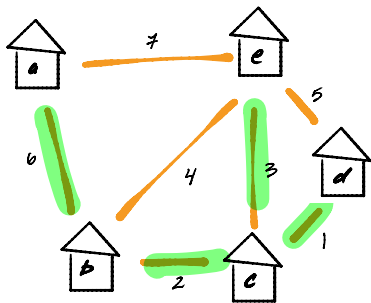# DSC 40B
Theoretical Foundations II

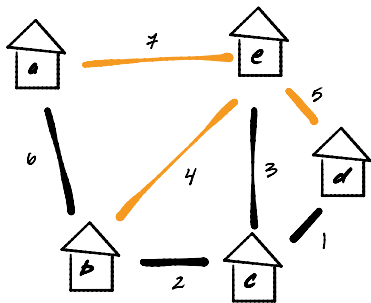Lecture 16 | Part 1

Minimum Spanning Trees

Today's Problem

$6 + 7 + 5 + 1 = 19$

$6 + 2 + 3 + 1 = 12$

▶ Choose a set of dirt roads to pave so that:
  ▶ can get between any two buildings only on paved roads;
  ▶ total cost is minimized.

▶ Solution: compute a **minimum spanning tree**.
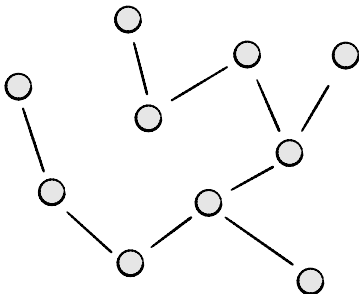
Today's Problem

- ▶ Choose a set of dirt roads to pave so that:
  - ▶ can get between any two buildings only on paved roads;
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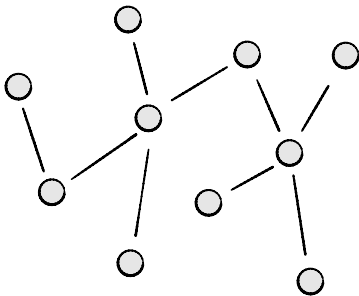- ▶ Solution: compute a **minimum spanning tree**.

Trees

An undirected graph $T = (V, E)$ is a **tree** if
- ▶ it is connected; and
- ▶ it is acyclic.

Example: a **tree**.

Trees

An undirected graph $T = (V, E)$ is a **tree** if
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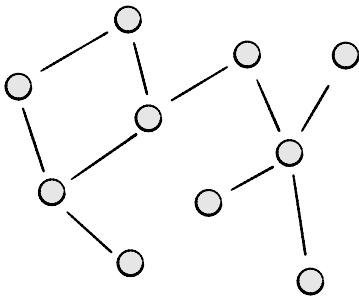- ▶ it is acyclic.

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Trees

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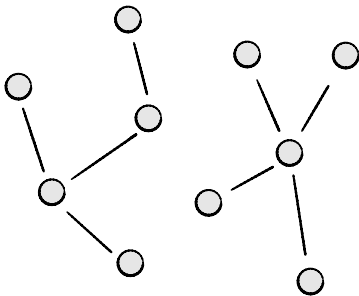
Example: **not** a tree.

Trees

An undirected graph $T = (V, E)$ is a **tree** if
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Trees: Equivalent Definition

An undirected graph $T = (V, E)$ is a **tree** if
- ▶ it is connected; and
- ▶ $|E| = |V| - 1$.

Example: a **tree**.

Trees: Equivalent Definition

An undirected graph $T = (V, E)$ is a **tree** if
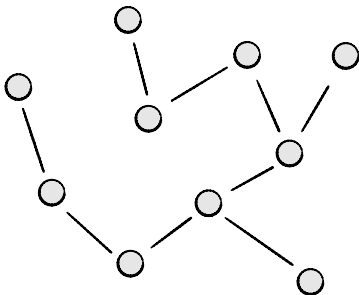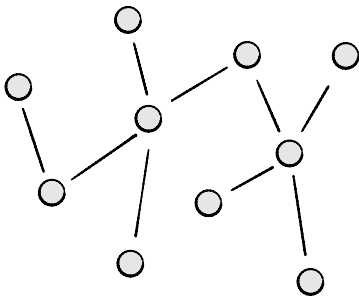- ▶ it is connected; and
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Example: a **tree**.

Trees: Equivalent Definition

An undirected graph $T = (V, E)$ is a **tree** if
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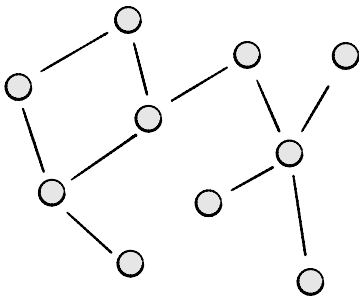Example: **not** a tree.

Trees: Equivalent Definition

An undirected graph $T = (V, E)$ is a **tree** if
- ▸ it is connected; and
- ▸ $|E| = |V| - 1$.

Example: **not** a tree.

Tree Properties
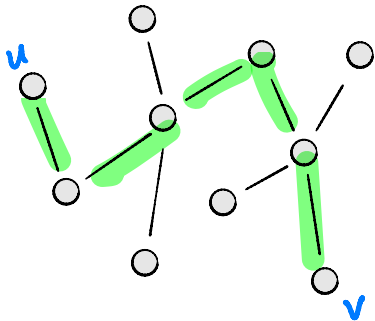
- ▶ There is a unique simple path between any two nodes in a tree.

- ▶ Adding a new edge to a tree creates a cycle (no longer a tree).

- ▶ Removing an edge from a tree disconnects it (no longer a tree).

Spanning Trees

Let $G = (V, E)$ be a **connected** graph. A **spanning tree** of G is a tree $T = (V, E_T)$ with the same nodes as G, and a subset of G's edges.
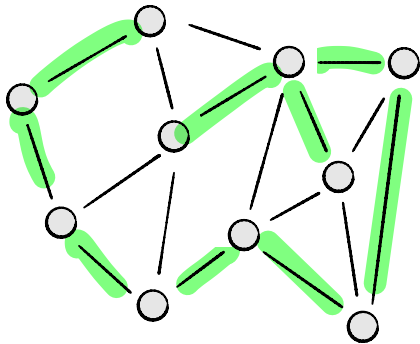
Many Spanning Trees

The same graph can have many spanning trees.
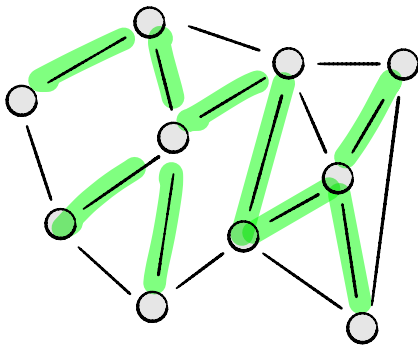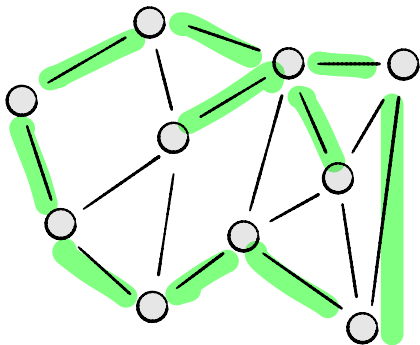
Spanning Tree Cost

If $G = (V, E, \omega)$ is a weighted undirected graph, the **cost** (or **weight**) of a spanning tree is the total weight of the edges in the spanning tree.
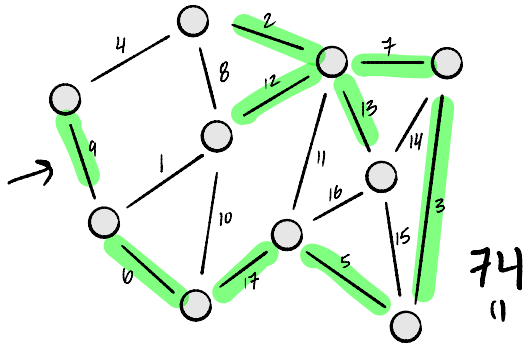
Cost: $9 + 6 + 17 + 5 + 3 + 7 + 13 + 12 + 2$

Spanning Tree Cost

Different spanning trees of the same graph can have different costs.
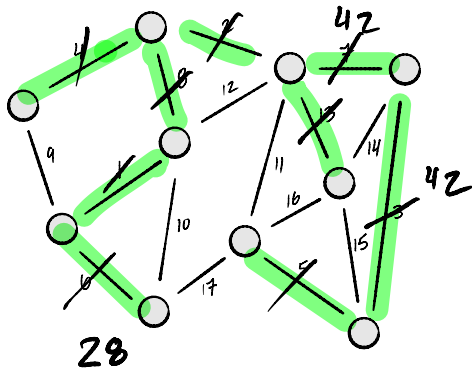
Cost: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 13 = 41$

Minimum Spanning Tree

▶ The **minimum spanning tree** problem is as follows:
  ▶ GIVEN: A weighted, undirected graph $G = (V, E, \omega)$.
  ▶ COMPUTE: a spanning tree of G with minimum cost (i.e., minimum total edge weight).

▶ For a given graph, the MST may not be unique.

Exercise

Suppose the edges of a graph $G = (V, e, \omega)$ all have the same weight. How can we compute an MST of the graph?

Today's Problem
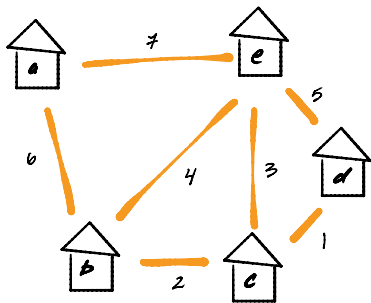
▶ Choose a set of dirt roads to pave so that:
  ▶ can get between any two buildings only on paved roads;
  ▶ total cost is minimized.

▶ Solution: compute a **minimum spanning tree**.

MSTs in Data Science?

▶ Do we need to find MSTs in data science?

▶ Actually, yes! (Next lecture)

DSC 40B
Theoretical Foundations II

Lecture 16 | Part 2

Prim's Algorithm

Building MSTs

▶ How do we build a MST efficiently?

▶ We'll adopt a **greedy** approach.
  ▶ Build a tree edge-by-edge.
  ▶ At every step, doing what looks best at the moment.

▶ This strategy isn't guaranteed to work in all of life's situations, but it works for building MSTs.

Two Greedy Approaches

- We'll look at two greedy algorithms:
    - Today: Prim's Algorithm
    - Next time: Kruskal's Algorithm

- Differ in the order in which edges are added to tree.

- Also differ in time complexity.

Prim's Algorithm, Informally
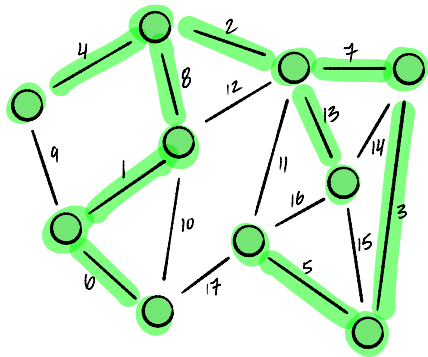
▶ Start by picking any node to add to "tree", *T*.

▶ While *T* is not a spanning tree, greedily add **lightest** edge from a node in *T* to a node not in *T*.
   ▶ "lightest" = edge of smallest weight

Prim's Algorithm, Informally
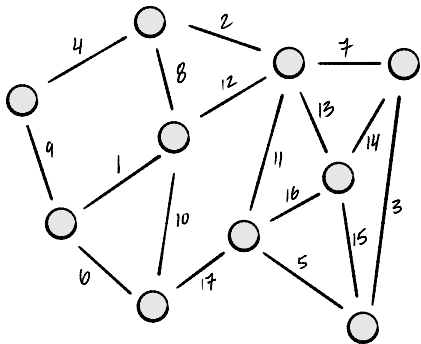
▶ Start by picking any node to add to "tree", *T*.

▶ While *T* is not a spanning tree, greedily add **lightest** edge from a node in *T* to a node not in *T*.
  ▶ "lightest" = edge of smallest weight

▶ **Is this guaranteed to work?** Yes, as we'll see.

Prim's Algorithm, Equivalently

▶ For each node u, store:
  ▶ estimated cost of adding node to tree;
  ▶ estimated "predecessor" v in the tree.

▶ At each step,
  ▶ Find node with smallest estimated cost.
  ▶ Add to tree T by including edge with estimated "predecessor".
  ▶ Update cost of neighbors.

▶ Same as adding lightest edge from T to outside T at every step!

Prim's Algorithm, Equivalently
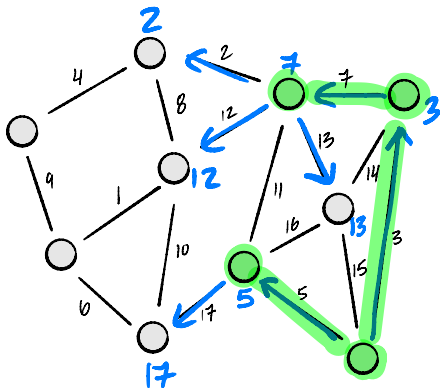
- ▶ While *T* is not a tree:
  - ▶ find the node *u* ∉ *T* with smallest cost
  - ▶ add the edge between *u* and its estimated "predecessor" to *T*
  - ▶ update estimated cost/pred. of *u*'s neighbors which aren't already in tree.

Recall: Priority Queues

- ▶ How do we efficiently find node with smallest cost?

- ▶ Priority Queues:
    - ▶ `PriorityQueue(priorities)`: creates priority queue from dictionary whose values are priorities.

    - ▶ `.extract_min()`: removes and returns key with smallest value.

    - ▶ `.decrease_priority(key, value)`: changes key's value.

- ▶ We'll use a priority queue to hold nodes not yet added to tree.

```python
def prim(graph, weight):
    tree = UndirectedGraph()

    estimated_predecessor = {node: None for node in graph.nodes}
    cost = {node: float('inf') for node in graph.nodes}
    priority_queue = PriorityQueue(cost)

    while priority_queue:
        u = priority_queue.extract_min()
        if estimated_predecessor[u] is not None:
            tree.add_edge(estimated_predecessor[u], u)
        for v in graph.neighbors(u):
            if weight(u, v) < cost[v] and v not in tree.nodes:
                priority_queue.decrease_priority(v, weight(u, v))
                cost[v] = weight(u, v)
                estimated_predecessor[v] = u
    return tree
```

Prim and Dijkstra

▶ This is a lot like Dijkstra's Algorithm for s.p.d.!

▶ Both: at each step, extract node with smallest cost, update its edges. (Prim: only those edges to nodes not in tree).

▶ Dijkstra update of (u, v):

```
cost[v] = min(cost[v], cost[u] + weight(u, v))
```

▶ Prim update of (u, v):

```
cost[v] = min(cost[v], weight(u, v))
```

DSC 40B
Theoretical Foundations II

Lecture 16 | Part 3

Time Complexity

Time Complexity

- A priority queue can be implemented using a **heap**.

- If a **binary min-heap** is used:
    - `PriorityQueue(est)` takes $\Theta(V)$ time.
    - `.extract_min()` takes $O(\log V)$ time.
    - `.decrease_priority()` takes $O(\log V)$ time.

Time Complexity

$$T(V, E) = O(\cancel{V} + E \log V + V \log V)$$

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def prim(graph, weight):
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        for v in graph.neighbors(u):
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                priority_queue.decrease_priority(v, weight(u, v))
                cost[v] = weight(u, v)
                estimated_predecessor[v] = u
    return tree
```

$\Theta(V)$

$\log V$ time to run once
runs V times = $V \log V$

$|E|$ execs

$\log V$ time to run once
potentially runs $|E|$ times
= $E \log V$

Time Complexity

▶ Using a **binary heap**...

▶ Overall: $\Theta(V \log V + E \log V)$.

▶ Since graph is assumed connected, $E = \Omega(V)$.

▶ So this simplifies to $\Theta(E \log V)$.

Fibonacci Heaps

- A priority queue can be implemented using a **heap**.

- If a **Fibonacci min-heap** is used:
  - `PriorityQueue(est)` takes $\Theta(V)$ time.
  - `.extract_min()` takes $\Theta(\log V)$ time[1].
  - `.decrease_priority()` takes $O(1)$ time.

---

[1]Amortized.

Time Complexity

$$O(V \log V + E)$$

Assume: $|E| \geq |V| - 1$

```python
def prim(graph, weight):
    tree = UndirectedGraph()

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            if weight(u, v) < cost[v] and v not in tree.nodes:
                priority_queue.decrease_priority(v, weight(u, v))
                cost[v] = weight(u, v)
                estimated_predecessor[v] = u
    return tree
```

Before:
E log V

Now:
E

Time Complexity

▶ Using a **Fibonacci heap**...

▶ Overall: $\Theta(V \log V + E)$.

Fibonacci vs. Binary Heaps

▶ Using Fibonacci heaps improves time complexity when graph is dense.

▶ E.g., if $E = \Theta(V^2)$:
  ▶ Prim's with Fibonacci: $\Theta(E) = \Theta(V^2)$
  ▶ Prim's with binary: $\Theta(E \log E) = \Theta(V^2 \log V)$. → slower

▶ But Fibonacci heaps are **hard** to implement; have large constants.

▶ Binary heaps used more in practice despite complexity.

DSC 40B
Theoretical Foundations II

Lecture 16 | Part 4
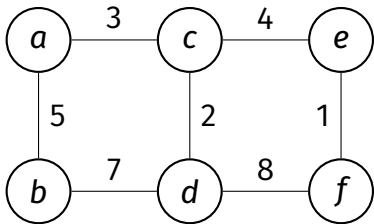
Correctness of Prim's Algorithm

Being Greedy

► At every step, we add the lightest edge.

► Is this "safe"?

Being Greedy

▶ At every step, we add the lightest edge.

▶ Is this "safe"?

▶ Yes! This is guaranteed to find an MST.

Promising Subtrees
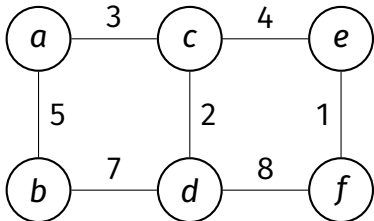

- Let $G = (V, E, \omega)$ be a weighted graph.

- A subgraph $T' = (V', E')$ is **promising** if it is "part" of some MST.
  - That is, it is an "MST in progress"
  - Not necessarily a tree!

- That is, there exists an MST $T = (V, E_{\text{mst}})$ such that $E' \subset E_{\text{mst}}$.

- Hint: a "promising subtree" where $V' = V$ is an MST!

Main Idea

Prim's starts with a promising subtree T. At each step, adds lightest edge from a node within T to a node outside of T.
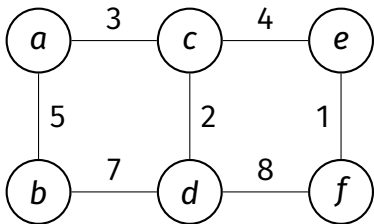
We'll show each new edge results in a larger promising subtree. Eventually the promising subtree becomes a full MST.

Claim

▶ Let $G = (V, E, \omega)$ be a weighted graph.

▶ Suppose $T' = (V', E')$ is a promising subtree for an MST of G.

▶ Let $e = (u, v)$ be a **lightest edge** from a node in T' to a node outside of T'. (Prim).

▶ Then adding (u, v) to T' results in another **promising subtree**.

Proof

- ▶ Suppose T_{mst} is an MST that includes T'.

- ▶ If T_{mst} includes e, we're done: $T' + e$ is promising.

- ▶ If it doesn't include e, it must have an edge f that connects T' to rest of the graph.

- ▶ Swap f with e in T_{mst}. The result is a tree, and it must be a MST since $\omega(e) \leq \omega(f)$.

- ▶ So there is an MST that contains $T' + e$.