Lecture 14

More Combinatorics Examples

DSC 40A

More examples

All we're going to do for this lecture is work through examples of combinatorics problems...

Ohh Joy!

Combinatoris = Counting

Summary

Suppose we want to select k elements from a group of n possible elements. The following table summarizes whether the problem involves sequences, permutations, or combinations, along with the number of relevant orderings.

	Yes, order matters	**No, order doesn't matter**
With replacement Repetition allowed	$\boxed{n^k}$ possible **sequences**	more complicated: watch **this video**, or see the domino example
Without replacement Repetition not allowed	$\boxed{\frac{n!}{(n-k)!}}$ possible **permutations**	$\boxed{\binom{n}{k}}$ **combinations**

Using the binomial coefficient, $\binom{n}{k}$

As an example, let's evaluate:

~~$9 \cdot 8 \cdot 7 \cdot \ldots \cdot 1$~~

$$\binom{9}{2} + \binom{9}{3}$$

$$= \frac{9!}{7!\,2!} + \frac{9!}{6!\,3!} = \frac{9 \cdot \overset{4}{\cancel{8}} \cdot \cancel{7!}}{\cancel{7!}\,\cancel{2!}} + \frac{9 \cdot 8 \cdot 7 \cdot \cancel{6!}}{\cancel{6!}\,3!}$$

$$= 36 + \frac{\overset{3}{\cancel{9}} \cdot \overset{4}{\cancel{8}} \cdot 7}{\cancel{3} \cdot \cancel{2} \cdot 1}$$

$$= 36 + 84 = 120$$

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

deal w/ double counting

from $P(n,k)$ $= \binom{n}{n-k}$

$$\binom{22}{10} = \binom{22}{12}$$

22 choose 10 (22 students I choose 10 for $\Sigma, C,)$

22 choose 12 (22 students I choose 12 **NOT** to get $\Sigma, C,)$

$$\binom{n}{k}\binom{n}{k+1}=\binom{n+1}{k+1}$$

Side Bar: Combinatorial Proofs

$120 = 120$

$$\binom{9}{2}+\binom{9}{3} = \binom{10}{3}$$

Previously $\binom{9}{2}+\binom{9}{3} = 120$

Look at $\binom{10}{3}$ $\quad \dfrac{10\cdot\cancel{9}^{3}\cdot\cancel{8}^{4}}{\cancel{3}\cdot\cancel{2}} = 120$

10 flips how many ways can we get 3 heads & 7 tails (Assume fair)

$\binom{10}{3}$ $\quad \{H\ H\ H\ T\ T\ T\ T\ T\ T\ T\}$

# of Permutations!

$$\frac{10!}{3!\,7!}$$

"total"

# of ways to rearrange the Heads

" " " for the tails.

Side Bar: Combinatorial Proofs Cont.

Proof: $\binom{9}{2}+\binom{9}{3}=\binom{10}{3}$

Claim: Both sides count the # of ways to get 3 heads from 10 flips

Both cases for a coin flip:

First flip is Heads

$\binom{9}{2} = \underset{1}{\boxed{H}}\ \{$ 2H and 7T $\underset{\ldots}{\_}\ \_\ \_\ \underset{5}{\_}\ \underset{\ldots}{\_}\ \_\ \_\ \underset{10}{\overset{3}{\_}}$

First flip is tails

$\binom{9}{3} = \underset{1}{\boxed{T}}\ \{$ 3H and 6T $\underset{\ldots}{\_}\ \_\ \_\ \underset{5}{\_}\ \underset{\ldots}{\_}\ \_\ \_\ \underset{10}{\overset{3}{\_}}$

The logical leap!

$\binom{9}{2}+\binom{9}{3} = \binom{10}{3}$

$\binom{n}{k}\binom{n}{k+1}\ \binom{n+1}{k+1} \longleftarrow$ generalize

Combinatorics as a tool for probability

Key idea

- If S is a sample space consisting of equally-likely outcomes, and A is an event, then $\mathbb{P}(A) = \frac{|A|}{|S|}$.

S "sample space" $\doteq$ set of all POSSIBLE outcomes

- In many examples, this will boil down to using permutations and/or combinations to count $|A|$ and $|S|$.

- **Tip**: Before starting a probability problem, always think about what the sample space S is!

Overview: Selecting students

We're going answer the same question using several different techniques.

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random without replacement. What is the probability that Avi is among the 5 selected students?

→ all of the students are equally likely to be choosen.

- w/o Replacement ~ so... no sequences
- no info about ordering ~ so... permutations & combination approaches are ok.

most complicated !

1. What is the Probability Space: S $|S|$

Selecting students

Method 1: Using permutations

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

$S :=$ {all permutations of 5 students selected from 20}

$|S|$: $\underset{1^{st}}{\underline{20}}$ $\underset{2^{nd}}{\underline{19}}$ $\underset{3^{rd}}{\underline{18}}$ $\underset{4^{th}}{\underline{17}}$ $\underset{5^{th}}{\underline{16}}$ $= 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16$

1: $\boxed{A}$ 19 18 17 16

2: 19 $\boxed{A}$ 18 17 16

3: 19 18 $\boxed{A}$ 17 16

4: 19 18 17 $\boxed{A}$ 16

5: 19 18 17 16 $\boxed{A}$

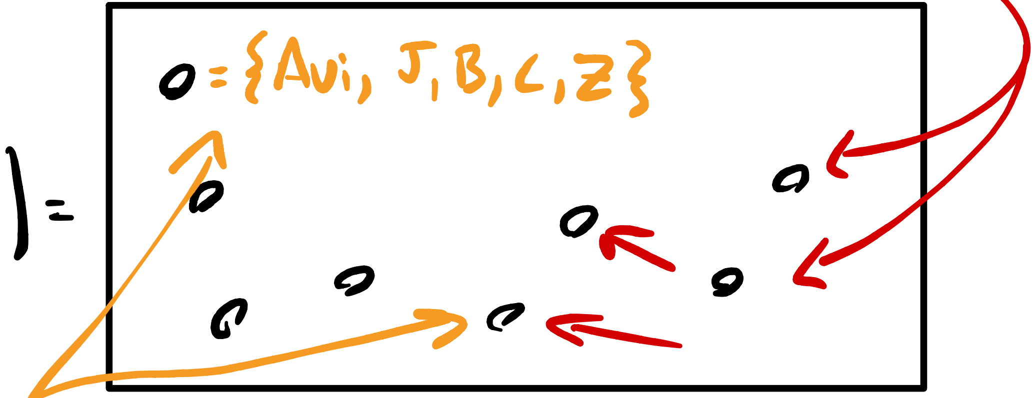
the Permutations w/ Avi : $(19 \cdot 18 \cdot 17 \cdot 16) \cdot 5$

$$\mathbb{P}(\text{Avi being selected}) = \frac{\#\text{ permutations including Avi}}{\#\text{ total Permutations}}$$

$$= \frac{\cancel{19 \cdot 18 \cdot 17 \cdot 16} \cdot 5}{20 \cdot \cancel{19 \cdot 18 \cdot 17 \cdot 16}} = \frac{5}{20} = \frac{1}{4}$$

$20 \cdot 19 \cdot 18 \cdot 17 \cdot 16$ events

$|S| =$



events where Avi is 1/5 students selected

Complement $|S|$

A   not A

$\neg A$ = not A

$\frac{1}{4}$ selected

$|S|$ $\neg$Avi   Avi

$1 - \frac{3}{4} = \frac{1}{4}$

$\frac{3}{4}$ Avi is not selected

Selecting students

Method 2: Using permutations and the complement

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

we want $\mathbb{P}$(Avi selected)   1 minus this

$$\mathbb{P}(\text{Avi selected}) = 1 - \mathbb{P}(\text{Avi not selected})$$

$$\mathbb{P}(\text{Avi not selected}) = \frac{\#\text{ permutations w/o Avi}}{\#\text{ total Permutations}} = \frac{\cancel{19 \cdot 18 \cdot 17 \cdot 16} \cdot 15}{20 \cdot \cancel{19 \cdot 18 \cdot 17 \cdot 16}} = \frac{15}{20} = \frac{3}{4}$$

$\underline{19}$ $\underline{18}$ $\underline{17}$ $\underline{16}$ $\underline{15}$

1st   2nd   3rd   4th   5th

only need to consider these 5

$$\Rightarrow \mathbb{P}(\text{Avi Selected}) = 1 - \frac{3}{4} = \frac{1}{4}$$

Permutations vs. combinations which $|S|$ is larger?

Selecting students

Method 3: Using combinations

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

$|S|$ = all combinations of 5 students selected from 20 students

$|S| = \binom{20}{5}$ "the # of ways to choose "5" from "20" knowing that order DOES NOT MATTER"

A = all combinations of 5 students including Avi.

now remove Avi such that $20-1=19$

$|A| = \binom{19}{4}$ "Avi has already been choosen, thus we only have 4 students left to choose from 19 total"

$$\mathbb{P}(\text{Avi selected}) = \frac{\#\text{ of Combinations w/ Avi}}{\#\text{ total Combinations}} \qquad \frac{|A|}{|S|}$$

$$\frac{A}{B} = A \cdot \frac{1}{B} \quad (\text{B is flipped})$$

$$= \frac{\binom{19}{4}}{\binom{20}{5}} = \frac{19!}{4!\,\cancel{15!}} \cdot \frac{5!\,\cancel{15!}}{20!} = \frac{\cancel{19!}(5 \cdot \cancel{4!})}{\cancel{4!}(20 \cdot \cancel{19!})} = \frac{5}{20} = \frac{1}{4}$$

$$\frac{20!}{5!\,15!}$$

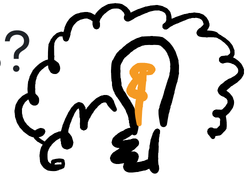
$\mathbb{P}(\text{Avi selected}) = \frac{5}{20} = \frac{1}{4}$

1. Replacement / order, #s
2. $|S|$
3. subset of $|S|$ = $P(A)$
4. Plug & chug

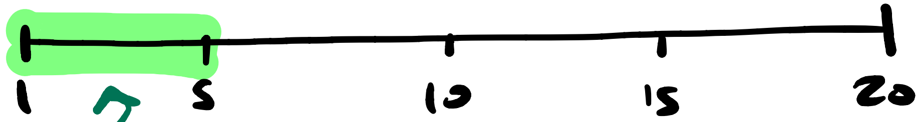
Selecting students

Method 4: The "easy" way

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?



shuffle all 20 students into a line



select first 5 as "good positions"

$|S|$ = all possible positions Avi could be in = 20

$|S| = 20$

$$\mathbb{P}(\text{Avi selected}) = \frac{\#\text{ good positions}}{\#\text{ total positions}} = \frac{5}{20} = \frac{1}{4}$$

w/o replacement P(selecting Avi) = 25%

Question 🤔

so... w/ replacement P(sel. Avi) = P(sel. Avi at least once)

= 1 − P(Avi is never selected)

$= 1 - \left(\frac{19}{20}\right)^5 \approx 22\%$

With vs. without replacement

We've determined that a probability that a random sample of 5 students from a class of 20 **without replacement** contains Avi (one student in particular) is $\frac{1}{4}$.

Suppose we instead sampled **with replacement**. Would the resulting probability be equal to, greater than, or less than $\frac{1}{4}$?

- A. Equal to.
- B. Greater than.
- C. Less than.

w/o Replacement

1st | 2nd | ... | ... | 5th

1/20 | 1/19 | 1/18 | 1/17 | 1/16

w/ rep.

J | J | J | J | J

1/20 | 1/20 | 1/20 | 1/20 | 1/20

Example: An unfair coin

Followup: An unfair coin

In the video you were asked to watch, we flipped an **unfair** coin 10 times, where the coin was biased such that for each flip, $\mathbb{P}(\text{heads}) = \frac{1}{3}$. $\longrightarrow \mathbb{P}(\text{Tails}) = \frac{2}{3}$ || Assume each flip is an independent event!

1. What is the probability that we see the specific sequence HHHHTTTTTT? 4H 6T

$\mathbb{P}(HHHHTTTTTT)$

Ind. Events

$$= \left(\tfrac{1}{3}\right)\cdot\left(\tfrac{1}{3}\right)\cdot\left(\tfrac{1}{3}\right)\cdot\left(\tfrac{1}{3}\right)\cdot\left(\tfrac{2}{3}\right)\cdot\left(\tfrac{2}{3}\right)\cdot\left(\tfrac{2}{3}\right)\cdot\left(\tfrac{2}{3}\right)\cdot\left(\tfrac{2}{3}\right)\cdot\left(\tfrac{2}{3}\right)$$

$$\boxed{=\left(\tfrac{1}{3}\right)^4\cdot\left(\tfrac{2}{3}\right)^6}$$

2. What is the probability that we see exactly 4 heads?

$A = \{$ HHHHTTTTTT, HHHTHTTTTT, HHHTTHTTTT, ⋮, TTTTTTHHHH $\}$ ← sequence of 4H/6T

? # of ways to arrange 4H, 6T: $\binom{10}{4}$

$$\mathbb{P}(4H, 6T) = \boxed{} \times \mathbb{P}(\text{one such sequence}) = \overset{A}{\binom{10}{4}}\left(\tfrac{1}{3}\right)^4\left(\tfrac{2}{3}\right)^6$$

$|S|$ = All sequences of 10 flips, including All H & All T

Followup: An unfair coin

3. What is the probability that we see exactly k heads, where $0 \leq k \leq 10$?

$\mathbb{P}(K \text{ heads}, 10-K \text{ Tails})$

$$\boxed{= \underbrace{\binom{10}{K}}_{\text{num of ways to arrange K H. \& 10-K T.}} \left(\frac{1}{3}\right)^{K} \left(\frac{2}{3}\right)^{10-K}}$$

4. What is the probability that we see *at least* k heads, where $0 \leq k \leq 10$?

$\mathbb{P}(\text{at least } k \text{ heads}) = \mathbb{P}(K \text{ heads}) + \mathbb{P}(K+1 \text{ heads}) + \dots + \mathbb{P}(10 \text{ heads})$

$$\boxed{= \sum_{i=K}^{10} \binom{10}{i} \underbrace{\left(\frac{1}{3}\right)^{i}}_{\text{Biased H}} \underbrace{\left(\frac{2}{3}\right)^{10-i}}_{\text{Biased T}}}$$