HT Homework 0

1 Probability and Statistics

1. (combinatorics)
   Let $C(N,K) = 1$ for $K = 0$ or $K = N$, and $C(N,K) = C(N-1,K) + C(N-1,K-1)$ $for N \geq 1$. Prove that $C(N,K) = \frac{N!}{K!(N-K)!}$ for $N \geq 1$ and $0 \leq K \leq N$.

2. (counting)
   What is the probability of getting exactly 4 heads when flipping 10 fair coins?

   What is the probability of getting a full house (XXXYY) when randomly drawing 5 cards out of a deck of 52 cards?

3. (conditional probability)
   If your friend flipped a fair coin three times, and tell you that one of the tosses resulted in head, what is the probability that all three tosses resulted in heads?

4. (Bayes theorem)
   A program selects a random integer X like this: a random bit is first generated uniformly. If the bit is 0, X is drawn uniformly from $\{0, 1, \ldots, 7\}$; otherwise, X is drawn uniformly from $\{0, -1, -2, -3\}$. If we get an X from the program with $|X| = 1$, what is the probability that X is negative?

5. (union/intersection)
   If $P(A) = 0.3$ and $P(B) = 0.4$,
   what is the maximum possible value of $P(A \cap B)$?
   what is the minimum possible value of $P(A \cap B)$?
   what is the maximum possible value of $P(A \cup B)$?
   what is the minimum possible value of $P(A \cup B)$?

6. (mean/variance)
   Let mean $\bar{X} = \frac{1}{N}\sum_{n=1}^{N} X_n$ and variance $\sigma_X^2 = \frac{1}{N-1}\sum_{n=1}^{N}\left(X_n - \bar{X}\right)^2$.
   Prove that

$$\sigma_X^2 = \frac{N}{N-1}\left(\frac{1}{N}\sum_{n=1}^{N} X_n^2 - \bar{X}^2\right).$$

7. (Gaussian distribution)
   If X_1 and X_2 are independent random variables, where $p(X_1)$ is Gaussian with mean 2 and variance 1 , $p(X_2)$ is Gaussian with mean -3 and variance 4. Let $Z = X_1 + X_2$. Prove $p(Z)$ is Gaussian, and determine its mean and variance.

2 Linear Algebra

1. (rank) What is the rank of $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{pmatrix}$?

2. (inverse) What is the inverse of $\begin{pmatrix} 0 & 2 & 4 \\ 2 & 4 & 2 \\ 3 & 3 & 1 \end{pmatrix}$?

3. (eigenvalues/eigenvectors) What are the eigenvalues and eigenvectors of $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$?

4. (singular value decomposition)