One Dimensional Random Variable

A random variable is a function ~~to~~ that assigns ~~the~~ a real number $X(s) = x$ to every element $s \in S$, corresponding to random experiment.

Range space (R_X): is the set of all possible values of x.

Eg: suppose we toss a pair of coins then $S = \{HH, TH, HT, TT\}$

We define: $X \to$ no. of head appear

$$X(H,H) = 2$$
$$X(HT) = X(TH) = 1$$
$$X(TT) = 0$$

Now Range space, $R_X = \{0, 1, 2\}$.

Two types of RV:- ① Discrete RV ② continuous RV

* Discrete RV:-

If X is a RV which takes finite number of or countably infinite number of values, then X is called discrete RV.

→ Prob. mass func. (Pmf) / Prob. func.:-

If X is discrete RV which can take the values $x_1, x_2, \ldots x_n$ such that $P(X = x_i) = P(x_i) = p_i$ then p_i is called pmf, provided the following cond^n satisfy,

① $p_i \geq 0$

② $\sum_{i=1}^{n} p_i = 1$

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Continuous RV :-

If X is a RV which takes all the values in an interval, then X is called CRV.

→ **Prob. density func. (pdf) :-**

If X is CRV, if f a func. $f(x)$ called pdf provided $f(x)$ satisfy the following cond^n :-

(1) $f(x) \geq 0 \quad x \in R_x$

(2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(3) for any a, b $\exists$ $-\infty < a < x < b < \infty$

$$P(a < x < b) = \int_a^b f(x)\,dx.$$

Cummulative Distribution Function (cdf)

If X is RV (discrete or continuous) then $P(X \leq x)$ is called cdf and is denoted by

(i) Discrete $F(x) = \sum_{x_j \leq x} P(x_j)$

(ii) continuous : $F(x) = \int_{-\infty}^{x} f(x)\,dx$

$= P(-\infty < X \leq x)$.

* Properties :-

1. $F(x)$ is non-decreasing fn. of x.

$x_1 < x_2$, $F(x_1) \leq F(x_2)$

2. $F(-\infty) = 0$, $F(\infty) = 1$

3. $$P(a < x < b) = \int_a^b f(x)\,dx$$

$$= \int_{-\infty}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx$$

$$= F(b) - F(a)$$

4. $f(x) = \dfrac{d}{dx} F(x)$

Q. A RV x has prob. func. $P(x = k) = \dfrac{c}{2^k}$; $k = 0, 1, 2 \ldots$ Then find

i) value of c

ii) ~~prob. of~~ $P(x \geq 5)$

iii) $P(x < 1/2)$

Since $\sum_{k=0}^{\infty} P(x = k) = 1 \Rightarrow \sum_{k=0}^{\infty} c/2^k = 1$

$$\frac{c}{1} + \frac{c}{2} + \frac{c}{4}$$

$$c\left(1 + \frac{1}{2} + \frac{1}{4} \ldots\right) = 1$$

$$c\left(\frac{1}{1 - 1/2}\right) = 1$$

$2c = 1 \Rightarrow c = 1/2$

$$\therefore P(x=k) = \frac{c}{2^k} = \frac{1}{2^{k+1}}$$

ii) $$P(x \geq 5) = \frac{1}{2^6} + \frac{1}{2^7} + \dots$$

$$= \frac{1}{2^6}\left(1 + \frac{1}{2} + \dots\right)$$

$$= \frac{1}{2^6}\left(\frac{1}{1-1/2}\right)$$

$$= \frac{2}{2^6}$$

$$= \frac{1}{2^5} = \frac{1}{32}$$

iii) $$P(x < 1/2) = P(x=0) = \frac{1}{2}$$

Q. If x be a R.V with pdf

$$f(x) = \begin{cases} kx^3, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

find value of (i) k

(ii) $P(1/4 < x < 3/4)$

(iii) $P(x > 1/2)$

(iv) $P(x < 0.8)$

(i) $$\cancel{\int_{-\infty}^{\infty} kx^3\,dx} \quad \int_0^1 kx^3\,dx = 1$$

$$\frac{k}{4}\left[x^4\right]_0^1 = 1$$

$$\frac{k}{4}(1) = 1 \Rightarrow k = 4$$

$$f(x) = \begin{cases} 4x^3 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$P(1/4 < x < 3/4) = \int_{1/4}^{3/4} 4x^3 \, dx$$

$$= \left[x^4\right]_{1/4}^{3/4}$$

$$= \frac{5}{16}$$

(iii) $$P(x > 1/2) = \int_{1/2}^{\infty} 4x^3 \, dx$$

$$= \int_{1/2}^{1} 4x^3 \, dx + \int_{1}^{\infty} 0 \, dx$$

$$= \left[x^4\right]_{1/2}^{1}$$

$$= 1 - (1/2)^4$$

$$= 1 - \frac{1}{16} = \frac{15}{16}$$

(iv) $$P(x < 0.8) = \int_{-\infty}^{0} 0 + \int_{0}^{0.8} 4x^3 \, dx$$

$$= \left[x^4\right]_{0}^{0.8}$$

$$= \frac{256}{625}$$

Q. Suppose that random variable x with pdf $f(x) = \begin{cases} 2x & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

find
(i) cdf
(ii) $F(0.5)$

(i) $F(x) = \int_{-\infty}^{x} f(x)\, dx$

for $x<0$
$F(x) = \int_{-\infty}^{x} f(x)\, dx = \int_{-\infty}^{x} 0\, dx = 0.$

for $0<x<1$

$F(x) = \int_{-\infty}^{0} f(x)\, dx + \int_{0}^{x} f(x)\, dx.$

$= 0 + \int_{0}^{x} 2x\, dx.$

$= [x^2]_0^x$

$= x^2$

for $x \geq 1$

$F(x) = \int_{-\infty}^{x} f(x)\, dx.$

$= \int_{-\infty}^{0} f(x)\, dx + \int_{0}^{1} f(x)\, dx + \int_{1}^{x} f(x)\, dx.$

$= 0 + \int_{0}^{1} 2x\, dx + \int_{1}^{x} 0\, dx.$

$= 1$

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

$$\frac{d(x^2)}{dx} = 2x \rightarrow \text{pdf}$$

$$F(0.5) = \int_{-\infty}^{0.5} f(x)\, dx$$

$$= \int_{-\infty}^{0} 0\, dx + \int_{0}^{0.5} 2x\, dx$$

$$= [x^2]_0^{0.5}$$

$$= 0.25$$

Q. A coin is known to come up 3 times head as often as tails. The coin is tossed 3 times. Let x denotes no. of heads that appear. Write the prob of dist. of x and also cdf.

$P(H) = 3/4$, $P(T) = 1/4$

S = { HHH, HHT, HTH, THH, HTT, THT, TTH, TTT }

X: no. of head appear

$R_X = \{0, 1, 2, 3\}$

$P(X=0) = P(TTT)$

$= (P(T))^3$

$= \left(\frac{1}{4}\right)^3$

X	0	1	2	3
P(X)	1/64	9/64	27/64	27/64

$$P(X=1) = P(HTT) + P(THT) + P(TTH)$$
$$= 3 \times \frac{3}{4} \times \left(\frac{1}{4}\right)^2$$
$$= \frac{9}{64}$$

$$P(X=2) = P(HHT) + P(HTH) + P(THH)$$
$$= 3 \times \left(\frac{3}{4}\right)^2 \times \frac{1}{4}$$
$$= \frac{27}{64}$$

$$P(x=3) = P(HHH)$$
$$= \left(\frac{3}{4}\right)^3$$
$$= \frac{27}{64}$$

cdf
$$F(a) = \sum_{x_j \le a} P(x = x_j)$$

$$F(x) = \begin{cases} 0 & x < 0 \\ 1/64, & 0 \le x < 1 \\ 1/64 + 9/64 = 10/64 & 1 \le x < 2 \\ 10/64 + 27/64 = 37/64 & 2 \le x < 3 \\ \frac{37}{64} + \frac{27}{64} = 1 & x \ge 3 \end{cases}$$

* Mean

Discrete :-

$$E(x) = \mu = \sum_i x_i \, p(x_i)$$

If $g(x)$ is a func. of ~~E(x)~~ x then

$$E(g(x)) = \sum_i g(x_i) \, p(x_i)$$

Continuous :-

$$E(x) = \int_{-\infty}^{\infty} x f(x) \, dx.$$

If $h(x)$ is func. of x then

$$E(h(x)) = \int_{-\infty}^{\infty} h(x) f(x) \, dx.$$

* Variance

Variance is denoted by $V(x)$ or σ_x^2 and defined as $V(x) = E(x - E(x))^2 = E(x^2) - (E(x))^2$

→ the square root of $V(x)$ is called standard deviation of x.

* Properties

→ $E(c) = c$

→ $E(ax+b) = a E(x) + b$

→ $E(E(x)) = E(x)$

→ $V(c) = 0$

→ $V(cx) = c^2 V(x)$

Proof:

$V(x) = E(x^2) - (E(x))^2$

~~$= c^2 E(x^2) - (c E(x))^2$~~

$V(cx) = E(c^2 x^2) - (E(cx))^2$

$= c^2 E(x^2) - (c E(x))^2$

$= c^2 (E(x^2) - (E(x))^2)$

$= c^2 V(x)$

Q. A RV x has following prob. distribution.

x	-2	-1	0	1	2	3
$P(x)$	0.1	K	0.2	2K	0.3	3K

find

(i) K

(ii) $P(-2 < x < 2)$

(iii) cdf of x

(iv) mean and variance of x

(v) mode & median

1. $\sum P(x) = 1$

$0.6 + 6K = 1$

$6K = 0.4$

$K = 1/15$

(ii) $P(-2 < x < 2) = K + 0.2 + 2K = 3K + 0.2$

$= \frac{3}{15} + 0.2$

$= 1/5 + 1/5$

$\frac{2}{5}$

(iii) $F(x) = \begin{cases} 0 & x < -2 \\ 0.1 & -2 \leq x < -1 \\ 1/6 & -1 \leq x < 0 \\ 11/30 & 0 \leq x < 1 \\ 1/2 & 1 \leq x < 2 \\ 4/5 & 2 \leq x < 3 \\ 1 & 3 \leq x \end{cases}$

(iv) $\mu = \sum x_i P(x_i)$

$= \frac{16}{15}$

~~Variance~~ $\cancel{E(x^2)}$ $\cancel{E(x^2) = \sum x_i^2 P(x}$

$E(x^2) = \sum_i x_i^2 P(x_i)$

$= 3.6 = \frac{18}{5}$

$V(x) = E(x^2) - (E(x))^2$

$= \frac{554}{225}$

(v) mode - ~~...~~ value of (x) for which value of $P(x)$ is max.

$\therefore$ mode = 2

median - $\frac{x_1 + x_2}{2}$ where x_1 is largest value of x such that $P(x \leq x_1) \leq 0.5$

$x_2 \rightarrow$ smallest value of x such that $P(x \leq x_2) \geq 0.5$

Q. RV X has pdf $f(x) = \begin{cases} k(1-x^2), & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$

find:

(i) K

(ii) $P(0.1<x<0.2)$

(iii) mean and variance

(iv) mode, median

i) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_{0}^{1} K(1-x^2)\,dx = 1$

$\frac{2}{3} K = 1$

$K = 3/2.$

(ii) $P(0.1<x<0.2)$

$\int_{0.1}^{0.2} \frac{3}{2}(1-x^2)\,dx = \frac{293}{2000} = 0.1465$

(iii) $E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

$= \int_{0}^{1} \frac{3x}{2}(1-x^2)\,dx.$

$= 0.375$

$= \frac{3}{8}$

$$E(x^2) = \int_0^1 \frac{3x^2}{2}(1-x^2)\,dx = 0.2 = \frac{1}{5}$$

$$V(x) = E(x^2) - (E(x))^2$$

$$= \frac{19}{320} = 0.059375$$

(iv) $\frac{d\,f(x)}{dx} = 0.$

$$\frac{d}{dx}\left(\frac{3}{2}(1-x^2)\right).$$

$$f'(x) = -3x = 0$$

$$x = 0.$$

$$f''x = -3 < 0$$

$x = 0$ is mode.

median, m: $\int_{-\infty}^{m} f(x)\,dx = 1/2$ or $\int_{m}^{\infty} f(x)\,dx = 1/2.$

$$\int_{-\infty}^{m} f(x)\,dx = 1/2.$$

$$\int_0^m \frac{3}{2}(1-x^2)\,dx = 1/2.$$

$$\frac{3}{2}\left[x - \frac{x^3}{3}\right]_0^m = \frac{1}{2}.$$

$$m - \frac{m^3}{3} = \frac{1}{3} \qquad \therefore m = 0.347.$$

$$3m - m^3 = 1$$

$$m^3 - 3m + 1 = 0$$

$$m = \underset{\times}{1.532},\ \underset{\times}{-1.87},\ 0.347$$

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Q. A CRV X has pdf $f(x) = k x^2 e^{-x}$, $x \geq 0$ find mean and standard deviation also find $P(|x| < 1)$

$$E(x) = \int_0^\infty x f(x)$$

$$\int_0^\infty k x^2 e^{-x} dx = 1$$

$$\left\{ \int_0^\infty e^{-x} x^{n-1} dx = \Gamma(n) \quad \Gamma(n+1) = n! \right\}$$

$$k \int_0^\infty x^2 e^{-x} dx = 1$$

$$k \Gamma(3) = 1$$

$$k (2) = 1$$

$$k = 1/2.$$

$$E(x) = \int_0^\infty x \frac{x^2 e^{-x}}{2} dx.$$

$$= \frac{1}{2} \int_0^\infty x^3 e^{-x} dx.$$

$$= \frac{1}{2} \Gamma(4)$$

$$= \frac{1}{2} (3!) = \frac{6}{2} = 3.$$

$$E(x^2) = \int_0^\infty x^2 \frac{x^2 e^{-x}}{2} dx.$$

$$= \frac{1}{2} \int_0^\infty x^4 e^{-x} dx.$$

$$= \frac{1}{2} \Gamma(5) = \frac{1}{2} 4! = \frac{24}{2} = 12$$

$\int uv\,dx = \cancel{u\int v\,dx - \int u'}$ $\quad \int v = v_1$

$uv_1 - u'v_2 + u''v_3 - \ldots$ $\quad \int v_1 = v_2$

~~std dev. = $\sqrt{12}$ = 3.4641~~

$V(x) = E(x^2) - (E(x))^2$

$= 12 - (3)^2$

$= 12 - 9$

$= 3.$

$\sigma = \sqrt{3}$

$P(|x| \le 1) = P(-1 \le x \le 1)$

$\int_{-1}^{1} f(x)\,dx = \cancel{\int_{-1}^{0} f(x)\,dx} + \int_{0}^{1} f(x)\,dx$

$= \frac{1}{2}\int_{0}^{1} x^2 e^{-x}\,dx.$

$= 0.0803$

Q. A box contains 12 items of which 4 are defective. A sample of 3 items is selected from the box. Let x denotes the no. of defective items in the sample. Find the prob. distribution of x. Also determine mean, standard dev., mode & median.

x: no. of def. items

$R_x = \{0, 1, 2, 3\}$

$P(X=0) = \frac{{}^{8}C_3}{{}^{12}C_3} = \frac{14}{55}$

~~$P(X=1) = \frac{{}^{4}C_1}{{}^{12}C_3} = \frac{1}{55}$~~

~~$P(X=2) = \frac{{}^{10}C_3}{{}^{12}C_3} = \frac{6}{11}$~~

$$P(x=1) = \frac{{}^4C_1 \times {}^8C_2}{{}^{12}C_3} = \frac{28}{55}$$

$$P(x=2) = \frac{{}^4C_2 \times {}^8C_1}{{}^{12}C_3} = \frac{12}{55}$$

$$P(x=3) = \frac{{}^4C_3}{{}^{12}C_3} = \frac{1}{55}$$

x	0	1	2	3
$P(x)$	14/55	28/55	12/55	1/55
	0.2545	0.5	0.2182	0.0182
	14/55	42/55	54/55	1

Mean $= E(x) = \sum x\,p(x)$

$= 1$

$E(x^2) = \sum x^2 p x$

$= \frac{17}{11}$

$V(x) = E(x^2) - (E(x))^2 = \frac{6}{11}$

std dev $= \sqrt{\frac{6}{11}}$

mode $= 1 = x$

median $= \frac{x_1 + x_2}{2} = \frac{0+1}{2} = \frac{1}{2}$

Q Given a CRV X has pdf $f(x) = \begin{cases} 6x(1-x), & 0\le x\le 1 \\ 0, & \text{otherwise} \end{cases}$

find mean, mode & median

~~f(x) =~~

$E(x) = \int_{-\infty}^{\infty} x f x \, dx$

$= \int_0^1 6x^2(1-x)\, dx.$

$= \frac{1}{2}.$

$E(x^2) = \int_0^1 x^2 f(x)$

$= \int_0^1 6x^3(1-x)\, dx$

$= \frac{3}{10}$

~~f(x)~~

$f(x)' = \frac{d}{dx}(6x - 6x^2)$

$\Rightarrow 6 - 12x = 0$

$\Rightarrow x = \frac{1}{2}$

$f(x)'' = -12 < 0.$

mode $= x = \frac{1}{2}.$

median, $m = \int_{-\infty}^{m} 6x(1-x)\, dx = \frac{1}{2}.$

$= 6\int_0^m x - x^2\, dx = \frac{1}{2} \Rightarrow \frac{x^2}{2} - \frac{x^3}{3} = \frac{1}{12}$

$\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{12} = 0$

$m = 1.3660,\ -0.8660,\ 0.5$

H.W

Q. Let X be a CRV with pdf $f(x) = y_0 e^{-|x|}$, $-\infty < x < \infty$

(i) P.T. $y_0 = 1/2$.

(ii) variance.

$$f(x) = \begin{cases} y_0 e^{-x}, & 0 < x < \infty \\ y_0 e^{x}, & -\infty < x < 0 \end{cases}$$

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* Markov's Inequality

If x is a RV assuming non-ve values. Then $P(x \geq a) \leq \frac{E(x)}{a}$, $a \neq 0$, $a > 0$

* Chebyshev's Inequality

Let x be a RV with mean $E(x) = \mu$ and $SD = \sigma$ then for any $K > 0$ we have

$$P(\mu - K\sigma < x < \mu + K\sigma) \geq 1 - \frac{1}{K^2}$$

$$= P(|x-\mu| < K\sigma) \geq 1 - \frac{1}{K^2}$$

$$= P(|x-\mu| \geq K\sigma) \leq \frac{1}{K^2}$$

* Use Chebyshev's inequality to find
(i) $P(|x-\mu| < 3)$
(ii) $P(-2 < x < 8)$ $E(x) = 3$
(iii) $P(|x-\mu| \geq 3)$ $E(x^2) = 13$

$$V(x) = E(x^2) - (E(x))^2$$
$$= 13 - 9$$
$$\sigma^2 = 4$$
$$\sigma = 2$$

(i) $P(|x-\mu| \leq K\sigma) \geq 1 - \frac{1}{K^2}$

$K\sigma = 3$
$K(2) = 3$
$K = 3/2$

$$P(|x-\mu| < 3) \geq 1 - \frac{4}{9}$$
$$= 5/9$$

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(ii) $P(\mu - k\sigma < x < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$

$\mu - k\sigma = -2$

$3 - 2k = -2$

$2k = 5$

$k = 5/2.$

$\therefore P(-2 < x < 8) \geq 1 - \frac{1}{k^2} = 1 - \frac{4}{25}$

$= \frac{21}{25}.$

Q: Suppose that it is known that the no. of items produced by a factory during a week is a RV with mean 50.

(1) What can be said about the prob. that this week's production which exceeds 75

(2) if the variance of a weeks produced is known to be 25, then what be said abt prob that this week's production will be ~~forto~~ 40 or 60.

$E(x) = \mu = 50$ $\quad a = 75$ $\quad V(x) = 25$

$\sigma = 5$

$P(x \geq 75) \leq \frac{E(x)}{a}$

$P(x \geq 75) \leq \frac{50}{75}$

$P(x \geq 75) \leq \frac{2}{3}$

$$P(40 < x < 60) \geq 1 - \frac{1}{K^2}$$

$$\mu - K\sigma = 40$$

$$50 - K(5) = 40$$

$$10 = 5K$$

$$K = 2.$$

$$P(40 < x < 60) \geq 1 - \frac{1}{4}$$

$$P(40 < x < 60) \geq 3/4$$

2-D Random Variable

Let S be the sample space assoc. with a RE.
Let $X = X(s) = x$, $Y = Y(s) = y$ be 2 func. each assigning a real number to each outcome $s \in S$ then (x, y) is called 2-D RV.

* If possible values of (x, y) are finite or countably infinite then x, y is called 2D discrete RV.

* If (x, y) can assume all the values in a specified region say R in xy plane, then (x, y) is called continuous 2-D RV.

* Joint Pmf.
$P(x = x_i, y = y_j) = P_{ij}$
• $P_{ij} \geq 0$
• $\sum_j \sum_i P_{ij} = 1$

If (x, y) is 2D discrete RV ~~such that~~ then P_{ij} is called joint Pmf if

* Joint Pdf.

If (x, y) is 2D continuous RV, if ∃ a func. $f(x, y)$ called joint pdf provided $f(x, y)$ ~~st~~ satisfy the following cond-n.

① $f(x, y) \geq 0 \quad \forall (x, y) \in R$

② $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

③ $P(a < x < b,\ c < y < d) = \int_{y=c}^{d}\int_{x=a}^{b} f(x,y)\,dx\,dy$

* cdf

If (x,y) is DRV, $F(x,y) = \sum_{y_j \le y}\sum_{x_i \le x} P(x_i, y_j)$

If (x,y) is CRV, $F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(x,y)\,dx\,dy$.

$\frac{\partial^2}{\partial x \partial y} F(x,y) = f(x,y)$

* Properties

→ $F(-\infty, y) = F(x, -\infty) = 0$

→ $F(\infty, \infty) = 1$

→ $P(a < x < b,\ Y \le y) = F(b,y) - F(a,y)$

→ $P(X \le x,\ c < Y < d) = F(x,d) - F(x,c)$

→ $P(a < x < b,\ c < y < d) = F(b,d) - F(b,c) - F(a,d) + F(a,c)$.

* Marginal ~~Pdf~~ ~~Cdf~~ Pmf

X \ Y	y_1	y_2	...	y_m	
x_1	P_{11}	P_{12}	...	P_{1m}	$f(x_1)$
x_2	P_{21}	P_{22}		P_{2m}	$f(x_2)$
⋮	⋮	⋮			⋮
x_n	P_{n1}	P_{n2}		P_{nm}	$f(x_n)$
	$g(y_1)$	$g(y_2)$	...	$g(y_m)$	1

~~Marginal pmf~~

Here x, y are DRV

It is defined as follows,

Marginal pmf of x:

$$p(x_i) = \sum_{j=1}^{m} p_{ij}$$

marginal pmf of y:

$$q(y_j) = \sum_{i=1}^{n} p_{ij}$$

* Marginal pdf.

Let x, y be CRV then,

marginal pdf of x:

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \;;\; -\infty < x < \infty$$

marginal pdf of y:

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \;;\; -\infty < y < \infty$$

* Cond^{n} Prob. distribution

Let (x,y) be DRV, pmf p_{ij} & $p(x_i)$ & $q(y_j)$ marginal pmf of x & y.

$$P(x = x_i \mid y = y_j) = \frac{p_{ij}}{q_j(y_j)} = P(y = y_j)$$

$\therefore P(y = y_j \mid x = x_i) = \dfrac{P_{ij}}{P(x_i) = P(x = x_i)}$

* Cond^n Pdf

If (x, y) are CRV
Pdf $f(x,y)$ & $g(x)$ & $h(y)$ marginal pdf of x & y resp.

Cond^n pdf of x, $g(x|y) = \dfrac{f(x,y)}{h(y)}$, $h(y) > 0$

" " y, $h(y|x) = \dfrac{f(x,y)}{g(x)}$, $g(x) > 0$

Q. Find
$P(x \leq 1)$, $P(y \leq 3)$, $P(x \leq 1, y \leq 3)$
$P(x \leq 1 \mid y \leq 3)$
$P(y \leq 3 \mid x \leq 1)$
$P((x+y) \leq 4)$
for the following prob. distribution.

X \ Y	1	2	3	4	5	6	Row sum
0	0	0	1/32	2/32	2/32	3/32	1/4
1	1/16	1/16	1/8	1/8	1/8	1/8	5/8
2	1/32	1/32	1/64	1/64	0	2/64	1/8
col sum							

i) $P(x \leq 1) = P(x=0) + P(x=1)$
$= P(x=0, y=y_i) + P(x=1, y=y_i)$
$= 7/8$

$$P(y \leq 3) = P(y=1) + P(y=2) + P(y=3)$$

$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64}$$

$$= \frac{23}{64}$$

$$P(x \leq 1, y \leq 3) = \frac{9}{32}$$

$$P(x \leq 1 \mid y \leq 3) = \frac{P(x \leq 1, y \leq 3)}{P(y \leq 3)}$$

$$= \frac{9/32}{23/64} = \frac{18}{23}$$

$$P(y \leq 3 \mid x \leq 1) = \frac{P(x \leq 1, y \leq 3)}{P(x \leq 1)} = \frac{9/32}{7/8}$$

x	y
0	1
0	2
0	3
0	4
1	1
1	2
1	3
2	1
2	2

$$P((x+y) \leq 4) = \frac{3}{32} + \frac{2}{16} + \frac{1}{8} + \frac{2}{32}$$

$$= \frac{13}{32}$$

Q. (X, Y) is 2D CRV

$$f(x,y) = \begin{cases} x^2 + \dfrac{xy}{3}, & 0 \le x \le 1,\ 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

(i) marginal pdf of x & y

(ii) $P(x > 1/2)$ (iii) $P(y < 1)$

(iv) $P((x+y) < 1)$ (v) $P(x > 1/2 \mid y > 1)$

i) $$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_0^2 x^2 + \frac{xy}{3}\, dy$$

$$= x^2 [y]_0^2 + \frac{x}{3} \frac{[y^2]_0^2}{2}$$

$$= 2x^2 + \frac{x}{6}(4)$$

$$= 2x^2 + \frac{2x}{3}, \quad 0 \le x \le 1$$

marginal pdf of y

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$= \int_0^1 x^2 + \frac{xy}{3}\, dx$$

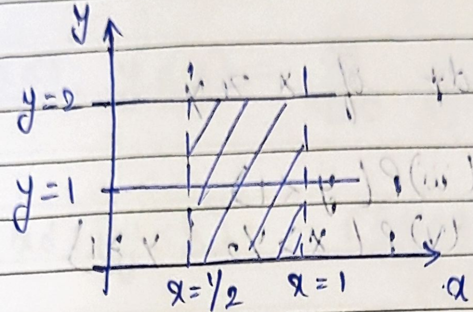
$$= \frac{[x^3]_0^1}{3} + \frac{y}{6}[x^2]_0^1$$

$$= 1/3 + \frac{y}{6}, \quad 0 \le y \le 2$$

$\frac{1}{2} x^2 = 1$

(ii) $\cancel{P(x > \tfrac{1}{2})} = \int_0^2 \int \;\; P(x > \tfrac{1}{2})$

$$P(a \le x \le b,\ c \le y \le d) = \int_a^b \int_c^d f(x,y)\, dy\, dx$$



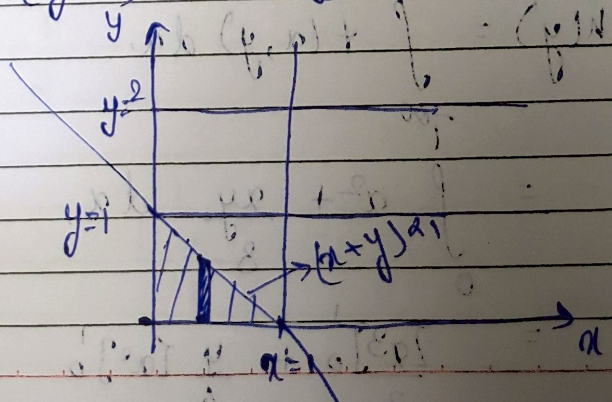
$$P(x > \tfrac{1}{2}) = P(\tfrac{1}{2} < x < 1,\ 0 \le y \le 2)$$

$$= \int_{1/2}^{1} \int_0^2 \left(x^2 + \frac{xy}{3} \right) dy\, dx$$

$$= \int_{1/2}^{1} 2x^2 + \frac{4x}{6}\, dx$$

$$= 5/6$$

(iii) $\cancel{P(y<1)}$ $P((x+y)<1)$



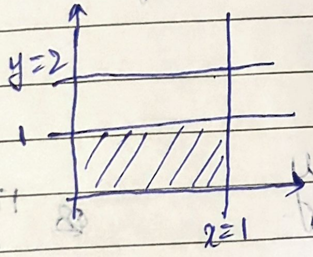
$$\cancel{\int_0^1 \int_0^2 \left(x^2 + \frac{xy}{3}\right)}$$

$$\int_0^1 \int_0^{1-x} \left(x^2 + \frac{xy}{3}\right) dy\, dx.$$

$$\int_0^1 x^2(1-x) + \frac{x}{6}(1-x)^2\, dx.$$

$$= \frac{7}{72}.$$

$P(y<1)$



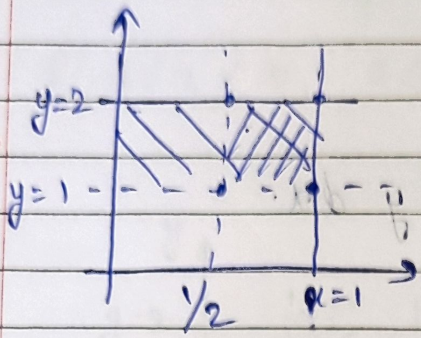
$$P(y<1) = \int_{y=0}^{1} \int_{x=0}^{1} \left(x^2 + \frac{xy}{3}\right) dx\, dy$$

$$= \int_0^1 \frac{[x^3]_0^1}{3} + \frac{y}{6}[x^2]_0^1\, dy$$

$$= \int_0^1 \frac{1}{3} + \frac{y}{6}\, dy$$

$$= \frac{5}{12}$$

$$P(x > 1/2 \mid y > 1) = \frac{P(x > 1/2 \cap y > 1)}{P(y > 1)}$$



$$= \frac{\int_1^2 \int_{x=1/2}^1 x^2 + \frac{xy}{3} \, dx\, dy}{\int_1^2 \int_{x=0}^1 x^2 + \frac{xy}{3} \, dx\, dy}$$

$$\frac{\int_1^2 \frac{[x^3]_{1/2}^1}{3} + \frac{y}{6}[x^2]_{1/2}^1 \, dy}{\int_1^2 \frac{[x^3]_0^1}{3} + \frac{y}{6}[x^2]_0^1 \, dy}$$

$1 - \frac{1}{8}$

$7/8$

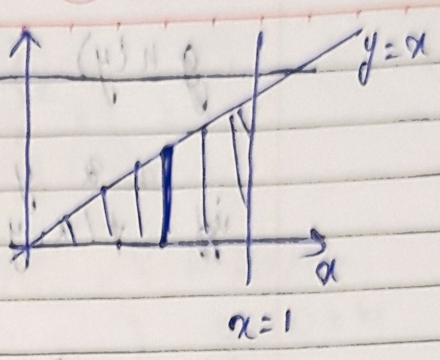
$1 - \frac{1}{4}$

$$\frac{\int_1^2 \frac{7}{8} + \frac{3y}{24} \, dy}{\int_1^2 \frac{1}{3} + \frac{y}{6} \, dy}$$

$$\frac{17/16}{7/12}$$

$$= \frac{23}{28}$$

$P(y<x)$



$$\int_{x=0}^{1}\int_{y=0}^{x}\left(x^2+\frac{xy}{3}\right)dy\,dx$$

$$\int_0^1 x^3 + \frac{x}{6}[y^2]_0^x \, dx$$

$$\int_0^1 x^3 + \frac{x^3}{6}\, dx$$

$$\int_0^1 \frac{7x^3}{6}\, dx.$$

$$\frac{7}{6}\,\frac{(x^4)_0^1}{4} = \frac{7}{24}$$

* Independent RV

Let (x,y) be 2-D ~~drv~~ DRV with joint pmf $p(x_i, y_j)$ and marginal pdf's of x and y $p(x_i)$ & $q(y_j)$ resp. then we say that x & y are independent iff $P(x_i, y_j) = p(x_i).P(y_j)$

$\cancel{f(x,y) = g(x).h(y)}$

Let x, y be a 2D continuous RV with joint pdf $f(x,y)$ & marginal pdf of x & y, $g(x)$ &

$g\ h(y)$ resp. then

iff $\Big[$ x & y and are independent RV
$f(x,y) = g(x) \cdot h(y)$ $\forall x, y$.

Let (x,y) be 2D DRV & joint pmf $p(x_i, y_j)$ & $p(x_i)$ & $q(y_j)$ then Expected value

$$E(x) = \sum_i x_i\, p(x_i)$$

$$E(y) = \sum_j y_j\, q(y_j) = P(y_j)$$

$$E(xy) = \sum x_i\, y_j\, P_{ij}$$

Let (x,y) be 2D CRV with joint pdf & marginal pdf's $g(x)$ & $h(y)$

$$E(x) = \int_{-\infty}^{\infty} x\, g(x)\, dx. \quad \text{where } g(x) = \int_{-\infty}^{\infty} f(x,y)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\, dy\, dx$$

$$E(y) = \int_{-\infty}^{\infty} y\, h(y)\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\, dx\, dy$$

$$E(xy) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$

$V(x) = E(x^2) - (E(x))^2$
$V(y) = E(y^2) - (E(y))^2$

$E(x+y) = E(x) + E(y)$ $\because E(ax+b) = aE(x)+b$

$E(xy) = E(x)E(y)$ if x & y independent RV.

$V(x+y) = V(x) + V(y)$

* Covariance of (x, y) $(\mathrm{cov}(x,y))$.

$$\mathrm{cov}(x,y) = E(xy) - E(x)E(y)$$

If x & y are independent, $\mathrm{cov}(x,y) = 0$

* Co-eff of correlation

$$\rho = \rho_{xy} = \frac{E(xy) - E(x)E(y)}{\sqrt{V(x)V(y)}}$$

$$= \frac{\mathrm{cov}(x,y)}{\sqrt{V(x)\cdot V(y)}} \qquad -1 \le \rho \le 1$$

Q. The marginal pdf of x & y
find
(i) E(x) & E(y)
(ii) cov(x,y)
(iii) σ_x, σ_y
(iv) ρ

x \ y	-4	2	7	Row sum (p(x))
1	1/8	1/4	1/8	1/2
5	1/4	1/8	1/8	1/2
col sum q(y)	3/8	3/8	1/4	1

marginal pmf of x

x	1	5
p(x)	1/2	1/2

marginal pmf

y	-4	2	7
q(y)	3/8	3/8	1/4

i) $E(x) = \sum_i x_i \, p(x_i)$

$= 1\left(\frac{1}{2}\right) + 5\left(\frac{1}{2}\right)$

$= \frac{6}{2} = 3$

$E(y) = \sum y_j \, q(y_j)$

$= -4\left(\frac{3}{8}\right) + 2 \times \left(\frac{3}{8}\right) + \frac{7}{4}$

$= \frac{7}{4} - \frac{6}{8} = 1$

$$E(xy) = \sum_{i,j} x_i y_j p_{ij}$$

$$= \left(1 \times -4 \times \frac{1}{8}\right) + \left(1 \times 2 \times \frac{1}{4}\right) + \left(1 \times 7 \times \frac{1}{8}\right)$$

$$+ \left(5 \times -4 \times \frac{1}{4}\right) + \left(5 \times 2 \times \frac{1}{8}\right) + \left(5 \times 7 \times \frac{1}{8}\right)$$

$$= 3/2$$

(ii) $\operatorname{cov}(x, y) = E(xy) - E(x)E(y)$

$$= \frac{3}{2} - 3 \times 1$$

$$= -3/2.$$

$$E(x^2) = \sum x_i^2\, p(x_i)$$

$$= 1^2 \times \frac{1}{2} + 5^2 \times \frac{1}{2}$$

$$= 13$$

$$E(y^2) = \sum_j y_j^2\, q(y_j) = (-4)^2 \times \frac{3}{8} + \left(2^2 \times \frac{3}{8}\right) + \left(7^2 \times \frac{1}{4}\right)$$

$$= \frac{79}{4}$$

$$V(x) = E(x^2) - (E(x))^2$$
$$= 13 - 3^2$$
$$= 4$$

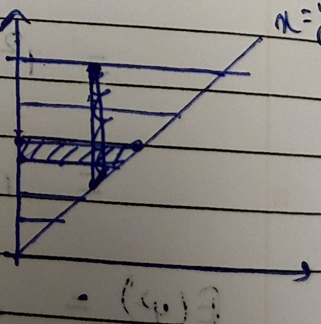
$$V(y) = E(y^2) - (E(y))^2$$
$$= \frac{79}{4} - 1$$
$$= \frac{75}{4}$$

$$\rho = \frac{\text{cov}(x,y)}{\sigma_x \sigma_y} = \frac{-3/2}{\sqrt{4}\sqrt{75/4}}$$

$$= -0.17320$$

Q. Find ~~coeff~~ coeff of corelation of x & y for the following pdf.

$$f(x,y) \begin{cases} 2, & (x,y) \in R \quad R: 0<x<y<1 \\ 0, & \text{otherwise} \end{cases}$$



Marginal pdf of x:

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_{y=x}^{1} 2\, dy = 2[y]_x^1$$

$$= 2(1-x), \quad 0<x<1$$

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{x=0}^{y} 2\,dx$$

$$= 2[x]_0^y = 2y, \quad 0<y<1$$

$$E(x) = \int_{x=0}^{1} x\,g(x)\,dx$$

$$= \int_0^1 (2x - 2x^2)\,dx$$

$$= [x^2]_0^1 - \frac{2[x^3]_0^1}{3}$$

$$= 1 - \frac{2}{3}$$

$$= 1/3$$

$$E(y) = \int_0^1 y\,h(y)\,dy$$

$$= \int_0^1 y(2y)\,dy$$

$$= \left[\frac{2y^3}{3}\right]_0^1$$

$$= 2/3$$

$$E(xy) = \int_0^1\int_0^y xy\cdot(2)\,dx\,dy$$

$$= \int_0^1 y\,[x^2]_0^y\,dy$$

$$= \int_0^1 y^3\,dy = 1/4$$

$\cancel{V(x) = \frac{1}{\sqrt{3}}}$ $\cancel{V(y) = \sqrt{8/3}}$

$\cancel{\rho = \frac{E(xy) - E(x)\,E(y)}{\sqrt{V(x)\,V(y)}}}$

$$E(x^2) = \int_{x=0}^{1} x^2 (2-2x)\,dx.$$

$$= 1/6$$

$$E(y^2) = \int_0^1 y^2 (2y)\,dy$$

$$= \frac{2[y^4]_0^1}{4}$$

$$= 1/2.$$

$$V(x) = E(x^2) - [E(x)]^2$$
$$= 1/6 - 1/3^2$$
$$= 1/18$$

$$V(y) = E(y^2) - [E(y)]^2$$
$$= \frac{1}{18}$$

$$\text{Cov} = E(xy) - E(x)\,Ey$$
$$= 1/4 - \frac{1}{8} \cdot \frac{2}{3}$$
$$= 1/36$$

$$\rho = \frac{1/36}{\sqrt{1/18 \cdot 1/18}} = \frac{1/36}{1/18} = \frac{1}{2}$$

Q. 5 A fair coin is tossed 3 times. Let x = 0 or 1 according as H or T occurs on the first toss. y: no. of heads. Find the joint prob. distribution and find cov of x & y.

$P(x=0, y=0) = 0$

$P(x=0, y=1) = P(HTT) = 1/8$

$P(x=0, y=2) = P(HHT) \text{ or } P(HTH) = 2/8$

$P(x=0, y=3) = P(HHH) = 1/8$

$P(x=1, y=0) = P(TTT) = 1/8$

$P(x=1, y=1) = P(TTH \text{ or } THT) = 2/8$

$P(x=1, y=2) = P(THH) = 1/8$

$P(x=1, y=3) = 0$

x \ y	0	1	2	3	p(x)
0	0	1/8	2/8	1/8	1/2
1	1/8	2/8	1/8	0	1/2
q(y)	1/8	3/8	3/8	1/8	1

$$E(x) = \sum_i x_i \, p(x_i) = 0 \cdot 1/2 + 1 \cdot 1/2 = \frac{1}{2}$$

$$E(y) = \sum y_j \, q(y_j) = 0 \cdot 1/8 + 1 \cdot 3/8 + 2 \cdot 3/8 + 3/8$$

$$= \frac{12}{8} = \frac{3}{2}$$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$$\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx$$

$$E(xy) = \sum_{i,j} x_i y_j P(x_i, y_j)$$

$$= 0 + 1.0.\tfrac{1}{8} + 1.1.\tfrac{2}{8} + 1.2 \times \tfrac{1}{8} + 0$$

$$= \frac{4}{8} = \frac{1}{2}$$

$$\mathrm{cov}(x,y) = E(xy) - E(x)E(y)$$

$$= \tfrac{1}{2} - \tfrac{1}{2}(\tfrac{3}{2})$$

$$= \frac{1}{2} - \frac{3}{4}$$

$$= -\frac{1}{4}$$

Q. joint pdf of RV (x, y) is given by

$$f(x,y) = \begin{cases} Kxy\, e^{-(x^2+y^2)}, & x \geq 0,\ y \geq 0 \\ 0 \end{cases}$$

find value of K and PT x & y are independent

$$\int_{y=0}^\infty \int_{x=0}^\infty Kxy\, e^{-(x^2+y^2)}\, dx\, dy = 1$$

$$\Rightarrow K \int_{y=0}^\infty y e^{-y^2} dy \cdot \int_{x=0}^\infty x \cdot e^{-x^2} dx = 1$$

$$\Rightarrow K \cdot \frac{1}{2}(\Gamma(1)) \times \frac{1}{2}(\Gamma(1)) = 1$$

$$\Rightarrow$$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$$

$$\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx$$

$$E(xy) = \sum x_i y_j\, P(x_i, y_j)$$

$$= 0 + 1\cdot 0\cdot 1/8 + 1\cdot 1\cdot 2/8 + 1\cdot 2 \times 1/8 + 0$$

$$= \frac{4}{8} = \frac{1}{2}$$

$$\mathrm{cov}(x,y) = E(xy) - E(x)\,E(y)$$

$$= 1/2 - 1/2\,(3/2)$$

$$= \frac{1}{2} - \frac{3}{4}$$

$$= \frac{-2}{4}$$

Q. joint pdf of RV (x,y) is given by

$$f(x,y) = \begin{cases} k\,xy\,e^{-(x^2+y^2)}, & x>0,\ y>0 \\ 0 \end{cases}$$

find value of k and PT x & y are independent.

$$\int_{y=0}^{\infty}\int_{x=0}^{\infty} k\,xy\,e^{-(x^2+y^2)}\,dx\,dy = 1$$

$$\Rightarrow k\int_{y=0}^{\infty} y e^{-y^2}\,dy \cdot \int_{x=0}^{\infty} x\cdot e^{-x^2}\,dx = 1$$

$$\Rightarrow k\cdot\frac{1}{2}(\Gamma(1)) \times \frac{1}{2}(\Gamma(1)) = 1$$

$$\Rightarrow k/4 = 1 \Rightarrow k = 4$$

$2n-1=1$
$n=0$

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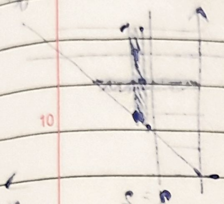
Mo Tu We Th Fr Sa Su

$$g(x) = \int_{y=0}^{\infty} f(x,y)\,dy$$

$$= \int_0^{\infty} 4xy\,e^{-(x^2+y^2)}\,dy$$

$$= 2xe^{-x^2}\int_0^{\infty} 2y\,e^{-y^2}\,dy$$



$$= 2xe^{-x^2}\Gamma(1)$$

$$= 2xe^{-x^2} \quad (0 < x < \infty)$$

$$h(y) = \int_0^{\infty} 4xy\,e^{-(x^2+y^2)}\,dx$$

$$= 2ye^{-y^2}\int_0^{\infty} 2 \cdot x \cdot e^{-x^2}\,dx$$

$$= 2ye^{-y^2}\,\Gamma(1)$$

$$= 2ye^{-y^2} \quad 0 < y < \infty$$

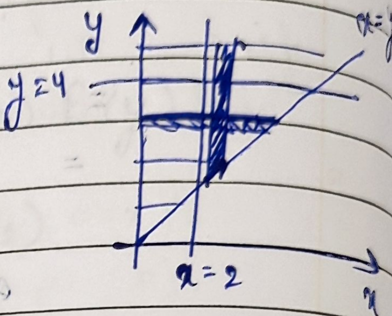
$$g(x)\cdot h(y) = 2xe^{-x^2} \cdot 2ye^{-y^2}$$

$$= 4xy\,e^{-(x^2+y^2)}$$

y-limit - vert
x- limit - hor.

Q. Suppose that joint pdf of (x, y) is given by

$$f(x,y) = \begin{cases} e^{-y}, & x>0,\ y \geq x \\ 0, & \text{elsewhere} \end{cases}$$



$P(x>2 \mid y<4)$

$$g(x) = \int_{y=x}^{\infty} f(x,y)\, dy$$

$$= \int_{x}^{\infty} e^{-y}\, dy$$

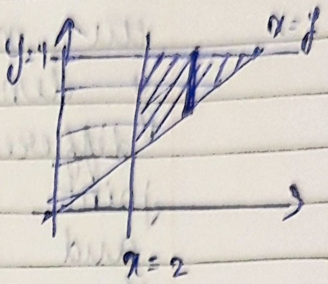
$$= -\left[e^{-y}\right]_{x}^{\infty}$$

$$= e^{-x} \qquad x \geq 0$$

$$h(y) = \int_{x=0}^{y} e^{-y}\, dx$$

$$= y e^{-y} \qquad y \geq x$$

$$P(x>2 \mid y<4) = \frac{P(x>2 \cap y<4)}{P(y<4)}$$



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$$= \int_2^4 \int_{y=x}^4 e^{-y}\, dy\, dx$$

$$= \int_2^4 -[e^{-y}]_x^4\, dx$$

$$= \int_2^4 -e^{-4} + e^{-x}\, dx$$

$$= -e^{-4}(2) - [e^{-x}]_2^4$$

$$= -2e^{-4} - [e^{-4} - e^{-2}]$$

$$= -2e^{-4} - e^{-4} + e^{-2}$$

$$= e^{-2} - 3e^{-4}$$

$$P(y>4) = \int_{x=0}^4 \int_{y=x}^4 e^{-y}\, dy\, dx$$

$$= \int_{x=0}^4 -[e^{-y}]_x^4\, dx$$

$$= \int_0^4 -e^{-4} + e^{-x}\, dx$$

$$= -4e^{-4} - [e^{-x}]_0^4$$

$$= -4e^{-4} - e^{-4} + 1$$

$$= 1 - 5e^{-4}$$

$$P(x>2 \mid y<4) = \frac{e^{-2} - 3e^{-4}}{1 - 5e^{-4}} = 0.0885$$

Q. A fair 6 sided die is rolled twice in succession. Let x denote the max of 2 outcomes and y denotes absolute value of the differences of 2 outcomes. Determine joint prob dist. of x & y and find variance of x and variance of y.

$R_x : \{1, 2, 3, 4, 5, 6\}$

$R_y : \{0, 1, 2, 3, 4, 5\}$

$x \backslash y$	0	1	2	3	4	5	$P(x)$
1	1/36	0	0	0	0	0	1/36
2	1/36	2/36	0	0	0	0	3/36
3	1/36	2/36	2/36	0	0	0	5/36
4	1/36	2/36	2/36	2/36	0	0	7/36
5	1/36	2/36	2/36	2/36	2/36	0	9/36
6	1/36	2/36	2/36	2/36	2/36	2/36	11/36
$q(y)$	6/36	10/36	8/36	6/36	4/36	2/36	1

$$E(x) = \sum_i x_i P(x_i)$$

$$= \frac{161}{36}$$

$$E(y) = \frac{35}{18}$$

$E(x^2) = \sum x^2 P(x)$

$= \frac{791}{36}$

$V(x) =$

Q. find cov of x, y. Test (x, y) are independent. justify.

$x \backslash y$	-1	0	1	$p(x)$
-1	0	0.2	0	0.2
0	0.1	0.2	0.1	0.4
1	0.1	0.2	0.1	0.4
$g(y)$	0.2	0.6	0.2	1

$E(x^2) = 0.6$

$E(y^2) = 0.4$

$E(xy) = \sum_{ij} x_i y_j P_{ij}$

$= -1(0.1) + 0.1 = 0$

$E(x) = \sum_i x_i P(x_i)$

$= -0.2 + 0.4 = 0.2$

$E(y) = -0.2 + 0.2 = 0$

$\text{cov} = E(x) E(y) - E(xy) = 0$

$P(x_i\, y_j) = P(x_i) \cdot P(y_j)$

$i = j = 1$

$P_{11} = 0$

$P(x_i)\, q(y_1) = 0.2 \times 0.2 = 0.04$

$\therefore$ not independent

Q. If x & y are 2D RV find $V(x+y)$ ~~(same above que)~~

$V(x+y) = E\left((x+y)^2\right) - [E(x+y)]^2$

$= E[x^2 + y^2 + 2xy] - [E(x) + E(y)]^2$

$= E(x^2) + E(y^2) + 2E(xy) - [E(x) + E(y)]^2$

$= E(x^2) + E(y^2) + 2E(xy) - (E(x))^2 - (E(y))^2 - 2E(x)E(y)$

$= E(x^2) - (E(x))^2 + E(y^2) - (E(y))^2 - 2[E(xy) - E(x)E(y)]$

$= V(x) + V(y) - 2\,cov(x,y)$

$\therefore \boxed{V(x) + V(y) - 2\,cov(x,y)}$

$V(3x+4) = 3^2 V(x)$

Q. 2 independent RV x_1 & x_2 have means 5 & 4 and variances 10 and 9. Find the cov b/w $U = 3x_1 + 4x_2$, $V = 3x_1 - x_2$ and also find coeff of correlation b/w U & V.

$E(x_1) = 5$ $\quad E(x_2) = 4$
$V(x_1) = 10$ $\quad V(x_2) = 9$

$COV(U, V) = E(UV) - E(U)E(V)$

$E(x_1 x_2) = E(x_1) \cdot E(x_2)$
$= 5 \times 4$
$= 20$

$V(x_1) = 10$
$E(x_1^2) - (E(x_1))^2 = 10$
$E(x_1^2) - 25 = 10$
$E(x_1^2) = 35$

$V(x_2) = 9$
$E(x_2^2) - (E(x_2))^2 = 9$
$E(x_2^2) = 25$

$E(UV) = E((3x_1 + 4x_2)(3x_1 - x_2))$
$= E(9x_1^2 + 9x_1x_2 - 4x_2^2)$
$= 9E(x_1^2) + 9E(x_1)E(x_2) - 4E(x_2^2)$
$= 9 \times 35 + 9.5.4 - 4.25$
$= 315 + 180 - 100$
$= 395$

$E(U)E(V) = E(3x_1 + 4x_2) \; E(3x_1 - x_2)$
$= [3E(x_1) + 4E(x_2)][3E(x_1) - E(x_2)]$
$= [3 \times 5 + 4 \times 4)(3 \times 5 - 4)$
$= (15 + 16)(11)$
$\cancel{31 \times 11} = 341$

$$\rho_{UV} = \frac{COV(U,V)}{\sqrt{V(U)\,V(V)}}$$

$$V(U) = V(3x_1 + 4x_2)$$
$$= 9V(x_1) + 16\,V(x_2)$$
$$= 9 \times 10 + 16 \times 9$$
$$= 90 + 144$$
$$= 234$$

$$V(V) = V(3x_1 - x_2)$$
$$= 9\,V(x_1) + V(x_2)$$
$$= 9 \times 10 + 9$$
$$= 99$$

$$COV(U,V) = E(UV) - E(U)E(V)$$
$$= 54$$

$$\rho = \frac{54}{\sqrt{234 \times 99}} = 0.3547$$

Q. x, y & z are uncorrelated RV with 0 mean and standard deviation 5, 12, 7 resp. If $U = x+y$ & $V = y+z$ find coeff of corr b/w U & V.

$E(x) = E(y) = E(z) = 0$

$\sigma x = 5 \quad \sigma y = 12 \quad \sigma_z = 7$

$V(x) = 25 \quad V(y) = 144 \quad V(z) = 49$

$E(UV) = E(UV) - E(U)E(y)$

$= E((x+y)(y+z)) - E(x+y)E(y+z)$

$= E(xy + xz + y^2 + zy) - [E(x)E(y) +$

$E(x)E(z) +$

$\because$ x, y, z are uncorrelated ($\rho = 0$) cov(x, y) = cov(y, z) = cov(y, z) = 0.

$\Rightarrow [E(xy) - E(x)E(y)] + [E(xz) - E(x)E(z)] +$

$[E(yz) - E(y)E(z)] + [E(y^2) - (E(y))^2]$

$= 0 + 0 + 0 + V(y)$
$= 144$

$V(U)V(V) = V(x+y)V(y+z)$

$= [V(x) + V(y)][V(y) + V(z)]$
$= 32617$

$$\rho = \frac{144}{\sqrt{32617}} = 0.7973$$

* uniformly distributed RV

Suppose x is continuous 1D RV then x is said to be uniformly dist. in an interval of [a, b] if pdf of x

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

Q If x is uniformly distributed RV in interval $-2 < x < 2$ then find

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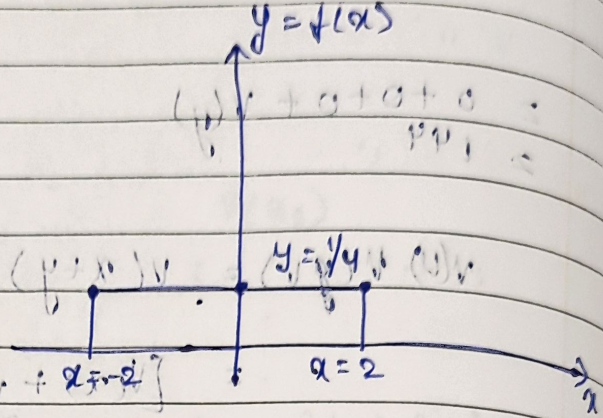
$P(x<1)$ $\quad$ $P(|x-1| \geq 1/2)$

$$f(x) = \begin{cases} \dfrac{1}{2-(-2)} = \dfrac{1}{4}, & -2 \leq x \leq 2 \\ 0 \end{cases}$$



$$P(x<1) = \int_{-2}^{1} f(x)\,dx$$

$$= \int_{-2}^{1} 1/4\,dx$$

$$= \frac{3}{4}$$

$$P(|x-1| \geq 1/2) = 1 - \underbrace{P(1/2 \leq x \leq 3/2)}_{P(|x-1| < 1/2)}$$

$$= 1 - \int_{1/2}^{3/2} 1/4\,dx$$

$$= \frac{1}{4}\left(\frac{3}{2} - \frac{1}{2}\right)$$

$$= 1 - \frac{1}{4}$$

$$= 3/4$$

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$$E(x) = \int x\, f(x)\, dx$$

$$= \int_{x=a}^{b} x \frac{1}{b-a} = \frac{1}{b-a} \cdot \frac{x^2}{2}\Big|_0^b$$

$$= \frac{b^2 - a^2}{2(b-a)}$$

$$= \frac{b+a}{2}$$

$$E(x^2) = \int_a^b x^2 \cancel{d} f(x)\, dx$$

$$= \int_a^b \frac{x^2}{b-a}\, dx$$

$$= \frac{[x^3]_a^b}{3(b-a)}$$

$$= \frac{b^3 - a^3}{3(b-a)}$$

$$= \frac{\cancel{(b-a)}(b^2 + a^2 + ab)}{3\cancel{(b-a)}}$$

$$= \frac{1}{3}(b^2 + a^2 + ab)$$

$$V(x) = E(x^2) - (E(x))^2$$

$$= \frac{b^2 + ab + a^2}{3} - \frac{(a+b)^2}{4}$$

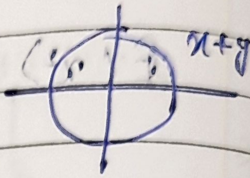
$$= \frac{4(b^2 + ab + a^2) - 3(a^2 + b^2 + 2ab)}{12}$$

$$= \frac{(a-b)^2}{12}$$

Q. Suppose a 2D continuous 2D RV is uniformly distributed over the region R then its joint pdf of

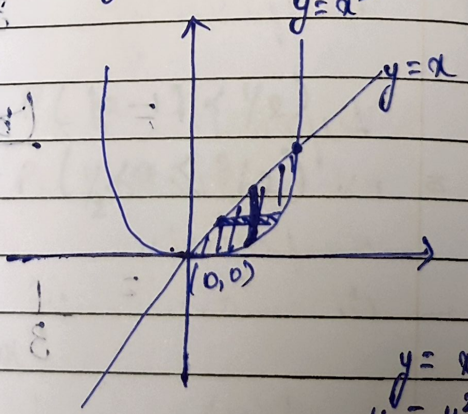
$$f(x,y) = \begin{cases} \dfrac{1}{\text{Area of Region R}}, & (x,y) \in R \\ 0, & \text{otherwise} \end{cases}$$

Ex: ~~xxx~~ if region is bounded by $x^2+y^2=1$



$$f(x) = \begin{cases} 1/\pi & (x,y) \in R \\ 0, & \end{cases}$$

Q. Find joint pdf of (x,y) which is uniformly dist. over a region R where R: $x=y$, $y=x^2$. Also find marginal pdf of x & y and variance of y.



$$\text{Area} = \int_0^1 \int_{x^2}^{x} dy\,dx.$$

$$= \int_0^1 (x-x^2)\,dx$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6}$$

$y = x^2$

$y = y^2$

$y^2 - y = 0$

$y(y-1) = 0$

$y = 1$

$y = 0.$

$$f(x,y) = \frac{1}{1/6} = 6$$

$$g(x) = \int_{y=x^2}^{x} f(x,y)\,dy$$

$$= 6(x - x^2) \qquad 0 \le x \le 1$$

$$h(y) = \int_{y}^{\sqrt{y}} f(x,y)\,dx$$

$$= 6[\sqrt{y} - y) \qquad 0 < y < 1$$

$$E(y) = \int_{y=0}^{1} y\, h(y)\,dy$$

$$= \int_{0}^{1} y\, 6(\sqrt{y} - y)\,dy$$

$$= 0.4 = 2/5$$

$$E(y^2) = \int_0^1 \int_{x^2}^{x} y^2\, 6\, dy\, dx = 3/14$$

$$\cancel{= \int_0^1 6(x - x^2)\,dx}$$

$$V(y) = 3/14 - (2/5)^2 = \frac{19}{350}$$

$$\cancel{= 6\left[\tfrac{1}{2} - \tfrac{1}{3}\right]}$$

$$= 1$$

$$\cancel{V(y) = E(y^2) - (E(y))^2}$$

$$= 1 - (2/5)^2 = 21/25$$

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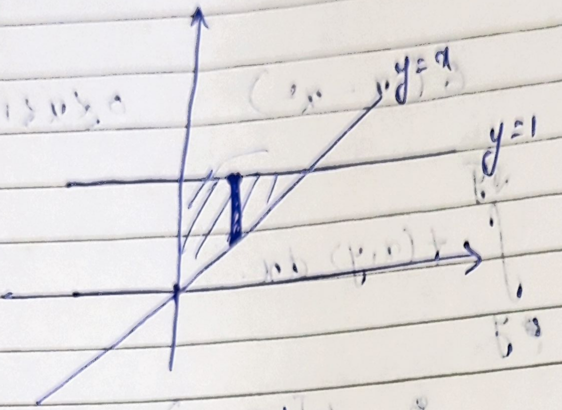
Q. Suppose that R.V. (x, y) is uniformly dist. in the triangular region R: $\{(x, y) \mid 0 \le x < y \le 1\}$

find ~~joint pdf~~.



$$\text{Area} = \int_{0}^{1}\int_{y=x}^{1} dy\, dx$$

$$= \int_{0}^{1} (1 - x)\, dx$$

$$= [x]_0^1 - \frac{[x^2]_0^1}{2}$$

$$= 1 - 1/2 = 1/2.$$

$$f(x,y) = 2$$

$$P(x > 1/2) = \int_{x=1/2}^{1}\int_{y=x}^{1} 2\, dy\, dx$$

$$\cancel{2(x-1)}$$

$$= \int_{1/2}^{1} 2(1-x)\, dx$$

$$= 2\left[(1/2) - \frac{1 - 1/4}{2}\right] = \left(1 - \frac{3}{4}\right)$$

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Q. If RV K is uniformly distributed over [0, 5], what is the prob that roots of equation $4x^2 + 4xK + (K+2) = 0$ are real

$$f(x) = \begin{cases} \frac{1}{5-0} = \frac{1}{5} & 0 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

$ax^2 + bx + c = 0$

$4x^2 + 4xK + (K+2) = 0$

$b^2 - 4ac \ge 0$

$(4K)^2 - 4(4)(K+1) \ge 0$

$16K^2 - 16K - 2(16) \ge 0$

$K^2 - K - 2 \ge 0$

$K^2 - K - 2 \ge 0$

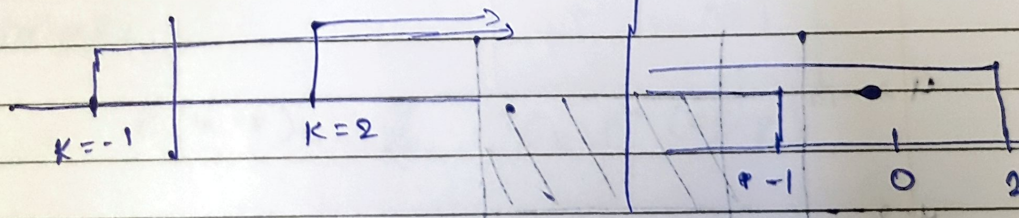
$K^2 - 2K + K - 2 \ge 0$

$K(K-2) + (K-2) \ge 0$

$(K-2)(K+1) \ge 0$

$(K+1) \ge 0$ & $(K-2) \ge 0$ | $(K+1) < 0$ & $K-2$

$K \ge -1$ | $K \ge 2$ | $K < -1$ | $K <$



$\therefore K \ge 2$ | $K < -1$ not possible

$$P(K \ge 2) = \int_2^5 \frac{1}{5}\, dK$$

$$= \frac{3}{5}$$

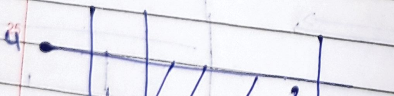
Q. Suppose that the dimensions x & y of rect metal plate maybe considered to be independent continuous RV

$$X: \quad g(x) = \begin{cases} x-1 & 1<x<2 \\ -x+3 & 2<x<3 \\ 0 & \text{elsewhere} \end{cases}$$

$$Y: \quad h(y) = \begin{cases} 1/2 & , 2<y<4 \\ 0 & , \text{elsewhere} \end{cases}$$

find pdf of area of plate, $A = xy$

$$f(x,y) = \begin{cases} \frac{1}{2}(x-1) & 1<x\le 2, \ 2<y<4 \\ \frac{1}{2}(3-x) & 2<x<3, \ 2<y<4 \\ 0 & \end{cases}$$



* Distributions.

$$X \swarrow \searrow$$

discrete	continuous
Binomial	gamma
Poisson	exponential
	chisquare
	Normal distribution

* Binomial Distribution.

Exp consists of

→ 'n' repeated trials

→ Each trial result maybe classified as success or failure.

→ Prob. of success is denoted by p and failure $1-p=q$.

→ Repeat trials are independent.

→ Here, [(RV) X] is no. of success in 'n' repeated trials.

Thm Let X be a BRV on n repeated trials

$$P(x=k) = {}^{n}C_{k}\, p^{k} (1-p)^{n-k}$$

$x=k$	0	1	2	...	n
$P(x=k)$	$1-p^n$	${}^{n}C_{1} p (p-1)^{n-1}$	${}^{n}C_{2} p^{2} (1-p)^{n-2}$	...	p^n

$$\sum_{k=0}^{n} P(x=k) = (1-p)^n + {}^{n}C_{1} p (1-p)^{n-1} \dots + p^n$$

$$= [p + (1-p)]^n = 1$$

$$E(x) = \sum x\,p(x)$$

$$= \sum_{k=0}^{n} k\,P(x=k)$$

$$= \sum_{k=0}^{n} k\,{}^{n}C_{k}\,p^{k}(1-p)^{n-k}$$

$$= \sum_{k=0}^{n} k\,\frac{n!}{k!\,(n-k)!}\,p^{k}(1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k\,\frac{n!}{k(k-1)!\,(n-k)!}\,p^{k}(1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n\,(n-1)!}{(k-1)!\,((n-1)-(k-1))!}\,p\,p^{k-1}(1-p)^{(n-1)-(k-1)}$$

$$= np\sum_{k=1}^{n-1} {}^{n-1}C_{k-1}\,p^{k-1}\,(1-p)^{(n-1)-(k-1)}$$

$$= np\,[p+(1-p)]^{n-1}$$

$$= np$$

$$\boxed{E(x) = np}$$

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$$E(x^2) = \sum_{k=0}^{n} k^2 P(x=k)$$

$$= \sum (k^2 - k + k) P(x=k)$$

$$= \sum_{0}^{n} (k^2 - k) P(x=k) + \sum_{0}^{n} k P(x=k)$$

$$= \sum k(k-1) \frac{n!}{(k)!(n-k)!} p^k (1-p)^{n-k} + np$$

$$\hookrightarrow k(k-1)(k-2)!$$

$=$

$$= \sum_{k=2}^{n} \frac{n(n-1)(n-2)!}{(k-2)!((n-2)-(k-2))!} p^2 \, p^{k-2} (1-p)^{(n-2)-(k-2)} + np$$

$$= n(n-1)p^2 \sum {}^{n-2}C_{k-2} \, p^{k-2} (1-p)^{(n-2)-(k-2)} + np$$

$$= n^2p^2 - np^2 \left[p + (1-p) \right]^{n-2} + np$$

$$= n^2p^2 - np^2 + np$$

$$V(x) = E(x^2) - (E(x))^2$$

$$= n^2p^2 - np^2 + np - (np)^2$$

$$= np - np^2$$

$$= np(1-p)$$

$$= npq$$

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Q. 6 fair coins are tossed find prob of getting

① Exactly 3 head
② atmost 3 heads
③ at least 3 heads
④ at least 1 head

x: no. of head

$p = 1/2 = q$

① $P(x=3) = {}^6C_3\, p^3 (1-p)^{6-3}$

$= {}^6C_3 (1/2)^3 (1/2)^3$

$= 5/16$

② $P(x \leq 3) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$= {}^6C_0 (1/2)^0 (1/2)^6 + {}^6C_1 (1/2)^1 (1/2)^5 + {}^6C_2 (1/2)^2 (1/2)^4 + {}^6C_3 (1/2)^3 (1/2)^3$

$= 21/32$

③ $P(x \geq 3) = P(x=3) + P(x=4) + P(x=5) + P(x=6)$

$= \sum_{k=3}^{6} {}^6C_k (1/2)^k (1/2)^{n-k}$

(4)

$$P(x \geq 1) = 1 - P(x<1)$$
$$= 1 - P(x=0)$$
$$= 1 - {}^6C_0 (1/2)^0 (1/2)^6$$
$$= \frac{63}{64}$$

Q. Determine the prob of getting 9 exactly twice in 3 throws with a pair of fair dice.

(6, 3) (3, 6) (4, 5) (5, 4)

P(getting 9 in single throw) $= \frac{4}{36} = \frac{1}{9} = p$, $q = \frac{8}{9}$

$$P(x=2) = {}^3C_2 (1/9)^2 (8/9)$$
$$=$$

Q. Out of 800 family with 5 children each how many would expect to have
(a) 3 boys
(b) 5 girls
Assume equal prob for boys and girls

X: no. of boys
$n = 5$
$p = 1/2$

a) $P(x=3) = {}^5C_3 (1/2)^3 (1/2)^2$
$= 5/16$

$\therefore$ out of 800 family having 3 boys
$= 800 \times 5/16 = 250$

b) $P(X=0) = {}^5C_0 (1/2)^0 (1/2)^5$
$= 1/32$

no. of family $= 800 \times \frac{1}{32} = 25$

Q. In sampling a large no. of parts manufactured by machine. The mean no. of defective in a sample of 20 is 2 out of 1000 such samples how many would be expected to contain at least 3 defective parts.

X : no. of defective parts
Given $p = 2/20 = 1/10 = 0.1$

$P(X \geq 3) = ?$

$P(X = k) = {}^nC_k (p)^k (q)^{n-k}$

$= {}^{20}C_k (0.1)^k (0.9)^{20-k}$

$P(X \geq 3) = 1 - P(X < 3)$
$= 1 - \sum_{k=0}^{2} {}^{20}C_k (0.1)^k (0.9)^{20-k}$

$= 0.3230$

Q. If x is binomially distributed RV with expected value 2 variance is 4/3. Find distribution x.

$E(x) = 2$ $V(x) = 4/3$

x	0	1	2	3	4	5	6
$P(x)$	$64/729$	$64/243$	$80/243$	$160/729$	$20/243$	$4/243$	$1/729$

$E = np$ $\quad V = npq$

$2 = np$ $\quad \frac{4}{3} = 2q$

$p = 1/3$ $\quad q = 2/3$

$n = 3 \times 2$
$= 6$

* Poisson Distribution.

A RV x taking on one of the values $0, 1, 2 \ldots$ is said to be poisson RV with the parameter $\alpha > 0$ if its pmf is given by

$$P(x = k) = \frac{e^{-\alpha} \alpha^k}{k!}, \quad k = 0, 1, 2 \ldots$$

$$\sum_{k=0}^{\infty} P(x = k) = \boxed{\sum_{k=0}^{\infty} \frac{e^{-\alpha} \alpha^k}{k!}} e^{-\alpha} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!}$$

$$= e^{-\alpha} (e^{\alpha})$$

$$E(x) = \sum x P(x) = \sum_{k=0}^{\infty} k (P(x = k))$$

$$= \sum_{k=0}^{\infty} \cancel{k} \frac{e^{-\alpha} \alpha^k}{\cancel{k}(k-1)!}$$

$$= e^{-\alpha} \sum_{k=1}^{\infty} \frac{\alpha . \alpha^{k-1}}{(k-1)!}$$

$$= \alpha e^{-\alpha} \sum_{k=1}^{\infty} \frac{\alpha^{k-1}}{(k-1)!}$$

$$= \alpha e^{-\alpha} e^{\alpha}$$
$$= \alpha$$

$$\boxed{\therefore E(x) = \alpha}$$

$$E(x^2) = \sum_{k=0}^{\infty} k^2 P(x=k)$$

$$= \sum_{k=0} k^2 \frac{e^{-\alpha}\alpha^k}{k(k-1)!}$$

$$= \sum_{k=1} k \frac{e^{-\alpha}\alpha^k}{(k-1)!}$$

$$= e^{-\alpha} \sum (k-1+1) \frac{\alpha^k}{(k-1)!}$$

$$= e^{-\alpha} \left[\sum (k-1) \frac{\alpha^k}{(k-1)!} + \sum \frac{\alpha^k}{(k-1)!} \right]$$

$$= e^{-\alpha} \left[\sum_{k=2} \frac{\alpha^2 \alpha^{k-2}}{(k-2)!} + \alpha e^{\alpha} \right]$$

$$= e^{-\alpha}\alpha^2 e^{\alpha} + e^{-\alpha}\alpha e^{\alpha}$$

$$= \alpha^2 + \alpha$$

$$\boxed{E(x^2) = \alpha^2 + \alpha}$$

$$V(x) = E(x^2) - (E(x))^2$$

$$= \alpha^2 + \alpha - \alpha^2$$

$$= \alpha$$

$n \to \infty$

$p \to 0$

$np \to \alpha$

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Q. Let x be binomially associated RV with parameter n & p. Suppose that $n \to \infty$ & $p \to 0$ such that $np \to \alpha$ under these cond^n we have prob. of $P(x=k) = \frac{e^{-\alpha}\alpha^k}{k!}$, $k = 0, 1, 2 ..$ $= P(\alpha)$ of RV x.

$$P(x=k) = {}^nC_k \, p^k (1-p)^{n-k}$$

$$= \frac{n!}{k!\,(n-k)!} \, p^k (1-p)^{n-k}$$

$$= \frac{n(n-1)(n-2)\dots(n-(k-1))\cancel{(n-k)!}}{k!\,\cancel{(n-k)!}} \, p^k (1-p)^{n-k}$$

$$= \frac{n^k}{k!} (1-1/n)(1-2/n)\dots\left(1-\frac{k-1}{n}\right) p^k (1-p)^{n-k}$$

$$P(x=k) = \lim_{n\to\infty} \frac{n^k p^k}{k!} (1-1/n) \dots \left(1-\frac{k-1}{n}\right)\left(1-\frac{\alpha}{n}\right)^{n-k}$$

$$= \frac{(\alpha)^k}{k!} \times \lim_{n\to\infty} \frac{(1-\alpha/n)^n}{(1-\alpha/n)^k}$$

$$= \frac{\alpha^k}{k!} \left[\left(1-\frac{\alpha}{n}\right)^{-n/\alpha}\right]^{-\alpha}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{\alpha}\right)^{\alpha} = e$$

$$= \frac{\alpha^k}{k!} e^{-\alpha}$$

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Gamma (in another ub)

Q. The no. of road acc. for a day in certain ~~ab~~ city is distributing as gamma variable with average of 6 and variance 18. find the prob. that there will be more than 8 accidents (ii) b/w 5 and accidents on a particular day.

$$E(x) = r/\alpha = 6$$

$$V(x) = r/\alpha^2 = 18$$

$$\frac{r/\alpha}{r/\alpha^2} = \frac{6}{18} \Rightarrow \alpha = 1/3$$

since $\frac{r}{\alpha} = 6 \Rightarrow r = 6 \times \frac{1}{3} = 2.$

$$f(x) = \begin{cases} \frac{\alpha}{\Gamma(r)} (\alpha x)^{r-1} e^{-\alpha x}, & x>0 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1/3}{\Gamma(2)} (1/3\, x)^{2-1} e^{-1/2\, x}, & x>0 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1/9\, x\, e^{-x/3}, & x>0 \\ 0 & \text{elsewhere} \end{cases}$$