The Banach Tarski Paradox

by

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Abstract

This dissertation studies the existence of paradoxical decompositions and in particular, Banach-Tarski Paradox. It discusses the paradoxes from algebraic point of view and introduces relationship between Congruence by Dissection and Equidecomposibility. The Banach-Tarski Paradox is proved using Hausdorff's Paradox and proof by absorbtion. In the end, the minimum number of pieces required in a paradoxical decomposition are discussed (without proofs).

Thesis Supervisor: Dr. Subiman Kundu
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Certificate

This is to certify that the report entitled The Banach Tarski Paradox submitted by Gaurav Mahajan (Entry No. 2008MT50448) in partial fulfillment of the requirement of the award of Integrated Masters in Technology in Mathematics and Computing, as a part of course MAD852 is a bona fide work under my supervision.

Dr. Subiman Kundu ....

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Chapter 1

Introduction

In 1924, S. Banach and A. Tarski proved a truly remarkable theorem: given a solid ball in R^3, it is possible to partition it into finitely many pieces and reassemble them to form *two* solid balls, each identical in size to the first.
At first such a duplication seems patently impossible. However, a moment's refection reminds us that the mathematical world doesn't always obey intuition.

The intention of this thesis is to provide a self-contained exposition of the proof of the Banach-Tarski Paradox and to introduce related topics. I will show, in fact, prove the stronger form of the Banach-Tarski Paradox : that any two bounded subsets of the plane with non-empty interior are such that one can be "cut" into a finite number of pieces and reassembled to form the other!

Two distinct themes arise when considering the refinements and ramifications of the Banach- Tarski Paradox. First, which we discuss in Chapter two, is the use of ingenious geometric and algebraic methods to construct paradoxes in situations where they seem impossible, and thereby getting proofs of the nonexistence of certain measures. Second, and this comprises the last section of Chapter three of this thesis, is minimizing the number of pieces in a paradoxical decomposition.

1.1 Historical Background of Paradoxical Decomposition

It has been known since antiquity that the notion of infinity leads very quickly to seemingly paradoxical constructions, many of which seem to change the size of objects by operations that appear to preserve size. In a famous example, Galileo observed that the set of positive integers can be put into a one-one correspondence with the set of even integers, even though the set of odd, and hence the set of all integers, seems more numerous than the evens. He deduced from this that " the attributes 'equal', 'greater' and 'less' are not applicable to infinite quantities," anticipating developments in the twentieth century, when paradoxes of this sort were used to prove the nonexistence of certain measures.

An important feature of Galileo's observation is its resemblance to a duplicating machine; his construction shows how, starting with the positive integers, one can produce two sets, each of which has the same size as the set of positive integers. The idea of duplication inherent in this example will be the main object of study in this thesis. The reason that this concept is so fascinating is that, soon after paradoxes such as Galileo's were being clarified by Cantor's theory of cardinality, it was discovered that even more bizarre duplications could be produced using rigid motions, which are distance-preserving (and hence also area-preserving) transformations. I refer to the Banach-Tarski Paradox on duplicating spheres or balls, which is often stated in the fanciful form: a pea may be taken apart into finitely many pieces that may be rearranged using rotations and translations to form a ball the size of the sun. The fact that the Axiom of Choice is used in the construction makes it quite distant from physical reality, though there are interesting examples that do not need the Axiom of Choice.

1.2 Formal Definition

We establish some terminology: G_n will denote the isometry group of R^n, and SO_n will denote the group of rotations of R^n. Since Dr.Subiman Kundu wants me to discuss 'B-T paradox' from algebraic perspective, I start with some definitions' needed for that approach.

We begin with a formal definition of the idea of duplicating a set using certain transformations. The general theory is much simplified if the transformations used are all bijections of a single set, and the easiest way to do this is to work in the context of group actions. Recall that,

Definition 1.1. *A group G is said to act on a set X if to each $g \in G$, there corresponds a bijection from X onto X also denoted by g s.t for any $g, h \in G$ and $x \in X$*

1. $g(h(x)) = (gh)(x)$ *and*

2. $I(x) = x$ *hold where I is the identity of G.*

Definition 1.2. *Let G be a group acting on a set X and suppose $E \subseteq X$. E is* G-paradoxical *(or paradoxical w.r.to G) if for some positive integers m, n there are pairwise disjoint subsets $A_1, ..., A_m$; $B_1, ..., B_n$ of E and $g_1, ..., g_m; h_1, ...h_n \in G$ s.t* $\bigcup_{i=1}^{m} g_i(A_i) = E = \bigcup_{j=1}^{n} h_j(B_j)$

Chapter 2

Examples of Paradoxical Actions

2.1 The Banach-Tarski Paradox

Any ball in R^3 is paradoxical with respect to the group of isometries of R^3.

More generally, we shall consider the possibility of paradoxes when X is a metric space and G is a subgroup of the group of isometries of X (an isometry is a bijection from X to X that preserves distance). In the case that G is the group of all isometries of X, we shall suppress G, using simply, E is paradoxical. We shall be concerned mostly with the case that X is one of the Euclidean spaces R^n.

2.2 Free Groups

Any group acts naturally on itself by left translation. The question of which groups are paradoxical with respect to this action turns out to be quite fascinating and is discussed at the end of this chapter. In this context the central example is the free group on two generators. Recall that

Definition 2.1. *The free group F with generating set M is the group of all finite words using letters from $\sigma, \sigma^{-1} : \sigma \in M$ where two words are equivalent if one can be transformed to the other by the removal or addition of finite pairs of adjacent-letters of the form $\sigma\sigma^{-1}$ or $\sigma^{-1}\sigma$.*

A word with no such adjacent pair is called a *reduced word* and to avoid the use of equivalence classes, F is usually taken to consist of all reduced words with the group operation being concatenation, the concatenation of 2 words is equivalent to a unique reduced word. The identity of F, denoted 1, is the empty word. Any two free generating sets for a free group have the same size, which is called the rank of the free group.

Theorem 2.2. *A free group F of rank 2 is F-paradoxical, where F acts on itself by left multiplication.*

Proof. Suppose σ, τ are free generators of F. If ρ is one of $\sigma^{\pm 1}, \tau^{\pm 1}$, let $W(\rho)$ be the set of elements of F whose representation as a word in $\sigma, \sigma^{-1}, \tau, \tau^{-1}$ begins, on the left, with ρ. Then $F = 1 \cup W(\sigma) \cup W(\sigma^{-1}) \cup W(\tau) \cup W(\tau^{-1})$, and these subsets are pairwise disjoint. Furthermore, $W(\sigma) \cup \sigma W(\sigma^{-1}) = F$ and $W(\tau) \cup \tau W(\tau^{-1}) = F$. For if $h \in F \setminus W(\sigma)$, then $\sigma^{-1}h \in W(\sigma^{-1})$ and $h = \sigma(\sigma^{-1}h) \in \sigma W(\sigma^{-1})$. Note that this proof uses only four pieces. $\square$

The preceding proof can be improved so that the four sets in the paradoxical decomposition cover all of F, rather than just $F \setminus \{1\}$.

Theorem 2.3. *If G is paradoxical and acts on X without nontrivial fixed points, then X is G-paradoxical. Hence X is F -paradoxical whenever F, a free group of rank 2, acts on X with no nontrivial fixed points.*

The main example of a paradoxical group is a free group of rank 2 (Theorem 2.2) and constructions such as the Banach-Tarski Paradox are based on the realization of such a group as a group of isometries of R^n. But actions of isometries on R^n have, in general, many fixed points, and so the main applications of Theorem 2.3 involve figuring out some way to deal with them. Nonetheless, the idea of lifting a paradox from a group to a set upon which it acts is, by itself, sufficient to obtain interesting pseudo- paradoxical results.

Chapter 3

The Banach-Tarski Paradox

The idea of cutting a figure into pieces and rearranging them to form another figure goes back at least to Greek geometry, where this method was used to derive area formulas for regions such as parallelograms. When forming such rearrangements, one totally ignores the boundaries of the pieces. The consideration of a notion of dissection in which every single point is taken into account, that is, a set-theoretic generalization of the classical geometric definition, leads to an interesting, and very general, equivalence relation. By studying the abstract properties of this new relation, Banach and Tarski were able to improve on Hausdorff's Paradox by eliminating the need to exclude a countable subset of the sphere. Since geometric rearrangements will be useful too, we start with the classical definition in the plane.

3.1 Congruence by dissection

Definition 3.1. *Two polygons in the plane are congruent by dissection if one of them can be decomposed into finitely many polygonal pieces that can be rearranged using isometries (and ignoring boundaries) to form the other polygon.*

It is clear that polygons that are congruent by dissection have the same area. The converse was proved in the early nineteenth century, and a simple proof can be given by efficiently making use of the fact that congruence by dissection is an equivalence relation (transitivity is easily proved by superposition, using the fact that the intersection of two polygons is a polygon.

3.1.1 Bolyai-Gerwien Theorem

Theorem 3.2. *Two polygons are congruent by dissection if and only if they have the same area.*

Proof. In order to prove the reverse direction. it suffices, because of transitivity to show that any polygon is congruent by dissection to a square (necessarily of the same area). We do this first for a triangle. Any triangle is congruent by dissection to a rectangle. A rectangle whose length is at most four times its width can be transformed to a square: the triangles to be moved are clearly similar to their images and the fact that the area of the square equals that of the rectangle implies that they are, in fact, congruent. For the situation when the length is greater than four times the width, the unbalanced rectangle can be transformed to one of the desired type by repeated halving and stacking. Hence any triangle is congruent by dissection with a square. The proof is concluded by observing that the Pythagorean Theorem can be proved in a way that can be used to transform two (or more) squares into one by dissection. Used repeatedly, this construction shows how any finite set of squares can be transformed by dissection to a single square. Now, any polygon can be split into triangles. Squaring the triangles and combining the squares as just indicated yields a square that is congruent by dissection to the original polygon. □

The theory of geometrical dissections in higher dimensions, or other geometries, is not at all as simple as in the plane. In fact, the third problem on Hilberts famous list asks whether a regular tetrahedron in R^3 is congruent by dissection (into polyhedra) with a cube. All proofs of the volume formula for a tetrahedron were based on a limiting process of one sort or another, such as the Devils staircase or Cavalieris Principle. Hopes for an elegant dissection proof were dashed when Dehn proved, in 1900, that a regular tetrahedron is not congruent by dissection with any cube. But it is possible that for a suitable generalization of dissection where a wider class of pieces is allowed, a regular tetrahedron is piecewise congruent to a cube. Indeed, one consequence of the Banach-Tarski Paradox is that a regular tetrahedron can be cubed if arbitrary sets are allowed as pieces.

3.2 Equidecomposability

The set-theoretic version of congruence by dissection may be stated in the context of an arbitrary group action.

Definition 3.3. *Suppose a group G acts on X and $A, B \subset X$, then A and B are* G-equidecomposable *(or called piece-wise G-congruent as in [1]) if A and B each can be partitioned into the same finite number of respectively G-congruent pieces. Formally* $A = \bigcup_{i=1}^{n} A_i$; $B = \bigcup_{i=1}^{n} B_i$; $A_i \cap A_j = \phi = B_i \cap B_j$ *if* $i < j \leq n$ *and there are* $g_1, ..., g_n \in G$ *s.t for each* $i \leq n$, $g_i(A_i) = B_i$.

The notation A $\sim_G$ B is used to denote the equidecomposibility relation. It is easy to prove that $\sim_G$ is an equivalence relation. We say $A \sim_{G_n} B$ if the disassemble can be affected with n pieces.

It is not immediately apparent that there is any connection between equidecomposability and congruence by dissection. Indeed, they differ in a most fundamental way. Since there is no restriction on the subsets that may be used to verify that $A \sim B$, there is no guarantee that A and B have the same area (or n-dimensional Lebesgue measure). For if the pieces are non-Lebesgue measurable, then the straightforward proof that works in the case of congruence by dissection cannot be used, since it involves summing the areas of the pieces.

In a different vein, one can ask whether polygons that are congruent by dissection are necessarily equidecomposable. The problem is that, somehow, the boundaries of the pieces in a geometrical dissection must be accounted for in a precise way. In a typical dissection the boundaries do double duty and so cannot simply be assigned to one of the pieces. This problem can be solved though and the main tool is a very important property of the equivalence relation $\sim_G$.

Whenever one has an equivalence relation on the collection of subsets of a set, one may define another relation $\preccurlyeq$ by $A \preccurlyeq B$ if and only if A is equivalent to a subset of B. Then $\preccurlyeq$ is really a relation on the equivalence classes and, in fact is reflexive and transitive. The Schroder-Bernstein Theorem of classical set theory states that if the cardinality relation is used - A and B are equivalent if there is a bijection from A to B - then $\preccurlyeq$ is antisymmetric as well; that is if $A \preccurlyeq B$ and $B \preccurlyeq A$, then A and B have the same cardinality. Thus $\preccurlyeq$ is a partial order on the equivalence classes. Banach realized that the proof of the Schroder-Bernstein Theorem could be applied to G-equidecomposability. From now on we use the notation $A \preccurlyeq B$ only in the context of equidecomposability: $A \preccurlyeq B$ means A is G-equidecomposable with a subset of B.

3.2.1 Banach-Schroder-Bernstein Theorem

Theorem 3.4. *Suppose G acts on X and $A, B \subseteq X$. If $A \preccurlyeq B$ and $B \preccurlyeq A$, then $A \sim_G B$. Thus $\preccurlyeq$ is a partial ordering of the $\sim_G$-classes in $P(X)$.*

This theorem eases dramatically the verification of equidecomposability. As an illustration, suppose a subset E of X is G-paradoxical, say A, B are disjoint subsets of E with $A \sim E \sim B$. Then $E \sim B \subseteq E \setminus A \subseteq E$, so the Banach-Schroder-Bernstein Theorem implies that $E \setminus A \sim E$. This proves the following result.

Theorem 3.5. *A subset E of X is G-paradoxical if and only if there are disjoint sets $A, B \subseteq E$ with $A \cup B = E$ and $A \sim E \sim B$.*

The following application of Banach-Schroder-Bernstein Theorem is important in that it shows that polygons that are congruent by (geometric) dissection are also equidecomposable, that is congruent by set-theoretic dissection.

3.2.2 Application to dissection of polygons

Theorem 3.6. *If the polygons P_1 and P_2 are congruent by dissection. then they are equidecomposable.*

Proof. Let Q_1, Q_2 be the open sets obtained by forming the union of all the interiors of the polygonal subsets of P_1, P_2 respectively, arising from the hypothesized dissection. Then $Q_1 \sim Q_2$, and so the proof will be complete once it is shown that $P_1 \sim Q_1$ and $P_2 \sim Q_2$, that is, that the boundary segments can be absorbed. This follows from the following fact (by setting $A = P_1$ and $T = P_1 \setminus Q_1$): If A is a bounded set in the plane with non- empty interior and T is a set, disjoint from A. consisting of finitely many (bounded) line segments, then $A \sim A \cup T$.

To prove this fact, let D be a disc contained in A, and let r be its radius. By subdividing the segments in T. we may assume that each one has length less than r. Let θ be any rotation of D about its center having infinite order, let R be any radius of D (excluding the center of D), and let $\bar{R} = R \cup \theta(R) \cup \theta^2(R) \cup \ldots$. Now, if $s \in T$ then $D \cup s \preccurlyeq D$. This is because $\theta(\bar{R})$is disjoint from R and $D \setminus \bar{R}$, so $D \cup s = (D \setminus \bar{R}) \cup \bar{R} \cup s \sim (D \setminus \bar{R}) \cup \theta(\bar{R}) \cup \sigma(s) \subseteq D$, where σ is any isometry taking s to a subset of R. Since, obviously, $D \preccurlyeq D \cup s$, the Banach-Schroder-Bernstein Theorem implies that $D \sim D \cup s$. Since each of the segments in T may thus be absorbed, one at a time into D, we have that $D \sim D \cup T$. Adding $A \setminus D$ to both sides yields $A \sim A \cup T$, as required. □

Because of the Bolyai-Gerwien Theorem, the preceding theorem implies that any two polygons of the same area are equidecomposable.

The preceding proof might be called a proof by absorption, since it shows how a troublesome set (the boundary segments) can be absorbed in a way that, essentially, renders it irrelevant. Now, we have seen that free groups of rank 2 cause paradoxes when they act without fixed points (Theorem 2.3), and so situations where the fixed points can be absorbed will be especially important.The following proof is typical of the absorption proofs, and immediately yields the Banach- Tarski Paradox.

3.3 Absorbing countable subset of Sphere

Theorem 3.7. *If D is a countable subset of S^2, then S^2 and $S^2 \setminus D$ are SO_3-equidecomposable (using two pieces).*

Proof. We seek a rotation, ρ, of the sphere such that the sets $D, \rho(D), \rho^2(D), \ldots$ are pairwise disjoint. This suffices, since then $S^2 = \bar{D} \cup (S^2 \setminus \bar{D}) \sim \rho(\bar{D}) \cup (S^2 \setminus \bar{D}) = S^2 \setminus D$, where $\bar{D} = \bigcup\{\rho^n(D) : n = 0, 1, 2, \ldots\}$. For the construction of ρ, let l be a line through the origin that misses the countable set D. Let A be the set of angles θ such that for some $n > 0$ and some $P \in D$, $\rho(P)$ is also in D where ρ is the rotation about l through $n\theta$ radians. Then A is countable, so we may choose an angle θ not in A; let ρ be the corresponding rotation about l. Then $\rho^n(D) \cap D = \emptyset$ if $n > 0$, from which it follows that whenever $0 \leq m < n$, then $\rho^m(D) \cap \rho^n(D) = \emptyset$ (consider $\rho^{n-m}(D) \cap D$); therefore ρ is as required. □

3.4 Duplicating spheres and balls with rotations and translations

Theorem 3.8. (The Banach-Tarski Paradox)(AC). *S^2 is SO_3-paradoxical, as is any sphere centered at the origin. Moreover, any solid ball in R^3 is G_3-paradoxical and R^3 itself is paradoxical.*

Proof. The Hausdorff Paradox states that $S^2 \setminus D$ is SO_3-paradoxical for some countable set D (of fixed points of rotations). Combining this with the previous theorem yields that S^2 is SO_3-paradoxical. Since, the previous result does not depend on the size of the sphere, spheres of any radius admit paradoxical decompositions. It suffices to consider balls centered at 0, since G_3 contains all translations. For definiteness, we consider the unit ball B, but the same proof works for balls of any size. The decomposition of S^2 yields one for $B \setminus \{0\}$ if we use the radial correspondence: $P \to \{\alpha P : 0 < \alpha \leq 1\}$. Hence it suffices to show that B is G_3 -equidecomposable with $B \setminus \{0\}$, that is, that a point can be absorbed. Let $P = (0, 0, \frac{1}{2})$ and let ρ be a rotation of infinite order about an axis through P but missing the origin. Then, as usual, the set $D = \{\rho^n(0) : n \geq 0\}$ may be used to absorb 0: $\rho(D) = D \setminus \{0\}$, so $B \sim B \setminus \{0\}$.If, instead, the radial correspondence of S^2 with all of $R^3 \setminus \{0\}$ is used, one gets a paradoxical decomposition of $R^3 \setminus \{0\}$ using rotations. Since, exactly as for the ball, $R^3 \setminus \{0\} \sim_{G_3} R^3$, R^3 is paradoxical via isometries. □

Rotations preserve volume, and this is why the result has come to be known as a paradox. A resolution is that there may not be a volume for the rotations to preserve; the pieces in the decomposition may be (indeed, will have to be) non-Lebesgue measurable.

3.5 Strong Form of the paradox

Theorem 3.9. (Banach-Tarski Paradox, Strong Form) (AC). *If A and B are any two bounded subsets of R^3 , each having nonempty interior, then A and B are equidecomposable.*

Proof. It suffices to show that $A \preccurlyeq B$, for then by the same argument, $B \preccurlyeq A$ and yields $A \sim B$. Choose solid balls K and L such that $A \subseteq K$ and $L \subseteq B$, and let n be large enough that K may be covered by n (overlapping) copies of L. Now, if S is a set of n disjoint copies of L, then using the Banach-Tarski Paradox to repeatedly duplicate L, and using translations to move the copies so obtained, yields that $L \succcurlyeq S$. Therefore $A \subseteq K \preccurlyeq S \preccurlyeq L \subseteq B$,so $A \preccurlyeq B$. □

3.6 Minimizing the Number of Pieces in a Paradoxical Decomposition

The techniques used to cut the number of pieces to a minimum lend to significant new ideas on how to deal with the fixed points of an action of a free group, adding to our ability to recognize when a group's action is paradoxical. First, we state precisely the way in which the pieces will be counted.

Definition 3.10. *Suppose G acts on X and $E \subseteq X$. Then E is G-paradoxical with r pieces if there are disjoint A,B with $E = A \cup B, A \sim_m E \sim_n B$,and $m + n = r$.*

Note that this definition is stronger than merely adding to Definition 1.2 the condition that m + n = r, because this definition requires $E = A \cup B$ rather than just

$E \supseteq A, B$. In fact, if r pieces are used in Definition 1.2, then by virtue of Theorem 3.5 and its proof, E is paradoxical using r + 1 pieces, in the sense of Definition 3.10. This distinction is strikingly illustrated by the free group, F, of rank two. The simple proof of Theorem 2.2 uses four pieces, but since the identity of F does not appear in any of the pieces, it yields only that F is paradoxical using five pieces. In order to do the same with four pieces, let σ and τ be the free generators of F and let $A_1 = W(\tau), A_2 = W(\tau^{-1}), A^3 = W(\sigma) \cup \{1, \sigma^{-1}, \sigma^{-2}, \dots\}$, and $A_4 = W(\sigma^{-1}) \setminus \{\sigma^{-1}, \sigma^{-2}, \dots\}$. Then $A_1 \cup \tau(A_2) = F = A_3 \cup \sigma(A_4)$. The following theorem shows how this construction can be modified to yield an important additional condition that is satisfied by the decomposition.

Theorem 3.11. *F, the free group generated by σ and τ, can be partitioned into $A_1, A_2, A_3,$ and A_4 such that $\sigma(A_2) = A_2 \cup A_3 \cup A_4$ and $\tau(A_4) = A_1 \cup A_2 \cup A_4$; therefore F is paradoxical using four pieces. Moreover, for any fixed $w \in F$, the partition can be chosen so that w is in the same piece as the identity of F.*

The action of a group G on a set X is called fixed-point free if $g(x) \neq x \forall x$ and $\forall g \in G - \{I\}$ (I is the identity of G). Note I(x) = x always holds; so actually a fixed-point free action of G on X is an action without non trivial fixed points.

Theorem 3.12. *If F, a free group of rank 2, acts on X without nontrivial fixed points, then X is F-paradoxical using four pieces. Hence any group having a free non-Abelian subgroup is paradoxical using four pieces.*

Definition 3.13. *E is called G-negligible if $\mu(E) = 0$ whenever μ is a finitely additive. G-invariant measure on P(X) with $\mu(E) < \infty$*

Theorem 3.14. *If E is G-paradoxical, then E is G-negligible.*

Although from a geometrical point of view, it is nicer to have decompositions into as few pieces as possible, such efficient decompositions do not have any new implications for the existence of finitely additive measures. If X is G-paradoxical, then X is G-negligible (Definition 3.13) no matter how many pieces are needed. But

Theorem 3.11, in its entirety, can be used to provide a way of dealing with fixed points of action of free groups that is entirely different from the method used earlier.

If G acts on X and $x \in X$, let $Stab(x)$, the stablizer of x, denote $(\sigma \in G : \sigma(x) = x)$; $Stab(x)$ is a subgroup of G.

Definition 3.15. *An action of a group G on X is called locally commutative if Stab(x) is commutative for every $x \in X$; equivalently if two elements of G have a common fixed point, then they commute.*

Of course, any action without any nontrivial fixed points is locally commutative. A more interesting example is the action of the rotation group SO_3 , on the sphere S^2. If two rotations of the sphere share a fixed point, they must have the same axis, and therefore they commute.

Theorem 3.16. *If the action of F on X is locally commutative, where F is freely generated by σ and τ, then X is F-paradoxical using four pieces.*

Theorem 3.17. *If G acts on X and X is G-paradoxical using four pieces, then G has two independent elements σ, τ such that the action of F, the group they generate, on X is locally commutative.*

Theorem 3.18. *A group G is paradoxical using four pieces if and only if G has a free subgroup of rank 2.*

Bibliography

[1] Karl Stromberg: The Banach Tarski Paradox - Amer. Math. Monthly pg(151-161), 1979.

[2] Stan Wagon: The Banach Tarski Paradox- Cambridge University Press, 1985.