Lecture 13

Combinatorics

DSC 40A

Question 🤔

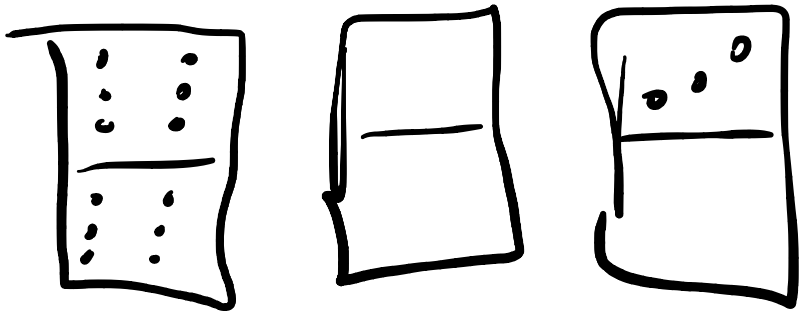
Take a moment to pause and reflect...

If you have any questions please post online to our forms/Q&A site.

Course staff will answer them ASAP!

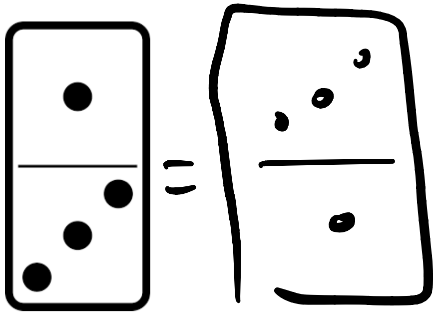
The domino problem

Example: Dominoes



(source: FiveThirtyEight)

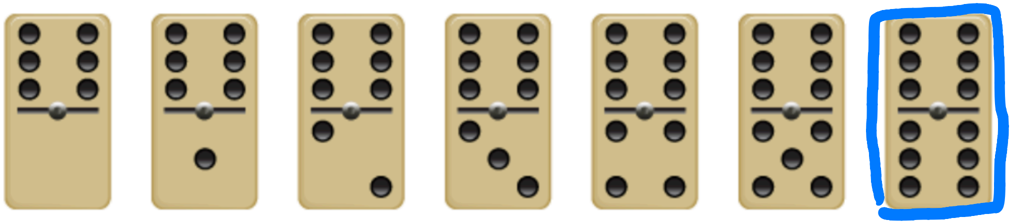
In a set of dominoes, each tile has two sides with a number of dots on each side: 0, 1, 2, 3, 4, 5, or 6. There are 28 total tiles, with each number of dots appearing alongside each other number (including itself) on a single tile.



(0,0)
(0,1) (1,1)
(0,2) (1,2) (2,2)
(0,3) (1,3) (2,3) (3,3)
(0,4) (1,4) (2,4) (3,4) (4,4)
(0,5) (1,5) (2,5) (3,5) (4,5) (5,5)
(0,6) (1,6) (2,6) (3,6) (4,6) (5,6) (6,6)

Example: Dominoes

Question 1: Now your friend picks a random tile from the set and tells you that at least one of the sides is a 6. What is the probability that your friend's tile is a double, with 6 on both sides?



$S = 28$

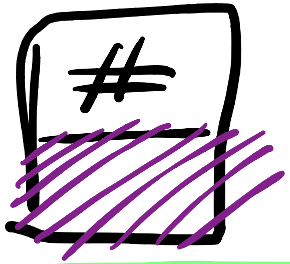
but

$S = 7$

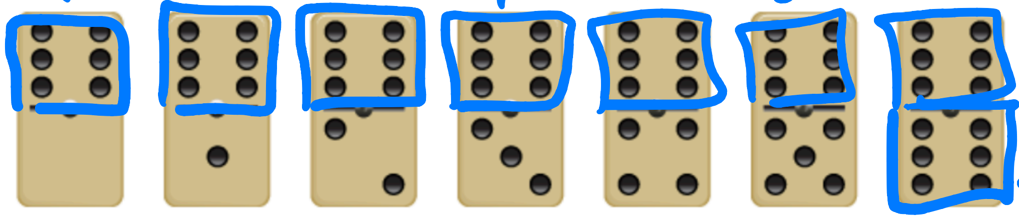
(0,0)
(0,1) (1,1)
(0,2) (1,2) (2,2)
(0,3) (1,3) (2,3) (3,3)
(0,4) (1,4) (2,4) (3,4) (4,4)
(0,5) (1,5) (2,5) (3,5) (4,5) (5,5)
(0,6) (1,6) (2,6) (3,6) (4,6) (5,6) $\boxed{(6,6)}$

$$P(\text{double} \mid \text{at least one side} = 6) = \frac{\#\text{ of Doubles w/ atleast one } 6}{\#\text{ at least one } 6} = \frac{1}{7}$$

Example: Dominoes



Question 2: Now you pick a random tile from the set and uncover only one side, revealing that it has 6 dots. What is the probability that this tile is a double, with 6 on both sides?



one tile but two sides 8

(0,0)
(0,1) (1,1)
(0,2) (1,2) (2,2)
(0,3) (1,3) (2,3) (3,3)
(0,4) (1,4) (2,4) (3,4) (4,4)
(0,5) (1,5) (2,5) (3,5) (4,5) (5,5)
(0,6) (1,6) (2,6) (3,6) (4,6) (5,6) (6,6)

28 Dominoes

28x2 56 sides

$P(\text{double} \mid \text{uncovered six})$

$\frac{2}{8} = \frac{1}{4}$

of the 8 half tiles only 2 are across from a six

See FiveThirtyEight's explanation here.

Don't believe me? Believe the simulation!

- To verify your answer to a probability problem, you can often run a simulation!

Probability of a double:

In [46]:
```
is_double / n
```
Out[46]: 0.250091

Probability of double 6s, given that at least one side is a 6:

In [47]:
```
double_6 / at_least_one_6
```
Out[47]: 0.14351561030844512

Probability of double 6s, given that we uncovered a single side and it was a 6:

In [48]:
```
double_6 / one_side_6
```
Out[48]: 0.25142644852869017

Combinatorics

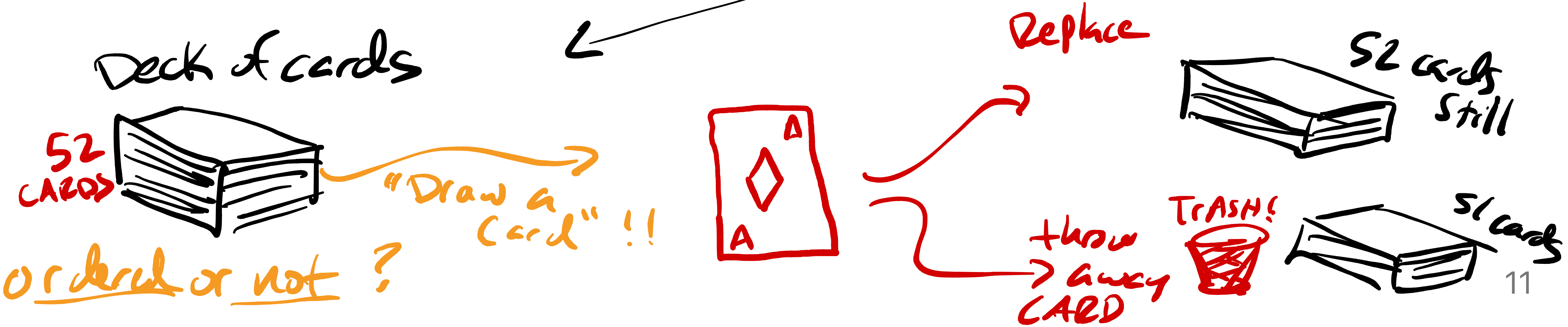
Motivation

- Many problems in probability involve counting.
  - Suppose I flip a fair coin 100 times. What's the probability I see 34 heads?
  - Suppose I draw 3 cards from a 52 card deck. What's the probability they all are all from the same suit?
- In order to solve such problems, we first need to learn how to count.
- The area of math that deals with counting is called **combinatorics**.

- How we count
- Counting
- # of ways we can do something.

Selecting elements (i.e. sampling)

- Many experiments involve choosing k elements randomly from a group of n possible elements. This group is called a **population**. "selecting "K" items from "N" possibilities.
  - If drawing cards from a deck, the population is the deck of all cards.
  - If selecting people from DSC 40A, the population is everyone in DSC 40A.
- Two decisions:
  - Do we select elements with or without **replacement**?
  - Does the **order** in which things are selected matter?



• ordered or not ?

Sequences

- A **sequence** of length k is obtained by selecting k elements from a group of n possible elements **with replacement (i.e. repetition is allowed)**, such that **order matters**.

n replacement

- **Example**: Draw a card (from a standard 52-card deck), put it back in the deck, and repeat 4 times. How many such sequences are there?

k

$n = 52$
$k = 4$

$\underset{1^{st}\text{ card}}{52} \xrightarrow{\text{replace}} \underset{2^{nd}}{52} \xrightarrow{\text{rep.}} \underset{3^{rd}}{52} \xrightarrow{\text{rep.}} \underset{4^{th}}{52} \Rightarrow 52^4$

- **Example**: A UCSD PID starts with "A" then has 8 digits. How many UCSD PIDs are possible?

$n = 10$
$k = 8$

$\boxed{A}\ \underset{\#1}{8}\ \underset{\#2}{7}\ \ldots\ \underset{\#10}{1}$

$\overbrace{10 \cdot 10 \cdots \cdot 10}^{8\text{ times}}$

10^8

Sequences

In general, the number of ways to select k elements from a group of n possible elements **with replacement (i.e. repetition is allowed)** and **order matters** is n^k.

$$\underset{1^{st}\text{ element}}{\underline{\;n\;}} \quad \underset{2^{nd}}{\underline{\;n\;}} \quad \cdots \quad \underset{(k-1)^{th}}{\underline{\;n\;}} \quad \underset{k^{th}\text{ element}}{\underline{\;n\;}} = \underbrace{n \times n \times \cdots \times n}_{k \text{ times}}$$

$$= n^k$$

Permutations

- A **permutation** is obtained by selecting k elements from a group of n possible elements **without replacement (i.e. repetition is not allowed)**, such that **order matters**.
- **Example**: Draw 4 cards, without replacement, from a standard 52-card deck. How many such permutations are there?

these "51" cards can be any card except the 1st "K" card

$n = 52$
$K = 4$

$$\underset{1st}{\underline{52}} \quad \underset{2nd}{\underline{51}} \quad \underset{3rd}{\underline{50}} \quad \underset{4th}{\underline{49}} = 52 \cdot 51 \cdot 50 \cdot 49$$

- **Example**: How many ways are there to select a president, vice president, and secretary from a group of 8 people?

$n = 8$
$K = 3$

$$\underset{P}{\underline{8}} \quad \underset{VP}{\underline{7}} \quad \underset{Sec.}{\underline{6}} = 8 \cdot 7 \cdot 6 = 336$$

Permutations

- In general, the number of ways to select k elements from a group of n possible elements **without replacement (i.e. repetition is not allowed)** and **order matters** is:

P = Permutation

not Probability

$$P(n,k) = \underbrace{(n)(n-1)\ldots(n-k+1)}$$

- To simplify: recall that the definition of $n!$ is:

$$n! = (n)(n-1)\cdots(2)(1)$$

$100!$

$= 100 \cdot 99!$

$= 100 \cdot 99 \cdot 98!$

$= 100 \cdot 99 \cdot 98 \cdot 97!$

$= 100 \cdot 99 \cdot \ldots \cdot 2 \cdot 1$

- Given this, we can write:

$$P(n,k) = \frac{n!}{(n-k)!} = \frac{(n)(n-1)\cdots(n-k+1)\cancel{(n-k)!}}{\cancel{(n-k)!}}$$

$$= (n)(n-1)\cdots(n-k+1)$$

Question 🤔

Pause the video and try to answer the question...

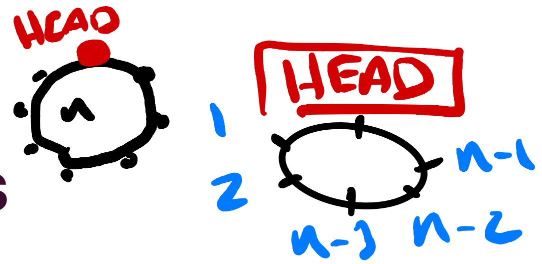
UCSD has 8 colleges. In how many ways can I rank my top 3 choices?

- A. 24. $8 \cdot 3$
- B. 336.
- C. 512. 8^3
- D. 6561. 3^8
- E. None of the above.

$n = 8$
$k = 3$

$$\underset{1^{st}\text{ choice}}{\underline{8}} \quad \underset{2^{nd}}{\underline{7}} \quad \underset{3^{rd}}{\underline{6}} = \frac{8!}{5!}$$

Circle Problem



$(n-1)(n-2)\cdots(2)(1)$

$(n-1)! = \frac{n!}{n}$

Special case of permutations

- Suppose we have n people. The total number of ways I can rearrange these n people in a line is:

$K=n$

$$\underset{1^{st}}{\underline{n}}\quad \underset{2^{nd}}{\underline{n-1}}\quad \underset{3^{rd}}{\underline{n-2}}\quad \cdots\quad \underset{(n-1)^{th}}{\underline{2}}\quad \underset{n^{th}}{\underline{1}} = (n)(n-1)\cdots(2)(1) = n!$$

- This is consistent with the formula:

$$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

HuH!?

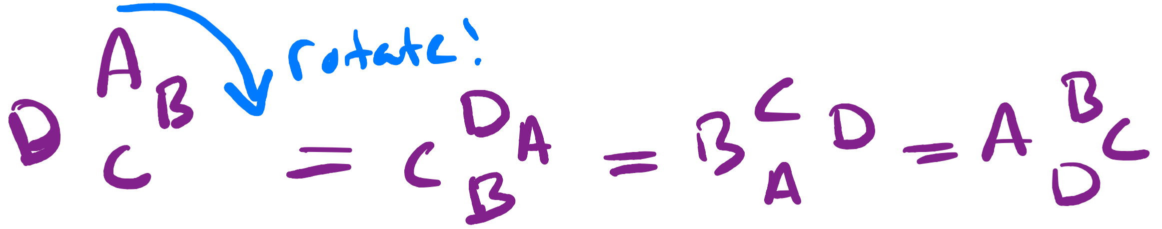
$0! = \cancel{0}\ 1$

- **Followup**: How many ways are there to arrange n people in a circle?

People: A, B, C, D

$n=4$

rotate!



$\frac{4!}{4} = 3!$

Combinations

- A **combination** is a set of k elements selected from a group of n possible elements **without replacement (i.e. repetition is not allowed)**, such that **order does not matter**.
- **Example**: There are 4 ice cream flavors. In how many ways can you pick two different flavors?

V C S T

vanilla choc. strawberry taro

VC CS ST
VS CT ~~TS~~
VT

6 combos

4 3

1st flavor 2nd flavor

$4 \cdot 3 = 12$ ~~flavor combos~~

Because of Double Counting ⁉⁉⁉

How about 3 flavors ^^ ? From our set of 4 flavors.

$$\underset{1^{st}\text{ Flavor}}{\underline{4}}\ \underset{2^{nd}}{\underline{3}}\ \underset{3^{rd}}{\underline{2}} = \frac{24 \text{ permutations}}{3! \text{ combinations}}$$

Consider 1 Combination...

CVT there are 3! ways of rearranging these flavors.

$$\frac{24}{6} = 4 \text{ combos.}$$

{CVT, CTV, VTC, VCT, TCV, TVC}

$n! =$ Permutation of n Distinct items $(n-k)! =$ Permutation of n items taken k times.

From permutations to combinations

w/o Replacement

- There is a close connection between: → order matters
  - the number of **permutations** of k elements selected from a group of n, and
  - the number of **combinations** of k elements selected from a group of n.

order does not matter

$$\#\text{ combinations} = \frac{\#\text{ permutations}}{\#\text{ orderings of } k \text{ elements}}$$

- Since $\#\text{ permutations} = \frac{n!}{(n-k)!}$ and $\#\text{ orderings of } k \text{ elements} = k!$, we have:

$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

→ Adjustment "for Double Counting"

Combinations

In general, the number of ways to select k elements from a group of n elements **without replacement (i.e. repetition is not allowed) and **order does not matter** is:

$$\boxed{\binom{n}{k} = \frac{n!}{(n-k)!k!}}$$

math 183

The symbol $\binom{n}{k}$ is pronounced "n choose k", and is also known as the **binomial coefficient**.

$\binom{n}{k} \neq \frac{n}{k}$ rather it is "n choose k"

Permutations: P VP S J A O $\neq$ P VP S A O J

Example: Committees

Combinations: P VP S = P VP S

"Roles" like order Does not Matter.

- How many ways are there to select a president, vice president, and secretary from a group of 8 people?

$$\underset{P}{8}\ \underset{VP}{7}\ \underset{S}{6} = 8 \cdot 7 \cdot 6 = 336$$

- How many ways are there to select a committee of 3 people from a group of 8 people?

$$\frac{336}{3!} = \binom{8}{3}$$

- If you're ever confused about the difference between permutations and combinations, **come back to this example**.
- More generally, don't jump straight to a formula: think about what the question is asking for.

Aside: Simplifying $\binom{n}{k}$

It's true that:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

However, when asked to simplify the value of $\binom{n}{k}$, do so strategically!

$$\binom{16}{3} = \frac{16!}{13!\ 3!} = \frac{16 \cdot 15 \cdot 14 \cdot \cancel{13!}}{3!\ \cancel{13!}} = \frac{\overset{8}{\cancel{16}} \cdot \overset{5}{\cancel{15}} \cdot 14}{\cancel{3} \cdot \cancel{2}}$$

$n-k$

$16-3=13$

$k=3$

$= 8 \cdot 5 \cdot 14$

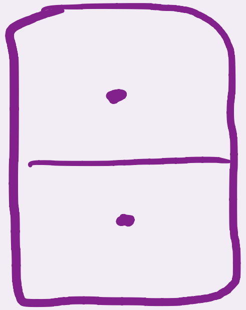
40

$40 \cdot$

160

$= 560$

Question 🤔



Pause the video and try to answer the question...

A domino consists of two faces, each with anywhere between 0 and 6 dots. A set of dominoes consists of every possible combination of dots on each face.

How many dominoes are in a set of dominoes?

- A. $\binom{7}{2}$
- B. $\binom{7}{1} + \binom{7}{2}$
- C. $\cancel{P(7,2)}$
- D. $\cancel{\frac{P(7,2)}{P(7,1)} 7!}$

n chose 1 = n

so... 7 choose 1 $\binom{7}{1} = 7$

$\binom{7}{2} = 21$

↓

implies only choosing w/o Replacement

$\binom{7}{1} + \binom{7}{2} = 7 + 21 = 28$

Summary

Suppose we want to select k elements from a group of n possible elements. The following table summarizes whether the problem involves sequences, permutations, or combinations, along with the number of relevant orderings.

	Yes, order matters	No, order doesn't matter
With replacement Repetition allowed	$\boxed{n^k}$ possible **sequences**	more complicated: watch this video* Domino example Previous slide
Without replacement Repetition not allowed	$\boxed{\dfrac{n!}{(n-k)!}}$ possible **permutations**	$\boxed{\dbinom{n}{k}}$ **combinations**

**or see the previous slide.*

Perfect for notes on a test "crib sheet"