In the name of women, life, freedom

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Project 1

Subject title: Uncapacitated facility location

Model of the optimization problem:

$$m \rightarrow \#\ origins, n \rightarrow \#\ destinations$$

$$f_i \rightarrow fixed\ cost\ of\ opening\ origin\ i$$

$$D_j \rightarrow the\ demand\ at\ destination\ j$$

$$\hat{c}_{ij} \rightarrow cost\ of\ transporting\ 1\ unit\ from\ origin\ i\ to\ destination\ j \Longrightarrow c_{ij} = D_j \hat{c}_{ij}$$

$$x_{ij} \rightarrow proportion\ of\ D_j\ transported\ from\ origin\ i$$

$$y_i \rightarrow 1\ if\ origin\ i\ is\ open, and\ 0\ otherwise$$

The goal of the metric facility location problem is to assign every client to a facility by opening a subset of them, as to minimize a cost function given by:

$$\min \textstyle\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{i=1}^{m} f_i y_i.$$

$$s.t. \sum_{i=1}^{m} x_{ij} = 1 \qquad j = 1, \dots, n$$

$$x_{ij} \leq y_i \qquad i = 1, \dots, m, \qquad j = 1, \dots, n$$

$$\sum_{i=1}^{m} y_i \geq 1$$

$$x_{ij} \geq 0 \quad \forall i, j$$

$$y_i = 0/1 \quad \forall i$$

Dual Sub-gradient Optimization

Phase 1: In the first phase, all unassigned clients uniformly raise their prices. When their offered price becomes larger than the connection cost to a given facility, they start contributing towards the opening cost of the facility. Once the opening cost of a facility is fully paid for, the facility is opened and all clients who had been contributing to its opening cost are assigned to it. If an unassigned client reaches an open facility, it is assigned to it and does not have to contribute to its opening cost. Phase 1 continues until there are no unassigned clients left. Note that at the end of this phase, clients do not retract their contributions to a given facility even if they end up assigned to another one. The process is shown in Figure 1.

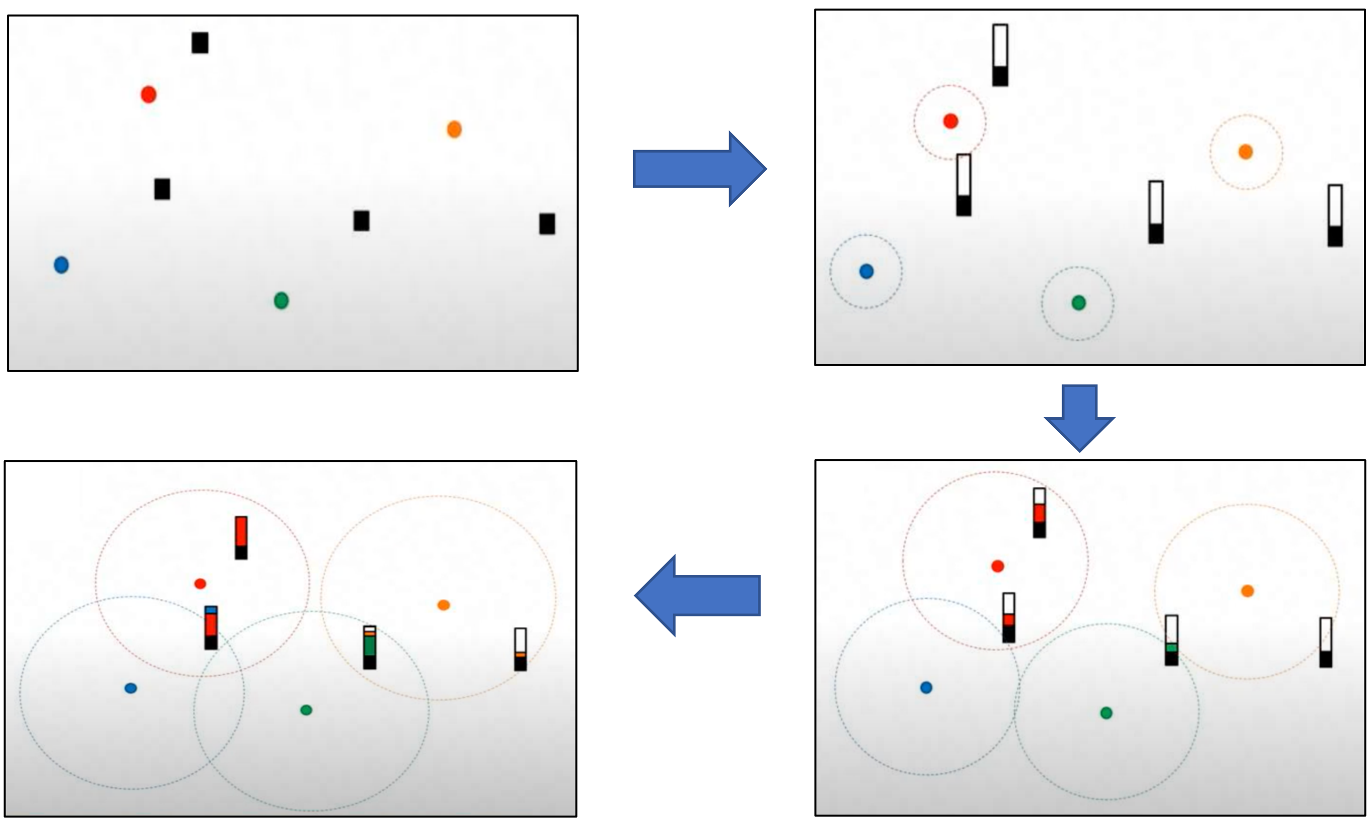

Figure 1: Illustration of dual sub-gradient process

Phase 2: at this point in the algorithm, every client is assigned to one facility. However, since clients do not retract their contributions to the opening cost of a facility even if they end up assigned to another, the open facilities are contributed more than they should at the end of phase

1. To resolve this problem, we stack the facilities in order of opening to give priority to those which were open sooner and implement the following steps:

$For\ any\ client\ j \in C$:

$If: \exists\ an\ open\ facility\ i\ such\ that\ p_j > c_{ij}, assign\ j\ to\ i$

$Else\ if: \exists\ an\ open\ facility\ i\ such\ that\ p_j = c_{ij}, assign\ j\ to\ i$

$If\ client\ j\ is\ unassigned, we\ can\ assign\ it\ to\ an\ open\ facility\ i\ such\ that\ c_{ij} < 3p_j$

The output of this method for #Customer = 30 and #Facilities = 10 is shown in Figure 2.

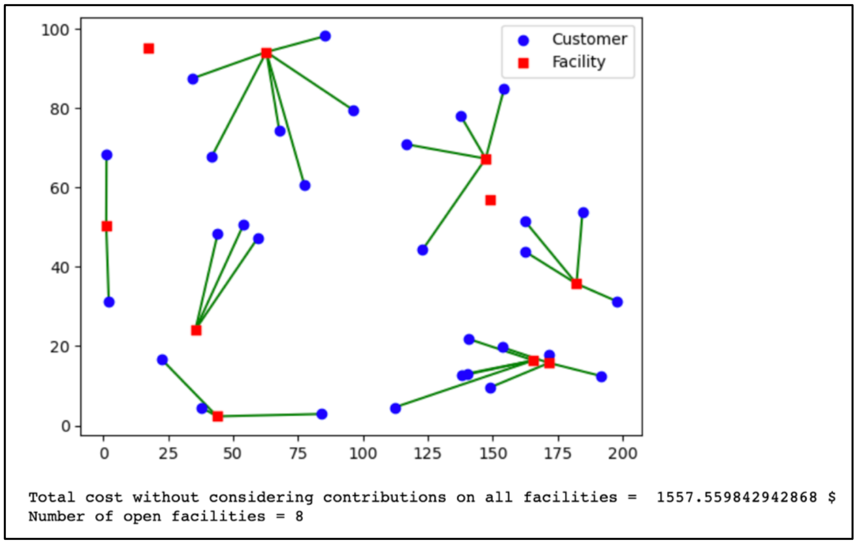

Figure 2: Assignments made by dual sub-gradient method for m = 10 and n = 40

Primal Sub-gradient Optimization

The primal structure of the problem can be written as following:

$$v^* = \min h(y)\ \ s.t. \sum_{i=1}^{m} y_i > 1\,, y \,\epsilon\, \{1,0\}^m$$

where for any $\bar{y}$, $h(\bar{y}) = \sum_j h_j(\bar{y}) + \sum_i f_i \bar{y}_i$, where $\forall j$:

$$h_j(\bar{y}) = min \sum_{i=1}^{m} c_{ij}x_{ij} \ s.t. \sum_{i=1}^{m} x_{ij} = 1, 0 \le x_{ij} \le \bar{y}_i , i = 1, \dots , m$$

The primal subproblem is solved to evaluate the function $h(y)$ at a certain point $\bar{y}$.

Assume that $\bar{y}$ is a given solution, $0 \le \bar{y}_i \le 1 \ \forall i.$ Now let $I = \{i: \bar{y}_i > 0\}$ and assume that $\sum_{i \epsilon I} \bar{y}_i > 1$. (This is necessary and sufficient for primal subproblem being feasible).

Then $x_{ij} = 0 \ \ \forall i \notin I, \forall j$, so:

$$h_j(\bar{y}) = min \sum_{i \in I} c_{ij}x_{ij} \quad s.t. \ \sum_{i \in I} x_{ij} = 1, \qquad 0 \le x_{ij} \le \bar{y}_i \ \ \forall i$$

This problem is trivially solved as follows, $\forall j$:

1) Set $b_j = 1$, $q = 1$, $\bar{I}(j) = \emptyset$.
2) Let $k(q) = argmin_{i \in I \setminus \bar{I}(j)} c_{ij}$.
3) Set $\hat{x}_{k(q)j} = \min(b_j, \bar{y}_{k(q)})$, $b_j = b_j - \hat{x}_{k(q)j}$ and $\bar{I}(j) = \bar{I}(j) \cup \{k(q)\}$.
4) If $b_j = 0$, stop; otherwise, let $q = q + 1$ and go to 1.

A sub-gradient of $h(y)$ is obtained as following:

1) Calculate $\widehat{\alpha}_j = c_{k(q),j} = \min_{i \in I \setminus \bar{I}(j)} c_{ij}$
2) $\hat{\beta}_{ij} = \max(0, \hat{\alpha}_j - c_{ij}) \ \ \forall i$
3) $\zeta_i = f_i - \sum_j \hat{\beta}_{ij} \ \ \forall i$
4) $y_i^{k+1} = P_i(y_i^k - \tau_P^k \zeta^k)$ where $P_i(y_i^k)$ is the projection on the feasible set $0 \le y_i \le 1$(simply setting negative elements equal to zero and elements larger than 1 equal to 1). The step-length τ_P^k is obtained by $\tau_P^k = \lambda_k \left(h(y^k) - \frac{\underline{v}}{\|\zeta^k\|^2} \right)$ where $\underline{v}$ is the best lower bound on v^*.

The output of this method for #Customer = 30 and #Facilities = 10 is shown in Figure 2

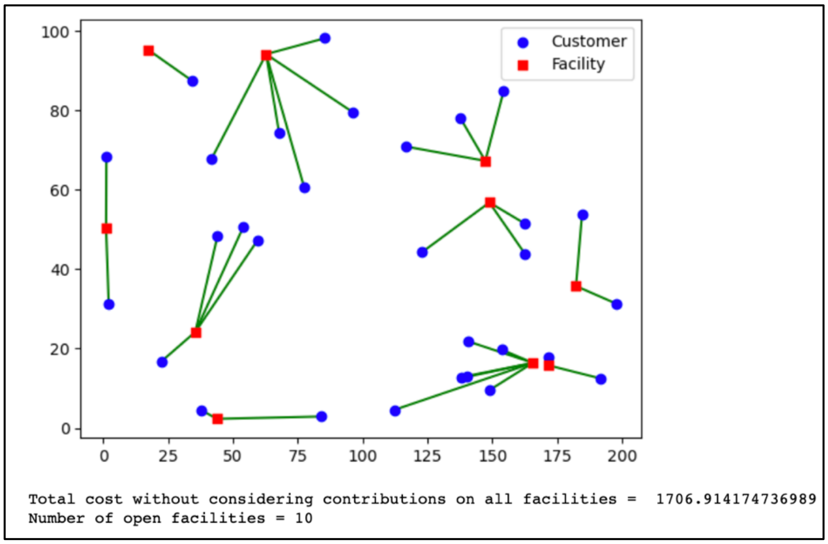

Figure 3: Assignments made by primal sub-gradient method for m = 10 and n = 40

Centralized Optimization

To implement centralized optimization, a procedure with the following steps, are proposed:

1) Construct all possible combinations of open facilities.
2) In each combination, assign a demand to a facility (among all the open facilities) where the cost of assigning the demand in minimum.
3) Calculate the total cost of the combination by taking the sum of assignments in step 2.
4) Repeat step 3 and 4 for all combinations.
5) The combination with overall minimum cost is the optimum combination and associated demand assignments, are the optimum assignments.

The output of this method for #Customer = 30 and #Facilities = 10 is shown in Figure 4.

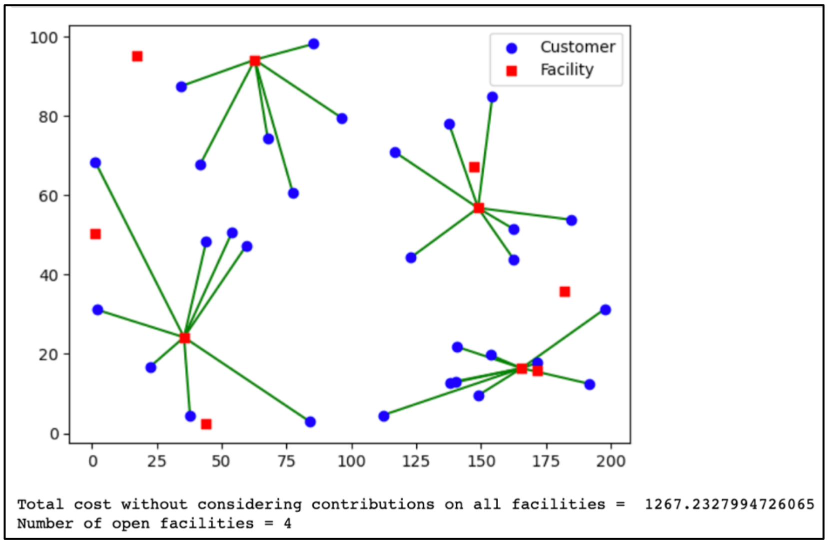

Figure 4: Assignments made by centralized method for m = 10 and n = 40

Alternating Direction Method of Multipliers

We rewrite the optimization problem as following:

$$L_P = f(x) + g(y) + z \times \left(\sum_i y_i - 1\right) + \frac{P}{2}\left\|\sum_i y_i - 1\right\|^2$$

$$where \quad f(x) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \ ; g(y) = \sum_{i=1}^{m} f_i y_i$$

the procedure is similar to primal sub-gradient method with some extra steps; meaning that we assume $x_i's$ and $y_i's$ are assumed to be integral and $\sum_{i \in I} x_{ij} = 1$ and $\sum_{i=1}^{m} y_i \geq 1$.

The algorithm is as following:

A. For several initializations for $y_i's$:

1) Value $y_i's$ randomly and make sure that none of them is greater than one or less than zeros.
2) z = 1, P = 2/3/4…

3) Find open facilities and store them in OF.
4) In a loop, do the following steps:
   a. $x^{k+1} = argmin\ L_P(x, y^k, z^k)$ → find the assigned faculties and their values
   b. Find the sub-gradient, ζ (discussed in primal sub-gradient section)
   c. In this part, we want to update $y_i's$ using ζ, but we need to make $g(y) + z^k \times (\sum_i y_i - 1) + \frac{P}{2}\|\sum_i y_i - 1\|^2$ as small as possible, in order to do that, we need to find the best length step to move in the direction of the gradient.
   d. Update $y_i's : Y = Y - step \times \zeta$
   e. Project $y_i's$ (discussed in primal sub-gradient section)
   f. Update $z : z^{k+1} = z^k + P(\sum_i y_i - 1)$
5) Save the open facilities and the how much it would cost to open them + the transportation cost.

B. Find the initialization that cost the less among all and name it, Min.
C. Find the open facilities in Min scenario and assign each customer to the nearest one.

The output of this method for #Customer = 30 and #Facilities = 10 is shown in Figure 5.

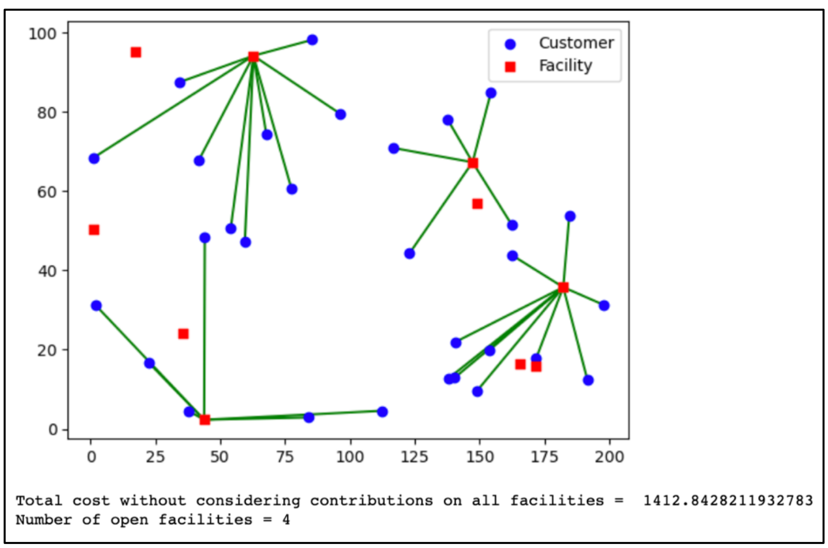

Figure 5: Assignments made by ADMM method for m = 10 and n = 40

Accelerated ADMM

The optimization problem for this method is the same as ADMM, and the algorithm is as following:

A. For several initializations for $y_i's$:

   1) Value $y_i's$ randomly and make sure that none of them is greater than one or less than zeros.
   2) z_0 = 0, z_1 = 1, a = 0, P = 2/3/4…
   3) Find open facilities and store them in OF.
   4) In a loop, do the following steps:

      a. $x^{k+1} = argmin\ L_P(x, \hat{y}^k, \hat{z}^k)$ → find the assigned faculties and their values
      b. Find the sub-gradient, ζ (discussed in primal sub-gradient section)
      c. In this part, we want to update $y_i's$ using ζ, but we need to make $g(y) + \hat{z}^k \times (\sum_i y_i - 1) + \frac{P}{2}\|\sum_i y_i - 1\|^2$ as small as possible, in order to do that, we need to find the best length step to move in the direction of the gradient.
      d. Update $y_i's : Y = Y - step \times \zeta$
      e. Project $y_i's$ (discussed in primal sub-gradient section)
      f. Update $z : z^{k+1} = z^k + P(\sum_i y_i - 1)$
      g. $a_{k+1} = \frac{1+\sqrt{1+a_k^2}}{2}$
      h. $\hat{z}^{k+1} = \hat{z}^k + \frac{a_k - 1}{a_k + 1}(z^k - z^{k-1})$
      i. $\hat{y}^{k+1} = argmin_y\ g(y) + \max(\hat{z}^{k+1}, -\sum_i y_i)$

   5) Save the open facilities and the how much it would cost to open them + the transportation cost.

B. Find the initialization that cost the less among all and name it, Min.

C. Find the open facilities in Min scenario and assign each customer to the nearest one.

The output of this method for #Customer = 30 and #Facilities = 10 is shown in Figure 6.

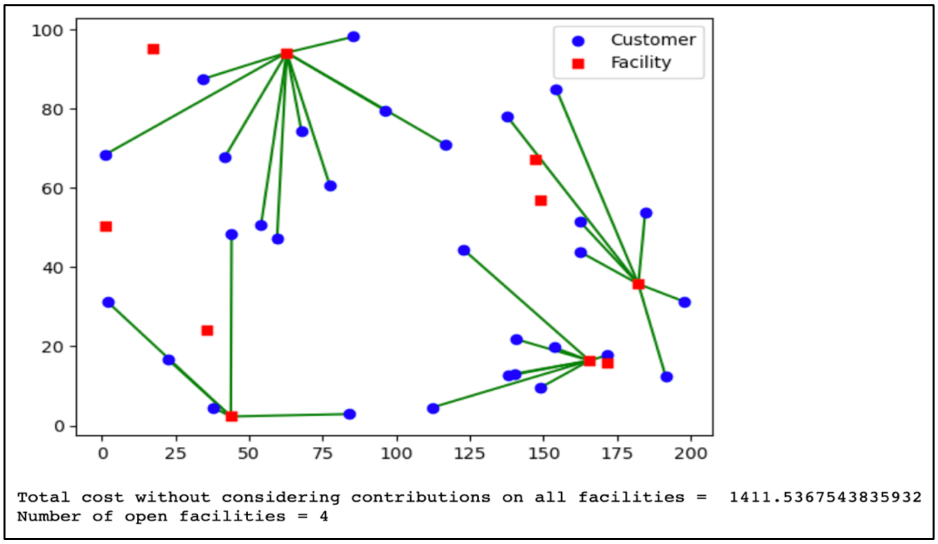

Figure 6: Assignments made by Accelerated ADMM method for m = 10 and n = 40

Resalts

Due to Figures 2,3, 4, and 5:

The least cost belongs to centralized method which sounds logical because this method searches among all possible combinations of open facilities and connect each customer to the closest one. After centralized method, ADMM happens to be the least costed method considering primal and dual sub-gradient method; as can be seen in Figure 5, it opened less facilities than two other methods and it is because of the term $\|\sum_i y_i - 1\|^2$ that wouldn't let all facilities to open. The dual sub-gradient method works better than the primal sub-gradient method because in the beginning of the latter method, all facilities are set to be open and updates $y_i's$ in the direction of sub-gradient, considering a small length for the step at each time, so none of them get to close at the end, therefore all the customers get assigned to the closest facility, which doesn't sound optimal at all; but in the dual sub-gradient method each client searches among the open facilities in order, and reach to the one which is affordable.

As an additional method, Accelerated ADMM is applied to this problem, which is similar to ADMM itself but updates the dual parameter $z^k$and $y_i's$ twice in each iteration which leads to a faster converging to the optimal solution; as can be seen, with same number of iterations, Accelerated ADMM has lower cost rather than ADMM.

As can be noticed, in dual sub-gradient method, we don't use sub-gradients in order to update $y_i's$, and in primal sub-gradient method and ADMM, learning rate changes depending on the situation.

To sum up, ADMM and its fast version are the best decomposition methods for uncapacitated facility location problem, but the dual sub-gradient method is the cheapest, because all it does is searching and adding, and the most expansive one is centralized method, because for n and m greater than some specific N, the number of combinations gets too much, which makes the whole process time-consuming.