2 Introduction to Quantum Codes (Week 2)

2.1 Repetition Code

We start by reviewing a basic classical error correcting code, known as repetition code. In classical world, our data consists of bits $b \in \{0, 1\}$. Suppose we wish to send a bit from one location to another through a noisy classical channel. The effect of noise in the channel is to flip the bit being transmitted with probability $p > 0$. To deal with this noise, we encode bit b by adding redundant bits.

Definition 2.1 (Repetition Code)**.** *Repetition code encodes $b \in \{0, 1\}$ into 3 bits.*

$$
\begin{aligned}
0 &\to 000 \\
1 &\to 111
\end{aligned}
$$

The bit strings 000 and 111 are referred to as *codewords*, and also sometimes referred to as the *logical 0* and *logical 1*, since they play the role of 0 and 1 respectively.

We now send all three bits 000 through the channel. Suppose our codeword experiences a single bit flip error, and the output of the channel is 010:

$$000 \xrightarrow{\text{noise}} 010$$

We can correct this bit flip by looking at the majority of the three bits. Since, only one bit has been flipped, this will give us the correct value for the three bits:

$$000 \xrightarrow{\text{noise}} 010 \xrightarrow{\text{majority}} 000$$

On the other hand, if there were 2 or more bit flips, this scheme would fail

$$000 \xrightarrow{\text{lots of noise}} 011 \xrightarrow{\text{majority}} 111 \neq 000$$

Hence, the 3-bit repetition code allows us to correct one bit flip error, and fails to correct 2 or more bit flips.

2.2 Quantum Noise

We consider a simplified model of quantum noise where only *bit flip* and *phase flip* error can occur.

Definition 2.2 (Bit flip error)**.** *A bit flip error is described by the unitary*

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which acts on the computational basis as

$$
\begin{aligned}
X\,|0\rangle &= |1\rangle \\
X\,|1\rangle &= |0\rangle
\end{aligned}
$$

Definition 2.3 (Phase flip error)**.** *A phase flip error is described by the unitary*

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

which acts on the computational basis as

$$Z|0\rangle = |0\rangle$$
$$Z|1\rangle = -|1\rangle$$

Observation 2.4. *Phase flip error acts like a bit flip error in the Hadamard basis since $Z = HXH$. We check by direct computation.*

$$Z|+\rangle = Z\frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |-\rangle$$
$$Z|-\rangle = Z\frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle$$

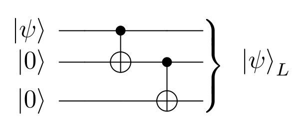
2.3 Fixing bit flip errors

We now describe our first quantum code to correct bit flip errors.

Definition 2.5 (3-qubit bit flip code)**.** *We define the code on the standard basis, and then extend via linearity.*

$$|0\rangle \to |000\rangle$$
$$|1\rangle \to |111\rangle$$
$$|\psi\rangle = a|0\rangle + b|1\rangle \to a|000\rangle + b|111\rangle = |\psi\rangle_L$$

We refer to the input state $|\psi\rangle$ as *physical qubits* and the output state $|\psi\rangle_L$ as *logical qubits*. To familarize us with quantum circuits, here is a circuit which implements this code:



The gate used above is a CNOT gate (Definition 1.5).

We encode $|\psi\rangle = a|0\rangle + b|1\rangle$, and send the logical qubits $|\psi\rangle_L = a|000\rangle + b|111\rangle$ through the quantum channel. Now, suppose there is a bit flip error, say on the second qubit.

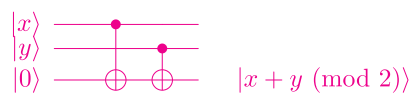
$$(I \otimes X \otimes I) \cdot |\psi\rangle_L = a|010\rangle + b|101\rangle = |\widetilde{\psi}\rangle_L$$

To detect a bit flip error, a natural strategy could be to measure the 3 qubits in the standard basis. This measurement gives string 010 with probability $|a|^2$, and string 101 with probability $|b|^2$. Using this, we can detect that a bit flip error occurred. However, in the process, we have collapsed our state $|\widetilde{\psi}\rangle_L$, and therefore, we can not recover the original state.

The issue is that we have measured too much. We should try to measure only “Has a bit flip error occurred, and on which qubits?” i.e. (1) Measure parity of first and second qubits, and (2) Measure parity of second and third qubits.

Both parities are 1 and 1, for both $|010\rangle$ and $|101\rangle$, so measuring it does not collapse the state. In other words, we perform the following circuit on $|\widetilde{\psi}\rangle_L$.

Scribe Task: Draw a circuit which computes the parity of 1st and 2nd qubit, and parity of 2nd and 3rd qubit. To begin, here is a circuit which computes parity of 1st and 2nd qubit.



For now, the circuit acts on $|\widetilde{\psi}\rangle_L = a\,|010\rangle + b\,|101\rangle$ as follows:

$$\begin{aligned} |\widetilde{\psi}\rangle_L \otimes |00\rangle &= a\,|010\rangle \otimes |00\rangle + b\,|101\rangle \otimes |00\rangle \\ &\xrightarrow[\text{parity on 2 ancilla}]{\text{circuit computing}} a\,|010\rangle \otimes |11\rangle + b\,|101\rangle \otimes |11\rangle \\ &= |\widetilde{\psi}\rangle_L \otimes |11\rangle \end{aligned}$$

The circuit measures the last two registers, which results in the measurement outcome 11. This is called syndrome, and it gives the required information about the error, i.e. in this case, a bit flip occured on the second qubit.

Syndrome	Error
00	no error
01	bit flip on 3rd qubit
10	bit flip on 1rd qubit
11	bit flip on 2rd qubit

In our case, we can fix the error by applying second bit flip.

$$(I \otimes X \otimes I) \cdot |\widetilde{\psi}\rangle_L = |\psi\rangle_L$$

Note that this code can not fix two bit flip errors. Moreover, it can not fix phase flip errors.

$$
\begin{aligned}
(Z \otimes I \otimes I) \cdot |\psi\rangle_L &= (Z \otimes I \otimes I)(a\,|000\rangle + b\,|111\rangle) \\
&= a\,|000\rangle - b\,|111\rangle
\end{aligned}
$$

We can not even detect this error because it is an encoding of state $a\,|0\rangle - b\,|1\rangle$.

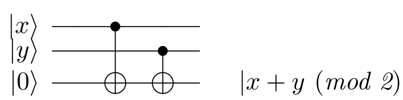
Observation 2.6 (A way to measure parity)**.** *We can measure "parity of 1st and 2nd qubit" and "parity of 2nd and 3rd qubit" by measurements of observables* Z_1Z_2 *and* Z_2Z_3*. Here the notation* Z_iZ_j *means Z gate applies on ith and jth qubit, and I on other qubits.*

In our case, Z_1Z_2 *has the spectral decomposition*

$$
Z_1Z_2 = (|00\rangle\langle 00| + |11\rangle\langle 11|) \otimes I - (|01\rangle\langle 01| + |10\rangle\langle 10|) \otimes I
$$

Notice how the eigenvalues characterize the parity, and therefore, measuring observable Z_1Z_2 *on* $|\widetilde{\psi}\rangle_L = a\,|010\rangle + b\,|101\rangle$ *does not change the state, and outputs* -1 *and identifies the syndrome.*

We need an extra ancilla qubit to transform this measurement into a unitary operation. Here is a circuit which computes parity of two qubits.



2.4 Fixing phase errors

The main observation to fix phase errors is that X and Z errors are switched by Hadamard transform

$$
X = HZH
$$

Definition 2.7 (3-qubit phase flip code)**.** *We define the code on the standard basis, and then extend via linearity.*

$$
\begin{aligned}
|0\rangle &\to |{+}{+}{+}\rangle \\
|1\rangle &\to |{-}{-}{-}\rangle \\
|\psi\rangle = a\,|0\rangle + b\,|1\rangle &\to a\,|{+}{+}{+}\rangle + b\,|{-}{-}{-}\rangle = |\psi\rangle_L
\end{aligned}
$$

Let's see what happens when there is a phase flip error, say on second qubit

$$
Z_2\,|\psi\rangle_L = a\,|{+}{-}{+}\rangle + b\,|{-}{+}{-}\rangle = |\widetilde{\psi}\rangle_L
$$

Now, we can locate the error just like before, by first transforming to $|0\rangle, |1\rangle$

bases. The circuit acts on $|\widetilde{\psi}\rangle_L = a\,|+-+\rangle + b\,|-+-\rangle$ as follows:

$$\begin{aligned}
|\widetilde{\psi}\rangle_L \otimes |00\rangle &= a\,|+-+\rangle \otimes |00\rangle + b\,|-+-\rangle \otimes |00\rangle \\
&\xrightarrow[\text{first 3 qubits}]{\text{Hadamard on}} a\,|010\rangle \otimes |00\rangle + b\,|101\rangle \otimes |00\rangle \\
&\xrightarrow[\text{parity on 2 ancilla}]{\text{circuit computing}} a\,|010\rangle \otimes |11\rangle + b\,|101\rangle \otimes |11\rangle \\
&\xrightarrow[\text{first 3 qubits}]{\text{Hadamard on}} |\widetilde{\psi}\rangle_L \otimes |11\rangle
\end{aligned}$$

In other words, the syndrome measurement can be performed by measuring the observable $H^{\otimes 3} Z_1 Z_2 H^{\otimes 3} = X_1 X_2$ and $H^{\otimes 3} Z_2 Z_3 H^{\otimes 3} = X_2 X_3$. In our case, we can fix the error by applying phase flip to the second qubit.

2.5 9 qubit Shor code

Definition 2.8 (9-qubit Shor code)**.** *We define the code on the standard basis, and then extend via linearity.*

$$\begin{aligned}
|0\rangle &\to \left(\frac{|000\rangle + |111\rangle}{\sqrt{2}}\right)^{\otimes 3} \stackrel{\text{def}}{=} |0\rangle_L \\
|1\rangle &\to \left(\frac{|000\rangle + |111\rangle}{\sqrt{2}}\right)^{\otimes 3} \stackrel{\text{def}}{=} |1\rangle_L \\
|\psi\rangle = a\,|0\rangle + b\,|1\rangle &\to a\,|0\rangle_L + b\,|1\rangle_L = |\psi\rangle_L
\end{aligned}$$

Now, suppose a bit flip error occurs on the second qubit of the first block

$$\begin{aligned}
I \otimes X \otimes I^{\otimes 7}\,|\psi\rangle_L &= (I \otimes X \otimes I^{\otimes 7}) \cdot (a\,|0\rangle_L + b\,|1\rangle_L) \\
&= a\left(\frac{|010\rangle + |101\rangle}{\sqrt{2}}\right) \cdot \left(\frac{|000\rangle + |111\rangle}{\sqrt{2}}\right)^{\otimes 2} \\
&+ b\left(\frac{|010\rangle - |101\rangle}{\sqrt{2}}\right) \cdot \left(\frac{|000\rangle - |111\rangle}{\sqrt{2}}\right)^{\otimes 2} \\
&= |\widetilde{\psi}\rangle_L
\end{aligned}$$

We can now apply the circuit from bit flip detection on first 3 qubits to identify the error. In other words, we measure $Z_1 Z_2$ and $Z_2 Z_3$, and in our case, it will give measurement outcome 11.

Instead, if we had a phase flip error occur on the second qubit of the first block

$$
\begin{aligned}
I \otimes Z \otimes I^{\otimes 7} |\psi\rangle_L &= (I \otimes X \otimes I^{\otimes 7}) \cdot (a |0\rangle_L + b |1\rangle_L) \\
&= a \left(\frac{|000\rangle - |111\rangle}{\sqrt{2}} \right) \cdot \left(\frac{|000\rangle + |111\rangle}{\sqrt{2}} \right)^{\otimes 2} \\
&+ b \left(\frac{|000\rangle + |111\rangle}{\sqrt{2}} \right) \cdot \left(\frac{|000\rangle - |111\rangle}{\sqrt{2}} \right)^{\otimes 2} \\
&= |\widetilde{\psi}\rangle_L
\end{aligned}
$$

To identify the phase flip error, we measure $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$. $X_1X_2X_3$ looks at the first block and returns $+1$ for $|000\rangle + |111\rangle$ and -1 for $|000\rangle - |111\rangle$. Similarly $X_4X_5X_6$ acts on the second block and therefore, $X_1X_2X_3X_4X_5X_6$ compares the sign of first two blocks and returns 1 if they are same and -1 otherwise.