R\hat{r}$ this implies that

$$\begin{aligned} -m\gamma\omega^2 R &= q\left(\frac{Q}{R^2} - \omega RB\right) \\ \Downarrow \\ \frac{q}{m} &= \frac{-\gamma\omega^2 R}{\left(\frac{Q}{R^2} - \omega RB\right)} \\ &= \frac{-\omega^2 R}{\sqrt{1 - (\omega R)^2}} \frac{1}{\frac{Q}{R^2} - \omega RB} \\ &= \frac{\omega^2}{\sqrt{1 - (\omega R)^2}} \frac{1}{\omega B - \frac{Q}{R^3}} \end{aligned}$$

Note that $\gamma = (1 - v^2)^{-1/2} = (1 - (\omega R)^2)^{-1/2}$.

2 Problem #2

(a) For an electromagnetic wave for which the electric field is given as

$$\mathbf{E} = E_0 \sin \omega t (\sin \omega z, \cos \omega z, 0) \quad (2.1)$$

where E_0 and ω are constant we can solve for \mathbf{B} by noting the maxwell equation

$$\begin{aligned} -\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times \mathbf{E} \\ &= \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_0 \sin \omega t \sin \omega z & E_0 \sin \omega t \cos \omega z & 0 \end{vmatrix} \\ &= (E_0 \omega \sin \omega t \sin \omega z, E_0 \omega \sin \omega t \cos \omega z, 0) \\ &\Downarrow \\ \mathbf{B} &= E_0 \cos \omega t (\sin \omega z, \cos \omega z, 0) \end{aligned}$$

Note that this satisfies one of the four source-free Maxwell equation. The remaining three are

$$\nabla \cdot \mathbf{E} = 0 \quad \nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}$$

We can quickly verify the first two equations by calculating

$$\begin{aligned} \nabla \cdot \mathbf{E} &= E_0 \sin \omega t \left(\frac{\partial}{\partial x}(\sin \omega z) + \frac{\partial}{\partial y}(\cos \omega z) + \frac{\partial}{\partial z}(0) \right) = 0 \\ \nabla \cdot \mathbf{B} &= E_0 \cos \omega t \left(\frac{\partial}{\partial x}(\sin \omega z) + \frac{\partial}{\partial y}(\cos \omega z) + \frac{\partial}{\partial z}(0) \right) = 0 \end{aligned}$$

and for the last we equation we calculate

$$\begin{aligned} \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_0 \cos \omega t \sin \omega z & E_0 \cos \omega t \cos \omega z & 0 \end{vmatrix} - \frac{\partial}{\partial t}(E_0 \sin \omega t) (\sin \omega z, \cos \omega z, 0) \\ &= E_0 \omega \cos \omega t (\sin \omega z, \cos \omega z, 0) - E_0 \omega \cos \omega t (\sin \omega z, \cos \omega z, 0) = 0 \end{aligned}$$

Therefore all the source-free Maxwell equations hold.

(b) For the electromagnetic wave in part (a) we can calculate the energy density, W , by

$$\begin{aligned} W &= \frac{1}{8\pi} (E^2 + B^2) = \frac{1}{8\pi} \left(E_0^2 \sin^2 \omega t (\sin^2 \omega z + \cos^2 \omega z) + E_0^2 \cos^2 \omega t (\cos^2 \omega z + \sin^2 \omega z) \right) \\ &= \frac{1}{8\pi} \left(E_0^2 (\sin^2 \omega t + \cos^2 \omega t) \right) \\ &= \frac{E_0^2}{8\pi} \end{aligned}$$

We can also calculate the *Poynting vector*, \mathbf{S} , as

$$\begin{aligned}\mathbf{S} &= \frac{1}{4\pi} \mathbf{E} \times \mathbf{B} = \frac{1}{4\pi} \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ E_0 \sin \omega t \sin \omega z & E_0 \sin \omega t \cos \omega z & 0 \\ E_0 \cos \omega t \sin \omega z & E_0 \cos \omega t \cos \omega z & 0 \end{vmatrix} \\ &= 0\hat{x} + 0\hat{y} + \frac{E_0^2}{4\pi} (\sin \omega t \cos \omega t \sin \omega z \cos \omega z - \sin \omega t \cos \omega t \sin \omega z \cos \omega z) \hat{z} \\ &= 0\end{aligned}$$

(c) We can repeat the process from parts (a) and (b) for the electric field

$$\mathbf{E} = E_0 \cos \omega z (\cos \omega t, -\sin \omega t, 0) \quad (2.2)$$

which gives us

$$\begin{aligned}-\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times \mathbf{E} \\ &= \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_0 \cos \omega t \cos \omega z & -E_0 \sin \omega t \cos \omega z & 0 \end{vmatrix} \\ &= (-E_0 \omega \sin \omega t \sin \omega z, -E_0 \omega \cos \omega t \sin \omega z, 0) \\ &\Downarrow \\ \mathbf{B} &= E_0 \sin \omega z (\cos \omega t, -\sin \omega t, 0)\end{aligned}$$

which allows us to calculate

$$\begin{aligned}\nabla \cdot \mathbf{E} &= E_0 \left(\cos \omega t \frac{\partial}{\partial x} (\cos \omega z) - \sin \omega t \frac{\partial}{\partial y} (\cos \omega z) + \frac{\partial}{\partial z} (0) \right) = 0 \\ \nabla \cdot \mathbf{B} &= E_0 \left(\cos \omega t \frac{\partial}{\partial x} (\sin \omega z) - \sin \omega t \frac{\partial}{\partial y} (\sin \omega z) + \frac{\partial}{\partial z} (0) \right) = 0\end{aligned}$$

and

$$\begin{aligned}\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_0 \cos \omega t \sin \omega z & -E_0 \sin \omega t \sin \omega z & 0 \end{vmatrix} - \frac{\partial}{\partial t} (E_0 \cos \omega z) (\cos \omega t, -\sin \omega t, 0) \\ &= E_0 \omega \cos \omega z (-\sin \omega t, -\cos \omega t, 0) - E_0 \omega \cos \omega z (-\sin \omega t, -\cos \omega t, 0) = 0\end{aligned}$$

We also calculate

$$\begin{aligned}W &= \frac{1}{8\pi} (E^2 + B^2) = \frac{1}{8\pi} \left(E_0^2 \cos^2 \omega z (\sin^2 \omega t + \cos^2 \omega t) + E_0^2 \cos^2 \omega z (\cos^2 \omega t + \sin^2 \omega t) \right) \\ &= \frac{1}{8\pi} \left(E_0^2 (\sin^2 \omega z + \cos^2 \omega z) \right) \\ &= \frac{E_0^2}{8\pi}\end{aligned}$$

and

$$\begin{aligned}
\mathbf{S} &= \frac{1}{4\pi} \mathbf{E} \times \mathbf{B} = \frac{1}{4\pi} \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ E_0 \cos \omega t \cos \omega z & -E_0 \sin \omega t \cos \omega z & 0 \\ E_0 \cos \omega t \sin \omega z & -E_0 \sin \omega t \sin \omega z & 0 \end{vmatrix} \\
&= 0\hat{x} + 0\hat{y} + \frac{E_0^2}{4\pi} (\sin \omega t \cos \omega t \sin \omega z \cos \omega z - \sin \omega t \cos \omega t \sin \omega z \cos \omega z) \hat{z} \\
&= 0
\end{aligned}$$

3 Problem #3

- (a) For constant \mathbf{E} and \mathbf{B} fields we can show that in general it is possible to find a new *Lorentz frame* related to the original by a boost velocity of the form

$$\mathbf{v} = \lambda \mathbf{E} \times \mathbf{B}$$

such that in the boosted frame \mathbf{E}' and \mathbf{B}' are parallel. We can do this by taking the boosted field equations

$$\begin{aligned}
\mathbf{E}' &= \gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \frac{\gamma - 1}{v^2} (\mathbf{v} \cdot \mathbf{E}) \mathbf{v} \\
\mathbf{B}' &= \gamma(\mathbf{B} - \mathbf{v} \times \mathbf{E}) - \frac{\gamma - 1}{v^2} (\mathbf{v} \cdot \mathbf{B}) \mathbf{v}
\end{aligned}$$

Now if we replace \mathbf{v} with the velocity we were given we find

$$\begin{aligned}
\mathbf{E}' &= \gamma(\mathbf{E} + (\lambda \mathbf{E} \times \mathbf{B}) \times \mathbf{B}) - \frac{\gamma - 1}{v^2} ((\lambda \mathbf{E} \times \mathbf{B}) \cdot \mathbf{E}) (\lambda \mathbf{E} \times \mathbf{B}) \\
&= \gamma(\mathbf{E} + \lambda((\mathbf{E} \cdot \mathbf{B})\mathbf{B} - (\mathbf{B} \cdot \mathbf{B})\mathbf{E})) \\
&= \gamma((1 - \lambda B^2)\mathbf{E} + \lambda(\mathbf{E} \cdot \mathbf{B})\mathbf{B})
\end{aligned}$$

And similarly we find

$$\begin{aligned}
\mathbf{B}' &= \gamma(\mathbf{B} - (\lambda \mathbf{E} \times \mathbf{E}) \times \mathbf{B}) - \frac{\gamma - 1}{v^2} ((\lambda \mathbf{E} \times \mathbf{B}) \cdot \mathbf{B}) (\lambda \mathbf{E} \times \mathbf{B}) \\
&= \gamma(\mathbf{B} - \lambda((\mathbf{E} \cdot \mathbf{E})\mathbf{B} + (\mathbf{B} \cdot \mathbf{E})\mathbf{E})) \\
&= \gamma((1 - \lambda E^2)\mathbf{B} + \lambda(\mathbf{E} \cdot \mathbf{B})\mathbf{E})
\end{aligned}$$

- (b)