DSC 40B

Theoretical Foundations II

Lecture 11 | Part 1

Adjacency Matrices (Recap)

Representations

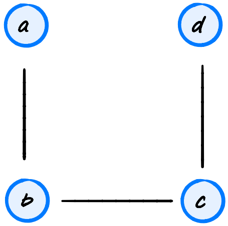
- How do we **store** a graph in a computer's memory?

- Three approaches:
  1. Adjacency matrices.
  2. Adjacency lists.
  3. "Dictionary of sets"

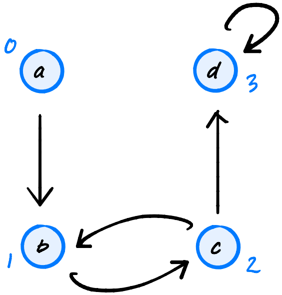
Adjacency Matrices

- Assume nodes are numbered 0, 1, ..., $|V| - 1$
- Allocate a $|V| \times |V|$ (Numpy) array
- Fill array as follows:
  - `arr[i,j] = 1` if $(i,j) \in E$
  - `arr[i,j] = 0` if $(i,j) \notin E$

Example



Example



$$
\begin{array}{c c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{array}
$$

Observations

- If G is undirected, matrix is symmetric.
- If G is directed, matrix may not be symmetric.

Time Complexity

operation	code	time
edge query	`adj[i,j] == 1`	$\Theta(1)$
degree(i)	`np.sum(adj[i,:])`	$\Theta(\lvert V\rvert)$

Space Requirements

- Uses $|V|^2$ bits, even if there are very few edges.
- But most real-world graphs are **sparse**.
  - They contain many fewer edges than possible.

Example: Facebook

- Facebook has 2 billion users.

$$
\begin{aligned}
(2 \times 10^9)^2 &= 4 \times 10^{18} \text{ bits} \\
&= 500 \text{ petabits} \\
&\approx 6500 \text{ years of video at 1080p} \\
&\approx 60 \text{ copies of the internet as it was in 2000}
\end{aligned}
$$

Adjacency Matrices and Math

- Adjacency matrices are useful mathematically.
- Example: (i, j) entry of A^2 gives number of hops of length 2 between i and j.

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Lecture 11 | Part 2

Adjacency Lists

What's Wrong with Adjacency Matrices?

- Requires $\Theta(|V|^2)$ storage.
- Even if the graph has no edges.
- **Idea**: only store the edges that exist.

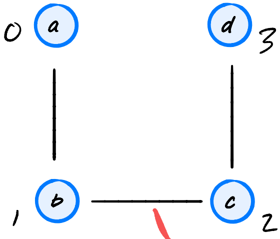
```
[
  [],
  [],
  []
]
```

Adjacency Lists

- Create a list `adj` containing $|V|$ lists.
- `adj[i]` is list containing the neighbors of node i.

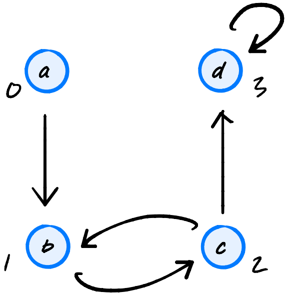
adj[0]

Example



[
0 [1],
1 [0,2],
2 [1,3],
3 [2]
]

Example



[
  [1],
  [2]
  [1,3],
  [3]
]

Observations

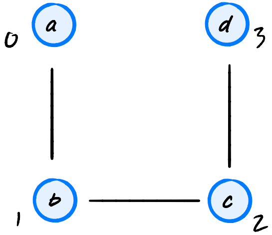
- If G is undirected, each edge appears twice.
- If G is directed, each edge appears once.

Time Complexity

operation	code	time
edge query	`j in adj[i]`	$\Theta(\text{degree}(i))$
degree(i)	`len(adj[i])`	$\Theta(1)$

Exercise

What does the following code do?




```
adj = [
        [1],
        [0, 2],
        [1, 3],
        [2]
]
```

```
for u in range(len(adj)):
    for v in adj[u]:
        print(f"({u}, {v})")
```

(0, 1)
(1, 0)
(1, 2)
⋮

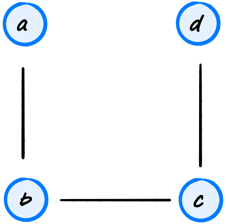
$G = (V, E)$

Exercise

How many times is the `print` statement executed in terms of $|V|$ and $|E|$?

$2|E|$ if undirected

$|E|$ if directed

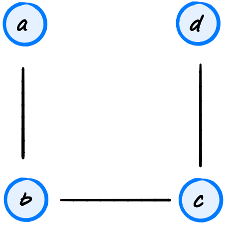



```
adj = [
    [1],
    [0, 2],
    [1, 3],
    [2]
]
```

```
for u in range(len(adj)):
    for v in adj[u]:
        print(f"({u}, {v})")
```

Exercise

What is the time complexity in terms of $|V|$ and $|E|$?




```
adj = [
    [1],
    [0, 2],
    [1, 3],
    [2]
]
```

```
for u in range(len(adj)):
    for v in adj[u]:
        print(f"({u}, {v})")
```

$\Theta(E+V)$

Exercise

What is the time complexity of the following code?

a d

b c

```
adj = [
    [],
    [],
    [],
    []
]
```

|V| iterations

```
for u in range(len(adj)):
    for v in adj[u]:
        print(f"({u}, {v})")
```

Looping Over Edges

- Looping over all edges in this way takes $\Theta(V + E)$ time.

- In aggregate, the `print` statement is executed:
  - $2|E|$ times if graph is undirected.
  - $|E|$ times if graph is directed.

- This is called an **aggregate analysis**.

Space Requirements

- Need $\Theta(|V|)$ space for outer list.
- Plus $\Theta(|E|)$ space for inner lists.
- In total: $\Theta(|V| + |E|)$ space.

$\Theta(V+E)$

Example: Facebook

- Facebook has 2 billion users, 400 billion friendships.

- If each edge requires 32 bits:

$$
\begin{aligned}
&(2 \text{ bits} \times 200 \times (2 \text{ billion}) \\
&= 64 \times 400 \times 10^9 \text{ bits} \\
&= 3.2 \text{ terabytes} \\
&= 0.04 \text{ years of HD video}
\end{aligned}
$$

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Lecture 11 | Part 3

Dictionary of Sets

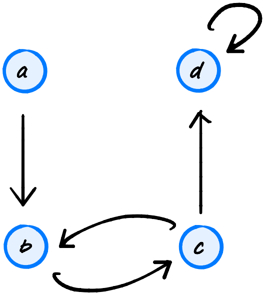
Tradeoffs

- Adjacency matrix: fast edge query, lots of space.
- Adjacency list: slower edge query, space efficient.
- Can we have the best of both?

Idea

- Use **hash tables**.
- Replace inner edge lists by `set`s.
- Replace outer list with `dict`.
  - Doesn't speed things up, but allows nodes to have arbitrary labels.

Example




```
{
    'a': {'b'},
    'b': {'c'},
    'c': {'b', 'd'},
    'd': {'d'}
}
```

Time Complexity

operation	code	time
edge query	`j in adj[i]`	$\Theta(1)$ average
degree(i)	`len(adj[i])`	$\Theta(1)$ average

Space Requirements

- Requires only $\Theta(V + E)$.
- But there is overhead to using hash tables.

Dict-of-sets implementation

- Install with `pip install dsc40graph`
- Import with `import dsc40graph`

networkx

- Docs: `https://eldridgejm.github.io/dsc40graph/`
- Source code: `https://github.com/eldridgejm/dsc40graph`
- Will be used in HW coding problems.

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Theoretical Foundations II

Lecture 11 | Part 4

Graph Search Strategies

How do we:

- determine if there is a path between two nodes?
- check if graph is connected?
- count connected components?

Search Stategies

- A **search strategy** is a procedure for exploring a graph.
- Different strategies are useful in different situations.

Node Statuses

At any point during a search, a node is in exactly one of three states:

- **visited**
- **pending** (discovered, but not yet visited)
- **undiscovered**

Rules

- At every step, next visited node chosen from among **pending** nodes.
- When a node is marked as **visited**, all of its neighbors have been marked as **pending**.

Choosing the next Node

How to choose among pending nodes?

- Idea 1: Visit **newest** pending (**depth-first search**).
- Idea 2: Visit **oldest** pending (**breadth-first search**).

Main Idea

Depth-first search (DFS) and breadth-first search (BFS) each discover different properties of the graph.

For example, we'll see that BFS is useful for finding shortest paths (DFS in general is not).

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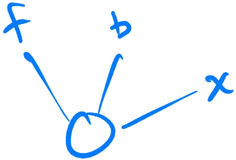
Lecture 11 | Part 5

Breadth-First Search



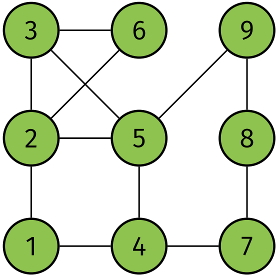
Breadth-First Search

- At every step:
  1. Visit **oldest** **pending** node.
  2. Mark its **undiscovered** neighbors as **pending**.

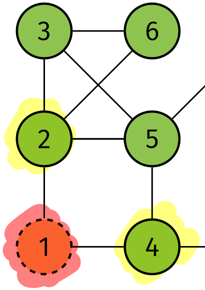
- Convention: in this class, neighbors produced in sorted order.[1]

[1] In general, the order in which a node's neighbors produced is arbitrary.

Example



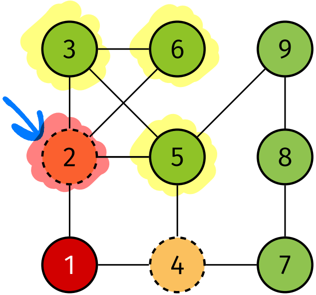
Example



pending = [~~1~~], 2, 4]

Before iterating.

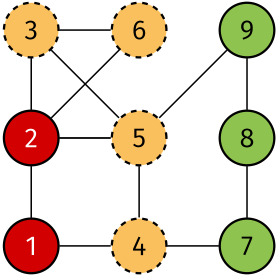
Example



pending = [~~2~~,4,3,5,6]

After 1st iteration.

Example

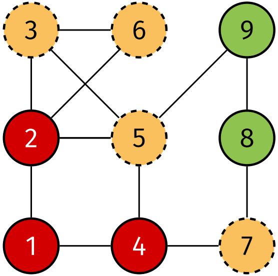


`pending = [4,3,5,6]`

After 2nd iteration.

Exercise: what will the picture look like after each of the next two iterations?

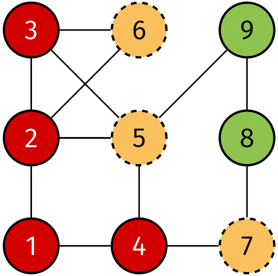
Example




```
pending = [3,5,6,7]
```

After 3rd iteration.

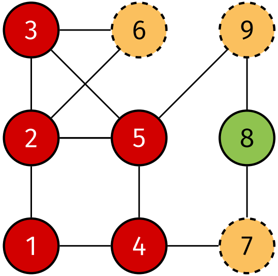
Example



`pending = [5,6,7]`

After 4th iteration.

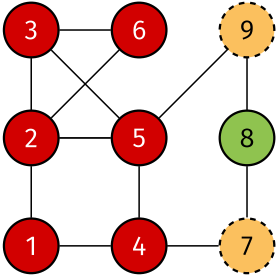
Example




```
pending = [6,7,9]
```

After 5th iteration.

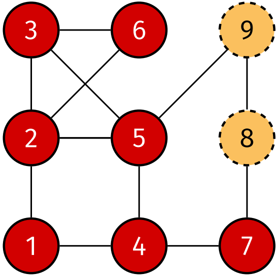
Example



`pending = [7,9]`

After 6th iteration.

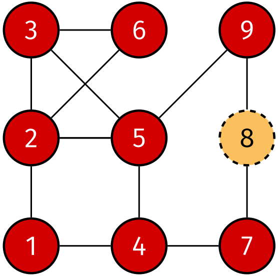
Example



pending = [9,8]

After 7th iteration.

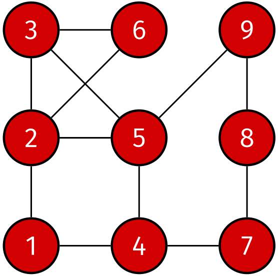
Example



pending = [8]

After 8th iteration.

Example



`pending = []`

After 9th iteration.

[1, 2, 3, 4]

Implementation

- To store pending nodes, use a FIFO **queue**.
  - FIFO = “First in, first out”.

- While queue is not empty:
  - Pop a node, u.
  - Add **undiscovered** neighbors to queue.

Queues in Python

- Want Θ(1) time pops/appends on either side.
- `from collections import deque` (“deck”).
  - `.popleft()` and `.pop()`
  - `list` doesn’t have right time complexity!
  - `import queue` isn’t what you want!
- Keep track of node status attribute using dictionary.

Exercise

```
for node in graph.nodes:
    status[node] = 'undiscovered'
```

```python
from collections import deque

def bfs(graph, source):
    """Start a BFS at `source`."""
    status = {node: 'undiscovered' for node in graph.nodes}

    status[source] = 'pending'
    pending = deque([source])

    # while there are still pending nodes
    while pending:
        # EXERCISE: fill this in...
        u = pending.popleft()
        for v in graph.neighbors(u):
            if status[v] == 'undiscovered':
                status[v] = 'pending'
                pending.append(v)
        status[u] = 'visited'
```

BFS

```python
from collections import deque

def bfs(graph, source):
    """Start a BFS at `source`."""
    status = {node: 'undiscovered' for node in graph.nodes}

    status[source] = 'pending'
    pending = deque([source])

    # while there are still pending nodes
    while pending:
        u = pending.popleft()
        for v in graph.neighbors(u):
            # explore edge (u,v)
            if status[v] == 'undiscovered':
                status[v] = 'pending'
                # append to right
                pending.append(v)
        status[u] = 'visited'
```

Note

- What does this code actually *return*?

Note

- What does this code actually *return*?
- Nothing, yet. It is a *foundation*.

Note

- BFS works just as well for directed graphs.

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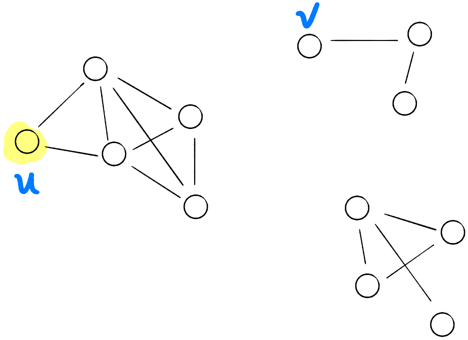
Lecture 11 | Part 6

Analysis of BFS

Exercise

What will `bfs` do when run on a disconnected graph?



Claim

- `bfs` with source u will visit all nodes reachable from u (and only those nodes).
- Useful!
  - Is there a path between u and v?
  - Is graph connected?

Exploring with BFS

- BFS will visit all nodes reachable from source.
- If **disconnected**, BFS will not visit all nodes.
- We can do so with a **full BFS**.
    - Idea: “re-start” BFS on undiscovered node.
    - Must pass statuses between calls.

Making Full BFS

Modify `bfs` to accept statuses:

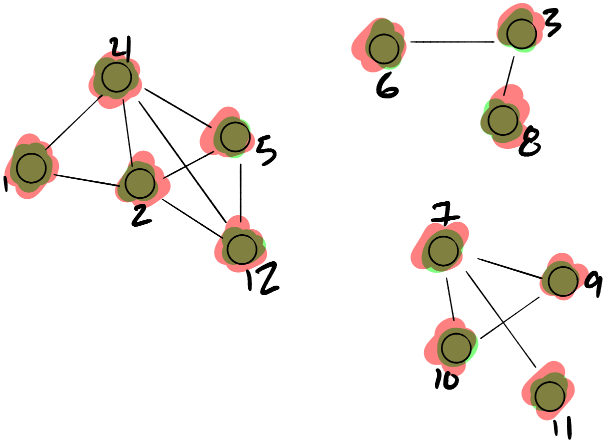
```
def bfs(graph, source, status=None):
    """Start a BFS at `source`."""
    if status is None:
        status = {node: 'undiscovered' for node in graph.nodes}
    # ...
```

Making Full BFS

Call `bfs` multiple times:

```
def full_bfs(graph):
    status = {node: 'undiscovered' for node in graph.nodes}
    for node in graph.nodes:
        if status[node] == 'undiscovered':
            bfs(graph, node, status)
```

Example



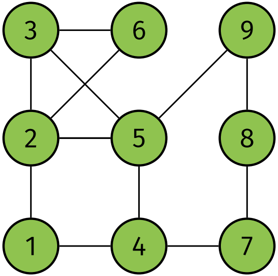
Observation

- If there are k connected components, `bfs` in line 5 is called exactly k times.

```
1 def full_bfs(graph):
2     status = {node: 'undiscovered' for node in graph.nodes}
3     for node in graph.nodes:
4         if status[node] == 'undiscovered':
5             bfs(graph, node, status)
```

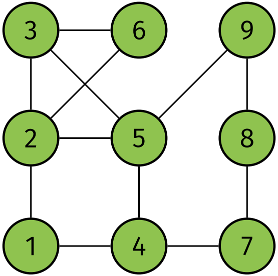
Exercise

How many times is each node added to the queue in a BFS of the graph below? Once.



Exercise

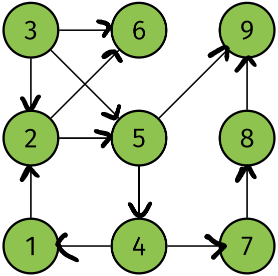
How many times is each edge "explored" in a BFS of the graph below? Twice



(9,8)
(8,9)

Exercise

How many times is each edge "explored" in a BFS of the *directed* graph below? Once



Key Properties of `full_bfs`

- Each node added to queue **exactly once**.

- Each edge is explored **exactly**:
    - **once** if graph is **directed**.
    - **twice** if graph is **undirected**.

Time Complexity of `full_bfs`

- Analyzing `full_bfs` is easier than analyzing `bfs`.
  - `full_bfs` visits all nodes, no matter the graph.
- Result will be **upper bound** on time complexity of `bfs`.
- We'll use an **aggregate analysis**.

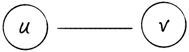
BFS

```python
def bfs(graph, source, status=None):
    """Start a BFS at `source`."""
    if status is None:
        status = {node: 'undiscovered' for node in graph.nodes}

    status[source] = 'pending'
    pending = deque([source])

    # while there are still pending nodes
    while pending:
        u = pending.popleft()
        for v in graph.neighbors(u):
            # explore edge (u,v)
            if status[v] == 'undiscovered':
                status[v] = 'pending'
                # append to right
                pending.append(v)
        status[u] = 'visited'
```

$\Theta(V)$

print

$\Theta(E)$



$\Theta(V+E)$

Time Complexity

```python
def full_bfs(graph):
    status = {node: 'undiscovered' for node in graph.nodes}
    for node in graph.nodes:
        if status[node] == 'undiscovered':
            bfs(graph, node, status)


def bfs(graph, source, status=None):
    """Start a BFS at `source`."""
    if status is None:
        status = {node: 'undiscovered' for node in graph.nodes}

    status[source] = 'pending'
    pending = deque([source])

    # while there are still pending nodes
    while pending:
        u = pending.popleft()
        for v in graph.neighbors(u):
            # explore edge (u,v)
            if status[v] == 'undiscovered':
                status[v] = 'pending'
                # append to right
                pending.append(v)
        status[u] = 'visited'
```

Time Complexity of Full BFS

- $\Theta(V + E)$
- If $|V| > |E|$: $\Theta(V)$
- If $|V| < |E|$: $\Theta(E)$
- Namely, if graph is **complete**: $\Theta(V^2)$.
- Namely, if graph is **very sparse**: $\Theta(V)$.

Notational Note

- We’ll often write $\Theta(V + E)$ instead of $\Theta(|V| + |E|)$.
- You can use whichever.

Next Time

- Finding **shortest paths** using BFS.