Physics 615

Methods of Theoretical Physics I

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Homework #1

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1 Problem #1

To evaluate the integral

$$I(k) = \int_0^5 \sin t e^{-k(\sinh t)^4} dt$$

to leading order in k we can use an asymptotic approach by taking t to be small. This implies that

$$I(k) \approx \int_0^5 t e^{-kt^4} dt$$

where we apply a change in variables as $t' = t^2$ which implies that

$$I(k) \approx \frac{1}{2} \int_0^{\sqrt{5}} e^{-kt'^2} dt'$$

Now we change variable again where $\tau = \sqrt{k} t'^2$ so

$$\begin{aligned} d\tau &= 2\sqrt{k} t' dt' \qquad \Downarrow \\ dt' &= \frac{1}{2} \frac{1}{\sqrt{k\tau}} d\tau \end{aligned}$$

So our integral becomes

$$\begin{aligned} I(k) &\approx \int_0^5 t e^{-kt^4} dt \\ &\Downarrow \\ &\approx \lim_{k\to\infty} \frac{1}{4\sqrt{k}} \int_0^{f(k)} \tau^{-1/2} e^{-\tau} d\tau \\ &\approx \frac{1}{4\sqrt{k}} \int_0^{\infty} \tau^{-1/2} e^{-\tau} d\tau \\ &\approx \frac{1}{4\sqrt{k}} \Gamma(1/2) \\ &\approx \frac{\sqrt{\pi}}{4\sqrt{k}} \end{aligned}$$

2 Problem #2

For the higher order expansion of the Gamma function given as

$$\Gamma(x) \approx \sqrt{2\pi} x^{x-1/2} e^{-x} \left(1 + \frac{A}{x} + \frac{B}{x^2}\right)$$

we can find A and B using the relation

$$\Gamma(x+1) = x\Gamma(x)$$

this implies that for large x we have

$$\begin{aligned}
\Gamma(x+1) &= x\Gamma(x) \\
\sqrt{2\pi}(x+1)^{x+1/2} e^{-(x+1)} \left(1 + \frac{A}{x+1} + \frac{B}{(x+1)^2}\right) &= x\sqrt{2\pi} x^{x-1/2} e^{-x} \left(1 + \frac{A}{x} + \frac{B}{x^2}\right) \\
\sqrt{2\pi}(x+1)^{x+1/2} e^{-(x+1)} \left(1 + \frac{A}{x+1} + \frac{B}{(x+1)^2}\right) &= \sqrt{2\pi} x^{x+1/2} e^{-x} \left(1 + \frac{A}{x} + \frac{B}{x^2}\right) \\
&\Downarrow \\
\left(\frac{x+1}{x}\right)^{x+1/2} e^{-1} \left(1 + \frac{A}{x+1} + \frac{B}{(x+1)^2}\right) &= 1 + \frac{A}{x} + \frac{B}{x^2} \\
\left(1 + \frac{1}{x}\right)^{x+1/2} e^{-1} \left(1 + \frac{A}{x+1} + \frac{B}{(x+1)^2}\right) &= 1 + \frac{A}{x} + \frac{B}{x^2}
\end{aligned}$$

This allows us to expand

$$\left(1 + \frac{1}{x}\right)^{x+1/2} = e\left(1 + \frac{1}{12x^2} - \frac{1}{12x^3} + \frac{113}{1440x^4} + \mathcal{O}(x^{-5})\right)$$

Which makes our equality become which allows us to solve for A and B by grouping terms of equal order.

$$\begin{aligned}
\left(1 + \frac{1}{12x^2} - \frac{1}{12x^3}\right)\left(1 + \frac{A}{x+1} + \frac{B}{(x+1)^2}\right) &= 1 + \frac{A}{x} + \frac{B}{x^2} \\
&\Downarrow \\
\left(1 + \frac{1}{12x^2} - \frac{1}{12x^3}\right)\left(1 + \frac{A}{x} - \frac{A}{x^2} + \frac{B}{x^2} - \frac{2B}{x^3}\right) &= 1 + \frac{A}{x} + \frac{B}{x^2} \\
1 + \frac{A}{x} + \frac{B - A + 1/12}{x^2} + \frac{13/12A - 2B - 1/12}{x^3} &= 1 + \frac{A}{x} + \frac{B}{x^2} \\
&\Downarrow \\
\frac{B - A + 1/12}{x^2} + \frac{13/12A - 2B - 1/12}{x^3} &= \frac{B}{x^2}
\end{aligned}$$

So we can say that

$$A = \frac{1}{12}$$

which leads to

$$\begin{aligned}
\frac{13}{144} - \frac{1}{12} - 2B &= 0 \\
2B &= \frac{1}{144} \\
B &= \frac{1}{288}
\end{aligned}$$

3 Problem #3

We can find the leading term of the integral

$$I = \int_0^\infty dt e^{kt-e^t}$$

by changing variables from $x = e^t/k$ which is small for large k. This implies that $dx = e^t/kdt$ or $dt = x^{-1}dx$. This allows us to change variables to

$$\begin{aligned} I &= \int_{1/k}^\infty e^{k\log(kx)-kx}\frac{1}{x}dx \\ &= \int_{1/k}^\infty e^{k\log(k)+k\log(x)-kx}\frac{1}{x}dx \\ &= e^{k\log(k)}\int_{1/k}^\infty e^{-k(x-\log(x))}\frac{1}{x}dx \end{aligned}$$

We can take $\phi(x) = x - \log(x)$ which has a minimum at $x = 1$ where $\phi''(x) > 0$ therefore we can take the asymptotic solution

$$I(k) \approx f(c)e^{-k\phi(c)}\sqrt{\frac{2\pi}{k\phi''(c)}}$$

which for $c = 1$ and $f(c) = 1$ we have

$$I(k) \approx k^k e^{-k}\sqrt{\frac{2\pi}{k}}$$

4 Problem #4

To find the sum

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2}$$

we can consider the complex integral

$$I = \oint_C \frac{\pi \cot(\pi z)}{(a+z)^2}$$

Note that a is not an integer. We note that $\pi \cot(\pi z)$ has a simple pole at integer values of z. They have a residue of value one. So if we take a contour that covers the positive half of the complex plane. This allows us to solve the integral as

$$I = 2\pi i \sum_{n=-N}^{N} \frac{1}{(z+a)^2} + 2\pi i \sum [\text{Res}\,[\pi \cot(\pi z), a]]$$

This implies that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2} = -\text{Res}\,[\pi \cot(\pi z), a] = -\pi^2 \csc^2(\pi a)$$