Lecture 15

Bayes' Theorem and Independence

DSC 40A

Agenda

- Law of Total Probability.
- Bayes' Theorem.
- Independence.

Remember, we've posted **many** probability resources on the resources tab of the course website. These will come in handy! Specific resources you should look at:

- The DSC 40A probability roadmap, written by Janine Tiefenbruck.
- The textbook Theory Meets Data, which explains many of the same ideas and contains more practice problems.

For combinatorics specifically, there are two supplementary videos Suraj Rampure (previously taught and developed major parts of 40a) created that you should watch. They are linked in the course cal..

Law of Total Probability

Example: Getting to school

You conduct a survey where you ask students two questions:

1. How did you get to campus today – trolley, bike, or drive? (Assume these are the only options.)
2. Were you late?

All six boxes sum to 1

	Late	Not Late
Trolley	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

$\mathbb{P}(\text{Trolley} \cap \text{not Late})$

$\cap$ = and

$\mathbb{P}(\text{Drive} \cap \text{Late})$

Question 🤔

Take a moment to pause and reflect...

$\mathbb{P}(\text{Bike} \cap \text{Late})$

	Late	Not Late
Trolley	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

$= 0.3$ $\mathbb{P}(\text{trolley})$

$= 0.1$

$= 0.6$

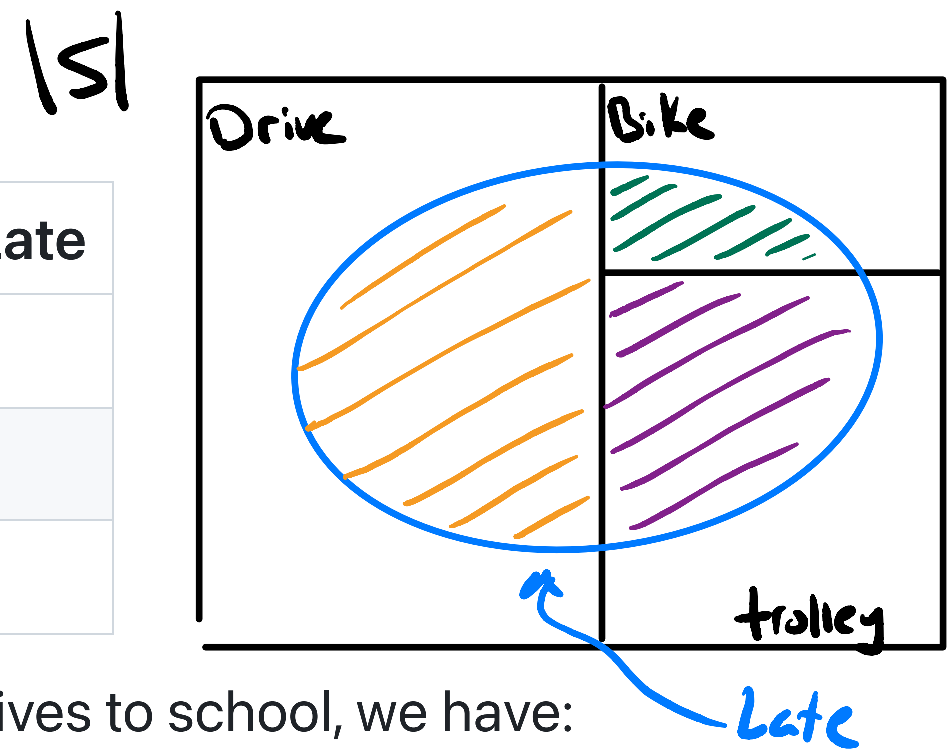
Σ $= 0.45$ $= 0.55$ $\mathbb{P}(\text{not Late})$

What's the probability that a randomly selected person was late?

A. 0.24 B. 0.30 C. 0.45 D. 0.50 E. None of the above.

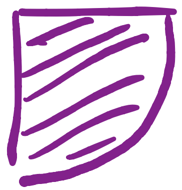
Example: Getting to school

	Late	Not Late
Trolley	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24



- Since everyone either takes the trolley, bikes, or drives to school, we have:

$$\underbrace{\mathbb{P}(\text{Late})}_{0.45} = \underbrace{\mathbb{P}(\text{Late} \cap \text{Trolley})}_{0.06} + \underbrace{\mathbb{P}(\text{Late} \cap \text{Bike})}_{0.03} + \underbrace{\mathbb{P}(\text{Late} \cap \text{Drive})}_{0.36}$$



$\mathbb{P}(x \mid y)$ → given

(Prob. of x given y)

Question 🤔

Take a moment to pause and reflect...

	Late	Not Late
Trolley	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

$$\mathbb{P}(\text{Late} \mid \text{trolley}) = \frac{\mathbb{P}(\text{Late} \cap \text{trolley})}{\mathbb{P}(\text{trolley})}$$

L = Late
NL = Not Late

$$= \frac{\mathbb{P}(\text{Late} \cap \text{trolley})}{\mathbb{P}(L \cap \text{trolley}) + \mathbb{P}(NL \cap \text{trolley})}$$

$$= \frac{0.06}{0.06 + 0.24}$$

$$= \frac{6}{30} = \frac{1}{5} = 0.2$$

Avi took the trolley to school. What is the probability that he was late?

A. 0.06 B. 0.20 C. 0.25 D. 0.45 E. None of the above.

Example: Getting to school

	Late	Not Late
Trolley	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

Intersection
"AND"

- Since everyone either takes the trolley, bikes, or drives to school, we have:

$$\mathbb{P}(\text{Late}) = \mathbb{P}(\text{Late} \cap \text{Trolley}) + \mathbb{P}(\text{Late} \cap \text{Bike}) + \mathbb{P}(\text{Late} \cap \text{Drive})$$

equivalent statements.

- Another way of expressing the same thing:

$$\mathbb{P}(\text{Late}) = \mathbb{P}(\text{Trolley})\,\mathbb{P}(\text{Late}|\text{Trolley}) + \mathbb{P}(\text{Bike})\,\mathbb{P}(\text{Late}|\text{Bike}) + \mathbb{P}(\text{Drive})\,\mathbb{P}(\text{Late}|\text{Drive})$$

$\cap$ = intersection i.e. "AND"

$\cup$ = union i.e. "OR"

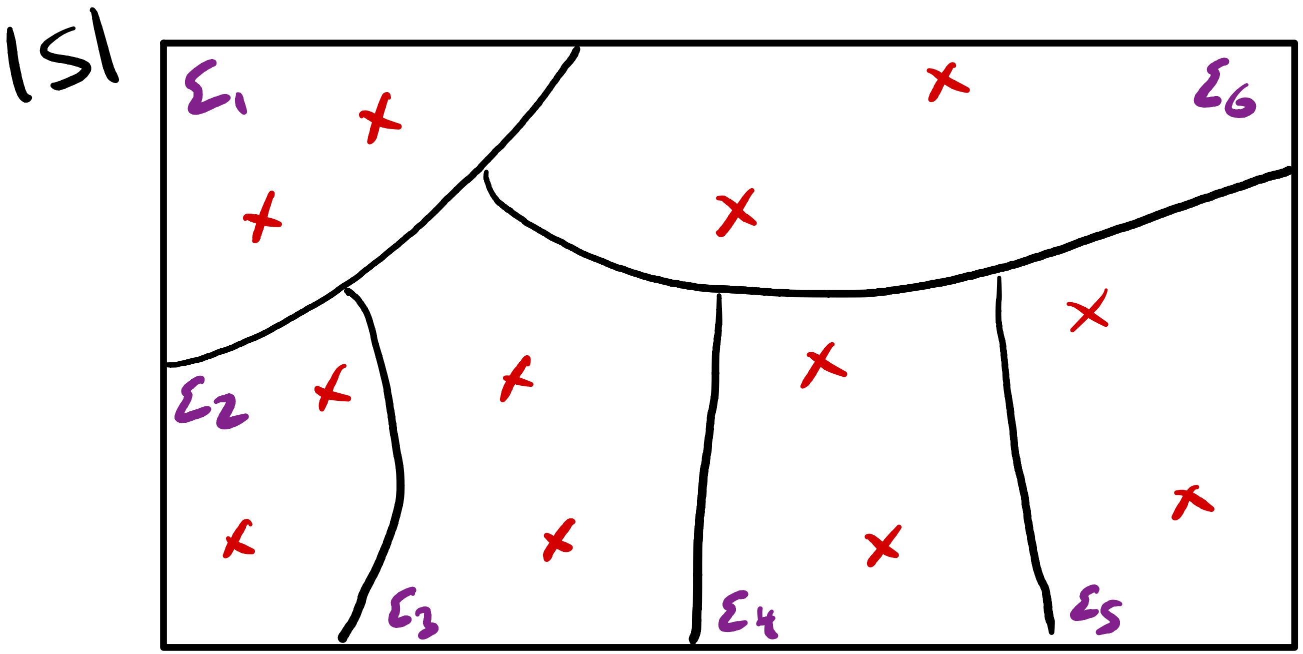
Partitions

- A set of events $E_1, E_2, \ldots, E_k$ is a **partition** of S if:
  - $\mathbb{P}(E_i \cap E_j) = 0$ for all pairs $i \neq j$.
  - $\mathbb{P}(E_1 \cup E_2 \cup \ldots \cup E_k) = 1$.
    - Equivalently, $\mathbb{P}(E_1) + \mathbb{P}(E_2) + \ldots + \mathbb{P}(E_k) = 1$.
- In other words, $E_1, E_2, \ldots, E_k$ is a partition of S if every outcome $s \in S$ is in **exactly** one event E_i.

|S| Sample Space

exclusive events i.e. you can't both Drive & Bike to School

e.g. CoinFlip H or T both are 0.5+0.5=1



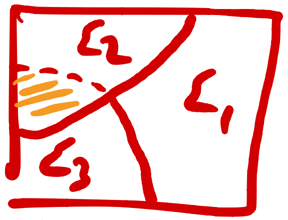
X = Individual outcome
e.g. taken trolley & late

For our outcomes "X" & E_k

(I.) no E overlaps w/ another E

(II.) every outcome "X" is in ONE & ONLY ONE E

NO!



OVERLAP!!!

Example partitions

- In getting to school, the events Trolley, Bike, and Drive.
- In getting to school, the events Late and Not Late.
- In selecting an undergraduate student at random, the events Freshman, Sophomore, Junior, and Senior, and Other. → super senior ^^ or HS
- In rolling a die, the events Even and Odd. CANT HAVE Eudd or an Oven
- In drawing a card from a standard deck of cards, the events Spades, Clubs, Hearts, and Diamonds. Suites
- Special case: Any event A and its complement $\bar{A}$.

Late (1 - Late)

LATE's complement is all students NOT LATE "on time"

A, B, C = pairwise mutually Exclusive, but Do not form a partition $P(A)+P(B)+P(C)\neq 1$

The Law of Total Probability

- If A is an event and $E_1, E_2, \ldots, E_k$ is a **partition** of S, then:

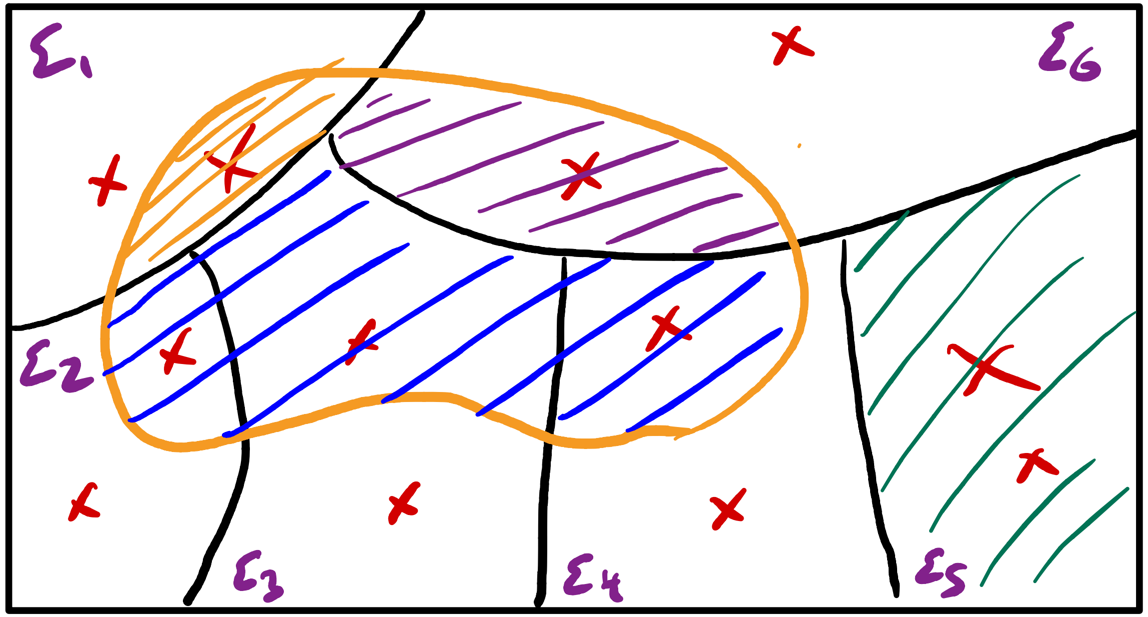
$$\mathbb{P}(A) = \mathbb{P}(A \cap E_1) + \mathbb{P}(A \cap E_2) + \ldots + \mathbb{P}(A \cap E_k)$$
$$= \sum_{i=1}^{k} \mathbb{P}(A \cap E_i)$$

$\mathbb{P}(\text{Late}) = \mathbb{P}(\text{Late} \cap \text{trolley}) + \mathbb{P}(\text{Late} \cap \text{Drive}) + \mathbb{P}(\text{Late} \cap \text{Bike})$

A : Late

E_1 : trolley , E_2 : Drive, E_3 : Bike

|S|



nothing in $\mathcal{E}_5$ in

$$\mathbb{P}(A \cap \mathcal{E}_1) = \mathbb{P}(A) \cdot \mathbb{P}(\mathcal{E}_1 \mid A) = \mathbb{P}(\mathcal{E}_1) \cdot \mathbb{P}(A \mid \mathcal{E}_1)$$

$$\boxed{\mathbb{P}(A)} = \mathbb{P}(A \cap \mathcal{E}_1) + \cdots + \overbrace{\mathbb{P}(A \cap \mathcal{E}_5)}^{0} + \mathbb{P}(A \cap \mathcal{E}_6)$$

$$\mathbb{P}(A) = \sum_{i=1}^{6} \mathbb{P}(A \cap \mathcal{E}_i)$$

The Law of Total Probability

"and"

- If A is an event and $E_1, E_2, \ldots, E_k$ is a **partition** of S, then:

"and" "and" "and"

$$\mathbb{P}(A) = \mathbb{P}(A \cap E_1) + \mathbb{P}(A \cap E_2) + \ldots + \mathbb{P}(A \cap E_k)$$
$$= \sum_{i=1}^{k} \mathbb{P}(A \cap E_i)$$

via multiplication Rule

Conditional Prob.

- Since $\mathbb{P}(A \cap E_i) = \mathbb{P}(E_i) \cdot \mathbb{P}(A|E_i)$ by the multiplication rule, an equivalent formulation is:

conditionals

$$\mathbb{P}(A) = \mathbb{P}(E_1) \cdot \mathbb{P}(A|E_1) + \mathbb{P}(E_2) \cdot \mathbb{P}(A|E_2) + \ldots + \mathbb{P}(E_k) \cdot \mathbb{P}(A|E_k)$$
$$= \sum_{i=1}^{k} \mathbb{P}(E_i) \cdot \mathbb{P}(A|E_i)$$

A is conditioned on E_2 "given" = | ≠ not 1 or L or I

$$P(\text{Trolley} \mid \text{Late}) = \frac{P(\text{Trolley} \cap \text{Late})}{P(\text{Late})} = \frac{P(\text{trolley} \cap \text{Late})}{P(T \cap L) + P(B \cap L) + P(D \cap L)}$$

Question 🤔

Take a moment to pause and reflect...

$$= \frac{0.06}{0.06 + 0.03 + 0.36}$$

	Late	Not Late
Trolley	0.06	0.24
Bike	0.03	0.07
Drive	0.36	0.24

$$\frac{0.06}{0.45} = \frac{6}{45} \approx 0.15$$

Lauren is late to school. What is the probability that she took the trolley? Choose the best answer.

A. About 0.05 B. About 0.15 C. About 0.30 D. About 0.40

Bayes' Theorem

$$P(A \cap B) = P(A) \cdot P(B|A)$$
or
$$= P(B) \cdot P(A|B)$$

Example: Getting to school

- Now, suppose we don't have that entire table. Instead, all you know is:
  - $\mathbb{P}(\text{Late}) = 0.45.$
  - $\mathbb{P}(\text{Trolley}) = 0.3.$
  - $\mathbb{P}(\text{Late}|\text{Trolley}) = 0.2.$ given $P(A|B)$
- Can we still find $\mathbb{P}(\text{Trolley}|\text{Late})$? can we find $P(B|A)$

we will Acomplish this via the Multiplication Rule

$$\mathbb{P}(\text{trolley}|\text{Late}) = \frac{\mathbb{P}(\text{trolley} \cap \text{late})}{\mathbb{P}(\text{Late})} = \frac{\mathbb{P}(\text{trolley}) \cdot \mathbb{P}(\text{Late}|\text{trolley})}{\mathbb{P}(\text{Late})}$$

$$= \frac{0.3 \cdot 0.2}{0.45} = \frac{0.06}{0.45} = \frac{6}{45}$$

AKA Bayes' Rule

Bayes' Theorem

- Recall that the multiplication rule states that:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B|A)$$

- It also states that: =

Because $P(A \cap B) = P(B \cap A)$

$$\mathbb{P}(B \cap A) = \mathbb{P}(B) \cdot \mathbb{P}(A|B)$$

- But since $A \cap B = B \cap A$, we have that:

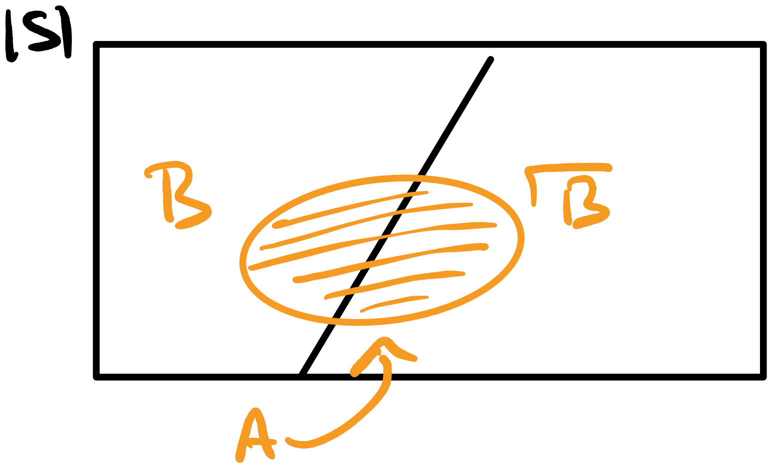
$$\mathbb{P}(A) \cdot \mathbb{P}(B|A) = \mathbb{P}(B) \cdot \mathbb{P}(A|B)$$

- Re-arranging yields **Bayes' Theorem**:

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B) \cdot \mathbb{P}(A|B)}{\mathbb{P}(A)}$$

Bayes' reverse engineer Conditional Probability Questions

Bayes' Theorem and the Law of Total Probability



$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap \bar{B})$

- Bayes' Theorem:

$$\boxed{\mathbb{P}(B|A) = \frac{\mathbb{P}(B) \cdot \mathbb{P}(A|B)}{\mathbb{P}(A)}}$$

- Recall from earlier, for any sample space S, B and $\bar{B}$ partition S. Using the Law of Total Probability, we can re-write $\mathbb{P}(A)$ as:

$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap \bar{B}) = \overbrace{\mathbb{P}(B) \cdot \mathbb{P}(A|B) + \mathbb{P}(\bar{B}) \cdot \mathbb{P}(A|\bar{B})}^{\text{using multiplication Rule.}}$$

- This means that we can re-write Bayes' Theorem as:

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B) \cdot \mathbb{P}(A|B)}{\mathbb{P}(B) \cdot \mathbb{P}(A|B) + \mathbb{P}(\bar{B}) \cdot \mathbb{P}(A|\bar{B})}$$

why?

Well we are not often Provided or Know $\mathbb{P}(A)$

95% of the time it works!

Example: Drug test

A manufacturer claims that its drug test will **detect steroid use**. What the company does not tell you is that 15% of all steroid-free individuals also test positive (the "false positive rate"). Suppose 10% of the Tour de France bike racers use steroids and your favorite cyclist just tested positive. What's the probability that they used steroids?

A: Positive Test

B: Cyclist Actually uses steroids

$\mathbb{P}(A|B) = 0.95$ "we have a pos. test, given cyclist uses steroids"

$\mathbb{P}(A|\bar{B}) = 0.15$ "we have a pos. test, given a cyclist DOES NOT use steroids"

$\mathbb{P}(B) = 0.1$ "How many cyclists % in Tour de France use steroids (real steroid Rate)"

What we want to know is $\mathbb{P}(B|A)$ "cyclist Actual uses steroids given Pos. test"

$$\mathbb{P}(A|B) = 0.95$$

$$\mathbb{P}(A|\bar{B}) = 0.15$$

$$\mathbb{P}(B) = 0.1$$

$$\mathbb{P}(A) = ?$$

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B) \cdot \mathbb{P}(A|B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B) \cdot \mathbb{P}(A|B)}{\mathbb{P}(A \cap B) + \mathbb{P}(A \cap \bar{B})}$$

(Bayes' Rule)

(Law of Total Prob.)

$$= \frac{\mathbb{P}(B) \cdot \mathbb{P}(A|B)}{\mathbb{P}(B)\,\mathbb{P}(A|B) + \mathbb{P}(\bar{B})\,\mathbb{P}(A|\bar{B})}$$

Insight that you are on the right track!

$$= \frac{0.1 \cdot 0.95}{0.1 \cdot 0.95 + (1-0.1) \cdot 0.15} = 40\%$$

{ what we want to know is $\mathbb{P}(B|A)$ "cyclist Actual uses steroids given Pos. test" } if test pos. 40% you actualy use

$|S|$ = prob. space

S = shake shack

Example: Taste test

- Your friend claims to be able to correctly guess what restaurant a burger came from, after just one bite.
- The probability that she correctly identifies an In-n-Out Burger is 0.55, a Shake Shack burger is 0.75, and a Five Guys burger is 0.6.
- You buy 5 In-n-Out burgers, 4 Shake Shack burgers, and 1 Five Guys burger, choose one of the burgers randomly, and give it to her.
- **Question**: Given that she guessed it correctly, what's the probability she ate a Shake Shack burger?

4 Events in this Problem!

I : in-n-out , S : shakeshack , F : five Guys , C : Correct Guess

$\mathbb{P}(I) = 0.5$, $\mathbb{P}(S) = 0.4$, $\mathbb{P}(F) = 0.1$

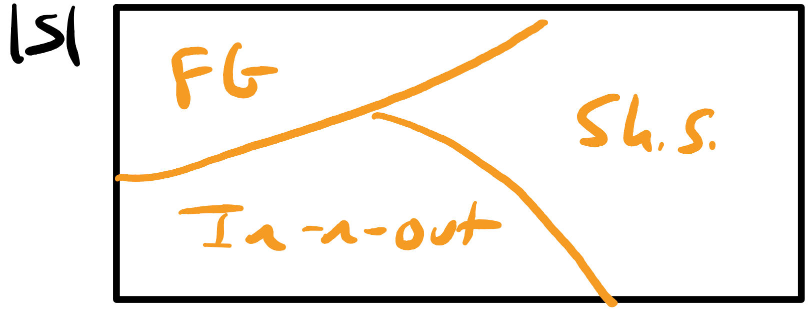
$\mathbb{P}(C|I) = 0.55$, $\mathbb{P}(C|S)\,0.75$, $\mathbb{P}(C|F) = 0.6$

$\boxed{\mathbb{P}(S|C)}$

$\mathbb{P}(I)=0.5$, $\mathbb{P}(S)=0.4$, $\mathbb{P}(F)=0.1$

$\mathbb{P}(c|I)=0.55$, $\mathbb{P}(c|s)\ 0.75$, $\mathbb{P}(c|F)=0.6$

$$\boxed{\mathbb{P}(s|c)}$$



$$\mathbb{P}(s|c) = \frac{\mathbb{P}(s)\cdot\mathbb{P}(c|s)}{\boxed{\mathbb{P}(c)}\,?} = \frac{\mathbb{P}(s)\cdot\mathbb{P}(c|s)}{\mathbb{P}(c\cap I)+\mathbb{P}(c\cap s)+\mathbb{P}(c\cap F)}$$

Bayes' theorem

Law of total Prob.

$\mathbb{P}(c|s)$

$\mathbb{P}(s|c)$

$$= \frac{\boxed{\mathbb{P}(s)\cdot\mathbb{P}(c|s)}}{\mathbb{P}(I)\mathbb{P}(c|I)+\boxed{\mathbb{P}(s)\mathbb{P}(c|s)}+\mathbb{P}(F)\mathbb{P}(c|F)}$$

multiplication Rule

$$= \frac{(0.4)(0.75)}{(0.5\cdot 0.55)+(0.4)(0.75)+(0.1\cdot 0.6)} \approx \frac{.3}{.255} \approx 45\%$$