Lecture 16

# Independence and Conditional Independence

DSC 40A

# Agenda

- Independence.
- Conditional independence.

# Question 🤔

Take a moment to pause and reflect...

If you have any questions please post online to our forms/Q&A site.

Course staff will answer them ASAP!

# Independence

# Updating probabilities

- Bayes' Theorem describes how to update the probability of one event, given that another event has occurred.

$$\underbrace{\mathbb{P}(B|A)}_{\text{new info}} = \frac{\underbrace{\mathbb{P}(B)}_{\text{old info}} \cdot \mathbb{P}(A|B)}{\mathbb{P}(A)}$$

ratio $\frac{P(A|B)}{P(A)}$

  - $\mathbb{P}(B)$ can be thought of as the "prior" probability of $B$ occurring, before knowing anything about $A$.
  - $\mathbb{P}(B|A)$ is sometimes called the "posterior" probability of $B$ occurring, given that $A$ occurred.

- What if knowing that $A$ occurred doesn't change the probability that $B$ occurs? In other words, what if:

$$\mathbb{P}(B|A) = \mathbb{P}(B)$$

## Independent events

$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

- $A$ and $B$ are **independent events** if one event occurring does not affect the chance of the other event occurring.

$$\mathbb{P}(B|A) = \mathbb{P}(B) \quad \text{Equivalent!} \quad \mathbb{P}(A|B) = \mathbb{P}(A)$$

Any extra knowledge or info about $A$ does not change $\mathbb{P}(B)$

- Otherwise, $A$ and $B$ are **dependent events**.

- Using Bayes' theorem, we can show that if one of the above statements is true, then so is the other.

Suppose $\mathbb{P}(B|A) = \mathbb{P}(B)$. Let's show $\mathbb{P}(A|B) = \mathbb{P}(A)$

Use Bayes' theorem!

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B)\mathbb{P}(A|B)}{\mathbb{P}(A)} = \mathbb{P}(B) \Rightarrow \cancel{\mathbb{P}(B)}\mathbb{P}(A|B) = \cancel{\mathbb{P}(B)}\mathbb{P}(A)$$

$$\Rightarrow \mathbb{P}(A|B) = \mathbb{P}(A)$$

# Independent events

- **Equivalent definition**: $A$ and $B$ are independent events if:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

- To check if $A$ and $B$ are independent, use whichever is easiest:
  - $\boxed{\mathbb{P}(B|A) = \mathbb{P}(B).}$
  - $\mathbb{P}(A|B) = \mathbb{P}(A).$
  - $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$

General multiplication Rule:

Holds for Independent & NON Ind.

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B|A)$$

only when events are independent

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

Ex.

52 Deck

Is A & B Ind.?

A = Drawing a Heart

B = Drawing a face Card

$\mathbb{P}(B) = \frac{12}{52} \approx .23$ $\quad$ $\mathbb{P}(B|A) = \frac{3}{13} \approx .23$

$\mathbb{P}(B|A) = \mathbb{P}(B)$ By: $\mathbb{P}(B|A) = \mathbb{P}(B)$

## Question 🤔

**Take a moment to pause and reflect...**

→ no overlap when events occur.

### Mutual exclusivity and independence

Suppose $A$ and $B$ are two events with non-zero probabilities. Is it possible for $A$ and $B$ to be both mutually exclusive and independent?

- A. Yes.
- B. No.

if $A, B$ ind. $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

if $A, B$ mutually Exclusive $\mathbb{P}(A \cap B) = 0$

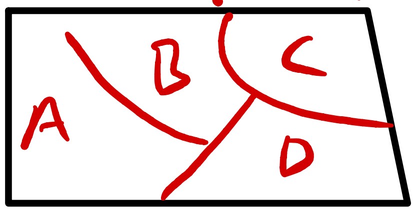
then $\Rightarrow \mathbb{P}(A) \cdot \mathbb{P}(B) = 0$

$\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$

??? can't do it!

## Example: Venn diagrams

For Partitions




} mutually exclusive & $\Sigma = 1$

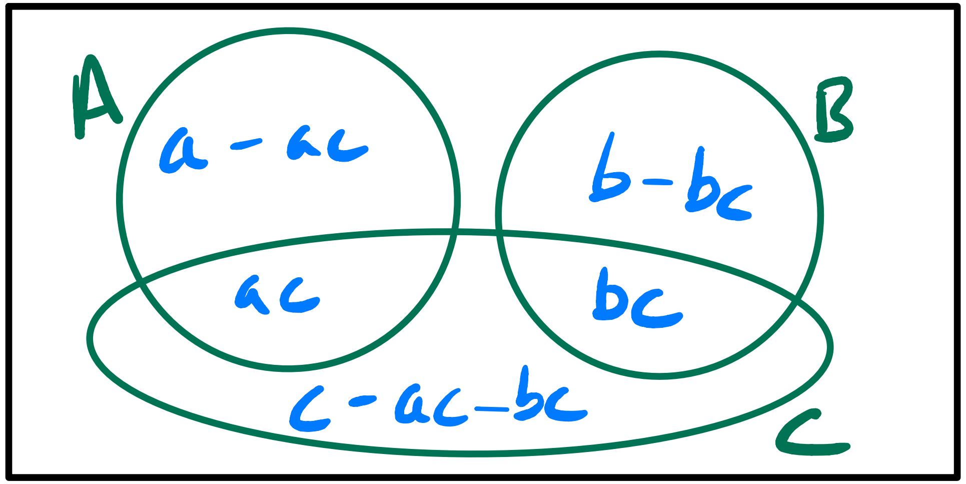
For three events $A$, $B$, and $C$, we know that:

- $A$ and $C$ are independent,
- $B$ and $C$ are independent,
- $A$ and $B$ are mutually exclusive,
- $\mathbb{P}(A \cup C) = \frac{2}{3}$, $\mathbb{P}(B \cup C) = \frac{3}{4}$, $\mathbb{P}(A \cup B \cup C) = \frac{11}{12}$.

$\mathbb{P}(A) = a$
$\mathbb{P}(B) = b$
$\mathbb{P}(C) = c$

Find $\mathbb{P}(A)$, $\mathbb{P}(B)$, and $\mathbb{P}(C)$.

A & B do not overlap!



$$\text{I. } \mathbb{P}(A \cup C) = \underbrace{a + c - ac}_{} = \tfrac{2}{3}$$

$$\text{II. } \mathbb{P}(B \cup C) = \underbrace{b + c - bc}_{} = \tfrac{3}{4}$$

$$\text{III. } \mathbb{P}(A \cup B \cup C) = a + b + c - ac - bc = \tfrac{11}{12}$$

$$\mathbb{P}(A \cap C) = \mathbb{P}(A)\,\mathbb{P}(C) = ac$$

ac are Ind.

$\mathbb{P}(A) = \frac{1}{3}$

$\mathbb{P}(B) = \frac{1}{2}$

$\mathbb{P}(C) = \frac{1}{2}$

$$\boxed{\text{I.} + \text{II.}} \quad a + b + 2c - ac - bc = \tfrac{2}{3} + \tfrac{3}{4}$$

$$\boxed{\text{III.}} \quad a + b + c - ac - bc = \tfrac{11}{12}$$

$$\boxed{\text{I.} + \text{II.} - \text{III.}} \quad c = \tfrac{2}{3} + \tfrac{3}{4} - \tfrac{11}{12} = \tfrac{8}{12} + \tfrac{9}{12} - \tfrac{11}{12} = \tfrac{6}{12} = \boxed{\tfrac{1}{2} = c}$$

need $c$

$$\text{I. } \underbrace{a + \tfrac{1}{2} - \tfrac{1}{2}a = \tfrac{2}{3}}_{} \Rightarrow \tfrac{1}{2}a = \tfrac{2}{3} - \tfrac{1}{2} = \tfrac{1}{6} \Rightarrow a = 2 \cdot \tfrac{1}{6} = \boxed{\tfrac{1}{3} = a}$$

$$\text{II. } \underbrace{b + \tfrac{1}{2} - \tfrac{1}{2}b = \tfrac{3}{4}}_{} \Rightarrow \tfrac{1}{2}b = \tfrac{3}{4} - \tfrac{1}{2} = \tfrac{1}{4} \Rightarrow b = 2\tfrac{1}{4} = \boxed{\tfrac{1}{2} = b}$$

then $S = S -$ "one card"

## Example: Cards

out

$S$

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Suppose you draw two cards, one at a time.
  - $A$ is the event that the first card is a heart.
  - $B$ is the event that the second card is a club.
- If you draw the cards **with** replacement, are $A$ and $B$ independent? yes
- If you draw the cards **without** replacement, are $A$ and $B$ independent? No!

$\mathbb{P}(B|A) = \frac{13}{52}$

$\mathbb{P}(B) = \frac{13}{52}$

$\mathbb{P}(B|A) = \frac{13}{51} \neq \frac{13}{52} = \mathbb{P}(B)$ if w/o replacement

$51 \neq 52$

$S$ = whole Deck.

## Example: Cards

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A | ♥: | 2, | 3, | 4, | 5, | 6, | 7, | 8, | 9, | 10, | J, | Q, | K, | A |
| | ♦: | 2, | 3, | 4, | 5, | 6, | 7, | 8, | 9, | 10, | J, | Q, | K, | A |
| | ♣: | 2, | 3, | 4, | 5, | 6, | 7, | 8, | 9, | 10, | J, | Q, | K, | A |
| | ♠: | 2, | 3, | 4, | 5, | 6, | 7, | 8, | 9, | 10, | J, | Q, | K, | A |

B

- Suppose you draw one card from a deck of 52.
  - $A$ is the event that the card is a heart.
  - $B$ is the event that the card is a face card (J, Q, K).
- Are $A$ and $B$ independent? yes

$\mathbb{P}(A) = \frac{13}{52} = \frac{1}{4}$ $\mathbb{P}(B) = \frac{12}{52} = \frac{3}{13}$

$\mathbb{P}(A \cap B) = \frac{3}{52} = \mathbb{P}(A) \cdot \mathbb{P}(B)$

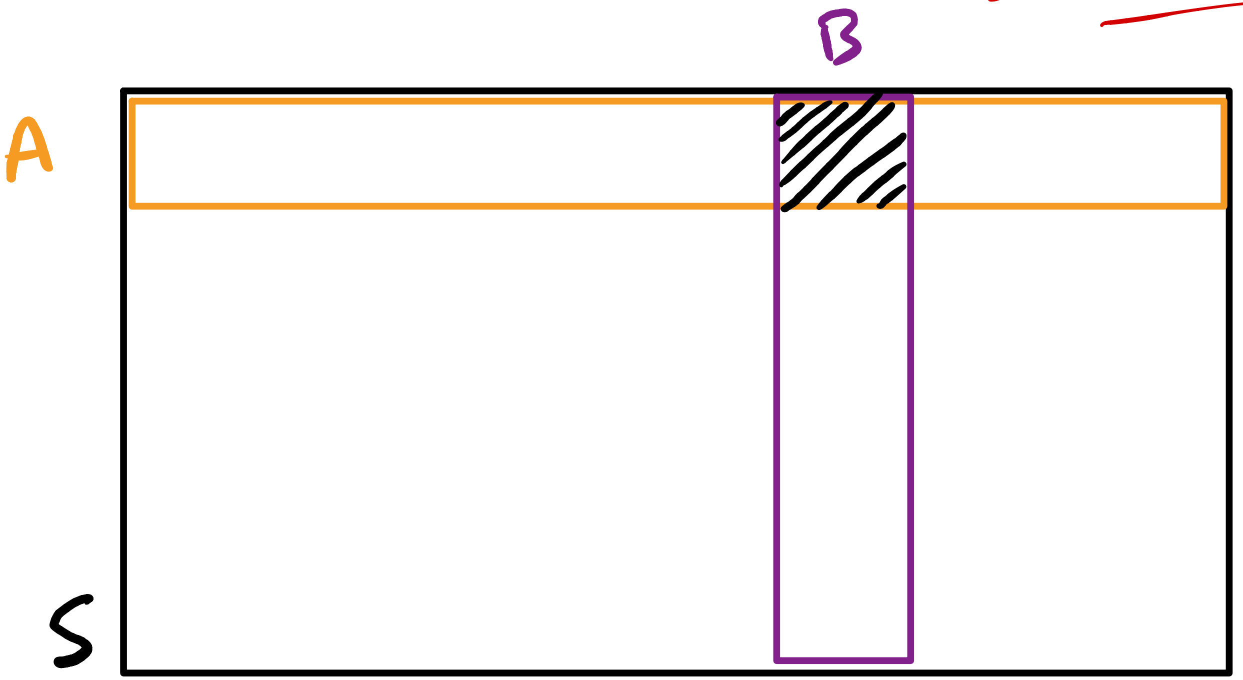
Prop. of face cards in $A$

$\mathbb{P}(B|A) = \frac{3}{13}$

$\mathbb{P}(B) = \frac{12}{52} = \frac{3}{13}$

Prop. of face cards in $S$

events equally likely = independent




the prop. of S taken by B

$$\mathbb{P}(B|A) = \mathbb{P}(B)$$

if so, then Independence

the proportion of A taken by B

## Assuming independence

- Sometimes we assume that events are independent to make calculations easier.
- Real-world events are almost never exactly independent, but they may be close.

makes the math manageable & easier to Assume!

## Example: Breakfast

1% of UCSD students are DSC majors. 25% of UCSD students eat avocado toast for breakfast. Assuming that being a DSC major and eating avocado toast for breakfast are independent:

1. What percentage of DSC majors eat avocado toast for breakfast?

$$\mathbb{P}(\text{Avo} \mid \text{DSC}) = \mathbb{P}(\text{Avo}) = 25\%$$

2. What percentage of UCSD students are DSC majors who eat avocado toast for breakfast?

$$\mathbb{P}(\text{Avo} \cap \text{DSC}) = \mathbb{P}(\text{Avo}) \cdot \mathbb{P}(\text{DSC})$$

$$= 0.25 \cdot 0.01 = 0.025 \in 0.25\%$$

# Conditional independence

## Conditional independence

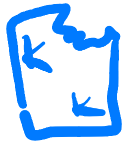
- Sometimes, events that are dependent **become** independent, upon learning some new information.
- Or, events that are independent can become dependent, given additional information.

$4J, 4Q, \cancel{4}3K$

## Example: Cards

| | | B | |
|---|---|---|---|
| A | ♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, | J, Q, K, | A |
| | ♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, | J, Q, K, | A |
| | ♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, | J, Q | A |
| | ♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, | J, Q, K, | A |

King of clubs is now gone




- Your dog ate the King of Clubs. Suppose you draw one card from a deck of 51.
  - $A$ is the event that the card is a heart.
  - $B$ is the event that the card is a face card (J, Q, K).
- Are $A$ and $B$ independent?

NO

$\mathbb{P}(B|A) = \frac{3}{13}$

$\mathbb{P}(B)\ \frac{11}{51}$

$\frac{11}{51} \neq \frac{3}{13}$

$\mathbb{P}(A) = \frac{13}{51}, \quad \mathbb{P}(B) = \frac{11}{51}$

$\mathbb{P}(A \cap B) = \frac{3}{51} \neq \underbrace{\frac{13}{51} \cdot \frac{11}{51}}_{\mathbb{P}(A) \cdot \mathbb{P}(B)}$

# Example: Cards

A ♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A } Drawn

♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

~~♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A~~

~~♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A~~

- Your dog ate the King of Clubs. Suppose you draw one card from a deck of 51.
  - $A$ is the event that the card is a heart.

    is Red
  - $B$ is the event that the card is a face card (J, Q, K).
- Suppose you learn that the card is red. Are $A$ and $B$ independent given this new information?

$$P(B|A) = \frac{3}{13} = \frac{6}{26} = P(B)$$

Within the Red Cards, A & B are Independent

## Conditional independence

- Recall that $A$ and $B$ are independent if:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

- $A$ and $B$ are **conditionally independent** given $C$ if:

$$\mathbb{P}((A \cap B)|C) = \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$$

C comes from the Definition of Reg. Independence.

- Given that $C$ occurs, this says that $A$ and $B$ are independent of one another.

$$\frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(C)} = \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} \cdot \frac{\mathbb{P}(B \cap C)}{\mathbb{P}(C)}$$

## Assuming conditional independence

- Sometimes we assume that events are conditionally independent to make calculations easier.
- Real-world events are almost never exactly conditionally independent, but they may be close.

## Example: Harry Potter and Discord

UCSD = C ("from previous slides)

Suppose that 50% of UCSD students like Harry Potter and 80% of UCSD students use Discord. What is the probability that a random UCSD student likes Harry Potter and uses Discord, assuming that these events are conditionally independent given that a person is a UCSD student?

$$\mathbb{P}(\heartsuit \text{HP} \cap \text{Discord} \mid \underbrace{\text{UCSD}}_{\text{"C"}}) = \mathbb{P}(\heartsuit \text{HP} \mid \underbrace{\text{UCSD}}_{\text{"C"}}) \cdot \mathbb{P}(\text{Discord} \mid \underbrace{\text{UCSD}}_{\text{"C"}})$$

$$= (0.5)\, 0.8$$

$$= 0.4$$

## Question 🤔

**Take a moment to pause and reflect...**

- Is it reasonable to assume conditional independence of:
  - liking Harry Potter
  - using Discord

  given that a person is a UCSD student?

yes!

- Is it reasonable to assume independence of these events in general, among all people?

no!

Age, location

**Which assumptions do you think are reasonable?**

A. Both.

B. Conditional independence only.

C. Independence (in general) only.

D. Neither.

# Independence vs. conditional independence

In general, **there is no relationship between independence and conditional independence**. All four scenarios before are possible:

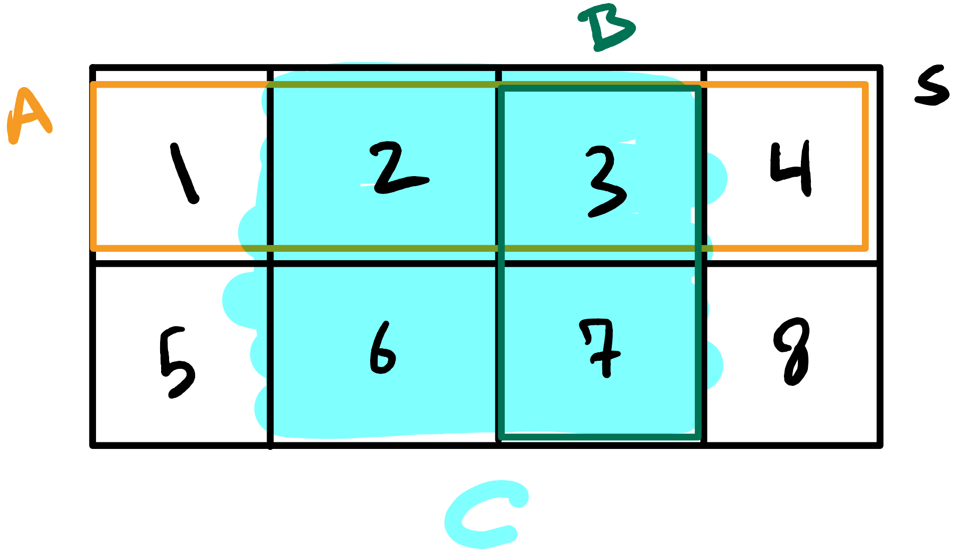
All True

1. $A$ and $B$ **are** independent, and **are** conditionally independent given $C$.
2. $A$ and $B$ **are** independent, but are **not** conditionally independent given $C$.
3. $A$ and $B$ are **not** independent, but **are** conditionally independent given $C$.
4. $A$ and $B$ are **not** independent, and are **not** conditionally independent given $C$.

# Example: Constructing events

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ where all outcomes are equally likely.
- Specify events $A$, $B$, and $C$ that satisfy the given conditions (e.g. $A = \{2, 5, 6\}$).
- Choose events that are neither impossible nor certain, i.e. $0 < \mathbb{P}(A), \mathbb{P}(B), \mathbb{P}(C) < 1$. $A = \{1,2,3,4\}$ $B = \{3,7\}$, $C = \{2,3,6,7\}$

**Scenario 1**: $A$ and $B$ **are** independent, and **are** conditionally independent given $C$.




A, B ind: $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

$$\frac{1}{8} = \frac{1}{2} \cdot \frac{1}{4}$$

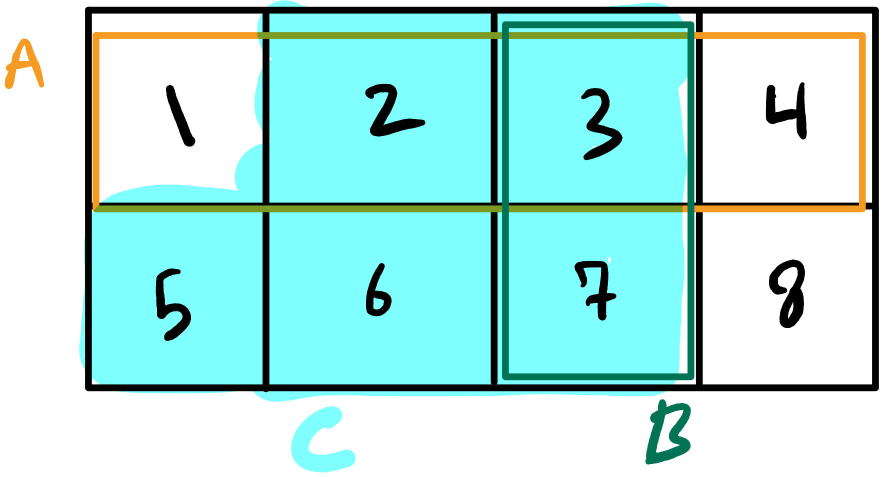
A, B cond. ind. given C: $\mathbb{P}((A \cap B) | C) = \mathbb{P}(A | C) \cdot \mathbb{P}(B | C)$

$$\frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2}$$

# Example: Constructing events

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ where all outcomes are equally likely.
- Specify events $A$, $B$, and $C$ that satisfy the given conditions (e.g. $A = \{2, 5, 6\}$).
- Choose events that are neither impossible nor certain, i.e. $0 < \mathbb{P}(A), \mathbb{P}(B), \mathbb{P}(C) < 1$.

**Scenario 2**: $A$ and $B$ **are** independent, but are **not** conditionally independent given $C$.




A, B Ind: $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

$\frac{1}{8} \quad \frac{1}{2} \quad \frac{1}{4}$

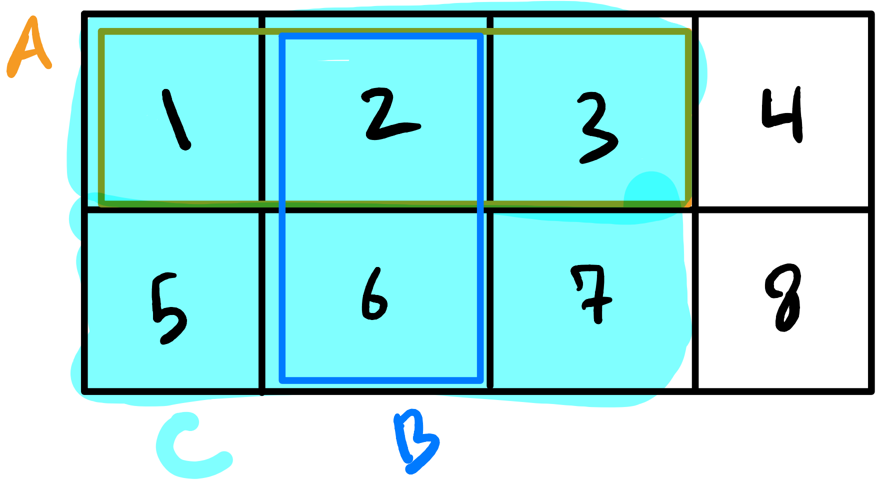
A, B not cond. Ind given C: $\mathbb{P}(A \cap B | C) \neq \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$

$\frac{1}{5} \neq \frac{2}{5} \cdot \frac{2}{5}$

## Example: Constructing events

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ where all outcomes are equally likely.
- Specify events $A$, $B$, and $C$ that satisfy the given conditions (e.g. $A = \{2, 5, 6\}$).
- Choose events that are neither impossible nor certain, i.e. $0 < \mathbb{P}(A), \mathbb{P}(B), \mathbb{P}(C) < 1$.

**Scenario 3**: $A$ and $B$ are **not** independent, but **are** conditionally independent given $C$.




A, B != Ind.

$$\mathbb{P}(A \cap B) \neq \mathbb{P}(A) \cdot \mathbb{P}(B)$$

$$\frac{1}{8} \neq \frac{3}{8} \cdot \frac{1}{4}$$

A, B cond. Ind given C :

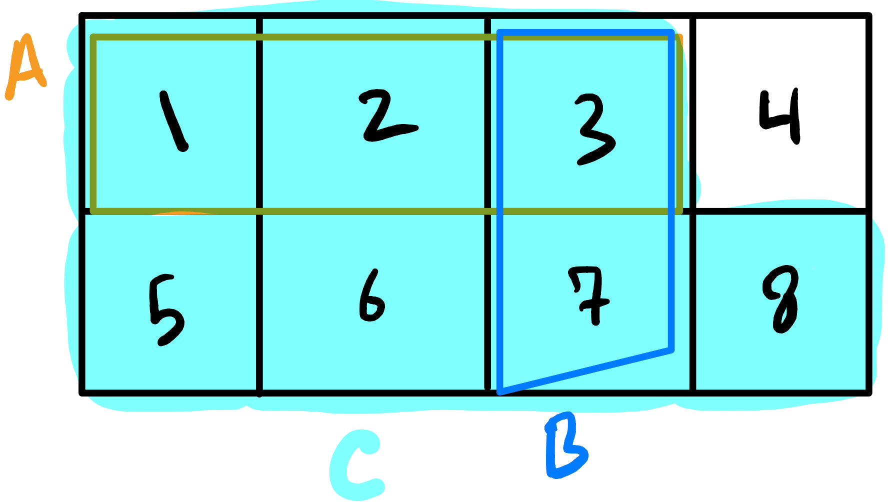
$$\mathbb{P}(A \cap B \mid C) = \mathbb{P}(A \mid C) \cdot \mathbb{P}(B \mid C)$$

$$\frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3}$$

## Example: Constructing events

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ where all outcomes are equally likely.
- Specify events $A$, $B$, and $C$ that satisfy the given conditions (e.g. $A = \{2, 5, 6\}$).
- Choose events that are neither impossible nor certain, i.e. $0 < \mathbb{P}(A), \mathbb{P}(B), \mathbb{P}(C) < 1$.

**Scenario 4**: $A$ and $B$ are **not** independent, and are **not** conditionally independent given $C$.




A, B not Ind.: $\mathbb{P}(A \cap B) \neq \mathbb{P}(A) \cdot \mathbb{P}(B)$

$$\frac{1}{8} \qquad \frac{3}{8} \cdot \frac{1}{4}$$

A, B not cond. Ind. given C: $\mathbb{P}((A \cap B) | C) \neq \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$

$$\frac{1}{7} \neq \frac{3}{7} \cdot \frac{2}{7}$$

# Summary

## Summary

- Two events $A$ and $B$ are **independent** when knowledge of one event does not change the probability of the other event.
  - Equivalent conditions: $\mathbb{P}(B|A) = \mathbb{P}(B)$, $\mathbb{P}(A|B) = \mathbb{P}(A)$, $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$.
- Two events $A$ and $B$ are **conditionally independent** given a third event, $C$, if they are independent given knowledge of event $C$.
  - Condition: $\mathbb{P}((A \cap B)|C) = \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$.
- In general, there is no relationship between independence and conditional independence.
- **Next time**: Using Bayes' Theorem and conditional independence to solve the **classification problem** in machine learning.