Chapter 1

Propositional Logic

We write the language of propositional logic as $\mathcal{L}_0$.

1.1 Syntax

Definition 1.1.1 (Syntax of $\mathcal{L}_0$)**.** Let P be a set of *propositional variables*:

$$P := \{p_i \mid i \in \mathbb{N}\}.$$

The language $\mathcal{L}_0$ is defined as the minimal set that satisfies the following equation:

$$\mathcal{L}_0 = P \cup \{\neg p \mid p \in \mathcal{L}_0\} \cup \{p_1 \wedge p_2 \mid p_1, p_2 \in \mathcal{L}_0\}.$$

Exercise 1.1.1. Construct a bijection between $\mathcal{L}_0$ and $\mathbb{N}$.

Remark 1.1.1. To be extremely pedantic, $\mathcal{L}_0$ should be parameterized by the choice of the base set P, i.e., $\mathcal{L}_0(P)$. But just like how we normally use x, y, z to represent numerical variables, it is standard to write propositional variables as p, q, r etc.

Remark 1.1.2. Equivalently, we can define $\mathcal{L}_0$ as the minimal set that contains any formula φ of the following form:

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi$$

where $p \in P$.

Exercise 1.1.2. Prove the equivalence between the two definitions.

Definition 1.1.2 (Disjunction and Implication)**.** Let $\varphi_1, \varphi_2 \in \mathcal{L}_0$ be arbitrary. Define:

$$\begin{aligned} \varphi_1 \vee \varphi_2 &:= \neg(\neg\varphi_1 \wedge \neg\varphi_2), \\ \varphi_1 \to \varphi_2 &:= \neg\varphi_1 \vee \varphi_2. \end{aligned}$$

Remark 1.1.3. This definition of implication was *invented* by Frege.

Exercise 1.1.3. What is a reasonable definition of the precendence order of the connectives?

Exercise 1.1.4. What are the other subsets of $\{\neg, \wedge, \vee, \to\}$ that can define all other connectives?

1.2 Semantics

Definition 1.2.1 (Interpretation)**.** An interpretation, or truth assignment, of $\mathcal{L}_0$ is any function $\sigma : P \to \{0, 1\}$.

Definition 1.2.2 (Evaluation)**.** Let σ be an interpretation of $\mathcal{L}_0$. The evaluation function

$$\widehat{\sigma} : \mathcal{L}_0 \to \{0, 1\}$$

is defined to satisfy the following conditions:

- For any $p \in P$, $\widehat{\sigma}(p) = \sigma(p)$.
- For any $\varphi \in \mathcal{L}_0$, $\widehat{\sigma}(\neg\varphi) = 1 - \widehat{\sigma}(\varphi)$.
- For any $\varphi_1, \varphi_2 \in \mathcal{L}_0$, $\widehat{\sigma}(\varphi_1 \wedge \varphi_2) = \widehat{\sigma}(\varphi_1) \cdot \widehat{\sigma}(\varphi_2)$.

Exercise 1.2.1. Prove the existence and uniqueness of this $\widehat{\sigma}$.

Remark 1.2.1. Because of the simplicity and uniqueness of the evaluation function, we often use "interpretation" and "evaluation" interchangeably, and write both as σ.

Remark 1.2.2. It is easy to see that such a $\widehat{\sigma}$ also satisfies that for any $\varphi_1, \varphi_2 \in \mathcal{L}_0$,

$$\widehat{\sigma}(\varphi_1 \vee \varphi_2) = \widehat{\sigma}(\varphi_1) + \widehat{\sigma}(\varphi_2) - \widehat{\sigma}(\varphi_1) \cdot \widehat{\sigma}(\varphi_2)$$
$$\widehat{\sigma}(\varphi_1 \to \varphi_2) = \widehat{\sigma}(\varphi_1) \cdot \widehat{\sigma}(\varphi_2) - \widehat{\sigma}(\varphi_1) + 1.$$

Exercise 1.2.2. Write down the truth table (the list of all possible evaluation functions) of the following formulas:

- $(p \to q) \to (\neg q \to \neg p)$
- $p \to (q \to r) \to ((p \to q) \to (p \to r))$

Definition 1.2.3 (Satisfiability)**.** We say $\varphi \in \mathcal{L}_0$ is satisfiable, if there exists an interpretation σ such that $\sigma(\varphi) = 1$. Otherwise we say it is unsatisfiable.

Definition 1.2.4 (Validity)**.** $\varphi \in \mathcal{L}_0$ is valid, if for any interpretation σ, $\sigma(\varphi) = 1$.

Proposition 1.2.1. φ is valid iff $\neg\varphi$ is unsatisfiable.

Example 1.2.1. The following formulas are valid:

- $p \to (q \to p)$
- $p \to (q \to r) \to ((p \to q) \to (p \to r))$

Definition 1.2.5 (The SAT Problem)**.** Given any $\varphi \in \mathcal{L}_0$, decide if φ is satisfiable. The dual problem of deciding whether φ is called the validity checking problem.

Remark 1.2.3. So far we have deliberately avoided giving any "logical" interpretation of the symbols "$\wedge, \vee, \to$." Indeed, they can be considered simply as convenient notations of certain functions defined on the value space of the propositional variables. However, it is easier to remember if we use our intuition about their interpretations as the logical "and, or, implies" respectively.

1.3 Complexity

Theorem 1.3.1. Satisfiability in $\mathcal{L}_0$ is NP-complete.

Corollary 1.3.1. Validity in $\mathcal{L}_0$ is co-NP-complete.

Corollary 1.3.2. Solving nonlinear equations over $\mathbb{F}_2$ (finite field of size 2) is NP-complete.

Exercise 1.3.1. Construct a bijection between $\mathcal{L}_0$ and polynomials over $\mathbb{F}_2$.