Banach Tarski Paradox

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Past Work

Worked on 'elementary' proof of Banach Tarski Paradox based on paper by Karl Stromberg

Axiom of Choice

Banach Tarski Paradox contrary to its name is actually a theorem and it depends in a critical way on the Axiom of Choice in Set Theory.

What is Axiom of Choice?

In 1904 Ernst Zermelo formulated the Axiom of Choice (abbreviated as *AC* throughout this presentation) in terms of what he called *coverings* (Zermelo 1904). He starts with an arbitrary set M and uses the symbol M' to denote an arbitrary nonempty subset of M, the collection of which he denotes by P. He continues:

Imagine that with every subset M' there is associated an arbitrary element $m1'$, that occurs in M' itself; let $m1'$ be called the 'distinguished' element of M'. This yields a 'covering' g of the set M by certain elements of the set M. The number of these coverings is equal to the product [of the cardinalities of all the subsets M'] and is certainly different from 0 (zero).

The last sentence of this quotation which asserts, in effect, that coverings always exist for the collection of nonempty subsets of any (nonempty) set is Zermelo's first formulation of the Axiom of Choice.

Variants of Axiom of Choice

The variant of axiom of choice used in Karl Strombergs proof is:

Given any set X of pairwise disjoint non-empty sets, there exists at least one set C that contains exactly one element in common with each of the sets in X.

Current Work

Introduction to Free Groups, Equidecomposable and Paradoxical Sets

Introduction

In 1924, S. Banach and A. Tarski proved a truly remarkable theorem: given a solid ball in R^3, it is possible to partition it into finitely many pieces and reassemble them to form *two* solid balls, each identical in size to the first.
At first such a duplication seems patently impossible. However, a moment's refection reminds us that the mathematical world doesn't always obey intuition.

Paradoxical Sets

One of the earliest paradoxes arose out of grappling with the notion of infinity. The set of integers can be put in 1-1 correspondence with all integers, for instance. This seems strange, at first: if set bijections are interpreted as "equality" in some sense, then $\mathbf{Z}$ is "equal" to a subset of itself. In fact, $\mathbf{Z}$ may be partitioned into two sets (even and odd integers), each "equal" to all of $\mathbf{Z}$.

And among the modern paradoxes, Russell's paradox on ordinary/extraordinary sets become quite famous and it dealt a stunning blow to Frege's Axiom V. Actually the discovery of such paradoxes in the beginning of this century gave birth to an important branch of modern mathematics-'Foundations of Mathematics' where in particular validity of some axioms and some mathematical procedures is debated.
Since Dr.Subiman Kundu wants me to discuss 'B-T paradox' from algebraic perspective, I start with some definitions' needed for that approach.

Definition 2.1: A group G is said to act on a set X if to each $g \in G$, there corresponds a bijection from X onto X also denoted by g s.t for any $g, h \in G$ and $x \in X$

1. $g(h(x)) = (gh)(x)$ and
2. $I(x) = x$ hold where I is the identity of G.

The reader is supposed to be familiar with these standard actions. We will now formulate the concept of paradoxicality.

Definition 2.2: Let G be a group acting on a set X and suppose E $\subseteq$ X. E is *G-paradoxical* (or paradoxical w.r.to G) if for some positive integers m, n there are pairwise disjoint subsets $A_1, \ldots.., A_m;\ B_1, \ldots.., B_n$ of E and $g_1, \ldots., g_m; h_1, \ldots.h_n \in G$ s.t $\bigcup_{i=1}^{m} g_i(A_i) = E = \bigcup_{j=1}^{n} h_j(B_j)$

Definition 2.3: An isometry of any metric spare X is a distance preserving bijection of X onto itself. It's clear that all the isometries of X form a group-multiplication is just the composition of two isometrics.

Free Groups

Definition 2.4: The free group F with generating set M is the group of all finite words using letters from $\sigma, \sigma^{-1} : \sigma \in M$ where two words are equivalent if one can be transformed to the other by the removal or addition of finite pairs of adjacent-letters of the form $\sigma\sigma^{-1}$ or $\sigma^{-1}\sigma$.

A word with no such adjacent pair is called a *reduced word* and to avoid the use of equivalence classes, F is usually taken to consist of all reduced words with the group operation being concatenation, the concatenation of 2 words is equivalent to a unique reduced word. The identity of F, denoted 1, is the empty word.
Any two free generating sets for a free group have the same size (i.e. same cardinality) which is called the rank of the free group.

Here are some important facts about free group:-

1. Any free group having rank $\geqslant 2$ must be non-abelian
2. A free group of rank 2 has a free subgroup of rank N_0: if σ, τ freely generate the group F of rank 2, then $\{\sigma^i \tau \sigma^{-i} : i = 0, 1, 2, ...\}$ is a set of free generation of a subgroup of F and the same is true of $\{\sigma^i \tau^i : i = 0, 1, 2, ...\}$.
3. For a group, the properties of being solvable and of containing a free non-abelian group are mutually exclusive.

Theorem 2.6 A free group F of rank 2 is F-paradoxical where F acts on itself by left multiplication.

In my presentation, in general, I will not give the proof of the theorems. For proofs, one is referred to [2].
The action of a group G on a set X is called fixed point free if $g(x) \neq x \; \forall x \in G - I$(I is the identity of G). Note $I(x) = x$ always holds; so actually a fixed point free action of G on X is an action without non trivial fixed points.

The action is called locally commutative if for each $x \in X = Stab(x)$ = the stabilizer of x is a commutative subgroup of G; equivalently , if 2 elements of G shares a fixed point, then they commute. Note that a fixed point free action is trivially locally commutative.

Various geometric constructions related to the Banach Tarski paradoxical decomposition of the sphere require the existence of free groups of isometries with one of above two properties.

Theorem 2.7 (AC): If a group G is G-paradoxical w.r.to the action of let translation and action of G on a set X is fixed-point free, then X is G- paradoxical

By Theorem 2.6 we know that a free group F of rank 2 is F-paradoxical w.r.to action of left translation. Hence Theorem 2.7 has an immediate important corollary.

Corollary 2.8: (AC): A set X is F-paradoxical whenever action of F, a free group of rank 2, on X is fixed-point free.

When we say that a group is paradoxical, we refer to the action of left translation. Now Theorem 2.6 actually says that the main example of a paradoxical group is a free group of rank 2 and constructions such as 'B-T Paradox' are based on the realization of such a group of isometries of R^n. But actions of isometries of R^n have, in general, many fixed points and so the main application of Theorem 2.7 is to figure out some ways to deal with them. Here the idea of using locally cumulative action, instead of fixed- point free action come up. However, note that one immediate contribution of Theorem 2.7 is the idea of lofting a paradox from a group to a set on which it acts. This idea of lifting has important implication in measure theory.

Since a subgroup of any group acts by test translation on the whole group and this action is fixed-point free, the following corollary is an immediate consequence of Theorem 2.6 and Theorem 2.7.

Corollary 2.9 A Group paradoxical subgroup is paradoxical. Hence, any group with a free subgroup of rank 2 (in particular, any nonabelian free group) is paradoxical.

Theorem 2.10 (Hausdorff's Paradox) (AC): There is a countable subset D of S^2 such that $S^2 - D$ is SO_3 - paradoxical.

But actually we have stated Hausdorff's paradox by using very sophisticated terminology. Hausdorff's own version was slightly different and as stated in theorem C of [1] : modulo a countable set of points, a 'half' of the sphere could be congruent to a 'third' of a sphere.

In [1], Stromberg has used Hausdorff's Paradox to show at the end of his paper in Remark 2 that not all subsets of the unit ball are Lebesgue measurable.

Now we are very close to 'B-T Paradox' "A countable subset of the sphere can be dense and so the paradoxical nature of 'Hausdorff Paradox' is not immediately clear. Still countable sets are very small in size compared to the whole (un-countable) sphere". Actually the smallness of D allows it to be eliminated completely and we get - 'B-T Paradox' - our ultimate goal - S^2 is SO_3 - paradoxical. But to have 'B-T Paradox', we need a definition and some results.

Equidecomposable Sets

Definition 2.11: Suppose a group G acts on X and A, B, C, then A and B are *G-equidecomposable* (or called piece-wise G-congruent as in [1]) if A and B each can be partitioned into the same finite number of respectively G-congruent pieces. Formally

$A = \bigcup_{i=1}^{n} A_i$; $B = \bigcup_{i=1}^{n} B_i$; $A_i \cap A_j = \phi = B_i \cap B_j$ if $i < j \leq n$ and there are $g_1, ..., g_n \in G$ s.t for each $i \leq n$, $g_i(A_i) = B_i$.

The notation A $\sim_G$ B is used to denote the equidecomposibility relation. It is easy to prove that $\sim_G$ is an equivalence relation. We say $A \sim_{G_n} B$ if the disassemble can be affected with n pieces.

Since '$\sim_G$' is an equivalence relation, we can define another relation '$\leq$' on P(X): $A \leq B$ $(A, B \subseteq X)$ iff $A \sim_G C$ where C is a subject of B. This relation is easily seen to be reflexive and transitive.

Note that Definition 2.2 actually says that $A \sim_G E \sim_G B$ where $A = \bigcup_{i=1}^{m} A_i$; $B = \bigcup_{j=1}^{n} B_j$; $A \cap B = \phi$ but $A \cup B$ may be proper subject of E.

Now $E \sim B \subseteq E - A \subseteq E \Rightarrow E \leq E - A$ and $E - A \leq E$ implying $E - A \sim_G E$. This proves the following result:-

Proposition 2.12: A subset E of X is G-paradoxical iff there are disjoint sets $A, B \subseteq E$ with $A \cup B = E$ and $A \sim_G E \sim_G B$.

The Banach-Tarski Paradox

In Stan Wagon's words: It is possible to cut up a pea into finitely many pieces that can be rearranged to form a ball the size of the sun. No doubt, this result, at first instance, seems patently impossible. More humorously, by using Stromberg's words, it can be stated this way: A billiard ball can be chopped into pieces which can be put back together to form a life -size statue of Banach.

Theorem 2.13 : If D is a countable subset of S^2, then S^2 and $S^2 - D$ are SO_3-equidecomposoable (using 2 pieces). Note that the proof of this theorem does not use AC. 'The B-T Paradox' - our goal - is a corollary to the above theorem and Hansdroff's Paradox (Theorem 2.10).

Corollary 2.14 (The Banach - Tarski Paradox) (AC) :- S^2 is SO_3 paradoxical, as is any sphere centered at the origin. Moreover, any solid ball in R^3 is G_3 paradoxical and is itself paradoxical. Part of the 2nd statement of the above corollary is Theorem E of Stromberg's paper [1].

Now the question is how do we relate corollary 2.16 (The B-T paradox) to the informal version of the paradox stated at the beginning of this section. Note that our informal version basically says about duplicating balls.

We know from Corollary 2.14 that any solid ball B in R^3 is G_3-paradoxical. Then by Proposition 2.12, there are two disjoint subsets $B_1, B_2 \subseteq$ B with $B_1 \cap B_2 =$ B and $B_1 \sim_{G_3} B \sim_{G_3} B_2$. Thus we get two balls B_1 and B_2 each identical to B in other words, we duplicate B. If one continues this process indefinitely, from a pea one can get a ball the size of the sun.

"Rotations preserve volumes and this is why the result has come to be known as a paradox. A resolution is that there may not be a volume for the rotations to preserve, the pieces in the decomposition may be(indeed, will have to be) non-Lebesgue measureable". I've already commented on this earlier.

Future Work

Hausdorff's Paradox and subsequently Banach Tarski Paradox

What I wish to achieve

1. Details of proof of Hausdorffs Paradox and use it to prove Banach Tarski Paradox/Theorem.
2. Establish connection between B-T paradox and free groups, and discuss its application in Measure Theory

References

1. Karl Stromberg: The Banach Tarski Paradox - Amer. Math. Monthly pg(151-161), 1979.
2. Stan Wagon: The Banach Tarski Paradox- Cambridge University Press, 1985.
3. Jan Mycielski and Stan Wagon: Large Free Groups of Isometries and their Geometrical Uses- L'Enseignemer Mathematique, t.30(1984), p-247-267.