ETH zürich

Stability of Faber-Krahn inequalities in time-frequency analysis

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June 20, 2023

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Abstract

We prove a sharp quantitative version of the Faber-Krahn inequality for the short-time Fourier transform in dimension 2, with its extension to the general d-dimensional case and the wavelet transform analogue, the Faber-Krahn inequality for wavelet transforms. In doing so, we give a brief historical overview and recall the main concepts from time-frequency and time-scale analysis, and Faber-Krahn concentration estimates.

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CHAPTER 1

Introduction

Geometric and functional inequalities play major roles in different problems across areas like geometric analysis, calculus of variations, partial differential equations or physics. Many of these inequalities are well understood, in the sense that we can answer the very basic questions about finding their sharp versions and characterizing equality cases. For instance, the Euclidean isoperimetric inequality provides an analytical expression of the variational principle stating that balls minimize perimeter under a certain volume constraint. For a measurable set $E \subset \mathbb{R}^n$, the inequality reads

$$P(E) \geq n\,|B|^{1/n}\,|E|^{(n-1)/n}\,,$$

where $P(E)$ is the perimeter of E (in the sense of De Giorgi [Ma2]), B is the unit ball in $\mathbb{R}^n$ and $|E|$ is the Lebesgue measure of E. Equality cases are characterized by balls (up to sets of measure zero).

However, in further studying these inequalities, one natural question that we are led to consider is whether extremals (or minimizers), i.e., the objects achieving equality, are stable or not. This is,

if an object fails to achieve the equality only by a small margin, does it have to be, in some sense, close to a minimizer?

This question has gained considerable interest in recent years, resulting in several contributions showing the stability of many well-known inequalities such as the isoperimetric inequality [EFT, FiMP, Fu, FMP2, Ma1], the Brunn-Minkowski inequality [FJ, HST], the Sobolev inequality [CFMP, FMP1], the Prékopa-Leindler inequality [BD, BFR] and Gagliardo-Nirenberg-Sobolev inequalities [BDNS, DT, Ng].

Quantitative stability results often take a specific form. Firstly, a deficit functional δ is extracted from the original inequality, indicating the failure of a particular object from achieving equality. Secondly, an appropriate method of measuring the difference between an object and its closest minimizer $\mathcal{D}$ is established. This part is crucial, since choosing the right $\mathcal{D}$ will result in the inclusion or exclusion of certain behaviors of the objects we are working with. Finally, the aim is to find an inequality of the type

$$\mathcal{D} \leq C\delta^{\alpha},$$

where C and α are positive constants that might depend on quantities related to the problem, like the dimension, or the measure of a set (cf. Theorem 4.1). Moreover, the larger α is, the stronger the result is, and we say it is sharp when α is the largest possible, i.e., when for a minimizing sequence, the convergence of δ^α and $\mathcal{D}$ to zero happens at the same rate.

Apart from their intrinsic mathematical interest, stability estimates often find applications both in other problems within mathematics (see for example [CF, DD]), and in the applications (see [FM]). In the computational aspect, it is often unfeasible to work without the use of approximations, and stability estimates help ensure their validity.

Let us bring back the example of the isoperimetric inequality. In this case the deficit functional is

$$\delta(E) := \frac{P(E)}{n\,|B|^{1/n}\,|E|^{(n-1)/n}} - 1,$$

making the inequality read $\delta(E) \geq 0$, and the right way to measure the distance between E and its closest ball turns out to be the *Fraenkel asymmetry* of E,

$$\mathcal{A}(E) := \inf\left\{\frac{|E\Delta B(x,r)|}{|E|} : x \in \mathbb{R}^n, |E| = |B(x,r)|\,, r > 0\right\}.$$

The sharp quantitative isoperimetric inequality on $\mathbb{R}^n$ then reads

$$\mathcal{A}(E) \leq C(n)\delta(E)^{1/2}.$$

It is often the case that in order to show stability, a useful approach is trying to make the proof of the original inequality into a quantitative one. In the case of geometric inequalities, where minimizers are a certain family of sets, reduction techniques to simplify the problem to sets or functions that possess some desirable qualities (such as boundedness, symmetries, etc.) have proven to be extremely useful [BFR, BPV, CFMP, FMP2, FMP1], and the theory of (optimal) mass transportation also presents itself as a strong tool that can encode the geometric properties of a set in terms of a transport map, allowing us to turn a geometric problem into an analytic one [FiMP, Ma1].

These points of view are present in many of the quantitative versions of the aforementioned inequalities, and they will play major roles in the work that we will discuss later in Chapter 4, where we will study the stability of the Faber-Krahn inequality for the short-time Fourier transform [NT]. This inequality provides a sharp bound for the concentration of functions in the time-frequency plane, and it is closely related to the theory of localization operators in time-frequency analysis, where the short-time Fourier plays a leading role.

The driving force of time-frequency (resp. time-scale) analysis is to represent functions in the time-frequency (resp. time-scale) plane with the aim of developing tools and techniques to analyze them in combination with the information given by their local frequency spectrum. Such tools are, for example, wavelet transforms and the short-time Fourier transform. In signal analysis, this corresponds to observing a specific frequency band of a signal around a certain point in time, in analogy with how a musical score describes the sounds that form part of a piece at every moment of its duration. Localization operators translate this process into a mathematical setting by reducing the energy of functions outside a certain, fixed set to a negligible amount.

The pioneering work in this circle of ideas was due to Landau and Pollack in the “Bell labs papers” (see Slepian’s survey [Sl]), but it was the work of I. Daubechies [Da2] that introduced an innovative perspective to time-frequency localization operators. She showed that for radially symmetric sets and Gaussian windows, the eigenfunctions of these operators are Hermite functions, and their eigenvalues can be computed explicitly. In general, it is very hard to find either of them when the domain is not a radially symmetric set, even if it is close to one (for example, an ellipse with very low eccentricity). Similarly, her work with Paul [DP] extended this to the time-scale plane and the associated wavelet transform.

Since then, both time-frequency and time-scale analysis have received a lot of attention and seen a great deal of contributions in the form of uncertainty principles (we refer to the surveys [BoD, FS, Gr2] and [BCO, De, FeG]) and concentration-type estimates [AS], among other things. The applications of localization operators are manifold and their literature is vast [BS, CG, Wo]. In particular, the operators associated to the short-time Fourier transform with Gaussian window have remained of considerable interest [AGR, APR, DFN, MR, Ol], and more recently, there has been an emerging body of ideas and techniques developed initially by F. Nicola and P. Tilli in [NT] in relation to the work of Abreu and Dörfler [AD] that are currently experiencing applications and extensions to time-scale analysis [RT] and various other areas.

The topic we will address in this thesis is the work by A. Guerra, J. P. G. Ramos, P. Tilli and the author in [GGRT], where a sharp quantitative stability version of the Faber-Krahn inequality for the short-time Fourier transform [NT] is established by means of a series of new ideas extending the original framework, along with its wavelet transform analogue [RT]. We aim to provide a relatively self-contained account and detailed insight on the techniques that proved to be useful in establishing the main results in the paper, often with a slightly varied point of view. The structure of the thesis will be the following. In Chapter 2 we introduce the preliminary notions and essential concepts from time-frequency and time-scale analysis. Chapter 3 will be a summary of the original work by F. Nicola and P. Tilli [NT], and the subsequent work by J. P. G. Ramos and P. Tilli [RT] where a Faber-Krahn inequality is established for the wavelet transform using the same methods. We will dedicate Chapter 4, the main part of the thesis, to showing the primary quantitative stability result, and finally, we extend these results to arbitrary dimension and wavelet transforms in Chapter 5.

Chapter 2

Time-Frequency Analysis

Classical Fourier analysis provides the harmonic analyst with a powerful tool: the Fourier transform. It enjoys a great deal of properties that allow us to look at a function at an additional level: its *frequency spectrum*. Operations like convolutions, multiplications, translations, modulations and dilations are switched by the Fourier transform, and information about the regularity of a function can be found in the decay properties of its Fourier transform.

In signal analysis, functions are interpreted as signals, and for $f \in L^2(\mathbb{R})$, the amount $f(x)$ can be thought of as the amplitude of f at time $x \in \mathbb{R}$. The Fourier transform of f at $\omega \in \mathbb{R}$ $\widehat{f}(\omega)$ is the amplitude of f at the frequency ω. Fourier analysis thus performs excellently whenever a signal f is comprised of a few dominating frequency bands that do not change as time goes on. However, when this stops being the case, the Fourier transform is no longer the appropriate tool to use, as frequency spectra might change with time.

Indeed, in order to compute $\widehat{f}(\omega)$ we need to know the entirety of f, and moreover, $\widehat{f}(\omega)$ only provides averaged information on the frequency ω in f over the whole duration of the signal. It becomes obvious at this point that a new tool is needed if we want to extract more detailed information on f. This is the point of view that time-frequency analysis brings to the table. Instead of just looking at a function f by itself or via its Fourier transform, the aim now is to use the joint information of both of them in order to improve the analysis. To do this, we need a two-dimensional representation that allows us to look at f at a certain frequency and a particular instant in time, very much like a music score of f.

The ideal time-frequency representation would, at least intuitively, run through looking at the "instantaneous frequency spectrum" of $f \in L^2(\mathbb{R})$ at $x \in \mathbb{R}$. To this end, let us define $f_\delta(y) = f(y)\mathbb{1}_{[x-\delta,x]}(y)$. We then seek to find where the function $\widehat{f_\delta}$ accumulates most of its mass, which will be the dominating frequencies of f at x.

Despite this being a reasonable approach, we readily encounter an obstruction in the form of uncertainty principles. Indeed, these roughly establish that a function and its Fourier transform cannot be simultaneously concentrated over small sets, making it impossible for the concept of an instantaneous frequency spectrum to be reasonable. Let us give a notion of concentration and follow with one such uncertainty principle. We will say that a function is ε-concentrated over a measurable set $\Omega \subset \mathbb{R}^d$ whenever

$$\left(\int_\Omega |f(x)|^{1/2}\,\mathrm{d}x\right)^2 \geq (1-\varepsilon)\|f\|_2.$$

The Donoho-Stark uncertainty principle then reads as follows [DoS].

Theorem 2.1 (Donoho-Stark uncertainty principle)**.** *Suppose that $f \in L^2(\mathbb{R}^d) \setminus \{0\}$ is ε_T-concentrated on $T \subset \mathbb{R}^d$ and $\widehat{f}$ is ε_Ω-concentrated on $\Omega \subset \mathbb{R}^d$. Then*

$$|T|\,|\Omega| \geq (1-\varepsilon_T-\varepsilon_\Omega)^2.$$

When we apply this result to our previously defined function f_δ, we find that if $\widehat{f_\delta}$ were to be μ-concentrated on $\Omega \subset \mathbb{R}$, then

$$|\Omega| \geq \frac{(1-\mu)^2}{\delta}.$$

In other words, we cannot expect $\widehat{f_\delta}$ to be very concentrated on small sets, and therefore the idea of an instantaneous frequency spectrum is not viable. Instead, we need to find a compromise between how reliably we can look at the frequencies of f while still analyzing over a small enough interval in time. This is where the *short-time Fourier transform* comes into play.

2.1 Time-Frequency and Time-Scale Representations

In order to circumvent the previous problem, we can choose a function (called a window function) through which to look at the frequency spectrum of f over a small window of time without losing too much to the uncertainty. Then, by sliding this window we can generate a representation of f in the time-frequency space. This will be the short-time Fourier transform. We refer to [Gr1] for a detailed description of the discussion we will present.

Definition 2.2 (Short-time Fourier transform)**.** Let $f, g \in L^2(\mathbb{R}^d)$ with $g \neq 0$. Then the short-time Fourier transform (STFT) of f with respect to the window g is defined as

$$V_g f(x,\omega) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i t\cdot\omega}\,\mathrm{d}t, \quad x,\omega \in \mathbb{R}^d. \tag{2.1}$$

The requirement that $f, g \in L^2(\mathbb{R}^d)$ ensures that $V_g f$ makes sense, but the definition can be extended to $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ for conjugate Hölder exponents $1 < p, q < \infty$ and, by duality, to tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ given a Schwarz window $g \in \mathcal{S}(\mathbb{R}^d)$.

The STFT itself has some very interesting properties that relate to those of the regular Fourier transform. It is absolutely continuous whenever $f, g \in L^2(\mathbb{R}^d)$, it enjoys the orthogonality property

$$\langle V_{g_1} f_1, V_{g_2} f_2\rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2\rangle\,\overline{\langle g_1, g_2\rangle},$$

analogous to the Parseval formula, that makes it an isometry from $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$ when $\|g\|_2 = 1$, and there is an inversion formula just like for the Fourier transform. Moreover, we also have uncertainty principles that provide lower bounds for the size of the sets where $V_g f$ is very concentrated. However, we will mostly be interested in how it interacts with a couple of operators that are inherently special in time-frequency analysis: translations and modulations.

Definition 2.3. For $x, \omega \in \mathbb{R}^d$, we define the *translation by* x

$$T_x f(t) := f(t - x),$$

and the *modulation by* ω

$$M_\omega f(t) := e^{2\pi i \omega \cdot t} f(t).$$

Operators of the form $M_\omega T_x$ or $T_x M_\omega$ are called time-frequency shifts, and they satisfy the following commutation relation

$$T_x M_\omega = e^{-2\pi i x \cdot \omega} M_\omega T_x. \tag{2.2}$$

Time-frequency shifts commute when $x \cdot \omega$ is an integer. They are isometries in $L^p(\mathbb{R}^d)$ for each $1 \leq p \leq \infty$, and the Fourier transform shifts their order in the following way

$$\widehat{T_x f} = M_{-x} \widehat{f}, \quad \widehat{M_\omega f} = T_\omega \widehat{f}.$$

In particular, it is not hard to prove that (2.2) describes how the short-time Fourier transform interacts with time-frequency shifts.

Lemma 2.4. *Whenever $V_g f$ is defined, it holds that*

$$V_g \left(T_u M_\eta f\right)(x, \omega) = e^{-2\pi i u \cdot \omega} V_g f(x - u, \omega - \eta), \quad \text{for } x, u, \omega, \eta \in \mathbb{R}^d. \tag{2.3}$$

Although we will later work with a fixed window function, so far we have not required that the g satisfy any especial properties. The choice of a window function is a whole topic with its own interest: it varies depending on the analysis and the dominating frequency bands of a signal – frequency resolution can only be obtained by trading time resolution, and the same occurs when trying to achieve good time resolution. Moreover, this phenomenon, called the *window effect*, takes place *uniformly* over all frequencies, which induces some undesirable properties as we will see later on.

In any case, a good choice for a window function would be one that balances both time and frequency resolution by minimizing uncertainty. Then, an obvious candidate is the Gaussian window, defined as

$$\varphi(x) := 2^{d/4} e^{-\pi |x|^2}. \tag{2.4}$$

The reason behind this is that it lies within a family of Gaussian functions that minimize the Heisenberg uncertainty. In dimension $d = 1$, let $\varphi_c(x) = e^{-\pi x^2/c}$.

Theorem 2.5 (Heisenberg uncertainty principle). *If $f \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$, then*

$$\|(x - a) f\|_2 \|(x - b) \widehat{f}\|_2 \geq \frac{1}{4\pi} \|f\|_2^2.$$

Equality occurs if and only if f is a multiple of $M_{-b} T_a \varphi_c(x)$.

By changing the parameter c we vary the size of the support of the window function, known as the *window size*. The wider the window, the better the frequency resolution and the greater the cost in time resolution, and vice-versa.

Now let us go back to the issue we mentioned. The time-frequency localization offered by the STFT is limited by the window effect, particularly in that it lacks a sense of the scale at which one looks at details around a point $x \in \mathbb{R}^d$. Indeed, notice that $V_g f = \langle f, M_\omega T_x g\rangle$, and when g is supported inside E and this is centered at the origin, then $\operatorname{supp} M_\omega T_x g \subset x + E$. Therefore the STFT has a locked time-frequency resolution. In the applications, it turns out that most of the signals that we find show high frequency components for short durations, and low frequency components for long durations. This is the case of heartbeat signals, partial discharges in electrical systems, and seismic signals, for example. It would then be desirable to localize signals in the time-frequency plane with different time-frequency resolution depending on where we are localizing. For high frequencies, it is more relevant to have good time resolution, whereas for lower ones, it is frequency resolution what we may want to focus on.

This is the starting point of time-scale analysis and wavelet theory, which provide a solution to this problem. In what follows, we will give a few of the basic ideas behind wavelet theory and the main results and properties that we will need later on (and for this reason we restrict to dimension $d = 1$), and we refer to [Ch, Da1, Gr1, Ml, Me] for a more comprehensive exposition of the theory of time-scale analysis.

In order to obtain different time-frequency resolution, wavelet theory substitutes frequency by scale, and modulations by dilations, i.e.

$$D_s f(x) = |s|^{-1/2} f(s^{-1}x), \quad s \in \mathbb{R} \setminus \{0\}\,. \tag{2.5}$$

The normalization $|s|^{-1/2}$ is chosen so that D_s is unitary on $L^2(\mathbb{R})$, and observe that if a function f is supported in E, then $D_s f$ is supported in sE. The time-scale representation that is the analogous to the short-time Fourier transform in wavelet theory is the *wavelet transform*.

Definition 2.6 (Wavelet transform)**.** Given a fixed function $g \in L^2(\mathbb{R})$, the wavelet transform of $f \in L^2(\mathbb{R})$ with respect to the window g is defined as

$$W_g f(x,s) = \frac{1}{s^{1/2}} \int_{\mathbb{R}} f(t) \overline{g\left(\frac{t-x}{s}\right)}\, \mathrm{d}t. \tag{2.6}$$

This is well-defined pointwise for each $x \in \mathbb{R}$, $s > 0$, and it enjoys a fair deal of desirable properties. As is the case with the STFT, the original function f can be recovered from its wavelet transform under certain admissibility conditions on g [Da1, Proposition 2.4.1], and in striking similarity with the Fourier transform, $W_g f$ reveals useful information on the Hölder regularity of f [Da1, Theorem 2.9.2].

The function g is usually called the *mother wavelet*, and the family $g^{x,s} = s^{-1/2} g\left(\frac{t-x}{s}\right)$ of translated and dilated copies of g are called the *wavelets*. For small values of s, the corresponding wavelets are "shrunken" versions of g, with frequency content concentrated mostly in the high frequency range. Conversely, when s attains large values, the frequency content of wavelets is concentrated on the low end of the spectrum. As a consequence of this, $W_g f$ will have good resolution in time when s is small, at the price of frequency resolution, and good frequency resolution for large values of s, and coarser time resolution.

Another way to look at this is by noting that $W_g f(x,s) = \langle f, T_x D_s g\rangle$. If g is supported inside E centered at the origin, then $\operatorname{supp} T_x D_s g \subset x + sE$, and we are gathering local information on a neighborhood of size s (roughly) around x. Therefore, the scale s indicates the resolution at which local details around x should be observed. As $s \to 0$ for any fixed x, the wavelet transform provides a "microlocal" view through g of the function f at x, a property which the short-time Fourier transform lacks. However, it can also be seen as a time-frequency representation, as $W_g f(x,s) = \left\langle \widehat{f}, M_{-x} D_{1/s}\widehat{g}\right\rangle$, providing information on f in the frequency band $\operatorname{supp} D_{1/s}\widehat{g}$.

As in the case of the STFT, the choice of window function is also a topic of its own interest. Different behaviours on the side of the signal motivate various choices for the mother wavelet, and each particular analysis suggests a possible approach. In our case, we will only be interested in a specific family of wavelets. For $\beta > 0$, define $\psi_\beta \in L^2(\mathbb{R})$ to be such that

$$\widehat{\psi_\beta}(t) = \frac{1}{c_\beta}\mathbb{1}_{[0,+\infty)}(t)t^\beta e^{-t}, \tag{2.7}$$

where the constant c_β is given by $c_\beta = \int_0^\infty 2^{2\beta-1}e^{-2t}\,\mathrm{d}t = 2^{2\beta-1}\Gamma(2\beta)$, and the Fourier transform is now normalized,

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{1/2}}\int_{\mathbb{R}} f(t)e^{-it\xi}\,\mathrm{d}t.$$

With this choice of window function, whenever $\widehat{f}$ vanishes over $(-\infty,0)$, it can be shown that the wavelet transform is actually an isometric inclusion from $H^2(\mathbb{C}^+)$, the Hardy space of the upper half plane, to $L^2(\mathbb{C}^+, s^{-2}\,\mathrm{d}x\,\mathrm{d}s)$ [Da1, Section 2.5]. Indeed, like the STFT, it satisfies the orthogonality relation [Gr1, Theorem 10.1]

$$\int_{\mathbb{R}}\int_{\mathbb{R}} W_{g_1}f_1(x,s)\overline{W_{g_2}f_2(x,s)}\frac{1}{s^2}\,\mathrm{d}x\,\mathrm{d}s = \left(\int_{\mathbb{R}}\overline{\widehat{g_1}(\omega)}\widehat{g_2}(\omega)\frac{1}{|\omega|}\,\mathrm{d}\omega\right)\langle f_1, f_2\rangle_{L^2(\mathbb{R})} \tag{2.8}$$

whenever $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R})$ and $\|\widehat{g}\|_{L^2(\mathbb{R},t^{-1})} < +\infty$, which in particular implies that W_g is an isometry from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}\times\mathbb{R}^+, s^{-2}\,\mathrm{d}x\,\mathrm{d}s)$ whenever $\|\widehat{g}\|_{L^2(\mathbb{R},t^{-1})} = 1$.

The Hardy space on the upper half plane is the space of holomorphic functions on $\mathbb{C}^+ = \{z \in \mathbb{C}\colon\ \operatorname{Im} z > 0\}$ and such that

$$\sup_{s>0}\int_{\mathbb{R}} |f(x+is)|^2\,\mathrm{d}x < +\infty.$$

This is precisely the growth condition that allows us to identify functions $f \in H^2(\mathbb{C}^+)$ with functions on the real line whose Fourier transform is supported on the positive line $[0,+\infty)$ via $f \leftrightarrow \lim_{\operatorname{Im} z\to 0} f =: \tilde{f}$ [St, Theorem 7.2.4]. If we then fix a function $g \in H^2(\mathbb{C}^+)\setminus\{0\}$ with $\|\widehat{g}\|_{L^2(\mathbb{R}^+,t^{-1})} < +\infty$, the wavelet transform of f with respect to g is

$$W_g f(z) = \langle f, \pi_z g\rangle_{H^2(\mathbb{C}^+)},$$

where $z = x + is$ and $\pi_z g(t) = s^{-1/2}g(s^{-1}(t-x))$. Moreover, since

$$\langle f_1, f_2\rangle_{H^2(\mathbb{C}^+)} = \left\langle \tilde{f}_1, \tilde{f}_2\right\rangle_{L^2(\mathbb{R})} = \left\langle \widehat{\tilde{f}_1}, \widehat{\tilde{f}_2}\right\rangle_{L^2(\mathbb{R}^+)},$$

then interpreting $\mathbb{R}\times\mathbb{R}^+$ as $\mathbb{C}^+$ we see that whenever $\|\widehat{g}\|_{L^2(\mathbb{R},t^{-1})} = 1$ holds, W_g is an isometry from $H^2(\mathbb{C}^+)$ to $L^2(\mathbb{C}^+, s^{-2}\,\mathrm{d}x\,\mathrm{d}s)$.

The reason why it is desirable to go from $L^2(\mathbb{R})$ to the more complicated Hardy space is because the latter can be seen as a particular type of L^2 space of holomorphic functions on the unit disk by means of the wavelet transform. However, this is not unique to the wavelet transform: the same occurs with the STFT, and this facilitates an approach to the problem of energy concentration in the time-frequency and time-scale planes. Since we will exploit this extensively in Chapters 3 and 4 when we talk about energy concentration in the time-frequency plane, let us introduce these transformations and see what roles V_g and W_g play in them.

2.2 The Bargmann and Bergman transforms

Let us begin by setting some notational grounds. For $t \in \mathbb{R}^d$, we will denote by t^2 the product $t \cdot t = |t|^2$, and similarly for a complex number $z = x + i\omega$, i.e. $z^2 = (x+i\omega)\cdot(x+i\omega)$. Its modulus will be $|z|^2 = z \cdot \overline{z} = x^2 + \omega^2$, and the product $z \cdot u$ of $z, u \in \mathbb{C}^d$ will just be zu.

For the reasons mentioned earlier, we will only be interested in the STFT with respect to the Gaussian window φ defined in (2.4), which we will denote by $\mathcal{V}$ instead of V_φ. With this particular choice, we can write

$$\begin{aligned} \mathcal{V}f(x,\omega) &= 2^{d/4} \int_{\mathbb{R}^d} f(t) e^{-\pi(t-x)^2} e^{-2\pi i \omega \cdot t} \,\mathrm{d}t \\ &= 2^{d/4} \int_{\mathbb{R}^d} f(t) e^{-\pi t^2} e^{2\pi x \cdot t} e^{-\pi x^2} e^{-2\pi i \omega \cdot t} \,\mathrm{d}t \\ &= 2^{d/4} e^{-\pi i x \cdot \omega} e^{-\frac{\pi}{2}(x^2+\omega^2)} \int_{\mathbb{R}^d} f(t) e^{-\pi t^2} e^{-2\pi t \cdot (x - i\omega)} e^{-\frac{\pi}{2}(x-i\omega)^2} \,\mathrm{d}t. \end{aligned} \tag{2.9}$$

If we turn $(x, \omega) \in \mathbb{R}^{2d}$ into a complex number $z = x + i\omega \in \mathbb{C}^d$, then (2.9) reads

$$\mathcal{V}f(x,-\omega) = 2^{d/4} e^{i\pi x\cdot\omega} e^{-\pi |z|^2/2} \int_{\mathbb{R}^d} f(t) e^{2\pi t \cdot z - \pi t^2 - \frac{\pi}{2} z^2} \,\mathrm{d}t. \tag{2.10}$$

For each $f \in L^2(\mathbb{R}^d)$, this integral converges uniformly over compact subsets of $\mathbb{C}^d$, and it defines an entire function, which we call the *Bargmann transform* of f.

Definition 2.7 (Bargmann transform)**.** Let $f \in L^2(\mathbb{R}^d)$. We define its Bargmann transform by

$$Bf(z) \coloneqq 2^{d/4} \int_{\mathbb{R}^d} f(t) e^{2\pi t\cdot z - \pi t^2 - \frac{\pi}{2} z^2} \,\mathrm{d}t, \quad z \in \mathbb{C}^d. \tag{2.11}$$

With this definition, (2.10) turns into

$$\mathcal{V}f(x,-\omega) = e^{i\pi x \cdot \omega} Bf(z) e^{-\pi |z|^2/2}, \tag{2.12}$$

from where we can readily see how the STFT can be looked at through the lenses of complex analysis. Furthermore, this relation is isometric when we choose the right space for Bf to live in. To this end, we introduce the *Bargmann-Fock space*.

Definition 2.8 (Bargmann-Fock space)**.** We denote by $\mathcal{F}(\mathbb{C}^d)$ the Hilbert space of all entire functions on $\mathbb{C}^d$ for which

$$\|F\|_{\mathcal{F}}^2 \coloneqq \int_{\mathbb{C}^d} |F(z)|^2 \, e^{-\pi |z|^2} \,\mathrm{d}z < +\infty,$$

endowed with the product

$$\langle F, G\rangle_{\mathcal{F}} := \int_{\mathbb{C}^d} F(z)\overline{G(z)} e^{-\pi|z|^2}\, \mathrm{d}z.$$

It is immediate that (2.12) implies that $Bf \in \mathcal{F}(\mathbb{C}^d)$ whenever $f \in L^2(\mathbb{R}^d)$, and that $\|Bf\|_{\mathcal{F}} = \|\mathcal{V}f\|_2$. Furthermore, as a map $B : L^2(\mathbb{R}^d) \longrightarrow \mathcal{F}(\mathbb{C}^d)$, the Bargmann transform is an isomorphism: it maps the basis of Hermite functions that diagonalize the Fourier transform into the basis of $\mathcal{F}(\mathbb{C}^d)$, given by the collection of all monomials of the form

$$e_\alpha(z) = \left(\frac{\pi^{|\alpha|}}{\alpha!}\right)^{1/2} z^\alpha = \prod_{j=1}^{d} \left(\frac{\pi^{\alpha_j}}{\alpha_j!}\right)^{1/2} z_j^{\alpha_j}$$

for $\alpha = (\alpha_1, ..., \alpha_d)$ with $\alpha_j \geq 0$.

The wavelet transform also enjoys a similar reinterpretation, although it requires a bit more work. To begin with, we introduce the Bergman spaces on the upper half plane and the unit disk.

Definition 2.9 (Bergman spaces)**.** For $\alpha > -1$, the Bergman space on the upper half plane, $\mathcal{A}_\alpha(\mathbb{C}^+)$, is the Hilbert space of analytic functions on $\mathbb{C}^+$ such that

$$\|f\|^2_{\mathcal{A}_\alpha(\mathbb{C}^+)} := \int_{\mathbb{C}^+} |f(z)|^2\, s^\alpha\, \mathrm{d}\mu^+ < +\infty, \tag{2.13}$$

where $\mathrm{d}\mu^+$ here is the normalized area measure on $\mathbb{C}^+$ [RT]. Similarly, the Bergman space on the disk, $\mathcal{A}_\alpha(D)$, is the Hilbert space of holomorphic functions $f\colon D \longrightarrow \mathbb{C}$ on the unit disk $D \subset \mathbb{C}^+$ such that

$$\|f\|^2_{\mathcal{A}_\alpha(D)} := \int_D |f(z)|^2\, (1-|z|^2)^\alpha\, \mathrm{d}z < +\infty. \tag{2.14}$$

These two spaces do not only resemble each other in their definitions, they are also connected via the unitary isomorphism

$$T_\alpha f(\omega) = \frac{2^{\alpha/2}}{(1-\omega)^{\alpha+2}} f\left(\frac{\omega+1}{i(\omega-1)}\right), \tag{2.15}$$

that maps $\mathcal{A}_\alpha(\mathbb{C}^+)$ into $\mathcal{A}_\alpha(D)$. Indeed, upon close inspection, the definition of T_α is motivated by a change of variables transporting D into $\mathbb{C}^+$ that also takes into account the weight s^α in (2.13). At the same time, $\mathcal{A}_\alpha(D)$ is connected to the Hardy space $H^2(\mathbb{C}^+)$ via the *Bergman transform*.

Definition 2.10 (Bergman transform)**.** Let $f \in H^2(\mathbb{C}^+)$ and $\alpha > -1$. The Bergman transform of order α of f is given by

$$B_\alpha f(z) = \frac{1}{s^{\frac{\alpha}{2}+1}} W_{\overline{\psi_{\frac{\alpha+1}{2}}}} f(-x, s) = c_\alpha \int_0^{+\infty} t^{\frac{\alpha+1}{2}} \widehat{f}(t) e^{izt}\, \mathrm{d}t, \tag{2.16}$$

where ψ is as in (2.7).

This always defines an analytic function, and since the wavelet transform with window ψ_β is an isometry from $H^2(\mathbb{C}^+)$ into $L^2(\mathbb{C}^+, s^{-2}\,\mathrm{d}x\,\mathrm{d}s)$, B_α maps $H^2(\mathbb{C}^+)$ into $\mathcal{A}_\alpha(\mathbb{C}^+)$ isometrically. Moreover, the range of $T_\alpha \circ B_\alpha$ is dense in $\mathcal{A}_\alpha(D)$.

Indeed, an orthonormal basis of $\mathcal{A}_\alpha(D)$ is given by the normalized monomials $z^n/\sqrt{c_n}$ for $n = 0, 1, 2, \ldots$, where

$$c_n = \int_D |z|^{2n} (1 - |z|^2)^\alpha \, \mathrm{d}z = 2\pi \int_0^1 r^{2n+1}(1 - r^2)^\alpha \, \mathrm{d}r = \frac{\Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(2+\alpha+n)}\pi.$$

Let now $\{\psi_n^\alpha\} \subset H^2(\mathbb{C}^+)$ be the family of functions defined by

$$\widehat{\psi_n^\alpha}(t) = b_{n,\alpha} l_n^\alpha(2t), \tag{2.17}$$

where $b_{n,\alpha}$ is chosen as a normalizing constant so that $\|\widehat{\psi_n^\alpha}\|_{L^2(\mathbb{R}^+,t^{-1})} = 1$, and l_n^α is

$$l_n^\alpha(x) := \mathbb{1}_{(0,+\infty)}(x) e^{-x/2} x^{\alpha/2} L_n^\alpha(x),$$

with $\{L_n^\alpha\}$ begin the sequence of generalized Laguerre polynomials of order α. Then

$$T_\alpha \left(B_\alpha \psi_n^\alpha \right)(\omega) = \omega^n/\sqrt{c_n}.$$

At this point, the Bargmann transform allows us to identify the STFT of $f \in L^2(\mathbb{R}^d)$ with an entire function in $\mathcal{F}(\mathbb{C}^d)$, in the same manner that $T_\alpha \circ B_\alpha$ lets us think of the wavelet transform of $f \in H^2(\mathbb{C}^+)$ as a holomorphic function in $\mathcal{A}_\alpha(D)$ on the unit disk. These identifications have been well know for a long time [Da1], and still are nowadays some of the most useful techniques when solving problems in time-frequency and time-scale analysis [Ab2, Ab1, AD, GGRT, NT, RT]. One of the keys to their utility is the structure that lies behind $\mathcal{F}(\mathbb{C}^d)$ and $\mathcal{A}_\alpha(D)$ as Hilbert spaces – they are reproducing kernel Hilbert spaces.

2.3 Reproducing Kernel Hilbert Spaces

We briefly pause at this point to introduce a particular type of spaces that reveal a lot of the structure behind the Fock and Bergman spaces. These are reproducing kernel Hilbert spaces. In them, evaluation functionals are continuous, with the vital implication that there exists a special function, called a reproducing kernel, that acts by realizing the evaluation functional through the inner product of the Hilbert space. As we will see later, this property is key in providing pointwise bounds in terms of the norm of the Hilbert space, which we will use extensively in chapters 3 and 4.

Definition 2.11. A reproducing kernel Hilbert space (RKHS) is a Hilbert space $\mathcal{H}$ where every evaluation functional $\delta_x,\ x \in X$, is continuous and bounded.

The intuition behind the implication we mentioned earlier goes as follows. The Riesz representation theorem gives one of the directions without much effort: if evaluation functionals are continuous, then there must exist a function taking a point $x \in X$ as an additional argument, which represents them through the inner product (this will be the kernel). Conversely, if such a function exists, then the inner product makes it continuous.

Definition 2.12. Let $\mathcal{H}$ be a Hilbert space of complex-valued functions over $X \neq \emptyset$. A function $K : X \times X \longrightarrow \mathbb{C}$ is called a reproducing kernel of $\mathcal{H}$ if it satisfies

(i) $\forall x \in X$, the function $K_x = K(\cdot, x) \in \mathcal{H}$.

(ii) $\forall x \in X$, $\forall f \in \mathcal{H}$, $\langle f, K(\cdot, x)\rangle = f(x)$.

A simple remark that follows immediately from the definition is that if $\mathcal{H}$ has a reproducing kernel K, then whenever $K_x \in \mathcal{H}$ is a function satisfying $K_x(y) = K(y, x)$, it has

$$K(x, y) = \langle K(\cdot, x), K(\cdot, y)\rangle = \langle K_x, K_y\rangle\,, \tag{2.18}$$

and therefore $K(x, y) = \overline{K(y, x)}$. Now we are prepared to show why these spaces hold their name.

Theorem 2.13. *Let $\mathcal{H}$ be a Hilbert space of functions $f\colon X \longrightarrow \mathbb{C}$. Then evaluation functionals are continuous if and only if $\mathcal{H}$ has a reproducing kernel.*

Proof. Let us assume first that $\mathcal{H}$ has a reproducing kernel. Then, using (2.18)

$$\begin{aligned} |\delta_x f| &= |\langle f, K(\cdot, x)\rangle| \le \|K(\cdot, x)\|\|f\| \\ &= \langle K(\cdot, x), K(\cdot, x)\rangle^{1/2}\,\|f\| = K(x, x)^{1/2}\|f\|. \end{aligned} \tag{2.19}$$

Hence δ_x is bounded.

Conversely, assume δ_x is bounded. Then the Riesz representation theorem implies that there exists $f_{\delta_x} \in \mathcal{H}$ such that $\delta_x f = \langle f, f_{\delta_x}\rangle$ for every $f \in \mathcal{H}$. If we define $K(x, y) := f_{\delta_x}(y)$ for each pair $x, y \in X$, then K is a reproducing kernel for $\mathcal{H}$. □

Now we explore two of the main properties of these spaces that we will rely on. The first important feature is the uniqueness of the reproducing kernel. Indeed, suppose $\mathcal{H}$ had two reproducing kernels, K_1 and K_2. Then

$$\langle f, K_1(\cdot, x) - K_2(\cdot, x)\rangle = f(x) - f(x) = 0$$

for all $f \in \mathcal{H}$ and all $x \in X$. Hence $\|K_1(\cdot, x) - K_2(\cdot, x)\| = 0$ and $K_1 = K_2$.

Secondly, as promised, we can extract pointwise bounds for functions in these spaces. Let $\mathcal{H}$ be a RKHS of functions $f\colon X \longrightarrow \mathbb{C}$, and let $f \in \mathcal{H}$. Then (2.19) is rearranged to read

$$|f(x)|^2\, K(x, x)^{-1} \le \|f\|^2. \tag{2.20}$$

The reason behind this rephrasing of (2.19) is merely notational, in order to agree with the treatment given in [NT] and [GGRT], and that we will use later.

An immediate consequence of this is that the continuity of evaluation maps also grants pointwise continuity for converging sequences of functions in $\mathcal{H}$. This is, if $(f_n)_{n\ge1} \subset \mathcal{H}$ converge to f in the norm of $\mathcal{H}$, then we can use (2.19) with $f_n - f$ to get

$$|f_n(x) - f(x)| \le \|K(\cdot, x)\|\|f_n - f\|.$$

Since $K(\cdot, x)$ is continuous, its norm is bounded, and the last term converges to 0. We can collect these observations into the following result on the basic properties of Hilbert spaces with reproducing kernels.

Proposition 2.14. *If $\mathcal{H}$ is a RKHS of functions $f\colon X \longrightarrow \mathbb{C}$ with reproducing kernel K, then*

(i) K is the unique reproducing kernel of $\mathcal{H}$.

(ii) $K(x, y) = \langle K(\cdot, x), K(\cdot, y)\rangle$ for all $x, y \in X$.

(iii) $K(x,y) = \overline{K(y,x)}$ *for all* $x, y \in X$.

(iv) *If* $(f_n)_{n\geq 1} \subset \mathcal{H}$ *converge to* f *in* $\mathcal{H}$*, i.e.* $\|f_n - f\| \to 0$ *as* $n \to \infty$*, then* $f_n(x) \to f(x)$ *as* $n \to \infty$ *for all* $x \in \mathcal{H}$.

Furthermore, we can already get ahead of ourselves and see exactly how these spaces will be useful in establishing stability results in Chapter 4. This is, (2.20) admits a quantitative version.

Let us start by fixing $x \in X$ and plugging $f = K_x$ into (2.20). Then,

$$|K(x,x)|^2 K(x,x)^{-1} \leq \|K_x\|^2 = K(x,x),$$

and equality is achieved. This suggests that equality occurs in (2.20) for multiples of the reproducing kernel. Motivated by this, we define

$$\Delta_f(x) = \frac{\|f\|^2 - |f(x)|^2 K(x,x)^{-1}}{\|f\|^2}.$$

Then, (2.20) reads $0 < \Delta_f(x)$, and this quantity measures the failure of (2.20) from being an equality.

Proposition 2.15. *Let* $\mathcal{H}$ *be a RKHS of complex-valued functions on* X *with kernel* K*, and let* $f \in \mathcal{H}$*. Then for every* $x \in X$ *there exists a constant* $c \in \mathbb{C}$ *with* $|c| = \|f\|$ *such that*

$$\frac{\|f - cf_x\|}{\|f\|} \leq \sqrt{2}\Delta_f(x)^{1/2}, \tag{2.21}$$

where $f_x = K(x,x)^{-1/2}K_x$ *is a multiple of the reproducing kernel.*

Proof. Fix $x \in X$ and consider the following normalization of K_x,

$$f_x(y) \coloneqq K(x,x)^{-1/2}K_x(y). \tag{2.22}$$

Then given any $f \in \mathcal{H}$, first assume that $f(x) \neq 0$ and choose $c = \|f\|\frac{f(x)}{|f(x)|}$. With this choice, and thanks to (2.20),

$$\begin{aligned}\mathbb{R} \ni \langle f, cf_x\rangle &= \bar{c}K(x,x)^{-1/2}f(x) = \|f\|K(x,x)^{-1/2}\,|f(x)| \\ &= \frac{|f(x)|^2 K(x,x)^{-1}}{|f(x)|\,K(x,x)^{-1/2}}\|f\| \geq \|f\|^2(1 - \Delta_f(x)).\end{aligned}$$

When we now look at the distance in the Hilbert space norm between f and our candidate, cf_x, we find that

$$\begin{aligned}\|f - cf_x\|^2 &= \|f\|^2 + |c|^2\,\|f_x\|^2 - 2\,\mathrm{Re}\,\langle f, cf_x\rangle \\ &\leq 2\|f\|^2 - 2\|f\|^2(1 - \Delta_f(x)) = 2\|f\|^2\Delta_f(x).\end{aligned}$$

Assume now that $f(x) = 0$. Then $\Delta_f(x) = 1$ and if $c = \|f\|$, the computation above shows that $\|f - cf_x\|^2 = 2\|f\|^2$. One obtains (2.21) by rearranging the terms. $\square$

Although this result might seem innocent at first glance, it is a crucial part of the ideas presented in Chapter 4, and it will be relevant in both the Fock and Bergman spaces. In more tangible terms, we have just shown that the closer that $|f(x)|^2 K(x,x)^{-1/2}$ is to $\|f\|^2$ over all $x \in X$, the closer f is to a multiple of the kernel in the Hilbert space.

2.3.1 The Bargmann-Fock and Bergman spaces

The spaces $\mathcal{F}(\mathbb{C}^d)$, $\mathcal{A}_\alpha(\mathbb{C}^+)$ and $\mathcal{A}_\alpha(D)$ are all reproducing Kernel Hilbert spaces. From now on, we will mostly be interested in $\mathcal{F}(\mathbb{C}^d)$ and $\mathcal{A}_\alpha(D)$, and we will refer to the latter simply by $\mathcal{A}_\alpha$.

Theorem 2.16. *The Bargmann-Fock space is a RKHS satisfying*

$$|F(z)| \leq \|F\|_{\mathcal{F}} e^{\pi|z|^2/2}, \quad \forall z \in \mathbb{C}^d. \tag{2.23}$$

The reproducing kernel is $K_\omega(z) = e^{\pi z \cdot \overline{\omega}}$, and equality holds in (2.23) *at some point $z_0 \in \mathbb{C}$ if and only if $F = cF_{z_0}$ for some $c \in \mathbb{C}$ with $|c| = \|F\|_{\mathcal{F}}$, where*

$$F_{z_0}(z) := e^{-\pi|z_0|^2/2} e^{\pi z \cdot \overline{z_0}}, \quad z \in \mathbb{C}. \tag{2.24}$$

Proof. Given that we can represent $F(z) = \sum_{\alpha \geq 0} \langle F, e_\alpha \rangle e_\alpha(z)$, the Cauchy-Schwarz inequality shows that evaluation functionals are bounded,

$$|F(z)| \leq \|F\|_{\mathcal{F}} \left(\sum_{\alpha \geq 0} \frac{\pi^\alpha}{\alpha!} |z^\alpha|^2 \right)^{1/2} = \|F\|_{\mathcal{F}} e^{-\pi|z|^2/2}.$$

The characterization of equality cases follows from equality in the Cauchy-Schwarz inequality, but we can also directly use Proposition 2.15. □

Of course, the same proof shows that $\mathcal{A}_\alpha$ is a RKHS.

Theorem 2.17. *The Bergman space on the disk is a RKHS satisfying*

$$|f(z)| \leq \|f\|_{\mathcal{A}_\alpha} \sqrt{\frac{\alpha+1}{\pi}} (1 - |z|^2)^{-(\alpha+2)/2}, \quad \forall z \in \mathbb{C}^d. \tag{2.25}$$

The reproducing kernel is $K_\omega(z) = \frac{\alpha+1}{\pi}(1 - \overline{w} z)^{-(\alpha+2)}$, and equality holds in (2.25) *at some point $\omega \in D$ if and only if $f = cf_\omega$ for some $c \in \mathbb{C}$ with $|c| = \|f\|_{\mathcal{A}_\alpha}$, where*

$$f_\omega(z) := \sqrt{\frac{\alpha+1}{\pi}} (1 - |z|^2)^{-\frac{\alpha+2}{2}} (1 - \overline{\omega} z)^{-(\alpha+2)}, \quad z \in D. \tag{2.26}$$

An immediate consequence of these facts is that, by Proposition 2.15, both (2.23) and (2.25) have quantitative versions of their own, which will appear later in chapters 4 and 5. We refer to [Gr1] for a more detailed exposition of the structure of the Fock space as a RKHS, and to [AD, Proposition 1] for general conditions on when radial measures give rise to a RKHS.

Chapter 3

Energy Concentration

Energy concentration of functions in the time-frequency plane is a topic of interest in both its mathematical and applied aspects. For a time-frequency representation like the STFT, it is natural to understand the energy of $V_g f$ as the square of its modulus, $|V_g f|^2$. This is known as the *spectrogram* of f, and in signal analysis it is a way of measuring the intensity of f at a point in the time-frequency plane.

The fraction of energy of $f \in L^2(\mathbb{R}^d)$ captured by a measurable set $\Omega \subset \mathbb{R}^{2d}$ in the time-frequency plane is given by the Rayleigh quotient [AGR, APR, Da2, MR]

$$\Phi_\Omega(f) \coloneqq \frac{\int_\Omega |\mathcal{V}f|^2}{\int_{\mathbb{R}^{2d}} |\mathcal{V}f|^2} = \frac{\langle \mathcal{V}^* \mathbb{1}_\Omega \mathcal{V} f, f\rangle}{\|f\|_2^2},$$

and the total energy that Ω accumulates is therefore

$$\Phi_\Omega \coloneqq \sup_{f\in L^2(\mathbb{R}^d)} \Phi_\Omega(f) = \sup_{f\in L^2(\mathbb{R}^d)} \frac{\langle \mathcal{V}^* \mathbb{1}_\Omega \mathcal{V} f, f\rangle}{\|f\|_2^2}. \tag{3.1}$$

This is precisely the norm of the localization operator $\mathcal{V}^* \mathbb{1}_\Omega \mathcal{V} \colon L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$, sometimes called Anti-Wick or Toeplitz operator. It has been thoroughly studied since it was first introduced in [Be] and [Da2], and its spectral properties were well known for a long time in the case where Ω is radially symmetric. Under this condition, it is diagonalizable in the basis of Hermite functions, and conversely, if a Hermite function is an eigenfunction and Ω is simply connected, then Ω is a ball centered at the origin (for a more detailed account, see [AD, AGR, APR, MR]). In general, whenever $\Omega \subset \mathbb{R}^{2d}$ is of finite measure, $\mathcal{V}^* \mathbb{1}_\Omega \mathcal{V}$ is a compact operator, and the supremum in (3.1) is attained.

The task of finding an upper bound for Φ_Ω and characterizing the functions maximizing the norm of $\mathcal{V}^* \mathbb{1}_\Omega \mathcal{V}$ remained, however, an open question (a conjecture, in fact, by L. D. Abreu and M. Speckbacher [AS]) before the work of F. Nicola and P. Tilli [NT]. In their paper, they give a complete answer in the form of their main result, the *Faber-Krahn inequality for the short-time Fourier transform.* In dimension $d = 2$, it reads as follows.

Theorem 3.1 (Faber-Krahn inequality for the STFT). *Among all measurable sets $\Omega \subset \mathbb{R}^2$ having prescribed (finite, nonzero) measure, Φ_Ω satisfies the inequality*

$$\Phi_\Omega \leq 1 - e^{-|\Omega|}, \tag{3.2}$$

achieves its maximum, and with it the equality in (3.2)*, if and only if Ω is equivalent, up to a set of measure zero, to a ball.*

Moreover, when Ω is a ball of center (x_0, ω_0), the only functions f that achieve the maximum in (3.1) *are those of the kind*

$$f(x) = c\varphi_{z_0}(x) = ce^{2\pi i\omega_0 x}\varphi(x - x_0), \quad c \in \mathbb{C} \setminus \{0\}, \quad z_0 = x_0 + i\omega_0, \tag{3.3}$$

that is, scalar multiples of the Gaussian window defined in (2.4)*, translated and modulated according to* (x_0, ω_0)*.*

When rewritten differently, the result shares a similar statement with the classical Faber-Krahn inequality, which states that among all open sets Ω of given measure, the ball minimizes the first Dirichlet eigenvalue,

$$\lambda_\Omega := \min_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u(z)|^2 \, dz}{\int_\Omega u(z)^2 dz}.$$

If we let $T_\Omega : H_0^1(\Omega) \longrightarrow H_0^1(\Omega)$ be the linear operator that to every $u \in H_0^1(\Omega)$ associates its weak solution $T_\Omega u \in H_0^1(\Omega)$ of the problem $-\Delta T_\Omega u = u$ in Ω, then integrating by parts,

$$\int_\Omega u^2 = -\int_\Omega u \Delta T_\Omega u = \int_\Omega \nabla u \cdot \nabla T_\Omega u = \langle T_\Omega u, u \rangle_{H_0^1},$$

and the statement is the same as claiming that

$$\lambda_\Omega^{-1} := \max_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\langle T_\Omega u, u \rangle_{H_0^1}}{\|u\|_{H_0^1}}$$

is maximized, among all open Ω of given measure, by a ball. This establishes a clear similarity between λ_Ω^{-1} and Φ_Ω, the operators T_Ω and $\mathcal{V}^* \mathbb{1}_\Omega \mathcal{V}$, and the classic Faber-Krahn inequality and Theorem 3.1.

In dimension $d \geq 2$, Theorem 3.1 is also valid after a few adaptations, and the proof remains the same up to a few minor changes. Apart from the intrinsic value of the result, Theorem 3.1 immediately comes with an interesting connection to uncertainty principles. In particular, it improves a result by Gröchenig [Gr1, Theorem 3.3.3] that follows from Lieb's uncertainty principle [Li], thus solving the long-standing question of finding the optimal bound for the measure of the set in which the STFT of a function is mostly supported.

Theorem 3.2 (Sharp uncertainty principle for the STFT)**.** *If $f \in L^2(\mathbb{R})\setminus\{0\}$ and $\Omega \subset \mathbb{R}^2$ satisfy*

$$\Phi_\Omega(f) \geq 1 - \varepsilon,$$

for some $\varepsilon > 0$, then

$$|\Omega| \geq \log \frac{1}{\varepsilon},$$

with equality if and only if Ω is a ball and f is as in (3.3)*, with (x_0, ω_0) being the center of Ω.*

The techniques involved in the proof of Theorem 3.1 take full advantage of the Bargmann transform and the Fock space in order to exploit properties like analyticity and the structure of $\mathcal{F}(\mathbb{C})$ as a RKHS. Because of this reason, these very same techniques can also be modified when approaching a wavelet concentration analogue problem. This was precisely the work done by J. P. G. Ramos and P. Tilli [RT], where the *Faber-Krahn inequality for the wavelet transform* is proved. For a set $\Delta \subset \mathbb{C}^+$, the quantity Φ_Δ (recall (3.1)) is now

$$C_\Delta^\beta := \sup\left\{ \int_\Delta \left| W_{\overline{\psi_\beta}} f(x,s) \right|^2 \frac{\mathrm{d}x \, \mathrm{d}s}{s^2} : f \in H^2(\mathbb{C}^+), \|f\|_{H^2} = 1 \right\}, \tag{3.4}$$

and the volume constraint under which we now consider the problem changes as well. Instead of the Lebesgue measure, we fix the hyperbolic measure of the set Δ, that is

$$\nu(\Delta) = \int_\Delta \frac{1}{s^2}\,\mathrm{d}x\,\mathrm{d}s < +\infty. \tag{3.5}$$

Then, the result for wavelet transforms reads as follows.

Theorem 3.3 (Faber-Krahn inequality for wavelet transforms)**.** *It holds that*

$$C_\Delta^\beta \leq C_{\Delta^*}^\beta, \tag{3.6}$$

where $\Delta^* \subset \mathbb{C}^+$ denotes any pseudohyperbolic disk so that $\nu(\Delta) = \nu(\Delta^*)$. Moreover, equality holds in (3.6) *if and only if Δ is a pseudohyperbolic disk of measure $\nu(\Delta)$.*

Moreover, equality holds in (3.6) *for a function $f_{\Delta,\beta}$ if and only if it is a multiple of $\Psi_{\beta,\omega}$, defined as*

$$\Psi_{\beta,\omega} = \frac{1}{y^{1/2}}\psi_0^{2\beta-1}\left(\frac{t-x}{y}\right),$$

where $\omega = x + iy$ is the center of Δ and $\psi_0^{2\beta-1}$ is as in (2.17)*.*

We will dedicate the current chapter to introducing the main ideas behind these results, and comparing the arguments involved in the STFT and wavelet versions. This will hopefully give a profound enough understanding of the techniques showcased in [NT, RT] that makes the approach to proving a sharp quantitative version of the Faber-Krahn inequalities in both contexts (which we will discuss in Chapter 4) rather natural. After this, we will present the d-dimensional version of Theorem 3.1, although we will omit the proof and we refer the reader to [NT, Section 4], since the same ideas are employed as in the one dimensional case.

3.1 The Faber-Krahn inequality for the STFT

Let us begin with a simple observation. Given $\Omega \subset \mathbb{R}^2$, the expression of $\Phi_\Omega(f)$ in terms of the Bargmann transform of f is the following,

$$\Phi_\Omega(f) = \frac{\int_\Omega |\mathcal{V}f(x,\omega)|^2\,\mathrm{d}x\,\mathrm{d}\omega}{\|f\|_2^2} = \frac{\int_{\Omega'} |Bf(z)|^2\,e^{-\pi|z|^2}\,\mathrm{d}z}{\|Bf\|_{\mathcal{F}}^2},$$

where $\Omega' = \{(x,-\omega) : (x,\omega) \in \Omega\}$. Since B is unitary, maximizing $\Phi_\Omega(f)$ over $L^2(\mathbb{R}) \setminus \{0\}$ turns out to be the same as solving the problem

$$\Phi_\Omega = \max_{F \in \mathcal{F}(\mathbb{C})\setminus\{0\}} \frac{\int_\Omega |F(z)|^2\,e^{-\pi|z|^2}\,\mathrm{d}z}{\|F\|_{\mathcal{F}}^2}. \tag{3.7}$$

On top of this, if f achieves the equality in (3.2), its Bargmann transform is

$$B(cM_\omega T_x\varphi) = ce^{\pi i x\omega}e^{-\pi|z_0|^2/2}e^{\pi z\overline{z_0}} = c' F_{z_0}, \quad z_0 = x + i\omega,\ c \in \mathbb{C}\setminus\{0\},$$

where $F_{z_0} = e^{-\pi|z_0|^2/2}e^{\pi z\overline{z_0}}$ is the normalized reproducing Kernel of $\mathcal{F}(\mathbb{C})$ (recall (2.24)). Therefore, we can reduce Theorem 3.1 to its Fock space counterpart.

Theorem 3.4. *For every $F \in \mathcal{F}(\mathbb{C}) \setminus \{0\}$ and every measurable set $\Omega \subset \mathbb{R}^2$ of finite measure, we have*

$$\frac{\int_\Omega |F(z)|^2 e^{-\pi|z|^2}}{\|F\|_{\mathcal{F}}^2} \,\mathrm{d}z \leq 1 - e^{-|\Omega|}. \tag{3.8}$$

Moreover, equality occurs (for some $F \in \mathcal{F}(\mathbb{C})$ and some $\Omega \subset \mathbb{C}$) if and only if $F = cF_{z_0}$ (for some $z_0 \in \mathbb{C}$ and some nonzero $c \in \mathbb{C}$) and Ω is equivalent, up to a set of measure zero, to a ball centered at z_0.

For a given $F \in \mathcal{F}(\mathbb{C})$ and under the condition that $|\Omega| = s > 0$, the left-hand side of (3.7) will be maximized when Ω is the *unique* super-level set of F with measure s, $A_{t(s)} = \{u > t(s)\}$. Its uniqueness is a feature of the analyticity of the integrand $u(z) = |F(z)|^2 e^{-\pi|z|^2}$, as we will see, and this motivates the study of integrals over super-level sets of u in terms of s. The issue of bounding these pointwise is the goal henceforth, and the main component in the proof of Theorem 3.4 is a property of these in connection to the distribution function of u.

We begin this approach to Theorem 3.4 by introducing the objects that will play major roles in the proof. These are the *distribution function* of u,

$$\mu(t) := |A_t| = |\{u > t\}|\,, \quad 0 < t \leq \max_{\mathbb{C}} u, \tag{3.9}$$

and its inverse, also known as the *decreasing rearrangement*,

$$u^*(s) := \sup\{t \geq 0 \colon \mu(t) > s\}\,, \quad s \geq 0. \tag{3.10}$$

Note that u is non-constant, and since $F \in \mathcal{F}(\mathbb{C})$ is holomorphic, u is also real analytic when seen as a function $u\colon \mathbb{R}^2 \longrightarrow [0, \infty)$. In particular, this implies that its level sets all have measure zero [KP, Mi],

$$|\{u = t\}| = 0 \quad \forall t \geq 0, \tag{3.11}$$

from which it follows that μ is continuous at every $t \in (0, \max u]$. Moreover, the same argument shows that

$$\{|\nabla u| = 0\} = 0. \tag{3.12}$$

We can extract some more information about the level sets of u using the Morse-Sard lemma. In fact, this tells us that for a.e. $t \in (0, \max u)$, A_t is open, bounded and has a smooth boundary,

$$\partial A_t = \{u = t\} \quad \text{for a.e. } t \in (0, \max u). \tag{3.13}$$

Finally, μ is strictly decreasing, which makes u^*, defined in (3.10) as its inverse function, map $[0, \infty)$ into $(0, \max u]$ decreasingly and continuously.

Condition (3.12) is known as the *coarea regularity* of u in the terminology of [AL, Section 3.6], and it is enough to grant the absolute continuity of μ, as the following lemma shows.

Lemma 3.5. *The function μ is absolutely continuous on the compact subintervals of* $(0, \max u]$ *and*

$$-\mu'(t) = \int_{\{u=t\}} |\nabla u|^{-1} \,\mathrm{d}\mathcal{H}^1, \quad \textit{for a.e. } t \in (0, \max u). \tag{3.14}$$

Similarly, u^* is absolutely continuous on the compact subintervals of $[0, \infty)$ *and*

$$-(u^*)'(s) = \left(\int_{\{u=u^*(s)\}} |\nabla u|^{-1} \,\mathrm{d}\mathcal{H}^1 \right)^{-1}, \quad \textit{for a.e. } s \geq 0. \tag{3.15}$$

Proof. We begin by noticing that u is locally Lipschitz, and therefore by an approximation argument we can use the coarea formula to conclude that for any Borel function $h : \mathbb{R}^2 \longrightarrow [0, \infty]$,

$$\int_{\mathbb{R}^2} h(z)\,|\nabla u(z)|\,\mathrm{d}z = \int_0^{\max u} \left(\int_{\{u=\tau\}} h\,\mathrm{d}\mathcal{H}^1 \right) \mathrm{d}\tau,$$

where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. Choosing

$$h(z) = \mathbb{1}_{A_t}(z)\,|\nabla u(z)|^{-1},$$

(with the convention that $0 \cdot \infty = 0$ in the integral and $|\nabla u(z)|^{-1}$ is ∞ if z is a critical point of u), we see that $h(z)\,|\nabla u(z)|$ equals $\mathbb{1}_{A_t}(z)$ for a.e. z thanks to (3.12). Then, recalling the definition of μ in (3.9),

$$\mu(t) = \int_t^{\max u} \left(\int_{u=\tau} |\nabla u|^{-1}\,\mathrm{d}\mathcal{H}^1 \right) \mathrm{d}\tau, \quad \forall t \in [0, \max u]. \tag{3.16}$$

In particular, this expression for μ implies that it is absolutely continuous on the compact subintervals of $(0, \max u]$, and (3.14) follows by straight computation.

Now we address the absolute continuity of u^*. For this, we will check that $(u^*)'$ exists almost everywhere on $(0, \infty)$ and satisfies (3.15), and that u^* maps sets of measure zero into sets of measure zero (known as Luzin's N property). This, together with the fact that it is a function of bounded variation, its continuity and the Banach-Zaretsky Theorem (see [He, Theorem 17]) will prove that it is absolutely continuous.

Let $D \subset (0, \max u]$ be the set where $\mu'(t)$ exists, coincides with (3.14) and is strictly positive, and set $D_0 = (0, \max u] \setminus D$. Notice two things: the integral in (3.14) is strictly positive for all t in $(0, \max u)$ (otherwise the isoperimetric inequality would give $|A_t| = 0$ and $t \geq \max u$); and μ is absolutely continuous, coinciding with (3.14) almost everywhere. This means that $|D_0| = 0$. Letting $\widehat{D} = \mu(D)$ and $\widehat{D_0} = \mu(D_0)$, we find that $|\widehat{D_0}| = 0$ by the absolute continuity of μ, and $\widehat{D} = [0, \infty) \setminus \widehat{D_0}$ since μ is invertible. On the other hand, using (3.14) we see that $(u^*)'(s)$ exists for every $s \in \widehat{D}$ and

$$-(u^*)'(s) = -\frac{1}{\mu'\left(\mu^{-1}(s)\right)} = \left(\int_{u=u^*(s)} |\nabla u|^{-1}\,\mathrm{d}\mathcal{H}^1 \right)^{-1}, \quad \forall s \in \widehat{D}.$$

This readily implies (3.15) since $|\widehat{D_0}| = 0$. Now, since u^* is differentiable everywhere on $\widehat{D}$, it maps sets of measure zero $N \subset \widehat{D}$ into sets of measure zero. Furthermore, since $u^*(\widehat{D_0}) = D_0$ and this set has measure zero, and because $\widehat{D} \cup \widehat{D_0}$ covers the domain of definition $[0, \infty)$ of u^*, it follows that u^* enjoys Luzin's N property. By the aforementioned argument, it is absolutely continuous on every compact interval $[0, a]$. □

Seeing $\mu'(t)\,\mathrm{d}t$ as a Radon measure on the compact subintervals of $(0, \max u]$, Lemma 3.5 shows how (3.12) rules out the possibility of it having a singular part. We now move on and present a key inequality that u^* and its derivative satisfy, which will play a pivotal role in the second part of the analysis. This is the following "convexity" property of u^*.

Lemma 3.6. *It holds that*

$$\left(\int_{\{u=u^*(s)\}} |\nabla u|^{-1}\,\mathrm{d}\mathcal{H}^1 \right)^{-1} \leq u^*(s) \quad \textit{for a.e. } s \geq 0 \tag{3.17}$$

and

$$(u^*)'(s) + u^*(s) \geq 0 \quad \textit{for a.e. } s \geq 0. \tag{3.18}$$

Proof. Let $t = u^*(s)$ and recall that for a.e. $t \in (0, \max u)$, the set A_t has a smooth boundary (3.13). Then we can use Hölder's inequality to obtain

$$\mathcal{H}^1(\{u=t\})^2 = \int_{\{u=t\}} \frac{|\nabla u|^{1/2}}{|\nabla u|^{1/2}} \, \mathrm{d}\mathcal{H}^1 \leq \left(\int_{\{u=t\}} |\nabla u|^{-1} \, \mathrm{d}\mathcal{H}^1 \right) \int_{\{u=t\}} |\nabla u| \, \mathrm{d}\mathcal{H}^1, \tag{3.19}$$

and the isoperimetric inequality reads

$$4\pi \, |\{u > t\}| \leq \mathcal{H}^1(\{u=t\})^2. \tag{3.20}$$

Combining these two expressions and dividing by t we come to find that

$$\frac{1}{t} \left(\int_{\{u=t\}} |\nabla u|^{-1} \, \mathrm{d}\mathcal{H}^1 \right)^{-1} \leq \frac{\int_{\{u=t\}} \frac{|\nabla u|}{t} \, \mathrm{d}\mathcal{H}^1}{4\pi \, |\{u > t\}|}. \tag{3.21}$$

Notice now that $\{u = t\}$ is the boundary of A_t, and if ν is the outer normal to $\{u = t\}$, then along this set, $|\nabla u| = \nabla u \cdot \nu$. Not only that, since u is equal to t on ∂A_t, the quotient $|\nabla u| / t$ can be interpreted as $-(\nabla u / u) \cdot \nu = -(\nabla \log u) \cdot \nu$, turning the integral on the right-hand side of (3.21) into

$$\int_{\{u=t\}} \frac{|\nabla u|}{t} \, \mathrm{d}\mathcal{H}^1 = -\int_{\partial A_t} (\nabla \log u) \cdot \nu \, \mathrm{d}\mathcal{H}^1 = -\int_{A_t} \Delta \log u(z) \, \mathrm{d}z.$$

Since $F(z)$ is a holomorphic function, $\log |F(z)|$ is harmonic and we obtain

$$\Delta(\log u(z)) = \Delta \left(\log |F(z)|^2 + \log e^{-\pi |z|^2} \right) = \Delta(-\pi \, |z|^2) = -4\pi. \tag{3.22}$$

Using this computation in (3.21) we see that the quotient on the right is 1, and recalling that $t = u^*(s)$, (3.17) follows. Lastly, (3.18) is simply (3.15) put together with (3.17). □

We can now address the main part of the analysis. That is, we can study the integrals of u over its super-level sets

$$I(s) = \int_{\{u > u^*(s)\}} u(z) \, \mathrm{d}z, \quad s \in [0, \infty) \tag{3.23}$$

in connection to the function u^*. Intuitively, since $u(z)$ is largest within A_t when compared to any other set of equal measure, proving the desired bound on $I(s)$ allow us to disregard the complexity of Ω.

Lemma 3.7. *The function I defined by* (3.23) *is of class C^1 on $[0, \infty)$, and*

$$I'(s) = u^*(s), \quad s \geq 0. \tag{3.24}$$

Moreover, I' is locally absolutely continuous and

$$I''(s) + I'(s) \geq 0 \quad \text{for a.e. } s \in [0, \infty). \tag{3.25}$$

Proof. Let $h > 0$ and $s \geq 0$. Then

$$I(s+h) - I(s) = \int_{\{u^*(s+h) < u \leq u^*(s)\}} u(z) \, \mathrm{d}z,$$

and by the definition of u^* in (3.10) and (3.9), $|A_{u^*(s)}| = s$. This means that the measure of the set over which we are integrating is just

$$|\{u^*(s+h) < u \leq u^*(s)\}| = |A_{u^*(s+h)}| - |A_{u^*(s)}| = s + h - s = h.$$

Therefore,

$$u^*(s+h) \leq \frac{I(s+h) - I(s)}{h} \leq u^*(s),$$

and the same inequality holds true for $h < 0$ whenever $s + h > 0$. Since u^* is continuous, it is enough to let $h \to 0$ when $s \geq 0$ and $h \to 0^+$ when $s = 0$ in order to prove (3.24).

The fact that I is C^1 follows from the continuity of u^*, and that I' is absolutely continuous on the compact subintervals of $[0, \infty)$ by Lemma 3.5. Finally, $I''(s) = (u^*)'(s)$ and (3.25) follows from (3.18). □

We are now ready to prove Theorem 3.4.

Proof of Theorem 3.4. We can assume that $\|F\|_{\mathcal{F}} = 1$ by homogeneity, which leaves us with the task of showing that

$$\int_\Omega u(z)\,\mathrm{d}z \leq 1 - e^{-s}$$

for every set $\Omega \subset \mathbb{R}^2$ of measure $s \geq 0$. Notice that if $A_{u^*(s)}$ is the super-level set of u that has measure s, then for any Ω of measure s,

$$\int_\Omega u(z)\,\mathrm{d}z \leq \int_{A_{u^*(s)}} u(z)\,\mathrm{d}z. \tag{3.26}$$

Indeed,

$$\int_{A_{u^*(s)}} u(z)\,\mathrm{d}z - \int_\Omega u(z)\,\mathrm{d}z = \int_{A_{u^*(s)}\setminus\Omega} u(z)\,\mathrm{d}z - \int_{\Omega\setminus A_{u^*(s)}} u(z)\,\mathrm{d}z,$$

and since $u > u^*(s)$ over $A_{u^*(s)} \setminus \Omega$, and $u < u^*(s)$ over $\Omega \setminus A_{u^*(s)}$, and they both have equal measure, it follows that this difference is positive. Therefore, we aim to prove

$$I(s) = \int_{A_{u^*(s)}} u(z)\,\mathrm{d}z \leq 1 - e^{-s}, \quad s \geq 0.$$

By letting $s = -\log\sigma$, this is equivalent to showing

$$G(\sigma) = I(-\log\sigma) \leq 1 - \sigma, \quad \sigma \in (0, 1]. \tag{3.27}$$

Notice that $G(1) = I(0) = 0$, and by monotone convergence,

$$\lim_{\sigma\to 0^+} G(\sigma) = \lim_{s\to\infty} I(s) = \|F\|_{\mathcal{F}} = 1.$$

Then G extends to a continuous function on $[0, 1]$. Moreover, in terms of G, (3.25) means that $G'(\sigma)$ is nondecreasing, which makes G a convex function. Since it agrees with $1 - \sigma$ on $\sigma = 0, 1$, its convexity implies the validity of (3.27).

Let us assume now that equality holds in (3.8) for some $F \in \mathcal{F}(\mathbb{C})$ with $\|F\|_{\mathcal{F}} = 1$ and some $\Omega \subset \mathbb{R}^2$ of measure $s_0 > 0$. Then, in particular $I(s_0) = 1 - e^{-s_0}$, and by setting $\sigma_0 = e^{-s_0}$, we have $G(\sigma_0) = 1 - \sigma_0$. This implies that $G(\sigma)$ coincides with $1 - \sigma$ everywhere by convexity, or equivalently, that $I(s) = 1 - e^{-s}$ for all $s \geq 0$. In particular, $I'(0) = 1$, which by (3.24) means that $\max u = u^*(0) = 1$. Inequality (2.23) reads

$$\max u \leq 1, \tag{3.28}$$

and since $1 = u^*(0) = \max u$, the characterization of equality cases for (2.23) implies that F has to be of the form $F = cF_{z_0}$ for some $z_0 \in \mathbb{C}$ with $|c| = 1$.

We are left to show that Ω is equivalent to a ball centered at z_0. To this end, notice that if equality holds in (3.8), then (3.27) is an equality as well, which means that Ω and $A_{u^*(s_0)}$ coincide up to a set of measure zero. Now, using the expression of F_{z_0} we find that

$$u(z) = |c|^2 \left|F_{z_0}\right|^2 e^{-1\pi|z|^2} = e^{-\pi|z-z_0|^2}.$$

Therefore u is radially symmetric about z_0 and radially decreasing, so $A_{u^*(s_0)}$ is a ball of measure s_0 centered at z_0.

The last step of the proof is to show that if F is of the form cF_{z_0} and Ω is equivalent to a ball of radius $r > 0$ centered at z_0, then equality in (3.8) holds. This can be done using polar coordinates,

$$\int_\Omega u(z)\,\mathrm{d}z = |c|^2 \int_{\{|z|<r\}} e^{-\pi|z|^2} = 2\pi\,|c|^2 \int_0^r \rho e^{-\pi\rho^2}\,\mathrm{d}\rho = |c|^2\,(1 - e^{-\pi r^2}),$$

which completes the proof, since $|c| = \|cF_{z_0}\|_{\mathcal{F}}$. □

3.2 The Faber-Krahn inequality for wavelet transforms

The techniques we described in the previous section can be adapted to show Theorem 3.3, as done in [RT]. Prior to being proven, this was conjectured by L. D. Abreu and M. Dörfler in [AD] after showing a result on an inverse problem on the related localization operator. Indeed, maximum wavelet concentration over $\Delta \subset \mathbb{C}^+$ can be seen as the norm of the localization operator $P_{\Delta,\beta} \colon L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$, which is (recalling (2.8)),

$$P_{\Delta,\beta} f = \left((W_{\overline{\psi_\beta}})^* \mathbb{1}_\Delta W_{\overline{\psi_\beta}}\right) f, \quad \|P_{\Delta,\beta}\|_{2\to 2} = \sup_{\|f\|_2=1} \int_\Delta \left|W_{\overline{\psi_\beta}} f\right|^2 \frac{1}{s^2}\,\mathrm{d}x\,\mathrm{d}s.$$

When trying to find bounds for this norm, one sees by [Wo, Propositions 12.1 and 12.2] that

$$\|P_{\Delta,\beta}\|_{2\to 2} \leq \begin{cases} 1 \\ \nu(D)/c_\beta. \end{cases}$$

We are led to consider two different problems. If we use the first bound, we impose no conditions on Δ, and aim to maximize C_Δ^β (recall (3.4)). But this is then trivially achieved by $\Delta = \mathbb{C}^+$. The second bound, however, leads us to consider the fixed ν-measure constraint (3.5), and therefore we now aim to maximize C_Δ^β over sets $\Delta \subset \mathbb{C}^+$ with $\nu(\Delta) = s > 0$ fixed. It is now that we encounter Theorem 3.3 as the solution.

The first step towards proving the theorem is reducing the problem, as in the STFT case, to one on concentration of analytic functions. This time, it is done through the Bergman transform and into the Bergman space on the disc. Indeed, recall that $T_\alpha \circ B_\alpha$ is a unitary isomorphism (see (2.15) and (2.16)) from $H^2(\mathbb{C}^+)$ into $\mathcal{A}_\alpha$, and observe that, if $\alpha = 2\beta - 1$,

$$\int_\Delta \left| W_{\overline{\psi_\beta}} f \right|^2 \frac{1}{s^2} \,\mathrm{d}x\,\mathrm{d}s = \int_\Omega \left| T_\alpha \left(B_\alpha f \right)(\omega) \right|^2 (1 - |\omega|^2)^\alpha \,\mathrm{d}\omega, \tag{3.29}$$

where $\Omega \subset D$ is the image of $\tilde{\Delta} = \{z = x + is\colon -x + is \in \Delta\}$ under the map $z \mapsto \frac{z-i}{z+1}$ on $\mathbb{C}^+$. On the other hand, we are naturally led to consider the hyperbolic measure μ on the unit disk defined by

$$\mu(\Omega) := \int_\Omega (1 - |z|^2)^{-2} \,\mathrm{d}z, \quad \Omega \subset D,$$

since

$$\mu(\Omega) = \int_D \mathbb{1}_\Delta \left(\frac{\omega + 1}{i(\omega - 1)} \right) (1 - |\omega|^2)^{-2} \,\mathrm{d}\omega = \frac{1}{4} \int_\Delta \frac{1}{s^2} \,\mathrm{d}x\,\mathrm{d}s = \frac{\nu(\Delta)}{4}.$$

Studying the problem we mentioned before is therefore the same as maximizing

$$R(f, \Omega) := \frac{\int_\Omega |f(z)|^2 (1 - |z|^2)^{\alpha+2} \,\mathrm{d}\mu(z)}{\|f\|^2_{\mathcal{A}_\alpha}} \tag{3.30}$$

among all $f \in \mathcal{A}_\alpha$ and under the constraint that for $\Omega \subset D$, its hyperbolic measure $\mu(\Omega) = s/4 > 0$ is fixed.

As before, we can rephrase Theorem 3.3 in terms of the Bergman space, which reads as follows.

Theorem 3.8. *Let $\alpha > -1$ and $s > 0$ be fixed. Among all functions $f \in \mathcal{A}_\alpha$ and among all measurable sets $\Omega \subset D$ such that $\mu(\Omega) = s$, the quotient $R(f, \Omega)$ defined in* (3.30) *satisfies the inequality*

$$R(f, \Omega) \le 1 - (1 + s/\pi)^{-(1+\alpha)}. \tag{3.31}$$

Moreover, equality in (3.31) *occurs if and only if Ω is a ball centered at some $\omega \in D$ and such that $\mu(\Omega) = s$, and f is a multiple of the reproducing kernel K_ω of $\mathcal{A}_\alpha$, i.e. there is $c \in \mathbb{C} \setminus \{0\}$ with $|c| = \|f\|_{\mathcal{A}_\alpha}$ and f is of the form* (2.26).

The function on the right-hand side of (3.30) is precisely $R(1, D_s)$, where D_s is a disk of measure $\mu(D_s) = s$. This can be easily seen by taking such a disk centered at the origin:

$$R(1, B_r) = \frac{\int_0^r \rho(1 - \rho^2)^\alpha \,\mathrm{d}\rho}{\int_0^1 \rho(1 - \rho^2)^\alpha \,\mathrm{d}\rho} = 1 - (1 - r^2)^{1+\alpha},$$

and since

$$\mu(B_r) = \int_{B_r} (1 - |z|^2)^{-2} \,\mathrm{d}z = 2\pi \int_0^r \rho(1 - \rho^2)^{-2} \,\mathrm{d}\rho = \pi \left(\frac{1}{1 - r^2} - 1 \right),$$

we deduce that $R(1, D_s) = 1 - (1 + s/\pi)^{-(1+\alpha)}$. The function

$$\theta(s) := 1 - (1 + s/\pi)^{-(1+\alpha)}, \quad s \ge 0 \tag{3.32}$$

will then be the wavelet case analogue of $1 - e^{-s}$ that appeared in the STFT case, and we will use it as our comparison function in what follows, eventually proving that $R(f, \Omega) \le \theta(\mu(\Omega))$. Again, the function

$$u(z) := |f(z)|^2 (1 - |z|^2)^{\alpha+2} \tag{3.33}$$

will play a major role in the proof, and this time the super-level sets

$$A_t = \{u > t\}, \quad t > 0$$

are all contained *strictly* within D, which, together with the fact that u is real-analytic (see [KP, Mi]), implies that the hyperbolic length of every level-set of u is zero,

$$L(\{u = t\}) := \int_{\{u=t\}} (1 - |z|^2)^{-1} \, \mathrm{d}\mathcal{H}^1(z) = 0, \quad \forall t > 0.$$

This follows from the fact that u can be extended to a continuous function on $\overline{D}$ by simply letting $u \equiv 0$ over ∂D. Indeed, consider any $z_0 \in D$ far from the origin, for example $|z_0| > 1/2$, and let $r = (1 - |z_0|)/2$. Then, for some constant $C > 1$ we have

$$\frac{1}{C}(1 - |z|^2) \leq r \leq C(1 - |z|^2), \quad \forall z \in B_r(z_0),$$

and from the analyticity of $|f(z)|^2$ it follows that it is a subharmonic function, which we use in estimating $u(z_0)$,

$$\begin{aligned} w(z_0) &:= \int_{B_r(z_0)} |f(z)|^2 (1 - |z|^2)^\alpha \, \mathrm{d}z \geq C_1 r^{\alpha+2} \frac{1}{\pi r^2} \int_{B_r(z_0)} |f(z)|^2 \, \mathrm{d}z \\ &\geq C_1 r^{\alpha+2} |f(z_0)|^2 \geq C_2 (1 - |z_0|)^{\alpha+2} |f(z_0)|^2 = C_2 u(z_0). \end{aligned}$$

Now, since $u(z) \in L^1(D)$, we find that $w(z_0) \to 0$ whenever $|z_0| \to 1$, and therefore

$$\lim_{|z_0| \to 1} u(z_0) = 0.$$

Naturally, as before, the *distribution function* of u and its *decreasing rearrangement* will play pivotal roles in the proof of Theorem 3.3. These are now

$$\rho(t) := \mu(A_t) = \mu(\{u > t\}), \quad t \in (0, \max u) \tag{3.34}$$

and

$$u^*(s) := \sup\{t \geq 0 \colon \rho(t) > s\}, \quad s \geq 0. \tag{3.35}$$

They are of course the analogues of (3.9) and (3.10) in the current setting, and as such, all their relevant properties for the analysis we carried out in Section 3.1 hold for ρ and u^* as well. In particular, (3.11), (3.11) and (3.13) are still valid, which readily renders the proof of the following result a matter of rewriting that of Lemma 3.5.

Lemma 3.9. *The function ρ is absolutely continuous on $(0, \max u]$, and*

$$\rho'(t) = \int_{\{u=t\}} |\nabla u|^{-1} (1 - |z|^2)^{-2} \, \mathrm{d}\mathcal{H}^1.$$

In particular, the function u^* is, as the inverse of ρ, locally absolutely continuous on $[0, +\infty)$, with

$$-(u^*)'(s) = \left(\int_{\{u=u^*(s)\}} |\nabla u|^{-1} (1 - |z|^2)^{-2} \, \mathrm{d}\mathcal{H}^1 \right)^{-1}.$$

At the same time, we can extract a differential inequality like (3.18) via a similar argument to that of Lemma 3.6. This time, however, the isoperimetric inequality will come in its hyperbolic variant (see [Iz, Os]). Furthermore, setting I to be as in (3.23),

$$I(s) = \int_{\{u>u^*(s)\}} u(z)\,\mathrm{d}z,$$

then Lemma 3.9 readily implies that

$$I'(s) = u^*(s), \quad I''(s) = -\left(\int_{\{u=u^*(s)\}} |\nabla u|^{-1}\,(1-|z|^2)^{-2}\,\mathrm{d}\mathcal{H}^1\right)^{-1},$$

and we can combine the wavelet analogues of Lemma 3.6 and Lemma 3.7 to obtain the following result.

Lemma 3.10. *The function I is of class C^1 on $[0,\infty)$ and $I' = u^*$ is locally absolutely continuous and satisfies*

$$I'(s) \geq -\frac{\alpha+2}{\pi+s} I'(s), \quad \textit{for a.e. } s \in [0,\infty). \tag{3.36}$$

Proof. The Cauchy-Schwarz inequality implies

$$\left(\int_{\partial A_{u^*(s)}} |\nabla u|^{-1}\,(1-|z|^2)^{-2}\,\mathrm{d}\mathcal{H}^1\right)\left(\int_{\partial A_{u^*(s)}} |\nabla u|\,\mathrm{d}\mathcal{H}^1\right) \geq \left(\int_{\partial A_{u^*(s)}} (1-|z|^2)^{-1}\,\mathrm{d}\mathcal{H}^1\right)^2.$$

The first factor on the left-hand side is $I''(s)$, and the right-hand side is the square of the hyperbolic length of $\partial A_{u^*(s)}$, so by rearranging, we have

$$I''(s) \geq -\left(\int_{\partial A_{u^*(s)}} |\nabla u|\,\mathrm{d}\mathcal{H}^1\right)^{-1} L\left(\partial A_{u^*(s)}\right)^{-2}. \tag{3.37}$$

In order to compute the first factor on the right-hand side, we note that

$$\Delta \log u(z) = \Delta \log(1-|z|^2)^{2+\alpha} = -4(\alpha+2)(1-|z|^2)^{-2},$$

and letting $w(z) = \log u(z)$,

$$\begin{aligned} -\frac{1}{u^*(s)} \int_{\partial A_{u^*(s)}} |\nabla u|\,\mathrm{d}\mathcal{H}^1 &= \int_{\partial A_{u^*(s)}} \nabla w \cdot u\,\mathrm{d}\mathcal{H}^1 = \int_{A_{u^*(s)}} \Delta w\,\mathrm{d}z \\ &= -4(\alpha+2)\int_{A_{u^*(s)}} (1-|z|^2)^{-2}\,\mathrm{d}z = -4(\alpha+2)\mu(A_{u^*(s)}) = -4(\alpha+2)s. \end{aligned}$$

Therefore, (3.37) turns into

$$I''(s) \geq -4(\alpha+2)su^*(s)L\left(\partial A_{u^*(s)}\right)^{-2} = -4(\alpha+2)sI'(s)L\left(\partial A_{u^*(s)}\right)^{-2}. \tag{3.38}$$

However, the hyperbolic isoperimetric inequality on the Poincaré disk reads

$$L\left(\partial A_{u^*(s)}\right)^2 \geq 4\pi s + 4s^2,$$

and we can plug this back in (3.38) to obtain (3.36). ∎

To conclude, we prove the theorem by means of a similar analysis to that of the proof of Theorem 3.4, but with slightly different techniques.

Proof of Theorem 3.8. We begin by checking that $\theta(0) = I(0) = 0$ and $\lim_{s\to+\infty} I(s) = \lim_{s\to+\infty} \theta(s) = 1$. If we now define the function $G(s) \coloneqq I(s) - \theta(s)$, then (3.6) is reduced to showing that $G(s) \leq 0$ for all $s > 0$. We will achieve this by a maximum principle type of argument.

First, we claim that $G'(0) \leq 0$. A computation reveals that

$$\theta'(s) = \frac{1+\alpha}{\pi}(1 + s/\pi)^{-(\alpha+2)}, \quad \theta''(s) = -2\frac{\alpha+2}{\pi}\theta'(s)(1 + s/\pi)^{-1},$$

so $\theta'(0) = (1 + \alpha)/\pi$ and by (2.25),

$$I'(0) = u^*(0) = \max_D u \leq \frac{1+\alpha}{\pi} = \theta'(0).$$

Consider now

$$m \coloneqq \sup\{r > 0\colon G(s) \leq 0 \quad \forall s \in [0, r]\}.$$

We will argue by contradiction, and assume that $m < +\infty$. Since $G(0) = G(m) = 0$, there exists $c \in (0, m)$ with $G'(c) = 0$. Now, define the function $h(s) = (\pi + s)^{\alpha+2}$, and notice that the differential inequalities that I and θ satisfy (i.e. (3.36)) can be put together to show that

$$(G'h)' \geq 0.$$

Notice that h is increasing on $\mathbb{R}$, and $G'h$ is nondecreasing. We could have two options for $G'(0)$. Either $G'(0) = 0$, which by (2.25) means that f is a multiple of the reproducing kernel and thus $G \equiv 0$, or $G'(0) < 0$. We can rule out the first by contradiction, and we are left with $G'(0) < 0$. In this case, we deduce that $G(c) < 0$. In particular, this implies that for some $d \in (c, m)$, G' has to be positive, $G'(d) > 0$.

As $G(m) = \lim_{s\to+\infty} G(s) = 0$, there is another point $c' \in (0, +\infty)$ with $G'(c') = 0$. However,

$$(G'h)(c) = 0,\ (G'h)(d) > 0,\ (G'h)(c') = 0$$

and since $G'h$ is nondecreasing, this is a contradiction. Hence $m = +\infty$ and (3.6) follows.

Regarding the characterization of equality cases, observe that the moment I and θ agree at some s, then $m < +\infty$ and the argument above shows that they must be equal everywhere and that f is as in (2.26). We deduce that Ω is a hyperbolic disc (up to a set of measure zero). $\square$

3.3 Higher dimensional version and related problems

The higher dimensional version of Theorem 3.1 sees a few changes. It will be necessary to introduce the (lower) incomplete gamma function γ, defined by

$$\gamma(k, s) \coloneqq \int_0^s \tau^{k-1} e^{-\tau}\, \mathrm{d}\tau, \tag{3.39}$$

where $s \geq 0$ and $k \geq 1$ is an integer, so that

$$\frac{\gamma(k,s)}{(k-1)!} = 1 - e^{-s}\sum_{j=0}^{k-1}\frac{s^j}{j!}. \tag{3.40}$$

The result then reads as follows.

Theorem 3.11 (Faber-Krahn inequality for the STFT in dimension d)**.** *For every measurable subset $\Omega \subset \mathbb{R}^{2d}$ of finite measure and for every $f \in L^2(\mathbb{R}^d) \setminus \{0\}$ it holds that*

$$\frac{\int_\Omega |\mathcal{V}f(x,\omega)|^2 \,\mathrm{d}x\,\mathrm{d}\omega}{\|f\|_{L^2}^2} \leq \frac{\gamma(d,c_\Omega)}{(d-1)!}, \tag{3.41}$$

where $c_\Omega := \pi \left(|\Omega| / \boldsymbol{\omega}_{2d}\right)^{1/d}$ is the symplectic capacity of the ball in $\mathbb{R}^{2d}$ having the same volume as Ω.

Moreover, equality occurs (for some f and some Ω such that $0 < |\Omega| < \infty$) if and only if Ω is equivalent, up to a set of measure zero, to a ball centered at some $(x_0,\omega_0) \in \mathbb{R}^{2d}$, and

$$f(x) = ce^{2\pi i x\cdot\omega_0}\varphi(x-x_0), \quad c \in \mathbb{C}\setminus\{0\}, \tag{3.42}$$

where φ is the Gaussian in (2.4)*.*

As previously mentioned, the proof of Theorem 3.11 follows the same steps as that of Theorem 3.1, with the appropriate adaptations to account for the change of dimension. In particular, an immediate consequence is the higher dimensional version of Theorem 3.2, which also follows by the exact same arguments.

Theorem 3.12 (Sharp uncertainty principle for the STFT in dimension d)**.** *If $f \in L^2(\mathbb{R}^d)\setminus \{0\}$ and $\Omega \subset \mathbb{R}^{2d}$ satisfy*

$$\int_\Omega |\mathcal{V}f(x,\omega)|^2 \,\mathrm{d}x\,\mathrm{d}\omega \geq (1-\varepsilon)\|f\|_2^2,$$

then

$$|\Omega| \geq \omega_{2d}\pi^{-d}\psi_d(\varepsilon)^d,$$

where $\psi_d(\varepsilon)$ is the inverse function of $s \mapsto 1 - \frac{\gamma(d,s)}{(d-1)!}$ for $s \geq 0$.

Moreover, equality occurs if and only if Ω is a ball and f s of the form (3.42)*, where (x_0,ω_0) is the center of the ball.*

Finally, we remark that Theorem 3.1 can also be seen as the quadratic case of two different generalizations [NT, Section 5]. These are a local version of the Lieb uncertainty principle and L^p-concentration estimates, where the same shape optimization problem is considered with L^p norms over functions in the modulation space

$$M^p(\mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{R})\colon \|f\|_{M^p} := \|\mathcal{V}f\|_p < \infty\}$$

of tempered distributions whose STFT (defined by duality) is a function in L^p [Fe, Gr1].

Chapter 4
Stability

We begin the main chapter of this thesis, where we will address the stability of Theorem 3.1. This corresponds to answering the following question: given $\Omega \subset \mathbb{R}^2$ and $f \in L^2(\mathbb{R})$ that are almost optimal, i.e., that fail to achieve the equality in (3.2) by a small margin, can we say that Ω is close to a ball and that f is close to a function of the form (3.3)?

We begin by choosing the appropriate method to measure *almost optimality* and *closeness* to an optimal set or function. For the first, we will consider the combined deficit

$$\delta(f;\Omega) := 1 - \frac{\displaystyle\int_\Omega |\mathcal{V}f(x,\omega)|^2 \,\mathrm{d}x\,\mathrm{d}\omega}{(1-e^{-|\Omega|})\|f\|^2_{L^2(\mathbb{R})}}, \tag{4.1}$$

allowing us to rewrite (3.2) as $\delta(f;\Omega) \geq 0$. Recalling (3.3), optimal functions will be Gaussians of the form $c\varphi_z$, and optimal sets will of course be balls. Then, closeness of $f \in L^2(\mathbb{R})$ and $\Omega \subset \mathbb{R}^2$ to their respective optimal counterparts will be measured in the normalized L^2-distance of f and its closest Gaussian (cf. (4.3)), and the *Fraenkel asymmetry* of Ω,

$$\mathcal{A}(\Omega) := \inf\left\{\frac{|\Omega \triangle B(x,r)|}{|\Omega|} := |B(x,r)| = |\Omega|\,, r>0, x\in\mathbb{R}^2\right\}. \tag{4.2}$$

As we remarked in the introduction, this is a natural notion of asymmetry and it is often used to formulate the stability of geometric and functional inequalities, such as the isoperimetric inequality [CL, FiMP, FMP2] or the Faber-Krahn inequality for the Dirichlet Laplacian [AKN, BPV].

With these choices, the question of stability can be positively answered, as shown by A. Guerra, J. P. G. Ramos, P. Tilli and the author in [GGRT], in the form of the sharp quantitative Faber-Krahn inequality for the STFT.

Theorem 4.1 (Stability of the Faber-Krahn inequality for the STFT)**.** *There is an explicitly computable constant $C>0$ such that, for all $\Omega \subset \mathbb{R}^2$ measurable and with finite measure and all $f \in L^2(\mathbb{R})\backslash\{0\}$, it holds that*

$$\inf_{z_0\in\mathbb{C},|c|=\|f\|_2} \frac{\|f - c\,\varphi_{z_0}\|_2}{\|f\|_2} \leq Ce^{|\Omega|/2}\delta(f;\Omega)^{1/2}, \tag{4.3}$$

and for some explicit constant $K = K(|\Omega|) > 0$ we also have

$$\mathcal{A}(\Omega) \leq K\delta(f;\Omega)^{1/2}. \tag{4.4}$$

We may remark a few facts about the nature of the result itself before explaining how it is obtained. First, the constants C and K are explicit, as the proof is purely quantitative and does not rely on compactness arguments, like previous results on the sharp quantitative Faber-Krahn inequality for the Dirichlet Laplacian [BPV, AKN], or any reduction steps through penalization methods [CL].

Moreover, the result is sharp in two different ways. Firstly, it is sharp in the decay rate, i.e. the factor $\delta(f;\Omega)^{1/2}$ cannot be substituted by $\delta(f;\Omega)^{\beta}$ for any $\beta > 1/2$. Secondly, it is also sharp in the dependence on Ω of the constant $e^{|\Omega|}$ in (4.3): the estimate

$$\inf_{z_0\in\mathbb{C},|c|=\|f\|_2} \frac{\|f - c\,\varphi_{z_0}\|_2}{\|f\|_2} \leq Ce^{\theta|\Omega|}\delta(f;\Omega)^{1/2}$$

no longer holds whenever $\theta < 1/2$. We will see this later in Section 4.5.

The stability estimate (4.3) holds for general functions $f \in L^2(\mathbb{R})$, and when looking at the problem from the point of view of localization operators, we can fix $\Omega \subset \mathbb{R}^2$, consider $L_{\Omega,\varphi} = \mathcal{V}^*\mathbb{1}_\Omega\mathcal{V}$, and focus on the specific case where $f = f_\Omega$ is the first eigenfunction of $L_{\Omega,\varphi}$. Then, we reach the following corollary.

Corollary 4.2. *Let $\Omega \subset \mathbb{R}^2$ be a measurable set of finite Lebesgue measure and define*

$$\tilde{\delta}(\Omega) := \delta(f_\Omega;\Omega),$$

where f_Ω is a unit-norm eigenfunction associated to the first eigenvalue Φ_Ω of $L_{\Omega,\varphi}$. There is an explicitly computable constant $C > 0$ such that

$$\inf_{z_0\in\mathbb{C},|c|=1} \|f - c\,\varphi_{z_0}\|_2 \leq Ce^{|\Omega|/2}\tilde{\delta}(\Omega)^{1/2}.$$

In this case, however, the sharpness of this corollary does not follow from that of Theorem 4.1. As a matter of fact, we do not currently know whether this result is sharp or not. To answer this question, it would be necessary to find an example that shows that it is. This would either run through computing the eigenfunctions of $L_{\Omega,\varphi}$ for domains Ω increasingly close to a ball (similarly to how ellipsoids and spherical harmonics are used for the isoperimetric inequality [Ma1, Fu]), or inversely, given a function f, construct a domain Ω_f such that f is the first eigenfunction of $L_{\Omega,\varphi}$. The first method seems rather hard, as the eigenfunctions of localization operators are not known even when Ω is an ellipse with low eccentricity: see [AD, Da2]. The second method, while more promising, remains an open topic that requires a new set of ideas which are essentially disjoint from the work we will over in the thesis.

Let us begin to outline the ideas and techniques involved in the proof of this result. As expected, a first step of translating the problem to the Fock space reduces Theorem 4.1 to the following statement.

Theorem 4.3 (Fock space version of Theorem 4.1)**.** *Let $\Omega \subset \mathbb{C}$ be a measurable subset of finite measure. There is an explicitly computable constant $C > 0$ such that, for all $F \in \mathcal{F}(\mathbb{C})\backslash\{0\}$, we have*

$$\inf_{\substack{|c|=\|F\|_{\mathcal{F}},\\ z_0\in\mathbb{C}}} \frac{\|F - cF_{z_0}\|_{\mathcal{F}}}{\|F\|_{\mathcal{F}}} \leq Ce^{|\Omega|/2}\delta(F;\Omega)^{1/2}, \tag{4.5}$$

where $F_{z_0} = \mathcal{B}\varphi_{z_0}$ as in (2.24)*, and we use the notation $\delta(F;\Omega) := \delta(f;\Omega)$, where $F = \mathcal{B}f$. Moreover, for some explicit constant $K = K(|\Omega|) > 0$ we also have*

$$\mathcal{A}(\Omega) \leq K(|\Omega|)\delta(f;\Omega)^{1/2}. \tag{4.6}$$

Upon close inspection of the proof of Theorem 3.1, one could expect that Theorem 4.1 is shown via adaptation of the original argument, this time running through quantitative versions of (3.25), (3.26) and (3.28). Although we are able to establish their stability counterparts, this is only directly achieved for (3.28) as a direct consequence of Proposition 2.15. In particular, (3.25) is achieved as a consequence of the proof itself, and (3.26) cannot be directly established in a quantitative form. Instead, we first prove the stability for the function by showing that $1 - \max u \le C(s)\delta(f; A_{u^*(s)})$ whenever this is small enough. We then use this to show that u behaves roughly like a Gaussian $v(z) = e^{-\pi|z|^2}$, which makes (3.26) easier to make quantitative.

This proof is, however, not the only one we present, as we also consider, although not in full detail, a *variational* approach to the stability of the function in the spirit of [Fu]. We will focus on the functional

$$F \mapsto \int_{\{u_F > u_F^*(s)\}} u_F(z)\, \mathrm{d}z. \tag{4.7}$$

Its second variation turns out to posses a few interesting properties around a neighbourhood of the constant function $F_0 \equiv 1$. In particular, studying it reveals that one needs to be especially precise when approximating F in the Fock space, as it vanishes on all approximations of F of order 1 (or 0).

These two directions are of independent interest but very closely related. The first method yields the optimal dependence on $|\Omega|$ of the constant in (4.3), whereas the variational one yields the *asymptotic optimal constant* as the deficit goes to zero, and provides the appropriate tools to show the sharpness of the result. The main difference between them is how through the first approach, one creates a comparison function for u^* which ends up showing the estimate $1 - \max u \le C\delta(f; A_{u^*(s)})$. The variational path instead addresses the super-level sets A_t of u when t is close to 1. In particular, through this analysis one finds many deep geometric links between the super-level sets of u around its maximum and spheres – they are in fact smooth and uniformly convex.

We will dedicate the rest of the manuscript to proving Theorem 4.3 along with its higher dimensional and wavelet counterparts. We begin in Section 4.1 with a discussion of the stability of the function. In Subsection 4.1.1 we will show a non-sharp version of it, which we require in order to find sharp estimates in Subsection 4.1.2; and we will follow with the stability of the set in Section 4.2. Later, we address the geometry of the super-level sets of u and the variational approach to the stability of the function in sections 4.3 and 4.4 respectively. We finish the discussion of Theorem 4.1 by showing how it is sharp in Section 4.5. Finally, in Chapter 5, we deal with the higher dimensional and wavelet case in Sections 5.1 and 5.2 respectively.

4.1 Stability for the function

We begin with a simple remark in the flavor of [Ma1]: we may always be able to assume that F and Ω have a small deficit $\delta(F; \Omega)$, as small as required. Indeed, if we look for a result of the type

$$\|F - cF_{z_0}\|_{\mathcal{F}} \le K\|F\|_{\mathcal{F}}\delta(F; \Omega)^{\alpha}$$

for some $K, \alpha > 0$, and there is already a constant C for which it holds, assuming that $\delta(F;\Omega)$ is smaller than some δ, then since $\|F - cF_{z_0}\|_{\mathcal{F}} \leq 2\|F\|_{\mathcal{F}}$, the constant K can be chosen as

$$K = \max\left\{C, 2\delta^{-\alpha}\right\}.$$

The same can be said about (4.4), since $\mathcal{A}(\Omega) \leq 2$ as well. For this reason, we will often look for an *asymptotic* result, that is, when necessary, we will assume that $\delta(F;\Omega)$ is smaller than some given δ. Additionally, by homogeneity, we can, and will assume that

$$\|F\|_{\mathcal{F}} = 1.$$

4.1.1 An initial stability estimate

We have already highlighted the importance of the super-level sets of u that are close to its maximum, but in order to establish a sharp result through volume estimates, we first need an initial non-sharp stability result that the sharp version builds upon. Although it is possible to reach such a bound with exponent $1/4$ via an analysis of the function G in the proof of Theorem 3.1 and through various uses of the Chebyshev inequality, we present a more beautiful –and sharper– proof. Let us introduce the setting.

We fix $s_0 > 0$ as the measure of Ω, i.e. $s_0 = |\Omega|$, and concern ourselves with the particular case when we take Ω to be a super-level set of u. For this reason, given $s > 0$, we define

$$\delta_s(F) \coloneqq \delta(F; \{u > u^*(s)\}) = \frac{1 - e^{-s} - I(s)}{1 - e^{-s}}. \tag{4.8}$$

This is, $\delta_s(F)$ is the deficit measuring the failure of F in achieving the equality in (3.8) over its unique super-level set of measure s.

The starting point of our argument will be the fact that

$$\int_0^\infty e^{-s}\,\mathrm{d}s = 1 = \lim_{s\to\infty} I(s) = \int_0^\infty I'(s)\,\mathrm{d}s = \int_0^\infty u^*(s)\,\mathrm{d}s.$$

Since $1 - \max u = u^*(0) \leq 1$, by the continuity of u^* there exists $s^* > 0$ for which $u^*(s^*) = e^{-s^*}$, and we may use this to prove the following lemma.

Lemma 4.4. *In the setting above, the following estimate holds:*

$$1 - \max u \leq \left(2e^{s_0}\delta_{s_0}(F)\right)^{1/2}$$

whenever $\max u > e^{-s_0}$. *Moreover, when* $\max u < e^{-s_0}$, *there is a sharper bound on the decay rate:*

$$1 - \max u \leq \frac{2\delta_{s_0}(F)}{1 - \delta_{s_0}(F)}.$$

Proof. In order to avoid the notation from being too condensed, we write

$$\delta_0 \coloneqq \delta_{s_0}(F) = \delta(F; \{u > u^*(s_0)\}). \tag{4.9}$$

STEP 1. We assume $s_0 > s^*$. In this first step we seek to exploit the fact that the functions $s \mapsto u^*(s) - e^{-s}$ and $s \mapsto e^{-s} - u^*(s)$ integrate the same over $[0, s^*]$ and $[s^*, +\infty)$

respectively. The first integral will be seen to control $(1 - \max u)^2$, whereas the second one will be directly controlled by $\delta_{s_0}(F)$.

Indeed, rearranging the expression for δ_0, we see that

$$\int_0^{s_0} (u^*(s) - e^{-s})\, \mathrm{d}s = -\delta_0(1 - e^{-s_0}),$$

which together with $\int_0^\infty (u^*(s) - e^{-s})\, \mathrm{d}s = 0$ implies that

$$A := \int_{s_0}^\infty (u^*(s) - e^{-s})\, \mathrm{d}s = \delta_0(1 - e^{-s_0}). \tag{4.10}$$

The quantity A represents the area in the region between the graphs of u^* and e^{-s}, starting at $s = s_0$. By controlling, in terms of δ_0, the area of the region between u^* and e^{-s} and limited by s^* and s_0, we can establish a control on $\int_{s^*}^\infty u^*(s) - e^{-s}\, \mathrm{d}s$. This, in turn, equals $\int_0^{s^*} (e^{-s} - u^*(s))\, \mathrm{d}s$, which in particular can be bounded from below in terms of $(1 - \max u)^2$. So in the end we get the desired control on $1 - \max u$.

We begin by defining the function

$$L(t) = \mu(t) + \log(t), \tag{4.11}$$

which is the length of the horizontal segment joining the graphs of e^{-s} and $u^*(s)$ at height $t < e^{s_0}$ (see Figure 4.1), and notice that it is a decreasing function since $L'(t) \leq 0$ by (3.18).

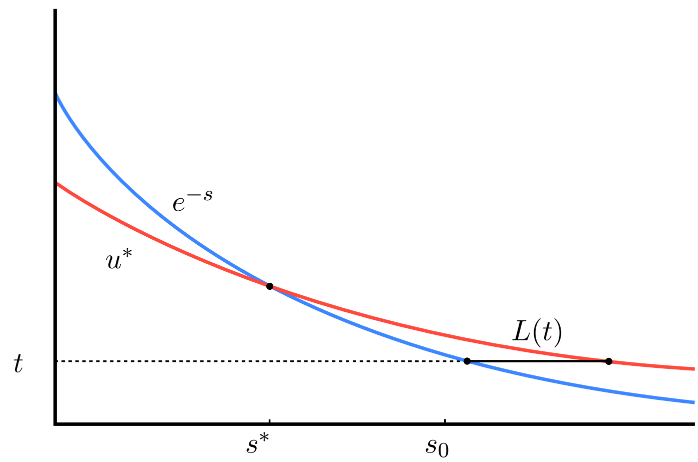

Figure 4.1: Definition of $L(t)$.

Define now A_1 to be the area of the region enclosed between the graphs of e^{-s} and $u^*(s)$, and below the horizontal line $t = e^{-s_0}$ (see Figure 4.2), that is

$$A_1 := \int_0^{e^{-s_0}} L(t)\, \mathrm{d}t.$$

Then $A \geq A_1$, and since L is decreasing,

$$A_1 \geq L(e^{-s_0})e^{-s_0} = (s_1 - s_0)e^{-s_0},$$

where s_1 is such that $u^*(s_1) = e^{-s_0}$. Combining this with (4.10) we get

$$s_1 - s_0 \leq \delta_0 e^{s_0}(1 - e^{-s_0}).$$

We now let A_2 be the area of the region enclosed between the graphs of e^{-s} and $u^*(s)$, but this time above the line $t = e^{-s_0}$. Then,

$$A_2 := \int_{e^{-s_0}}^{e^{-s^*}} L(t)\,\mathrm{d}t \le L(e^{-s_0})(e^{-s^*} - e^{-s_0})$$
$$= (s_1 - s_0)(e^{-s^*} - e^{-s_0}) \le \delta_0 e^{s_0}(1 - e^{-s_0})(e^{-s^*} - e^{-s_0}).$$

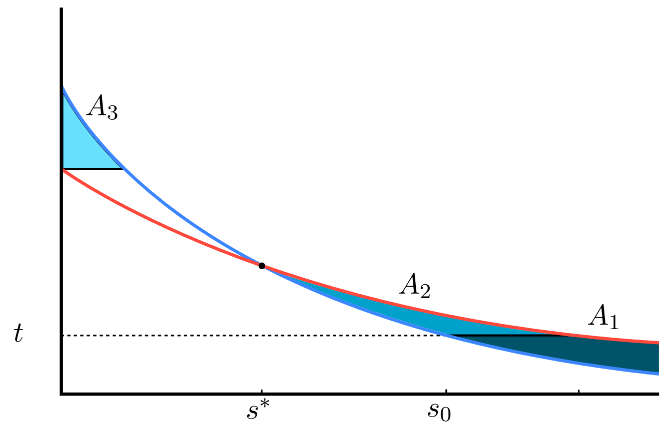

Figure 4.2: Regions with areas A_1, A_2 and A_3.

If we put (4.10) and the above estimate together, we obtain

$$\begin{aligned}\int_{s^*}^{\infty} (u^*(s) - e^{-s})\,\mathrm{d}s &= \int_{s^*}^{s_0} (u^*(s) - e^{-s})\,\mathrm{d}s + \int_{s_0}^{\infty} (u^*(s) - e^{-s})\,\mathrm{d}s \\ &\le A_2 + \delta_0(1 - e^{-s_0}) \le \delta_0(1 - e^{-s_0})e^{s_0 - s^*}.\end{aligned} \tag{4.12}$$

At this point, we can switch the region of integration while still preserving the new-found bound, as we indicated earlier,

$$\int_0^{s^*} (e^{-s} - u^*(s))\,\mathrm{d}s = \int_{s^*}^{\infty} (u^*(s) - e^{-s})\,\mathrm{d}s \le \delta_0(1 - e^{-s_0})e^{s_0 - s^*}. \tag{4.13}$$

It remains to bound the left-hand side from below by $1 - \max u$. Writing $T := \max u = u^*(0)$ and setting A_3 to be the area of the region between the graph of e^{-s} and the horizontal line $t = T$ for $0 < s < -\log T$, we have

$$A_3 := \int_0^{-\log T} e^{-s}\,\mathrm{d}s - T(-\log T) = 1 - T + T\log T.$$

Alternatively, by convexity of e^{-s} and since its derivative at $s = 0$ is equal to -1, the region between $u^*(s)$ and e^{-s} from $s = 0$ to $s = -\log T$ contains a triangle of area $(1 - T)^2/2$. Then, since $A_3 \le \int_0^{s^*} e^{-s} - u^*(s)\,\mathrm{d}s$, we find

$$(1 - T)^2 \le 2A_3 \le 2\delta_0(1 - e^{-s_0})e^{s_0 - s^*} \le 2\delta_0(e^{s_0} - 1) \le 2\delta_0 e^{s_0}.$$

Step 2. Assume now that $s^* > s_0$. We can treat this situation as two separate cases. Either $T \ge e^{-s_0}$ or $T < e^{-s_0}$. The first case is very similar to Step 1. Indeed, it is not necessary to switch the integration region now, since the deficit δ_0 can be rewritten to read

$$\int_0^{s_0} (e^{-s} - u^*(s))\,\mathrm{d}s = \delta_0(1 - e^{-s_0}),$$

allowing us to argue as before to obtain $A_3 \leq \delta_{s_0}(1-e^{-s_0})$. This yields the desired control on $1 - \max u$, that is,

$$(1-T)^2 \leq 2\delta_0(1-e^{-s_0}) \leq e^{s_0}\delta_0.$$

We are left to deal with the case where $T < e^{-s_0}$. For this, we define (see Figure 4.3)

$$A_4 := \int_0^{s_0} (e^{-s} - T)\,\mathrm{d}s = 1 - e^{-s_0} - s_0 T, \quad A_5 := \int_0^{s_0} (T - u^*(s))\,\mathrm{d}s, \tag{4.14}$$

so that $A_4 + A_5 = \delta_0(1-e^{-s_0})$. An immediate lower bound can be obtained by ignoring $A_5 \geq 0$ and using (4.14) for A_4,

$$1 - e^{-s_0} - s_0 T \leq \delta_0(1-e^{-s_0}). \tag{4.15}$$

This, coupled with the fact that $T < e^{-s_0}$, yields, after rearranging,

$$e^{s_0} \leq 1 + \frac{s_0}{1-\delta_0}.$$

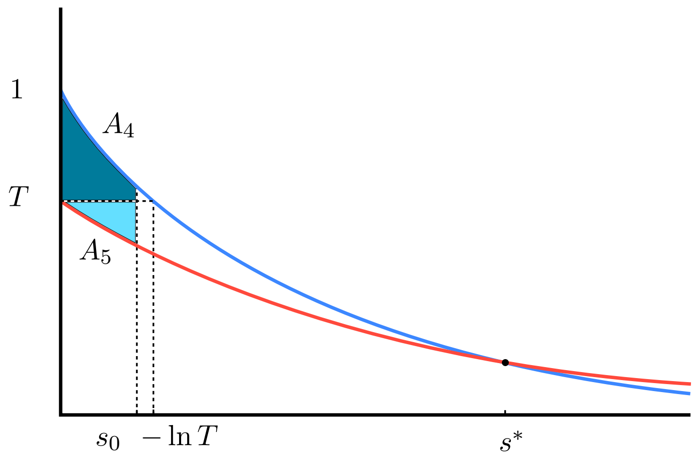

Figure 4.3: Definition of A_4 and A_5.

At this point, it suffices to notice that by convexity, this can only happen if s_0 is smaller than x_0 that satisfies $e^{x_0} = 1 + \frac{x_0}{1-\delta_0}$. Using the second order Taylor expansion of the exponential,

$$1 + x_0 + \frac{x_0^2}{2} < e^{x_0} = 1 + \frac{x_0}{1-\delta_0},$$

which implies that

$$x_0 < \frac{2\delta_0}{1-\delta_0}.$$

Coming back to (4.15), since $\delta_0 < 1$ and $s_0 \leq x_0$ we have

$$T \geq (1-\delta_0)\frac{1-e^{-s_0}}{s_0} \geq (1-\delta_0)\frac{1-e^{-x_0}}{x_0} = e^{-x_0}(1-\delta_0)\frac{e^{x_0}-1}{x_0} = e^{-x_0}.$$

Then $1 - T \leq 1 - e^{-x_0}$, and in particular

$$1 - T \leq 1 - e^{-x_0} \leq x_0 \leq \frac{2\delta_0}{1-\delta_0}. \qquad \square$$

This far into the discussion, we remark that we can already show a quantitative result for the function. In fact, in the Fock space, Proposition 2.15, the quantitative version of [NT, Proposition 2.1] and the next natural step of the proof, reads as follows.

Proposition 4.5. *Let $F \in \mathcal{F}(\mathbb{C})$. Then for every $z_0 \in \mathbb{C}$ there exists a constant $c \in \mathbb{C}$ with $|c| = \|F\|_{\mathcal{F}}$ such that*

$$\frac{\|F - cF_{z_0}\|_{\mathcal{F}}}{\|F\|_{\mathcal{F}}} \leq \sqrt{2}\Delta_F(z_0)^{1/2}.$$

This result combines with Lemma 4.4 to show that (under the normalization $\|F\|_{\mathcal{F}} = 1$) there are $z_0 \in \mathbb{C}$ and $c \in \mathbb{C}$, $|c| = 1$ with

$$\|F - cF_{z_0}\|_{\mathcal{F}} \leq 8(e^{|\Omega|} - 1)^{1/4}\delta_{|\Omega|}(F)^{1/4}.$$

Additionally, if one is willing to pay the price of halving the exponent of $\delta(F;\Omega)$, it is possible to obtain a quantitative estimate for the set without much effort by a direct application of the Chebyshev inequality. Indeed, the inequality above provides a bound for the difference $|u(z) - v(z)|$, where

$$v(z) := e^{-\pi|z|^2}, \tag{4.16}$$

which already shows that, whenever Ω is a super-level set of u, its asymmetry can be easily bounded in terms of $\delta_{|\Omega|}(F)^{1/4}$. For a general Ω, the Chebyshev inequality then shows the bound

$$\left|\left\{z \in A_\Omega \setminus \Omega : |T(z)|^2 - |z|^2 > d(\Omega)^{1/2}\right\}\right| \leq c\frac{1 - e^{-s_0}}{e^{-s_0}} d(\Omega)^{1/2},$$

where $A_\Omega = \{v > v^*(|\Omega|)\}$, T is any map transporting $A_\Omega \setminus \Omega$ into $\Omega \setminus A_\Omega$ and

$$d(\Omega) = \frac{1}{1 - e^{-|\Omega|}}\left[\int_{A_\Omega \setminus \Omega} v(z) - v(T(z))\right].$$

This, in turn, implies that whenever $\Omega \subset \mathbb{R}^2$, there is a constant $K(|\Omega|) > 0$ such that

$$\frac{|A_\Omega \Delta \Omega|}{|\Omega|} \leq K(|\Omega|)d(\Omega)^{1/2}.$$

Curiously enough, this is actually sharp in the decay rate (as can be seen by looking at the ellipsoids from [Ma1]), but to no use: this approach eventually yields an exponent 1/8 on $\delta(F;\Omega)$, and we will see a different proof in Section 4.2 that avoids using the Chebyshev inequality and preserves the exponent from the stability estimate of the function while employing the same ideas, although through refined techniques.

4.1.2 Sharp stability for the function

As discussed earlier, the first method of showing sharp stability relies on a sharp bound for the measure of the super-level sets A_t, i.e. $\mu(t)$ (recall (3.9)) for t close to 1.

Lemma 4.6. *Let $F \in \mathcal{F}(\mathbb{C})$ be such that $\|F\|_{\mathcal{F}} = 1$. There is an absolute, computable constant $C > 0$ such that the following holds. If $T = \max_{\mathbb{C}} u(z)$ satisfies $1 - T < \frac{1}{C}$, then*

$$\mu(t) < (1 + C(1 - T))\log\left(\frac{T}{t}\right),$$

whenever $\frac{t}{T} > 1 - \frac{1}{C}$.

Proof. Up to translating and rotating F, we may assume that the maximum of $u(z)$ occurs at $z = 0$ and that $F(0)$ is a real number, so we have $T = F(0)^2$.

STEP 1. We will first prove the sub-optimal inclusion

$$\{z \in \mathbb{C} \colon u(z) > (1-\varepsilon)^2 T\} \subset D(0, C\sqrt{\varepsilon}),$$

for some absolute constant $C > 0$. Write F in terms of its Fock space series representation,

$$F(z) = \sum_{k=0}^{\infty} a_k \frac{\pi^{k/2}}{\sqrt{k!}} z^k. \tag{4.17}$$

A direct application of the Cauchy-Schwarz inequality yields

$$|F(z)|e^{-\pi\frac{|z|^2}{2}} \leq e^{-\pi\frac{|z|^2}{2}} \left(\sum_{k=0}^{\infty} |a_k| \frac{\pi^{k/2}}{\sqrt{k!}} |z|^k \right), \tag{4.18}$$

and in addition to this, one easily observes that if u has a maximum at $z = 0$, then $|F(z)|^2$ must have a critical point there as well. On the other hand,

$$|F(z)|^2 = T + \pi^{1/2} T^{1/2} \cdot \mathrm{Re}(a_1 z) + O(|z|^2)$$

holds in a neighbourhood of the origin, which implies that $a_1 = 0$; see also the discussion at the beginning of Section 4.3. Hence, by (4.18), if we restrict z to the set $\{z \in \mathbb{C} \colon u(z) > (1-\varepsilon)^2 T\}$, using that $a_1 = 0$ and the Cauchy-Schwarz inequality,

$$\begin{aligned} (1-\varepsilon)T^{1/2} &\leq |F(z)|e^{-\pi\frac{|z|^2}{2}} \\ &\leq T^{1/2} \cdot e^{-\pi\frac{|z|^2}{2}} + e^{-\pi\frac{|z|^2}{2}} \left(\sum_{k=2} |a_k| \frac{\pi^{k/2}}{\sqrt{k!}} |z|^k \right) \\ &\leq T^{1/2} \cdot e^{-\pi\frac{|z|^2}{2}} + e^{-\pi\frac{|z|^2}{2}} (1-T)^{1/2} \left(e^{\pi|z|^2} - 1 - \pi|z|^2 \right)^{1/2} \\ &\leq T^{1/2} \cdot e^{-\pi\frac{|z|^2}{2}} + (1-T)^{1/2} \left(1 - (1+\pi|z|^2) \cdot e^{-\pi|z|^2} \right)^{1/2}. \end{aligned} \tag{4.19}$$

Let $\tau := \pi|z|^2$, so that (4.19) may be rewritten as

$$(1-\varepsilon) \leq e^{-\tau/2} + \left(\frac{1-T}{T} \right)^{1/2} \left(1 - (1+\tau)e^{-\tau}\right)^{1/2}.$$

We may use the inequality $(1-(1+\tau)e^{-\tau})^{1/2} \leq 1 - e^{-\tau}$, valid for all $\tau \geq 0$, on the factor in the right-hand side, and setting $\delta = \left(\frac{1-T}{T}\right)^{1/2}$, we conclude that

$$1 - \varepsilon \leq e^{-\tau/2} + \delta(1 - e^{-\tau}).$$

Finally, if we set $x := 1 - e^{-\tau/2}$, the last inequality can be rewritten as

$$x - \varepsilon \leq \delta(2x - x^2) \leq 2\delta x,$$

which, in turn, implies $x \leq \frac{\varepsilon}{1-2\delta}$ or, equivalently,

$$\tau \leq -2\log\left(1 - \frac{\varepsilon}{1-2\delta}\right) \leq C\frac{\varepsilon}{1-2\delta}.$$

Since $1-T<1/C$ we have $\delta \leq (\frac{1}{C-1})^{1/2}$ and, as $\tau=\pi|z|^2$, one easily concludes the assertion of the first step.

Step 2. We now complete the proof of the lemma by refining the computations from Step 1. We first assume that t is arbitrary, but in a moment it will be chosen to be so that T/t is very small, depending on δ, as in the statement. Suppose that $u(z)>t$. We use the power-series expansion (4.17) in explicit form:

$$R(z) := \frac{F(z)}{F(0)} - 1 = \sum_{k=2}^{\infty} a_k \frac{\pi^{k/2}}{\sqrt{k!}} z^k.$$

Notice first that, by the estimates in (4.19), we have

$$|R(z)|^2 \leq (1-T)\left(e^{\pi|z|^2} - 1 - \pi|z|^2\right) \leq \delta^2\left(e^{\pi|z|^2} - 1 - \pi|z|^2\right), \tag{4.20}$$

where $\delta = \left(\frac{1-T}{T}\right)^{1/2}$ as in Step 1. The condition $u(z)>t$ may be rewritten as

$$\pi|z|^2 + \log(t/T) < \log\left(\frac{|F(z)|^2}{T}\right),$$

and so

$$\pi|z|^2 + \log(t/T) < \log\left(1 + 2\operatorname{Re} R(z) + |R(z)|^2\right) \leq |R(z)|^2 + 2\operatorname{Re} R(z). \tag{4.21}$$

We now relabel $h(z) := 2\operatorname{Re} R(z)$. This, together with (4.20), implies

$$\pi|z|^2 + h(z) < \log(T/t) + \delta^2\left(e^{\pi|z|^2} - 1 - \pi|z|^2\right). \tag{4.22}$$

Choosing t so that

$$\frac{t}{T} = (1-\varepsilon_0)^2 T,$$

Step 1 readily implies that $\{u>t\} \subset D(0, C\sqrt{\varepsilon_0})$; hence, if ε_0 is chosen sufficiently small, then $\pi|z|^2 \leq 1$ whenever $u(z)>t$. Thus, for such z we have $e^{\pi|z|^2} - 1 - \pi|z|^2 < \pi|z|^2$, and from (4.22),

$$(1-\delta^2)\pi|z|^2 + h(z) < \log(T/t). \tag{4.23}$$

At this point, we have shown the set inclusion

$$\{u>t\} \subset E_1 = \left\{z \in \mathbb{C} \colon (1-\delta^2)\pi|z|^2 + h(z) < \log(T/t)\right\}.$$

We now aim to bound the measure of the larger set so as to obtain an estimate for $\mu(t)$. To do this, we show that it is a star-shaped set, and compare it to the case when h does not contribute to it,

$$E_0 = \left\{z \in \mathbb{C} \colon (1-\delta^2)\pi|z|^2 < \log(T/t)\right\}.$$

The reason behind this is because a similar estimate can be extracted easily for E_0, and then shown to be inherited by every (appropriate) deformation of E^1 into E_0 through the contribution of h,

$$E_\sigma = \left\{z \in \mathbb{C} : (1-\delta^2)\pi\,|z|^2 + \frac{\sigma}{\delta} h(z) < \log(T/t)\right\},$$

exploiting a cancellation effect from the fact that it is a harmonic function.

In order to put this strategy into practice, we will need a few bounds on h and its derivatives. We group these into the following lemma.

Lemma. *There is an absolute constant $C > 0$ such that, for $z \in B(0, 1/\sqrt{\pi})$, we have*

$$|h(z)| \leq C\delta|z|^2, \quad |\nabla h(z)| \leq C\delta|z|, \quad |\nabla^2 h(z)| \leq C\delta. \tag{4.24}$$

Note that the bound (4.20) yields

$$|R(z)| \leq \pi\delta|z|^2. \tag{4.25}$$

The first inequality in (4.24) follows directly from the definition of h and (4.25). In order to show the estimates for the derivatives of h, note that $|\nabla h| \leq 2|R'(z)|$ and $|\nabla^2 h| \leq 4|R''(z)|$. But, by the Cauchy integral formula,

$$|R^{(n)}(z)| \leq \frac{n!}{2\pi} \int_{\gamma_z} \frac{|R(w)|}{|w - z|^{n+1}} \,\mathrm{d}w, \tag{4.26}$$

where γ_z is a closed, simple curve containing z. Applying (4.26) with $\gamma_z = \partial D(0, 2|z|)$ and using (4.25) we obtain the estimate for ∇h, while applying (4.26) with $\gamma_z = \partial B(0, 2/\sqrt{\pi})$ and using again (4.25), the estimate for $\nabla^2 h$ also follows.

Following the previous discussion, we let

$$V_\sigma(r, \theta) := (1 - \delta^2)\pi r^2 + \frac{\sigma}{\delta} h(re^{i\theta}), \tag{4.27}$$

where $\sigma \in (-2\delta, 2\delta)$. By (4.24), we see that

$$\partial_r V_\sigma \geq 2r(1 - \delta^2) - C\delta r > 0 \tag{4.28}$$

as long as $0 < r < 1/\sqrt{\pi}$ and δ is sufficiently small, which shows that E_σ is *starshaped*. Moreover, $\lim_{r\to 0} V_\sigma(r, \theta) = 0 < \log(T/t)$, and, for each fixed $\theta \in [0, 2\pi]$, we have

$$V_\sigma(1/\sqrt{\pi}, \theta) > (1 - \delta^2) - C\delta > \log(T/t)$$

since we are assuming that $\log(T/t)$ is sufficiently small. Therefore, given $t > 0$ with T/t sufficiently close to 1, for each $\theta \in [0, 2\pi)$ there is a unique number $r_\sigma(\theta) > 0$ such that

$$V_\sigma(r_\sigma(\theta), \theta) = \log(T/t). \tag{4.29}$$

Due to (4.28), the implicit function theorem shows that $\sigma \mapsto r_\sigma(\theta)$ is a smooth function of σ for each fixed $\theta \in [0, 2\pi)$. We can thus differentiate (4.29) with respect to σ, and in fact we will need the following estimates during the proof.

Lemma. *Let r_σ be defined through* (4.29)*. Then there is an absolute, computable constant $C > 0$ such that, for $\sigma \in (-2\delta, 2\delta)$, we have*

$$|r'_\sigma| + |r''_\sigma| \leq C r_\sigma. \tag{4.30}$$

If we differentiate (4.29), we obtain

$$2(1 - \delta^2)\pi r_\sigma r'_\sigma + \frac{1}{\delta} h(r_\sigma e^{i\theta}) + \frac{\sigma}{\delta}\langle \nabla h(r_\sigma e^{i\theta}), e^{i\theta}\rangle r'_\sigma = 0. \tag{4.31}$$

Solving for r'_σ,

$$r'_\sigma = -\frac{\frac{1}{\delta} h(r_\sigma e^{i\theta})}{2(1 - \delta^2)\pi r_\sigma + \frac{\sigma}{\delta}\langle \nabla h(r_\sigma e^{i\theta}), e^{i\theta}\rangle} \tag{4.32}$$

and so, by (4.24), we have $|r'_\sigma| \leq Cr_\sigma$. We now compute the second derivative of r_σ. Differentiating (4.31) once more with respect to σ, we obtain

$$\begin{aligned}2(1-\delta^2)\pi\left((r'_\sigma)^2 + r_\sigma r''_\sigma\right) + \frac{2}{\delta}\langle\nabla h(r_\sigma e^{i\theta}), e^{i\theta}\rangle r'_\sigma \\ + \frac{\sigma}{\delta}D^2 h(r_\sigma e^{i\theta})(e^{i\theta}, e^{i\theta})r'_\sigma + \frac{\sigma}{\delta}\langle\nabla h(r_\sigma e^{i\theta}), e^{i\theta}\rangle r''_\sigma = 0.\end{aligned} \tag{4.33}$$

Solving for r''_σ in (4.33) and using (4.24) and $|r'_\sigma| \leq Cr_\sigma$ repeatedly, we see that $|r''_\sigma| \leq Cr_\sigma$.

These new estimates are necessary in order to bound the area of the super-level set $\{u > t\}$. Indeed, as we remarked before, $\{u > t\} \subset \{z \in \mathbb{C}\colon 0 < |z| < r_\delta(\theta)\} = E_\sigma$, and since this set is starshaped,

$$\mu(t) \leq \frac{1}{2}\int_0^{2\pi} r_\delta(\theta)^2 \,\mathrm{d}\theta.$$

This motivates the definition of

$$f(\sigma) \coloneqq \frac{1}{2}\int_0^{2\pi} r_\sigma(\theta)^2 \,\mathrm{d}\theta,$$

where as before, we assume that $\sigma \in (-2\delta, 2\delta)$. Differentiating f and passing the derivative inside the integral (which, due to the bound the bound (4.30), is allowed), we obtain

$$|f''(\sigma)| \leq C\int_0^{2\pi} r_\sigma(\theta)^2 \,\mathrm{d}\theta. \tag{4.34}$$

Furthermore, notice that $r'_\sigma|_{\sigma=0} = \frac{h(r_0 e^{i\theta})}{\delta(1-\delta^2)\pi r_0}$ by (4.32). We come now to the crucial step in the proof: recall that, by definition, h is harmonic and so, by the mean value property,

$$f'(0) = \frac{1}{\pi\delta(1-\delta^2)}\int_0^{2\pi} h(r_0 e^{i\theta})\,\mathrm{d}\theta = \frac{2}{\delta(1-\delta^2)}h(0) = 0.$$

The Taylor expansion (with remainder) of f, together with (4.34) provides us with the bound

$$f(\delta) = f(0) + \frac{f''(\sigma)}{2}\delta^2 \leq f(0) + C\delta^2\int_0^{2\pi} r_\sigma(\theta)^2 \,\mathrm{d}\theta, \quad \text{for some } \sigma \in (0,\delta). \tag{4.35}$$

On the other hand, by (4.29) evaluated at 0 and σ, we find

$$(1-\delta^2)\pi(r_\sigma^2 - r_0^2) = -\frac{\sigma}{\delta}h(r_\sigma e^{i\theta}).$$

By using (4.24) and taking δ sufficiently small, we have $r_\sigma^2 \leq 2r_0^2$ for all $\sigma \in (-2\delta, 2\delta)$. Thus,

$$\mu(t) \leq f(\delta) \leq \frac{1+C\delta^2}{2}\int_0^{2\pi} r_0^2\,\mathrm{d}\theta = \frac{1+C\delta^2}{1-\delta^2}\log\frac{T}{t},$$

since $(1-\delta^2)\pi r_0^2 = \log\frac{T}{t}$. □

We can now use the previous bound to prove a sharp stability estimate for the function.

Lemma 4.7. *Let $F \in \mathcal{F}(\mathbb{C})$ be such that $\|F\|_{\mathcal{F}} = 1$ and $|\Omega| = s_0$. There is an absolute, computable constant $C > 0$ such that*

$$1 - \max u \leq Ce^{s_0}\delta_{s_0}(F).$$

Proof. Before starting, note that we can, and do assume that $T < e^{-s_0}$, since in the complementary regime Lemma 4.4 already yields optimal bounds.

STEP 1. With the aid of Lemma 4.6, we show uniqueness and we find a suitable lower bound on s^*, as defined just before the proof of Lemma 4.4. Indeed, assume there exist $s_1^* < s_2^*$ such that $u^*(s_i^*) = e^{-s_i^*}$, $i = 1, 2$. Then, recall that the function $L(t)$ defined in (4.11) is decreasing, which implies that everywhere in $[s_1^*, s_2^*]$, it must happen that $u^*(s) = e^{-s}$. This, in turn, means that $L'(t) = 0$ for $t \in [e^{-s_2^*}, e^{-s_1^*}]$, and the proof of Lemma 3.6 shows that there is equality in $L'(t) \geq 0$ in a point where μ is differentiable if, and only if, the set $\{u > t\}$ is a ball, and $|\nabla u|$ is constant on the boundary $\partial\{u > t\} = \{u = t\}$. However, under these conditions, u must be $e^{-\pi|z-z_0|^2}$ for some $z_0 \in \mathbb{C}$, and $u^*(s) = e^{-s}$ for each $s \geq 0$. Since we are ruling out this case, s^* must be unique with the property that defines it.

By Lemma 4.6 we have:

$$\mu(t) \leq (1 + C(1-T)) \log\left(\frac{T}{t}\right). \tag{4.36}$$

Set then $t = e^{-s^*}$ in (4.36) above. As $t = u^*(s^*)$, we obtain

$$s^* \leq (1 + C(1-T)) \cdot \log\left(\frac{T}{t}\right) = (1 + C(1-T)) \cdot (\log T + s^*).$$

Rearranging, one obtains

$$C(1-T) \cdot s^* \geq (1 + C(1-T)) \log\left(\frac{1}{T}\right), \tag{4.37}$$

which implies that $s^* \geq c_0$ for some $c_0 > 0$ absolute and computable.

STEP 2. We will now use the bound from Lemma 4.6 to find a comparison function for e^{-s} that can play the role of u^* in the proof of Lemma 4.4 and improves the control on $1 - \max u$ found before.

Notice that the estimate reads $s \leq (1 + C(1-T)) \log(T/u^*(s))$, which may be rewritten as

$$u^*(s) \leq T e^{-\frac{s}{1+C(1-T)}}. \tag{4.38}$$

Define the point $\tilde{s}$ such that $Te^{-\frac{\tilde{s}}{1+C(1-T)}} = e^{-\tilde{s}}$, that is,

$$\tilde{s} = \frac{1 + C(1-T)}{C(1-T)} \log\left(\frac{1}{T}\right).$$

In particular, (4.37) implies that $\tilde{s}$ can be bounded from above and below by absolute constants c_1 and c_0,

$$s^* \geq c_1 \geq \tilde{s} \geq c_0$$

whenever T satisfies the conditions of Lemma 4.6. If we now do the comparison step, we find

$$\begin{aligned} \int_0^{\tilde{s}} \left(e^{-s} - T \cdot e^{-\frac{s}{1+C(1-T)}}\right) \mathrm{d}s &= (1-T)(1 + C(e^{-\tilde{s}} - T)) \\ &= (1-T)(1 + C \cdot T(e^{\frac{\log T}{C(1-T)}} - 1)), \end{aligned} \tag{4.39}$$

and our goal is to show that the last factor is bounded from below by an absolute, positive constant. To this end, consider the function

$$\mathcal{R}(C, T) = C \cdot T(1 - e^{\frac{\log T}{C(1-T)}}). \tag{4.40}$$

Notice that, for fixed $C > 0$,

$$\lim_{T\to 1} \mathcal{R}(C,T) = C(1-e^{-1/C}) < 1, \qquad T \mapsto \mathcal{R}(C,T) \text{ is decreasing.}$$

Since C is computable, absolute and finite, if we suppose that $1-T$ is small enough (depending only on C from Lemma 4.6), then $\mathcal{R}(C,T) < 1-\rho_0$, where $\rho_0 > 0$ is some absolute and computable constant. Thus, by (4.39),

$$\begin{aligned}\int_0^{\tilde{s}} \left(e^{-s} - T\cdot e^{-\frac{s}{1+C(1-T)}}\right) \mathrm{d}s &\geq (1-T)\left(1-T\cdot\mathcal{R}(C,T)\right) \\ &\geq (1-T)\left((1-T)+\rho_0 T\right) > \frac{\rho_0}{2}(1-T).\end{aligned} \tag{4.41}$$

STEP 3. To conclude the desired estimate, we divide the analysis depending on the relative positions between $s^*, \tilde{s}, s_0$. Recall that $s^* \geq \tilde{s}$, and before beginning the case analysis, recall also from the proof of Lemma 4.4, and (4.10) in particular, that the deficit δ_0 satisfies the relation

$$\int_0^{s_0} \left(e^{-s} - u^*(s)\right) \mathrm{d}s = \delta_0(1-e^{-s_0}). \tag{4.42}$$

Case 1: $s_0 \geq s^* \geq \tilde{s}$. Putting (4.38) and (4.41) together yields

$$\int_0^{s^*} \left(e^{-s} - u^*(s)\right) \mathrm{d}s \geq \int_0^{\tilde{s}} \left(e^{-s} - T\cdot e^{-\frac{s}{1+C(1-T)}}\right) \mathrm{d}s \geq \frac{\rho_0}{2}(1-T),$$

and the conclusion follows from (4.13), since

$$\delta_0(1-e^{-s_0})e^{s_0-s^*} \geq \int_0^{s^*} \left(e^{-s} - u^*(s)\right) \mathrm{d}s.$$

Case 2: $s^* \geq s_0 \geq \tilde{s}$. This case follows directly from (4.41) and (4.42), since

$$\int_0^{s_0} \left(e^{-s} - u^*(s)\right) \mathrm{d}s \geq \int_0^{\tilde{s}} \left(e^{-s} - T\cdot e^{-\frac{s}{1+C(1-T)}}\right) \mathrm{d}s \geq \frac{\rho_0}{2}(1-T).$$

Case 3: $s^* \geq \tilde{s} \geq s_0$. In this case, notice that if $s \in [0,\tilde{s}]$, then

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(e^{-s} - Te^{-\frac{s}{1+C(1-T)}}\right) = -e^{-s} + \frac{T}{1+C(1-T)}e^{-\frac{s}{1+C(1-T)}} < 0.$$

Hence, again by (4.41), and by concavity,

$$\begin{aligned}(1-e^{-s_0})\delta_0 &\geq \int_0^{s_0} \left(e^{-s} - u^*(s)\right) \mathrm{d}s \\ &\geq \frac{s_0}{\tilde{s}} \int_0^{\tilde{s}} \left(e^{-s} - Te^{-\frac{s}{1+C(1-T)}}\right) \mathrm{d}s \geq s_0 \frac{\rho_0}{c_1}(1-T).\end{aligned}$$

This finishes the proof, since $s_0 \mapsto \frac{1-e^{-s_0}}{s_0}$ is bounded from above and below by positive, absolute constants. $\square$

4.2 Stability for the set

After the discussion in the previous section, it remains to prove stability of the set. The ideas in this part are based on a mass transportation argument after taking the point of view of a perturbation problem. We aim to find a quantitative version of (3.26), but we know very little about the function u a priori, and so we need to exploit the result proved in the last section. We do this by arguing that u has to be very close to a Gaussian v (recall (4.16)). We can measure this difference and work with the Gaussian explicitly in (3.26), which allows us to find a proof of (4.4).

Proof of Theorem 4.1. Let $F \in \mathcal{F}(\mathbb{C})$ with $\|F\|_{\mathcal{F}} = 1$. We write $\delta = \delta(F;\Omega)$ for simplicity, and we may assume that $\delta \leq \delta_0$, for some arbitrarily small constant δ_0. We may assume that u attains its maximum at the origin and the stability result for the function suggests the decomposition $F = 1 + \rho G$, where

$$\rho = \|F - 1\|_{\mathcal{F}} \leq Ce^{|\Omega|/2}\delta^{1/2}, \quad \|G\|_{\mathcal{F}} = 1.$$

Given the uniqueness of the super-level sets with respect to a fixed measure, we will write $A_\Omega \coloneqq A_{u^*(|\Omega|)}$, and through the rest of the proof the sets $\Omega \setminus A_\Omega$ and $A_\Omega \setminus \Omega$ will play a pivotal role: take T to be any transport map $T\colon A_\Omega \setminus \Omega \to \Omega \setminus A_\Omega$, that is,

$$\mathbb{1}_{\Omega\setminus A_\Omega}(T(x)) \det \nabla T(x) = \mathbb{1}_{A_\Omega \setminus \Omega}(x),$$

cf. [FG, Section 1.4.1], and define

$$B \coloneqq \left\{ x \in A_\Omega \setminus \Omega \colon |T(x)|^2 - |x|^2 > C_{|\Omega|}\gamma \right\},$$

where $C_{|\Omega|}, \gamma$ are constants to be chosen later. Notice that, by virtue of (2.23),

$$\begin{aligned} |u - v| &= \left| (|F|^2 - 1)e^{-\pi|z|^2} \right| \\ &\leq \left(|F - 1|e^{-\pi|z|^2/2} \right) \left(|F|e^{-\pi|z|^2/2} + e^{-\pi|z|^2/2} \right) \leq \rho(F)(\|F\|_{\mathcal{F}} + \|1\|_{\mathcal{F}}) = 2\rho, \end{aligned} \tag{4.43}$$

which yields the set inclusions

$$\{v > u^*(s) + 2\rho\} \subset \{u > u^*(s)\} \subset \{v > u^*(s) - 2\rho\}. \tag{4.44}$$

Since T is a transport map, the inclusion above guarantees that the inequality

$$\int_B u(z) - u(T(z))\,\mathrm{d}z \leq \int_{A_\Omega\setminus\Omega} u(z) - u(T(z))\,\mathrm{d}z = \int_{A_\Omega} u - \int_\Omega u \eqqcolon d(\Omega) \tag{4.45}$$

holds true. Note too that from the proof of Theorem 3.1, we have the bound

$$d(\Omega) \leq 1 - e^{-|\Omega|} - \int_\Omega u = (1 - e^{-|\Omega|})\delta. \tag{4.46}$$

STEP 1. Having set the context up, we begin by showing the following lower bound on B,

$$u(x) - u(T(x)) \geq 5\gamma \quad \text{for } x \in B, \tag{4.47}$$

whenever $C_{|\Omega|}$ and γ are chosen adequately. Notice that

$$\begin{aligned} u(x) - u(T(x)) = e^{-\pi|x|^2} &- e^{-\pi|T(x)|^2} \\ &+ 2\rho\left(\mathrm{Re}(G(x)e^{-\pi|x|^2}) - \mathrm{Re}(G(T(x))e^{-\pi|T(x)|^2})\right) \\ &+ \rho^2\left(|G(x)|^2e^{-\pi|x|^2} - |G(T(x))|^2e^{-\pi|T(x)|^2}\right), \end{aligned}$$

and since $|G(y)|^2e^{-\pi|y|^2} \leq 1$ for each $y \in \mathbb{R}^2$, we can bound the difference from below by

$$\begin{aligned} u(x) - u(T(x)) &\geq e^{-\pi|x|^2} - e^{-\pi|T(x)|^2} - (4\rho + \rho^2) \\ &= e^{-\pi|x|^2}\left(1 - e^{-\pi\left(|T(x)|^2 - |x|^2\right)}\right) - 4\rho - \rho^2. \end{aligned} \tag{4.48}$$

This holds for any $x \in A_\Omega \setminus \Omega$. If, in particular, $\pi(|T(x)|^2 - |x|^2) \geq 1$ then we have

$$u(x) - u(T(x)) \geq \frac{e^{-|\Omega|}}{2}(1 - e^{-1}) - 4\rho - \rho^2 \geq \frac{e^{-|\Omega|}}{4} - 4\rho - \rho^2.$$

On the other hand, if $\pi(|T(x)|^2 - |x|^2) \leq 1$, we argue as follows: since $x \in B \subset A_\Omega$, (4.43) shows that

$$e^{-\pi|x|^2} \geq u(x) - 2\rho \geq u^*(|\Omega|) - 2\rho \geq e^{-|\Omega|} - 4\rho > \frac{e^{-|\Omega|}}{2},$$

where we also used that

$$e^{-|\Omega|} - 2\rho \leq u^*(|\Omega|) \leq e^{-|\Omega|} + 2\rho. \tag{4.49}$$

Thus, from (4.48),

$$u(x) - u(T(x)) \geq \frac{e^{-|\Omega|}\left(|T(x)|^2 - |x|^2\right)}{2} - 4\rho - \rho^2 \geq C_{|\Omega|}e^{-|\Omega|}\frac{\gamma}{2} - 4\rho - \rho^2.$$

Choosing $C_{|\Omega|} = 20e^{|\Omega|}$ and $\gamma \geq \rho$, the previous estimates yield the desired (4.47).

STEP 2. With this bound for $u(x) - u(T(x))$ we can estimate the size of B, which already shows that A_Ω and Ω must be close. Observe that by the definition of T, the identities

$$|\Omega| - |B| = |\Omega| - |T(B)| = |\Omega \setminus T(B)| = |\Omega| - |A_\Omega \setminus \Omega| + |(\Omega \setminus T(B)) \setminus A_\Omega|,$$

hold true, and thus

$$\frac{1}{2}|\Omega \Delta A_\Omega| = |A_\Omega \setminus \Omega| = |B| + |(\Omega \setminus T(B)) \setminus A_\Omega|. \tag{4.50}$$

By estimating the terms on the right-hand side we find the desired bound. The estimate for the first term follows by combining (4.45), (4.46) and (4.47):

$$|B| \leq \frac{d(\Omega)}{5\gamma} \leq \frac{\delta(1 - e^{-|\Omega|})}{5\gamma}. \tag{4.51}$$

To estimate the second term, note that $\Omega \setminus T(B)$ is contained in a $C' \cdot C_{|\Omega|}\gamma$-neighborhood of A_Ω (for C' that depends only on $|\Omega|$ and that we may drop), but we know by (4.44) that A_Ω is in nested between two concentric balls:

$$\{z\colon v(z) > u^*(|\Omega|) + 2\rho\} \subset A_\Omega \subset \{z\colon v(z) > u^*(|\Omega|) - 2\rho\} =: E_\Omega. \tag{4.52}$$

Combining this information with (4.49), we can estimate

$$
\begin{aligned}
|(\Omega \setminus T(B)) \setminus A_\Omega| &\le 4C_{|\Omega|}\gamma\sqrt{-\log(e^{-|\Omega|} - 4\rho)} \\
&\le 4C_{|\Omega|}\gamma\sqrt{|\Omega| + 8\rho e^{|\Omega|}} \\
&\le 4C_{|\Omega|}\gamma\left(|\Omega|^{1/2} + 4\rho\frac{e^{|\Omega|}}{|\Omega|^{1/2}}\right) \le 4C_{|\Omega|}\gamma\left(|\Omega|^{1/2} + 4C\delta^{1/2}\frac{e^{2|\Omega|}}{|\Omega|^{1/2}}\right),
\end{aligned}
\tag{4.53}
$$

provided that ρ is sufficiently small, depending on $|\Omega|$. Choosing $\rho \le \gamma = C\delta^{1/2}$, where C is the constant provided by Theorem 4.3, and combining (4.50), (4.51) and (4.53), we get

$$
|\Omega \Delta A_\Omega| \le C\delta^{1/2},
$$

for some new but still explicitly computable constant C.

STEP 3. To conclude, we just need to compare Ω with the ball $S_\Omega \coloneqq \{z \colon v(z) \ge e^{-|\Omega|}\}$. In fact, as $S_\Omega \subset E_\Omega$, and $|E_\Omega \setminus S_\Omega| \le C\delta^{1/2}$, we naturally have

$$
\begin{aligned}
|S_\Omega \Delta \Omega| &\le |\Omega \setminus E_\Omega| + |E_\Omega \setminus S_\Omega| + |S_\Omega \setminus \Omega| \\
&\le |\Omega \setminus E_\Omega| + |E_\Omega \setminus S_\Omega| + |E_\Omega \setminus \Omega| \le |E_\Omega \Delta \Omega| + C\delta^{1/2}.
\end{aligned}
$$

It suffices to estimate $|E_\Omega \Delta \Omega|$. We apply the estimate from the previous step twice, together with (4.52)

$$
|E_\Omega \Delta \Omega| = |E_\Omega \setminus \Omega| + |\Omega \setminus E_\Omega| \le |E_\Omega \setminus A_\Omega| + |A_\Omega \setminus \Omega| + |\Omega \setminus A_\Omega| \le C\delta^{1/2},
$$

and the conclusion follows. □

4.3 The geometry of super-level sets

We will dedicate this section to exploring the geometric properties of the super-level sets of u. In particular, we have seen that the function $V_\sigma(r, \theta)$ defined in (4.27) is monotone increasing in r, which in particular implies that its sub-level sets are star-shaped. By doing a more careful analysis, we can show that this happens for super-level sets $A_t = \{u > t\}$ of u too, as long as t is sufficiently large. Moreover, these sets actually satisfy even better properties, as explained by the following proposition.

Proposition 4.8. *There exist small explicit constants $\delta_0, c > 0$ such that for all $F \in \mathcal{F}(\mathbb{C})$ with*

$$
\delta(F; A_{u^*(s)}) \le e^{-s}\delta_0
$$

and for all $s < c\log(1/\delta_0)$, the super-level set

$$
A_{u^*(s)} = \{z \in \mathbb{C} : u(z) > u^*(s)\}
$$

has smooth boundary and convex closure.

The core result in this section will be Lemma 4.9. It shows that, above a certain threshold, all level sets of the function u behave like those of the standard Gaussian, as long as F is sufficiently close to 1 in the Fock space – which happens if the deficit is small enough. Before we present it, let us introduce a few normalizations.

We will assume that $\|F\|_{\mathcal{F}} = 1$ and that F attains its maximum at $z = 0$. Furthermore, it will be useful to consider the decomposition for F that we used in the proof of the set stability, namely

$$F = 1 + \varepsilon G, \quad \|G\|_{\mathcal{F}} = 1.$$

Lemma 4.9. *Let F satisfy the normalizations above. Then there exist constants $\varepsilon_0, c_1 > 0$ with the following property: if $\varepsilon \leq \varepsilon_0$, then for any $\alpha \in [0, 2\pi]$, the function*

$$g_\alpha(r) \coloneqq u(re^{i\alpha}) = |F(re^{i\alpha})|^2 e^{-\pi r^2}$$

is strictly decreasing on the interval $\left[0, c_1\sqrt{\log(1/\varepsilon)}\right]$.

Proof. Without loss of generality we will take $\alpha = 0$, and we will separate the proof in two cases: when r is close to 0, and when it is away from it.

Case 1: $1/10 < r < c_1\sqrt{\log(1/\varepsilon)}$**.** We differentiate the function g_0 in terms of r, which gives us

$$\begin{aligned} g_0'(r) &= -2\pi r|F(r)|^2 e^{-\pi r^2} + 2\operatorname{Re}(F'(r)\overline{F(r)})e^{-\pi r^2} \\ &= -2\pi r(1 + 2\varepsilon\operatorname{Re}(G(r)) + \varepsilon^2|G(r)|^2)e^{-\pi r^2} + 2\varepsilon\operatorname{Re}(G'(r)\overline{(1+\varepsilon G(r))})e^{-\pi r^2}, \end{aligned} \tag{4.54}$$

The last term can be bounded using the Cauchy integral formula,

$$|G'(w)| \leq \frac{2}{|w|}\sup_{|z|=2|w|}|G(z)| \leq 2\frac{e^{2\pi|w|^2}}{|w|}, \tag{4.55}$$

since $|G(z)| \leq e^{\pi|z|^2/2}$, and by (2.23),

$$|1 + \varepsilon G(r)| \leq e^{\frac{\pi}{2}r^2}.$$

The ε^2-term in (4.54) is negative, and $|\operatorname{Re} G(r)|e^{-\pi r^2} \leq e^{-\pi r^2/2} \leq 1$, so we can estimate

$$g_0'(r) \leq -2\pi r(e^{-\pi r^2} - 2\varepsilon) + \frac{4\varepsilon}{r}e^{\frac{3\pi}{2}r^2}.$$

Now, given that $r < c_1\sqrt{\log(1/\varepsilon)}$, it holds that $e^{\pi r^2} \leq e^{\pi c_1^2 \log(1/\varepsilon)} = \varepsilon^{-\pi c_1^2}$, so for all ε small enough, we have $e^{-\pi r^2} - 2\varepsilon \geq \varepsilon^{\pi c_1^2} - 2\varepsilon > 0$ provided that $\pi c_1^2 < 1$. Since also $1/10 \leq r$, we can further bound

$$g_0'(r) \leq -2\pi r(\varepsilon^{\pi c_1^2} - 2\varepsilon) + 40\varepsilon^{1-\frac{3\pi}{2}c_1^2},$$

which reveals that the dominating term will be $-2\pi r\varepsilon^{\pi c_1^2}$ as long as ε is small enough and

$$\frac{5\pi}{2}c_1^2 < 1, \tag{4.56}$$

resulting in $g_0'(r) < 0$, as we claimed.

Case 2: $0 < r \leq 1/10$**.** This case is more subtle, and the reason for dividing the proof in two parts, as $g_0'(r) \to 0$ when $r \to 0$. We need to analyze the second derivative of g_0

and show that it is strictly negative for $r \in (0, 1/10)$. This implies that the first derivative decreases in $(0,1)$ and, as $g_0'(0) = 0$, it follows that $g_0'(r) < 0$ in this interval, thus proving the claim.

Starting from (4.54), we compute:

$$\begin{aligned} g_0''(r) = &- 2\pi(1 + 2\varepsilon \operatorname{Re}(G(r)) + \varepsilon^2|G(r)|^2)e^{-\pi r^2} \\ &- 4\pi\varepsilon(\operatorname{Re}(G'(r)) + 2\varepsilon \operatorname{Re}(G'(r)\overline{G(r)}))e^{-\pi r^2} \\ &+ 4\pi^2 r^2(1 + 2\varepsilon \operatorname{Re}(G(r)) + \varepsilon^2|G(r)|^2)e^{-\pi r^2} \\ &+ 2\varepsilon \operatorname{Re}(G''(r)\overline{(1+\varepsilon G(r))})e^{-\pi r^2} + 2\varepsilon^2|G'(r)|^2 e^{-\pi r^2} \\ &- 2\pi r\varepsilon \operatorname{Re}(G'(r)\overline{(1+\varepsilon G(r))})e^{-\pi r^2}. \end{aligned}$$

We now follow the same strategy as before. For $|w| \leq 1$, the Cauchy integral formula yields

$$|G'(w)| \leq \frac{1}{2\pi} \max_{|z|=2} |G(z)| \leq 2e^{2\pi}, \qquad |G''(w)| \leq \frac{1}{\pi} \max_{|z|=2} |G(z)| \leq 4e^{2\pi}, \tag{4.57}$$

and we have

$$g_0''(r) \leq -2\pi(1 - 2\pi r^2)e^{-\pi r^2} + \varepsilon f(\varepsilon),$$

where $f\colon \mathbb{R} \longrightarrow [0,\infty)$ is a smooth function. Since $r < 1/10$, it holds that $1-2\pi r^2 > 0$ and so the first term above is negative. The second term will be small enough to not be the leading term as soon as ε is sufficiently small, resulting in $g_0''(r) < 0$ for all $r \in (0, 1/10)$, from where the conclusion follows. ☐

Lemma 4.9 already provides intuition on the next result: since u is radially decreasing close to its maximum, given any z in some super-level set A_t of u, the segment $[0,z]$ should be contained in A_t, as for any of its points w, it must happen that $u(w) > u(z) > t$.

Lemma 4.10. *Under the same hypotheses of Lemma 4.9, one may find a small constant $c_2 > 0$ such that, for $t > \varepsilon^{c_2}$, the level sets*

$$\{z \in \mathbb{C} \colon u(z) > t\}$$

are all star-shaped with respect to z_0. Moreover, for such t, the boundary $\partial\{u > t\} = \{u = t\}$ is a smooth, closed curve.

Proof. STEP 1. Let $c_1 > 0$ be given by Lemma 4.9. We claim that if $|z| > c_1\sqrt{\log(1/\varepsilon)}$, then

$$u(z) < 4\varepsilon^{\pi c_1^2}. \tag{4.58}$$

As before, we use the decomposition $F = 1 + \varepsilon G$ to write

$$u(z) = (1 + 2\varepsilon \operatorname{Re}(G(z)) + \varepsilon^2|G(z)|^2)e^{-\pi|z|^2}.$$

For $|z| > c_1\sqrt{\log(1/\varepsilon)}$, and ε sufficiently small, since $\|G\|_{\mathcal{F}} = 1$, we can bound as in the last proof to show that

$$u(z) \leq \varepsilon^{\pi c_1^2}(1 + 2\varepsilon + \varepsilon^2) < 4\varepsilon^{\pi c_1^2},$$

since we can choose $\pi c_1^2 \leq \frac{1}{2}$ to force the choice of a leading term (cf. (4.56)).

STEP 2. We now claim that the conclusion of the lemma holds with $c_2 = \frac{\pi c_1^2}{2}$. Suppose that this were not the case: then there is $t_0 > \varepsilon^{c_2}$ such that $A_{t_0} := \{z \in \mathbb{C} \colon u(z) > t_0\}$ is not star-shaped with respect to 0, i.e. there exists a point $w_0 \in A_{t_0}$ for which, for some $r \in (0,1)$, it happens that $rw_0 \notin A_{t_0}$. Then, (4.58) implies that

$$|w_0| < c_1\sqrt{\log(1/\varepsilon)}. \tag{4.59}$$

Indeed, if $|w_0| > c_1\sqrt{\log(1/\varepsilon)}$ then, for a small enough ε, we would have

$$u(w_0) < 4\varepsilon^{\pi c_1^2} < \varepsilon^{\pi c_1^2/2} < t_0,$$

contradicting the fact that $w_0 \in A_{t_0}$. However, (4.59) contradicts Lemma 4.9: if we let α_0 be such that $e^{i\alpha_0} = \frac{w_0}{|w_0|}$, then the function $s \mapsto |F(se^{i\alpha_0})|^2 e^{-\pi s^2}$ is strictly decreasing for $s < c_1\sqrt{\log(1/\varepsilon)}$ and thus

$$t_0 > u(rw_0) > u(w_0) > t_0,$$

which is absurd. Hence A_{t_0} is star-shaped with respect to the origin.

STEP 3. Finally, the smoothness of the boundary $\partial\{u > t\} = \{u = t\}$ is guaranteed by the Inverse Function Theorem. Thanks to (4.58), we see that if z has $u(z) = t > \varepsilon^{c_2}$, then $|z| < c_1\sqrt{\log(1/\varepsilon)}$, and by Lemma 4.9, it holds that $\nabla u(z) \neq 0$. Then t is a regular value of u and the set $\{u = t\}$ is a smooth curve. □

Lemmata 4.9 and 4.10 reveal some of the desirable geometric properties of the super-level sets of u. However, the complete picture is even more surpising: these sets are actually convex close to the maximum of u.

Proposition 4.11. *Under the same assumptions as in Lemma 4.9, there are small constants $\varepsilon_0, c_3 > 0$ such that, as long as $\varepsilon \leq \varepsilon_0$ and $s < -c_3\log(\varepsilon)$, the set*

$$A_{u^*(s)} := \{z \in \mathbb{C} \colon u(z) > u^*(s)\}$$

has convex closure.

The proof of Proposition 4.11 is technical and we refer the reader to [GGRT] for the complete argument. It essentially relies on a series of estimates on the curvature of the set $A_{u^*(s)}$ compared to that of the circle $\{v > u^*(s)\}$. These ensure that the sets A_t are locally convex, which imply that their closure is convex through the Tietze–Nakajima theorem [Na, Ti]. This proposition, together with the previous lemma, shows the validity of Proposition 4.8.

4.4 A variational approach

Despite the fact that we have already essentially given a proof for the stability of the function in Subsection 4.1.2, this result can also be established through a variational approach, with the benefit that this provides tools to easily show how Theorem 4.1 is sharp. At the same time, it is much more technically involved, with many results having long and technical proofs that escape the scope of this manuscript. For this reason, we will limit our exposition of this approach to presenting the main results and ideas comprising the variational proof, and we refer to the original work [GGRT] for a fully detailed account.

Through the discussion we present in this section, it will be useful to introduce a series of normalizations. In what follows, we consider

$$\rho := \min_{z_0 \in \mathbb{C}, c \in \mathbb{C}} \frac{\|F - c \cdot F_{z_0}\|_{\mathcal{F}}}{\|F\|_{\mathcal{F}}}.$$

By appropriately translating, we can assume that u attains its maximum at $z_0 = 0$, and by normalizing, we can set $F(0) = 1$. In particular, we have $F'(0) = 0$ and by Proposition 4.5, $F_{z_0} \equiv 1$ is the best contender for ρ, so that

$$\rho = \min_{c \in \mathbb{C}} \frac{\|F - c\|_{\mathcal{F}}}{\|F\|_{\mathcal{F}}}.$$

Observe that ρ differs slightly from the distance to the extremizing class used in (4.5), due to the condition on c. However, it is equivalent to this distance. Since

$$0 \leq \|c - F(z_0)e^{-\pi|z_0|^2/2}\|_{\mathcal{F}} \leq |c|^2 - 2 \operatorname{Re} \overline{c} F(z_0) e^{-\pi|z_0|^2/2} + \max_{z \in \mathbb{C}} |F(z)|^2 e^{-\pi|z|^2},$$

it holds that

$$\begin{aligned} \|F - c \cdot F_{z_0}\|_{\mathcal{F}}^2 &= \|F\|_{\mathcal{F}}^2 + |c|^2 - 2 \operatorname{Re}(\overline{c} F(z_0)) e^{-\pi|z_0|^2/2} \\ &\geq \|F\|_{\mathcal{F}}^2 + |c|^2 - 2|c||F(z_0)|e^{-\pi|z_0|^2/2} \geq \|F\|_{\mathcal{F}}^2 - \max_{z \in \mathbb{C}} |F(z)|^2 e^{-\pi|z|^2}, \end{aligned} \tag{4.60}$$

and hence

$$\rho \leq \min_{z_0 \in \mathbb{C}, |c| = \|F\|_{\mathcal{F}}} \frac{\|F - c \cdot F_{z_0}\|_{\mathcal{F}}}{\|F\|_{\mathcal{F}}} = \min_{|c| = \|F\|_{\mathcal{F}}} \frac{\|F - c\|_{\mathcal{F}}}{\|F\|_{\mathcal{F}}} \leq \sqrt{2}\rho. \tag{4.61}$$

In addition to ρ, it will be convenient to consider the slightly different quantity

$$\varepsilon := \|F - 1\|_{\mathcal{F}}.$$

This allows us to write

$$F = 1 + \varepsilon G, \quad \text{where} \quad \|G\|_{\mathcal{F}} = 1, \tag{4.62}$$

and the above assumptions on F and F' are translated as

$$\langle G, 1 \rangle_{\mathcal{F}} = \langle G, z \rangle_{\mathcal{F}} = 0. \tag{4.63}$$

Having discussed these normalizations, we introduce the following functional, which plays a pivotal role in the entirety of the current section. Fix $s > 0$, and consider

$$\mathcal{K} \colon \mathcal{F}(\mathbb{C}) \to \mathbb{R}, \quad \mathcal{K}[F] := \frac{I(s)}{\|F\|_{\mathcal{F}}},$$

where $I(s)$ is the integral of u over its super-level set of measure s (recall (3.23)). The main result in this section will be the following theorem.

Theorem 4.12. *Fix $s \in (0, \infty)$. There are explicit constants $\varepsilon_0(s), C(s) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, we have*

$$\mathcal{K}[1] - \mathcal{K}[1 + \varepsilon G] \geq C(s)\varepsilon^2$$

whenever $\|G\|_{\mathcal{F}} = 1$ satisfies (4.63).

Its proof is essentially disjoint from the previous discussion, with the exception that we will need the sub-optimal stability estimate from Lemma 4.4, and this theorem directly implies the sharp function stability part of Theorem 4.3, although without the optimal dependence in terms of $|\Omega|$.

Proof of (4.5) *through Theorem 4.12.* Without loss of generality, we may assume that $\|F\|_{\mathcal{F}} = 1$, $F(0) \in \mathbb{R}$ and $F'(0) = 0$. If the deficit is sufficiently small, Lemma 4.4 implies

$$\|F - 1\|_{\mathcal{F}} = \varepsilon = \|F\|_{\mathcal{F}}\rho \le C\big(e^{|\Omega|}\delta(F;\Omega)\big)^{1/4},$$

together with Proposition 4.5, and $\Omega = A_{u^*(t)} = \{u > u^*(t)\}$. Hence, (4.62) is in force and we can assume that ε is sufficiently small so that we may apply Theorem 4.12. This then implies that

$$(1 - e^{-|\Omega|})\delta(F;\Omega) = \mathcal{K}[1] - \mathcal{K}[F] \ge C(|\Omega|)\varepsilon^2 = C(|\Omega|)\|F - 1\|_{\mathcal{F}}^2.$$

To complete the proof, note that $F(0) = 1 \le \|F\|_{\mathcal{F}}$. Thus

$$\|F\|_{\mathcal{F}} \ge \rho \ge 2^{-1/2} \min_{z_0\in\mathbb{C}, |c|=\|F\|_{\mathcal{F}}} \frac{\|F - c\cdot F_{z_0}\|_{\mathcal{F}}}{\|F\|_{\mathcal{F}}},$$

where the last inequality follows from (4.61). □

The proof of Theorem 4.12 is based on a series of preliminary results. The first of these is Lemma 4.13, which shows that $\mathcal{K}[1] - \mathcal{K}[1 + \varepsilon G]$ is essentially controlled by the second variation of $\mathcal{K}$ at 1, in the direction of G.

Lemma 4.13. *There is $\varepsilon_0 = \varepsilon_0(s)$ and a modulus of continuity η, depending only on s, such that*

$$|\mathcal{K}[1 + \varepsilon G] - \mathcal{K}[1]| \le \left|\frac{\varepsilon^2}{2}\nabla^2\mathcal{K}[1](G,G)\right| + \eta(\varepsilon)\varepsilon^2$$

for all $0 \le \varepsilon \le \varepsilon_0(t)$ and $G \in \mathcal{F}(\mathbb{C})$ such that $\|G\|_{\mathcal{F}} = 1$ and which satisfy (4.63)*. Here we have defined*

$$\nabla^2\mathcal{K}[1](G,G) := \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\Big|_{\varepsilon=0}\mathcal{K}[1 + \varepsilon G].$$

Since 1 is a local maximum for $\mathcal{K}$, its second variation is negative definite, but to prove Theorem 4.12 we need to show that this happens uniformly, as the next proposition indicates.

Proposition 4.14. *For all $G \in \mathcal{F}(\mathbb{C})$ such that $\|G\|_{\mathcal{F}} = 1$ and which satisfy* (4.63)*, it holds that*

$$\frac{1}{2}\nabla^2\mathcal{K}[1](G,G) \le -se^{-s}.$$

These two results then provide a direct proof of the theorem.

Proof of Theorem 4.12. Combining Lemma 4.13 and Proposition 4.14, we have

$$\mathcal{K}[1] - \mathcal{K}[1 + \varepsilon G] \ge -\varepsilon^2\left(\frac{1}{2}\nabla^2\mathcal{K}[1](G,G) + \eta(\varepsilon)\right) \ge \varepsilon^2\left(\frac{C(s)}{2} - \eta(\varepsilon)\right).$$

The conclusion now follows by choosing $\varepsilon_0 = \varepsilon_0(s)$ even smaller so that $\frac{C(s)}{4} \ge \eta(\varepsilon_0)$. □

What is now left to do is to prove Proposition 4.14. This involves three independent results that build up to show an explicit expression for the second variation of $\mathcal{K}$ in the form of Lemma 4.17 below. They rely on the following strategy.

The first step is to consider the sets

$$\Omega_\varepsilon := \{u_\varepsilon > u_\varepsilon^*(s)\}, \quad u_\varepsilon := |1+\varepsilon G|^2 e^{-\pi|\cdot|^2}, \tag{4.64}$$

and to write

$$\Omega_\varepsilon = \Phi_\varepsilon(\Omega_0),$$

for a suitable volume-preserving flow Φ_ε. This is constructed using a general lemma that allows to deform the unit disk into a given family of graphical domains over the unit circle through a suitable flow.

Lemma 4.15. *Denote by $D_0 \subset \mathbb{R}^2$ the unit disk, and suppose that we are given a one-parameter family $\{D_\varepsilon\}_{\varepsilon\in[0,\varepsilon_0]}$ of domains, whose boundaries are given by smooth graphs over the unit circle:*

$$\partial D_\varepsilon = \{(1+g_\varepsilon(\omega))\omega\colon \omega \in \mathbb{S}^1\}.$$

Suppose, additionally, that the family $\{g_\varepsilon\}_{\varepsilon\in[0,\varepsilon_0]}$ depends smoothly on ε. Then there exists a family $\{Y_\varepsilon\}_{\varepsilon\in[0,\varepsilon_0]}$ of smooth vector fields, which depends smoothly on the parameter ε, such that, if Ψ_ε denotes the flow associated with Y_ε, i.e. if

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Psi_\varepsilon = Y_\varepsilon(\Psi_\varepsilon),$$

then $\Psi_\varepsilon(D_0) = D_\varepsilon$. In addition, Y_ε is of the form

$$Y_\varepsilon(r,\omega) = \frac{1}{r}(1+g_\varepsilon(\omega))\partial_\varepsilon g_\varepsilon(\omega)\omega$$

and such that $\operatorname{div}(Y_\varepsilon) = 0$ *in a neighbourhood of* $\mathbb{S}^1$.

These types of results are well-known, and we refer the reader to [GGRT] for the details, and to [AFM, Theorem 3.7] for a more general statement. Then, using this result on Ω_ε yields a globally defined vector field X_ε that depends smoothly on ε, with associated flows Φ_ε.

Lemma 4.16. *Let $G \in \mathcal{F}(\mathbb{C})$ satisfy* (4.63). *There is $\varepsilon_0 = \varepsilon_0(s, \|G\|_{\mathcal{F}}) > 0$ such that, for all $\varepsilon \in [0,\varepsilon_0]$, there are globally defined smooth vector field X_ε, with associated flows Φ_ε, such that*

$$\Omega_\varepsilon = \Phi_\varepsilon(\Omega_0)$$

and moreover X_ε depends smoothly on ε and is divergence-free in a neighborhood of $\partial\Omega_0$. We also have

$$\int_{\partial\Omega_\varepsilon} \langle X_\varepsilon, \nu_\varepsilon\rangle = 0, \tag{4.65}$$

where ν_ε denotes the outward-pointing unit vector field on $\partial\Omega_\varepsilon$.

Finally, proving Lemma 4.17, although by far not a simple task, amounts to exploiting the properties of $\mathcal{K}, \mu$ and X_ε in relation to G on $\partial\Omega_0$.

Lemma 4.17. *For all $G \in \mathcal{F}(\mathbb{C})$ which satisfy* (4.63)*, we have*

$$\frac{1}{2}\nabla^2\mathcal{K}[1](G,G) = \int_{\Omega_0} |G|^2 e^{-\pi|z|^2}\,\mathrm{d}z - \|G\|_{\mathcal{F}}^2 \int_{\Omega_0} e^{-\pi|z|^2}\,\mathrm{d}z + e^{-|\Omega_0|} \fint_{\partial\Omega_0} |G|^2\,\mathrm{d}\mathcal{H}^1(z). \tag{4.66}$$

With this expression for the second variation of $\mathcal{K}$ at hand, we are now able to prove Proposition 4.14.

Proof of Proposition 4.14. Since $G \in \mathcal{F}(\mathbb{C})$ satisfies (4.63), we can write

$$G(z) = \sum_{k=2}^{\infty} a_k \Big(\frac{\pi^k}{k!}\Big)^{\frac{1}{2}} z^k, \quad \|G\|_{\mathcal{F}}^2 = \sum_{k=2}^{\infty} |a_k|^2.$$

Then, (4.66) can be rewritten in terms of the power series of G as

$$\frac{1}{2}\nabla^2\mathcal{K}[1](G,G) = \sum_{k=2}^{\infty} |a_k|^2 V_k(s), \tag{4.67}$$

where

$$V_k(s) := \frac{\pi^k}{k!}\int_{B(0,\sqrt{s/\pi})} |z|^{2k} e^{-\pi|z|^2} - \int_{B(0,\sqrt{s/\pi})} e^{-\pi|z|^2} + e^{-s}\frac{s^k}{k!}. \tag{4.68}$$

We now claim that $V_k(s) \le 0$ for all $s \ge 0$. Indeed, from (4.68),

$$\begin{aligned} V_k(s) &= -\frac{\pi^k}{k!}\int_{\mathbb{C}\setminus B(0,\sqrt{s/\pi})} |z|^{2k} e^{-\pi|z|^2}\,\mathrm{d}z + \left(1 + \frac{s^k}{k!}\right) e^{-s} \\ &= -\frac{\Gamma(k+1,s)}{k!} + \left(1+\frac{s^k}{k!}\right)e^{-s} = -\left(\sum_{j=1}^{k-1}\frac{s^j}{j!}\right)e^{-s}, \end{aligned}$$

where $\Gamma(a,s) := \int_s^\infty r^{a-1}e^{-r}\,\mathrm{d}r$ is the upper incomplete Gamma function. We conclude by remarking that $\lim_{k\to\infty} V_k(s) = e^{-s} - 1 < 0$, and, given that $V_k(s)$ is decreasing in k for $s > 0$ fixed,

$$\inf_{k\ge 2}(-V_k(s)) = -V_2(s) = se^{-s}.$$

Using this in (4.67) yields

$$\frac{1}{2}\nabla^2\mathcal{K}[1](G,G) = \sum_{k=2}^{\infty}|a_k|^2 V_k(s) \le V_2(s)\|G\|_{\mathcal{F}} = -se^{-s},$$

which concludes the proof. □

4.5 Sharpness

We now address the sharpness of the stability estimates in Theorem 4.3. This is where the variational approach turns out to be quite useful, since the computations are greatly simplified. Our strategy in this section will be to use a particular example to show that it is not possible to hope for a better exponent for the deficit and a better dependence on $|\Omega|$ of the constant. In order to do this, we begin by introducing the example.

Proposition 4.18. *Let $s > 0$ be a fixed positive real number. Given $\varepsilon_0 > 0$ sufficiently small, there is a constant $C > 0$ and, for each $\varepsilon < \varepsilon_0$ a set Ω_ε and a function $\tilde{F}_\varepsilon \in \mathcal{F}(\mathbb{C})$ such that* *(i)* Ω_0 *a ball and* $|\Omega_\varepsilon| = s$*;*

(ii) $\inf_{c,z_0\in\mathbb{C}} \|\tilde{F}_\varepsilon - cF_{z_0}\|_{\mathcal{F}} \geq \frac{\varepsilon}{C}$*;*

(iii) the deficit satisfies $\delta(\tilde{F}_\varepsilon;\Omega_\varepsilon) \leq C\frac{se^{-s}}{\pi^2(1-e^{-s})}\varepsilon^2$.

Proof. Let $F_\varepsilon(z) = 1 + \varepsilon z^2$ and as usual let us write $u_\varepsilon(z) := |F_\varepsilon(z)|^2 e^{-\pi|z|^2}$. Consider the domains

$$\Omega_\varepsilon = \{z \in \mathbb{C} \colon u_\varepsilon(z) > u_\varepsilon^*(s)\},$$

where $s > 0$ is fixed. We then have

$$\begin{aligned}(1-e^{-s})\delta(F_\varepsilon;\Omega_\varepsilon) &= \mathcal{K}[1] - \mathcal{K}[F_\varepsilon] \\ &\leq -\frac{\varepsilon^2}{2}\nabla^2\mathcal{K}[1](z^2,z^2) + \varepsilon^2\eta(\varepsilon) = 2\frac{se^{-s}}{\pi^2}\varepsilon^2 - \eta(\varepsilon)\varepsilon^2.\end{aligned} \tag{4.69}$$

where we used Lemma 4.13 to pass to the second line and also (4.66) in the last equality. Now note that taking ε sufficiently small yields the desired upper bound if we choose $\tilde{F}_\varepsilon = \frac{F_\varepsilon}{\|F_\varepsilon\|_{\mathcal{F}}}$. For the lower bound on $\|\tilde{F}_\varepsilon - cF_{z_0}\|_{\mathcal{F}}$, we recall from (4.60) that

$$\|\tilde{F}_\varepsilon - c\cdot F_{z_0}\|_{\mathcal{F}}^2 \geq 1 - \max_{z_0\in\mathbb{C}} |\tilde{F}_\varepsilon(z_0)|^2 e^{-\pi|z|^2}.$$

In order to finish, we only need to show that the only global maximum of $|F_\varepsilon(z)|^2 e^{-\pi|z|^2}$ occurs at $z = 0$, which is equivalent to showing that

$$(1 + 2\varepsilon(x^2 - y^2) + \varepsilon^2\,|z|^4) < e^{\pi|z|^2},$$

for each $z \in \mathbb{C}$. As $1 + \pi|z|^2 + \frac{\pi^2}{2}|z|^4 < e^{\pi|z|^2}$, this inequality is true if $\varepsilon < \frac{\pi}{4}$. Thus, for such ε,

$$\|\tilde{F}_\varepsilon - cF_{z_0}\|_{\mathcal{F}}^2 \geq 1 - \frac{1}{1 + \frac{2}{\pi^2}\varepsilon^2} \geq \frac{\varepsilon^2}{\pi^2},$$

which concludes the proof. □

With this we immediately see how Theorem 4.1 is sharp.

Corollary 4.19. *The following assertions hold:*

(i) the factor $\delta(f;\Omega)^{1/2}$ *cannot be replaced by* $\delta(f;\Omega)^\beta$*, for any* $\beta > 1/2$*, in* (4.3) *and* (4.4)*;*

(ii) there is no $c \in (0,1)$ *such that, for all measurable sets* $\Omega \subset \mathbb{C}$ *of finite measure, we have*

$$\min_{z_0\in\mathbb{C},|c|=\|f\|_2} \frac{\|f - c\varphi_{z_0}\|_2}{\|f\|_2} \leq C\Big(e^{c|\Omega|}\delta(f;\Omega)\Big)^{1/2}.$$

Proof. Notice that (ii) follows directly from combining (ii) and (iii) in the statement of Proposition 4.18 and taking $s \to \infty$; so we just have to prove (i).

The fact that one cannot improve the exponent in (4.3) follows directly from Proposition 4.18 above. To see it, we argue as follows. As long as ε is small enough, Lemma 4.16 provides X_ε and Ψ_ε, by means of which we can write

$$\Omega_\varepsilon = \varphi_\varepsilon(\Omega_0).$$

Additionally, since Φ_ε is the flow of X_ε, we have

$$\Phi_\varepsilon(z) = \Phi_0(z) + \varepsilon X_0(\Phi_0(z)) + O(\varepsilon^2) = z + \varepsilon X_0(z) + O(\varepsilon^2), \tag{4.70}$$

and we may write $X_0(z) = h_0(z)z$ for some scalar function $h_0 \colon \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{R}$. Indeed, the fact that X_0 is of this form follows from its explicit formula in a neighbourhood of $\mathbb{S}^1$

$$X_\varepsilon(r, \omega) = \frac{1}{r}(1 + f_\varepsilon(\omega))\partial_\varepsilon f_\varepsilon(\omega)\omega, \tag{4.71}$$

granted by Lemma 4.15. Lastly, by comparing the expansions of $u_\varepsilon(\Phi_\varepsilon(z))$ and $u_\varepsilon^*(s)$ as a power series in ε for $z \in \partial\Omega_0 = \partial\{u_0(z) > u_0^*(s)\}$, we infer (see [GGRT]) that

$$\pi\langle X_0, z\rangle = \operatorname{Re}(z^2) \text{ on } \partial\Omega_0. \tag{4.72}$$

Writing (4.72) in terms of h_0 then yields the explicit expression

$$h_0(z) = \frac{\cos(2\theta)}{\pi}, \quad z = r_0 e^{i\theta},$$

where r_0 is the radius of the ball Ω_0. Therefore,

$$|\Omega_\varepsilon \triangle \Omega_0| \geq |\Omega_\varepsilon \setminus \Omega_0| \geq \left|\left\{re^{i\theta} \colon r_0 < r < r_0 + \varepsilon\frac{\cos(2\theta)}{\pi} - C\varepsilon^2\right\}\right| > c\, r_0^2 \varepsilon,$$

which concludes the proof. $\square$

CHAPTER 5

Generalizations

The arguments that show stability in sections 4.1 and 4.2 in the previous chapter all feature techniques that are based on estimates on the distribution function μ and rearrangement u^*, bounds given by the RKHS structure of $\mathcal{F}(\mathbb{C})$, and mass transportation arguments. None of these are, a priori, specific to the one-dimensional case, and as such, we are able to extend them to both the setting of dimension $d > 1$ and the wavelet case.

5.1 Stability in higher dimensions

As hinted by Theorem 3.11, slightly modified versions of the deficit δ defined in (4.1) and the asymmetry (4.2) are now required in the higher dimensional setting. We can simply replace $\mathbb{R}^2$ by $\mathbb{R}^{2d}$ in (4.2) to find the latter, and given Ω and f as in Theorem 3.11, the new deficit will be

$$\delta(f;\Omega) \coloneqq 1 - \frac{\int_\Omega |\mathcal{V}f(x,\omega)|^2 \, \mathrm{d}x \, \mathrm{d}\omega}{\|f\|_2^2 \gamma(d, c_\Omega)}(d-1)!, \tag{5.1}$$

where, again, $c_\Omega = \pi(|\Omega| / \boldsymbol{\omega}_{2d})^{1/d}$ and γ is the lower incomplete gamma function. Then, the higher dimensional counterpart of Theorem 4.1 can be stated as follows.

Theorem 5.1 (Stability of the FK inequality for the STFT in dimension d)**.** *There is an explicitly computable constant $C = C(d) > 0$ such that, for all measurable sets $\Omega \subset \mathbb{R}^{2d}$ with finite measure $|\Omega| > 0$ and all functions $f \in L^2(\mathbb{R}^d)\backslash\{0\}$, we have*

$$\inf_{\substack{|c|=\|f\|_2, \\ z_0 \in \mathbb{C}^d}} \frac{\|f - c\varphi_{z_0}\|_2}{\|f\|_2} \leq C e^{c_\Omega} \delta(f;\Omega)^{1/2}, \tag{5.2}$$

and moreover, for some explicit constant $K = K(d, |\Omega|)$ we also have

$$\mathcal{A}(\Omega) \leq K \delta(f;\Omega)^{1/2}. \tag{5.3}$$

As one could expect, the first step to tackling Theorem 5.1 is to reduce the problem to the Fock space over $\mathbb{C}^d$. After this initial reduction, the rest of the proof follows the exact same strategy as the case $d = 1$, with only a few minor changes having to be taken into account.

Theorem 5.2 (Fock space version of Theorem 5.1)**.** *There is a computable constant $C = C(d) > 0$ such that, for all measurable sets $\Omega \subset \mathbb{R}^{2d}$ with finite measure $|\Omega| > 0$ and all functions $F \in \mathcal{F}(\mathbb{C}^d)\backslash\{0\}$, we have*

$$\inf_{\substack{|c|=\|F\|_{\mathcal{F}},\\ z_0\in\mathbb{C}^d}} \frac{\|F - cF_{z_0}\|_{\mathcal{F}}}{\|F\|_{\mathcal{F}}} \leq Ce^{c_\Omega}\delta(F;\Omega)^{1/2}, \tag{5.4}$$

where F_{z_0} is of the form (2.24)*. Moreover, for some explicit constant $K = K(d, |\Omega|)$ we also have*

$$\mathcal{A}(\Omega) \leq K\delta(F;\Omega)^{1/2}. \tag{5.5}$$

Proof. The proof of Lemma 4.4 remains virtually unchanged, with s^* now defined by

$$\int_0^\infty v^*(s)\,\mathrm{d}s = \int_0^\infty u^*(s)\,\mathrm{d}s,$$

where $v^*(s) = e^{-\pi(s/\boldsymbol{\omega}_{2d})^{1/d}}$ instead of e^{-s}, and the function L now being

$$L(t) = \mu(t) - \nu(t), \quad \text{where} \quad \nu(t) := \frac{\boldsymbol{\omega}_{2d}}{\pi^d}\left(\log\frac{1}{t}\right)^d.$$

The fact that it is decreasing is again shown via the d-dimensional version of the convexity inequality (3.18), i.e.

$$\mu'(t) + \left[\frac{d\boldsymbol{\omega}_{2d}^{1/d}}{\pi}\mu(t)^{1-1/d}\right]\frac{1}{t} \leq 0, \tag{5.6}$$

and instead of e^{s_0}, the constant that we find in front of the deficit is now e^{c_Ω}. The main change regarding the case $\max u = T > v^*(s_0)$ is found when estimating A_3 from below: we come to a bound of the type $A_3 \geq C\frac{(1-T)^{d+1}}{(d+1)!}$, hence the non-sharp estimate reads

$$1 - T \leq C(d)\left(e^{c_\Omega}\delta(F;\Omega)\right)^{1/(d+1)}.$$

When $s_0 < s^*$ and $T < v^*(s_0)$, one finds that

$$e^{c_\Omega} - \sum_{j=0}^{d-1}\frac{c_\Omega^j}{j!} \leq \gamma(d, c_\Omega)v^*(s_0) \leq \frac{s_0(d-1)!}{(1-\delta_0)}.$$

Therefore $c_\Omega \leq x$, where x is defined by

$$e^x - \sum_{j=0}^{d-1}\frac{x^j}{j!} = \frac{x^d}{d(1-\delta_0)}.$$

With the same strategy as before, we find a bound of the type

$$1 - T \leq K(d)\frac{\delta_0}{1-\delta_0},$$

Moving on to the sharp stability estimates for the function, the conclusion of Lemma 4.6 now reads

$$\mu(t) \leq \frac{1}{d!}\left(1 + C(d)(1-T)\right)\left(\log T/t\right)^d,$$

and $C(d)$ depends on the dimension. The proof is the same, except every instance of $\log(T/t)$ has to be substituted by $\frac{1}{d!}\log(T/t)$, r_σ is now defined over $\mathbb{S}^{2d-1}$, and instead of using the Cauchy integral formula in (4.26) to bound the derivatives of R, and consequently of h, we need to use the fact that h is harmonic. Indeed, seeing h as a function $h(x_1, y_1, \dots, x_d, y_d)$ with $z = (x_1 + iy_1, \dots, x_d + iy_d)$, its partial derivatives with respect to x_j and y_j are also harmonic functions, and by a direct application of the mean value property of harmonic functions together with the divergence theorem, we obtain the bound,

$$\left|\frac{\partial}{\partial x_j} h(x)\right|, \left|\frac{\partial}{\partial y_j} h(x)\right| \leq \frac{d}{r} \max_{B_r(x)} |h|, \quad 1 \leq j \leq d.$$

Indeed, if e_i is the i-th unit vector of the usual basis in $\mathbb{R}^d$ and u has the mean value property, then, for any $R > 0$,

$$\frac{\partial}{\partial x_i} u(y) = \frac{1}{|B_R(y)|} \int_{B_R(y)} \operatorname{div}(e_i u)(x) \, \mathrm{d}x = \frac{1}{\boldsymbol{\omega}_d R^d} \int_{\partial B_R(y)} u e_i \cdot \nu \, \mathrm{d}\mathcal{H}^{d-1},$$

and therefore

$$\left|\frac{\partial}{\partial x_i} u(y)\right| \leq \frac{d \boldsymbol{\omega}_d R^{d-1}}{\boldsymbol{\omega}_d R^d} \sup_{\partial B_R(y)} |u| \leq \frac{d}{R} \sup_{\partial B_R(y)} |u|.$$

A similar type of bound can be obtained for the k-th order derivatives of h by induction on the order of the multi-index, together with the above result. We only use the bound on the second order derivatives.

This shows that (4.24) reads the same in the current setting, with C now depending on the dimension. Finally, the estimate for $\mu(t)$ in Lemma 4.6 changes in that the function f is now defined by

$$f(\sigma) = \frac{1}{2d} \int_{\mathbb{S}^{2d-1}} r_\sigma(z')^{2d} \, \mathrm{d}S(z').$$

Lemma 4.7 follows exactly the same proof, but again with $v^*(s)$ instead of e^{-s}. In STEP 1 it is only necessary to accommodate for dimensional changes, resulting in $c_0(d), c_1(d) > 0$ depending on the dimension. In STEP 2, the estimate (4.41) now follows from an analysis of

$$\int_0^{\tilde{s}} v^*(s) - Te^{-\left(\frac{d!s}{1+C(1-T)}\right)^{1/d}} = \\ \frac{1}{(d-1)!}\left[\gamma(d, (d!\tilde{s})^{1/d}) - T(1 + C(1-T))\gamma\left(d, \left(\frac{d!\tilde{s}}{1+C(1-T)}\right)^{1/d}\right)\right]. \tag{5.7}$$

Indeed, when T is close enough to 1, the above expression behaves like $(1-T)\rho_0$ for a positive constant ρ_0. STEP 3 is identical to the case $d = 1$.

Finally, Proposition 4.5 and the proof of set stability remain unchanged, and the constant C in (4.4) now depends on d as well. This concludes the proof. ☐

5.2 Stability for wavelet transforms

Since the proof of Theorem 3.3 is done, as discussed in Chapter 3, in the framework introduced in [NT] that shows Theorem 3.1, the techniques we have introduced can be adapted to show stability in this context as well. The deficit and asymmetry both require the following changes. We redefine δ so that it reads

$$\delta(f;\Delta,\beta) := 1 - \frac{\displaystyle\int_\Delta \left|W_{\overline{\psi_\beta}} f(x,s)\right|^2 \frac{\mathrm{d}x\,\mathrm{d}s}{s^2}}{C^\beta_{\Delta^*}\|f\|_2^2},$$

where $\Delta^* \subset \mathbb{C}^+$ is any pseudohyperbolic disc of measure $\nu(\Delta)$, and the asymmetry now substitutes the Lebesgue measure by the measure ν (recall (3.5)),

$$\mathcal{A}(\Delta) = \inf\left\{\frac{\nu\left((\Delta\setminus\Delta^*)\cup(\Delta^*\setminus\Delta)\right)}{\nu(\Delta)} : \Delta^*\subset\mathbb{C}^+, \nu(\Delta^*) = \nu(\Delta)\right\}.$$

Theorem 5.3 (Stability of the FK inequality for Wavelet transforms)**.** *Let $\Delta \subset \mathbb{C}^+$, $f \in H^2(\mathbb{C}^+)$ and $\beta > 0$. Then there exists an explicit, computable constant $C = C(s,\beta) > 0$ such that*

$$\inf_{\substack{|c|=\|f\|_2,\\ \xi\in\mathbb{C}^+}} \frac{\|f - c\Psi_{\beta,\xi}\|_2}{\|f\|_2} \le C\delta(f;\Delta,\beta)^{1/2}. \tag{5.8}$$

Moreover, for some explicit constant $K = K(s,\beta)$ we also have

$$\mathcal{A}(\Delta) \le K\delta(f;\Delta,\beta)^{1/2}. \tag{5.9}$$

Of course, this result can be stated in terms of Bergman spaces and the Bergman transform. To this end, for $\alpha = 2\beta - 1$, a function $f = T_\alpha \circ B_\alpha(\tilde{f})$ for $\tilde{f} \in L^2$, and a set $\Omega \subset D$, the image of some $\Delta \subset \mathbb{C}^+$ under the map $z \mapsto \frac{z-i}{z+1}$, we put

$$\delta(f;\Omega,\alpha) := \delta(\tilde{f};\Delta,\beta).$$

Additionally, we define the asymmetry $\mathcal{A}_D$ to be the equivalent of $\mathcal{A}$ for the disc, that is

$$\mathcal{A}_D(\Omega) = \mathcal{A}(\Delta), \quad \text{where } \Omega = \left\{\frac{z-i}{z+1} : z \in \Delta\right\}.$$

In turn, this definition is equivalent to

$$\mathcal{A}_D(\Omega) = \inf\left\{\frac{\mu(\Omega\Delta B(x,r))}{\mu(\Omega)} : \mu(B(x,r)) = \mu(\Omega),\ x\in D\right\}.$$

Then, Theorem 5.3 reads as follows in terms of Bergman functions.

Theorem 5.4 (Bergman space version of Theorem 5.3)**.** *Let $\alpha > -1$, $\Omega \subset D$ such that $\mu(\Omega) = s > 0$ and let $f \in \mathcal{A}_\alpha(D)$ denote a function in the Bergmann space on the disc. Then there exists an explicit, computable constant $C = C(s,\alpha) > 0$ such that*

$$\inf_{\substack{|c|=\|f\|_{\mathcal{A}_\alpha},\\ \omega\in\mathbb{C}}} \frac{\|f - cf_\omega\|_{\mathcal{A}_\alpha}}{\|f\|_{\mathcal{A}_\alpha}} \le C\delta(f;\Omega,\alpha)^{1/2}, \tag{5.10}$$

where f_ω is as in (2.26)*. Moreover, for some explicit constant $K = K(s,\alpha)$ we also have*

$$\mathcal{A}_D(\Omega) \le K\delta(f;\Omega,\alpha)^{1/2}. \tag{5.11}$$

The outline of the proof is, once again, the same as that of Theorem 4.1. Since the current setting is now considerably different when compared to the higher dimensional case, we will carefully go over and highlight the necessary changes and considerations. We start by fixing the index $\alpha > -1$, for which reason we shall omit this in the upcoming definitions. Throughout this section, we will assume, by homogeneity, that $\|f\|_{\mathcal{A}_\alpha} = 1$. We then set (recall Section 3.2) $u(z) = |f(z)|^2 (1 - |z|^2)^{\alpha+2}$, with which we define

$$I(s) = \int_{\{u > u^*(s)\}} u(z)\,\mathrm{d}\mu(z),$$

and in turn

$$\delta_s(f) := \delta(f; \{u > u^*(s)\}, \alpha) = \frac{\theta(s) - I(s)}{\theta(s)}.$$

5.2.1 Non-sharp stability

We begin by addressing Lemma 4.4. Notice that we now have

$$v^*(s) = \frac{1+\alpha}{\pi}\left(1 + \frac{s}{\pi}\right)^{-(\alpha+2)},$$

where v^* denotes the rearrangement according to the measure $\mathrm{d}\mu$. The existence of s^* such that $v^*(s^*) = u^*(s^*)$ follows as usual, which allows us to rephrase Lemma 4.4 as follows.

Lemma 5.5. *The following estimates hold true*

$$\frac{1+\alpha}{\pi} - \max u \le \begin{cases} \sqrt{2\frac{(\alpha+1)(\alpha+2)\theta(s_0)}{\pi^2 v^*(s_0)}\delta_{s_0}(f)} & \text{if } s_0 > s^*, \\ \sqrt{2\frac{(\alpha+1)(\alpha+2)}{\pi^2}\theta(s_0)\delta_{s_0}(f)} & \text{if } s_0 \le s^*. \end{cases}$$

whenever $\max u \ge v^*(s_0)$. *Moreover, when* $\max u < v^*(s_0)$*, the estimate is sharper on the decay rate,*

$$\frac{1+\alpha}{\pi} - \max u \le \frac{2\pi\delta_{s_0}(f)}{(\alpha+2)(1-\delta_{s_0}(f))}.$$

Proof. The distribution function of u is now denoted by ρ, and we set $\delta_0 = \delta_{s_0}(f)$ again, for brevity. Inequality (3.18), which in this context is obtained through the hyperbolic isoperimetric inequality (recall Section 3.2), now reads

$$\rho'(t) + \frac{\pi + \rho(t)}{\alpha+2}\cdot\frac{1}{t} \le 0 \tag{5.12}$$

in terms of ρ. The function $L(t)$ is defined by

$$L(t) = \rho(t) - \pi\left[\left(\frac{\pi}{\alpha+1}t\right)^{-\frac{1}{\alpha+2}} - 1\right],$$

and a direct computation shows that $L'(t) \le 0$ for $t \le t^*$. The constant that accompanies the deficit is now

$$2\frac{(\alpha+1)(\alpha+2)}{\pi^2}\cdot\frac{\theta(s_0)}{v^*(s_0)},$$

where the factor on the right comes from estimating the area A_1 from above, and the one on the left comes from bounding A_3 from below.

The rest of the proof is a mechanical adaptation of the original one, with the only subtlety being that the definition of x in the case $\max u < v^*(s_0)$ should now be replaced by

$$v^*(x)\theta(x) = \frac{\pi}{\alpha+1}\left(1+\frac{x}{\pi}\right)^{\alpha+2}\left[1-\left(1+\frac{x}{\pi}\right)^{-(1+\alpha)}\right] = \frac{x}{1-\delta_0}. \tag{5.13}$$

By means of the same argument one achieves the bound

$$x \leq \frac{2\pi\delta_0}{(1-\delta_0)(\alpha+2)},$$

which together with (5.13) yields the result. □

5.2.2 Sharp stability for the function

The next milestone is the sharp stability result for the function. We now need sharp estimates for the distribution function of u according to ρ, in analogy to Lemma 4.7. This is achieved by the following result.

Lemma 5.6. *Let $f \in \mathcal{A}_\alpha$ be such that $\|f\|_{\mathcal{A}_\alpha} = 1$. There is an absolute, computable constant $C > 0$ such that if $T := \max_{z\in\mathbb{C}} |f(z)|^2 (1-|z|^2)^{\alpha+2}$ satisfies $\delta^2 := \frac{1-f(0)^2}{T} < \frac{1}{C}$, then*

$$\rho_f(t) < \pi(1+C\delta^2)\left[\left(\frac{T}{t}\right)^{\frac{1}{\alpha+2}} - 1\right]$$

whenever $t/T > 1-1/C$.

Proof. We may assume that u achieves its maximum at $z=0$ and that $f(0)$ is a positive number. Denote then $u(0) = T$. We can prove the non-optimal set inclusion of Step 1 in the proof of Lemma 4.6 using the same method. Indeed, write

$$f(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{\sqrt{c_k}}.$$

Then

$$|f(z)|(1-|z|^2)^{\frac{\alpha+2}{2}} \leq (1-|z|^2)^{\frac{\alpha+2}{2}} \sum_{k=0}^{\infty} |a_k| \frac{|z|^k}{\sqrt{c_k}}.$$

We deduce that $a_1 = 0$, and when restricting to the set $\{u > (1-\varepsilon)^2 T\}$, we can use the previous inequality to get

$$1-\varepsilon \leq \left(1-|z|^2\right)^{\frac{\alpha+2}{2}} + \left(\frac{1-a_0}{T}\right)^{1/2}\left[\frac{1+\alpha}{\pi} - (1-|z|^2)^{\alpha+2}\left(\frac{1}{c_0}+\frac{|z|^2}{c_1}\right)\right]^{1/2}.$$

We call $\delta = \sqrt{(1-a_0^2)/T}$ and let F be the factor in brackets in the expression above. Then it is not hard to find a bound for F^2 by expanding its expression, which yields

$$F^2 \leq \frac{\alpha+1}{\pi} C_\alpha |z|^4$$

whenever $|z| < 1$. In particular, this completes STEP 1,

$$\varepsilon \geq C(1 - \tilde{C}\delta)\, |z|^2 .$$

STEP 2. Let $R(z) = f(z)/f(0) - 1$. Then by the previous estimates, $|R(z)|^2 \leq C\delta^2 |z|^4$, and the condition $u(z) > t$ can be rewritten to read

$$(T/t)^{\frac{1}{\alpha+2}} > \frac{1}{1 - |z|^2} \cdot \frac{1}{(1 + |R(z)|^2 - h(z))^{\frac{1}{\alpha+2}}}, \tag{5.14}$$

where now $h = -2 \operatorname{Re} R$. Since

$$1 > h(z) - |R(z)|^2 = 1 - \frac{|f(z)|^2}{|f(0)|^2} = \frac{|f(0)|^2 - |f(z)|^2}{|f(0)|^2} > 0,$$

we can expand the last factor in (5.14) in its series representation to obtain the bound

$$\frac{1}{(1 + |R(z)|^2 - h(z))^{\frac{1}{\alpha+2}}} \geq 1 + (\alpha + 2) \left(h(z) - |R(z)|^2 \right).$$

This, combined with the bound we found earlier for $|R(z)|^2$, implies that

$$(T/t)^{\frac{1}{\alpha+2}} - 1 > \frac{1}{1 - |z|^2} \left[|z|^2 + (\alpha + 2)h(z) - C(\alpha + 2)\delta^2 |z|^4 \right],$$

which motivates the definition of

$$V_\sigma(r, \theta) := \frac{1}{1 - r^2} \left[r^2 + (\alpha + 2)\frac{\sigma}{\delta} h(re^{i\theta}) - C(\alpha + 2)\delta^2 r^4 \right], \quad \sigma \in (-2\delta, 2\delta). \tag{5.15}$$

We then check that for each $\theta \in [0, 2\pi)$, and for each $\sigma \in (-2\delta, 2\delta)$, there exists $r_\sigma(\theta)$ such that $V_\sigma(r_\sigma(\theta), \theta) = (T/t)^{1/(\alpha+2)} - 1$. By the implicit function theorem, $\sigma \mapsto r_\sigma(\theta)$ is smooth and, as before, the next step is to bound the derivatives of $r_\sigma(\theta)$ w.r.t. σ in order to eventually bound $\rho(t)$. Differentiating

$$r_\sigma^2 + (\alpha + 2)\frac{\sigma}{\delta} h(r_\sigma e^{i\theta}) - (\alpha + 2)C\delta^2 r_\sigma^4 = \left[(T/t)^{\frac{1}{\alpha+2}} - 1 \right] (1 - r_\sigma^2)$$

implicitly with respect to σ twice we obtain bounds for $|r_\sigma'|$ and $|r_\sigma''|$, where, again, r_σ' denotes the derivative with respect to σ. Indeed, we find

$$r_\sigma' = -\frac{\frac{\alpha+2}{\delta} h(r_\sigma e^{i\theta})}{\frac{\alpha+2}{\delta} \partial_r h(r_\sigma e^{i\theta}) e^{i\theta}\sigma - C\delta^2(\alpha + 2) r_\sigma^3 + 2r_\sigma (T/t)^{\frac{1}{\alpha+2}}},$$

and

$$\begin{aligned} r_\sigma'' = &-\frac{\frac{\alpha+2}{\delta} \left(\partial_r h(r_\sigma e^{i\theta})(e^{i\theta} + r_\sigma') + \sigma e^{i\theta} \partial_r^2 h(r_\sigma e^{i\theta})(r_\sigma')^2 \right)}{\frac{\alpha+2}{\delta} \partial_r h(r_\delta e^{i\theta}) e^{i\theta}\sigma - C\delta^2(\alpha + 2) r_\sigma^3 + 2(T/t)^{\frac{1}{\alpha+2}} r_\sigma} \\ &- \frac{C\delta^2(\alpha + 2) r_\sigma^2 (r_\sigma')^2 + 2(T/t)^{\frac{1}{\alpha+2}} (r_\sigma')^2}{\frac{\alpha+2}{\delta} \partial_r h(r_\delta e^{i\theta}) e^{i\theta}\sigma - C\delta^2(\alpha + 2) r_\sigma^3 + 2(T/t)^{\frac{1}{\alpha+2}} r_\sigma}. \end{aligned}$$

These yield the bounds $|r_\sigma'|, |r_\sigma''| \leq C r_\sigma$. By the previous considerations,

$$\{u > t\} \subset \left\{ z = re^{i\theta} : \theta \in [0, 2\pi), 0 < r < r_\delta(\theta) \right\},$$

which implies that

$$\begin{aligned}\rho(t) &\leq \mu\left(\left\{z = re^{i\theta} : \theta \in [0, 2\pi), 0 < r < r_\delta(\theta)\right\}\right)\\ &= \int_0^{2\pi}\int_0^{r_\delta(\theta)} \frac{r}{(1-r^2)^2}\,\mathrm{d}r\,\mathrm{d}\theta \leq \frac{1}{2}\int_0^{2\pi} \frac{r_\delta(\theta)^2}{1-r_\delta(\theta)^2}\,\mathrm{d}\theta.\end{aligned}$$

This suggests that we define

$$F(\sigma) = \frac{1}{2}\int_0^{2\pi} \frac{r_\sigma(\theta)^2}{1-r_\sigma(\theta)^2}\,\mathrm{d}\theta.$$

We can argue as in the STFT case in order to show that $|F''(\sigma)| \leq CF(\sigma)$, and that $F'(0) = 0$. Using Taylor's theorem with remainder,

$$F(\delta) = F(0) + \frac{\delta^2}{2}F''(\sigma) \leq F(0) + C\delta^2 F(\sigma). \tag{5.16}$$

Now, evaluating $V_\sigma(r_\sigma(\theta), \theta)$ at 0 and σ, we find

$$\left|\frac{r_0^2}{1-r_0^2} - \frac{r_\sigma^2}{1-r_\sigma^2}\right| = \left|\frac{\sigma}{\delta}\frac{\alpha+2}{1-r_0^2}h(r_\sigma e^{i\theta}) + (\alpha+2)C\delta^2\left(\frac{r_0^4}{1-r_0^2} - \frac{r_\sigma^4}{1-r_\sigma^2}\right)\right|,$$

and by taking δ small enough, we have

$$\frac{r_\sigma^2}{1-r_\sigma^2} \leq 2\frac{r_0^2}{1-r_0^2}.$$

Plugging this back in (5.16) yields

$$\rho(t) \leq F(\delta) \leq \frac{1}{2}(1+C\delta^2)\int_0^{2\pi} \frac{r_0^2}{1-r_0^2}\,\mathrm{d}\theta \leq \pi\frac{1+C\delta^2}{1-K\delta^2}\left[\left(\frac{T}{t}\right)^{\frac{1}{\alpha+2}} - 1\right],$$

and this is precisely

$$\rho(t) \leq \pi\left(1 + \frac{(C+K)\delta^2}{1-K\delta^2}\right)\left[\left(\frac{T}{t}\right)^{\frac{1}{\alpha+2}} - 1\right] \leq \pi(1+\tilde{C}\delta^2)\left[\left(\frac{T}{t}\right)^{\frac{1}{\alpha+2}} - 1\right]$$

for a small enough δ. □

Finally, we can show the sharp stability for the function. First, we briefly remark that Proposition 2.15 can be applied to Bergman spaces: setting

$$\Delta_f(z) = \frac{\|f\|_{\mathcal{A}_\alpha}^2 - |f(z)|^2(1-|z|^2)^{\alpha+2}}{\|f\|_{\mathcal{A}_\alpha}^2},$$

then it reads as follows.

Proposition 5.7. *Let $f \in \mathcal{A}_\alpha$. Then for every $w \in \mathbb{C}$ there exists a constant $c \in \mathbb{C}$ with $|c| = \|f\|_{\mathcal{A}_\alpha}$ such that*

$$\frac{\|f - cf_w\|_{\mathcal{A}_\alpha}}{\|f\|_{\mathcal{A}_\alpha}} \leq \sqrt{2}\Delta_f(w)^{1/2}.$$

We complete the stability for the function with the following result.

Lemma 5.8. *There is an absolute, computable constant $C_\alpha = C(\alpha) > 0$ such that*

$$\frac{\alpha+1}{\pi} - \max u \leq C_\alpha \cdot \max\left\{\frac{\theta(s_0)}{v^*(s_0)}, 1\right\} \cdot \delta_{s_0}(F).$$

Proof. We can replicate the steps in the proof of Lemma 4.7, obtaining the bound

$$u^*(s) \leq T\left(1 + \frac{s}{\pi(1+C\delta^2)}\right)^{-(\alpha+2)} =: w(s).$$

Given that $w(0) = u^*(0)$, there exists $\tilde{s} > 0$ such that $v^*(\tilde{s}) = w(\tilde{s})$. With this we can bound

$$\int_0^{s^*} (v^*(s) - u^*(s))\,\mathrm{d}s \geq \int_0^{\tilde{s}} (v^*(s) - w(s))\,\mathrm{d}s = \theta(\tilde{s}) - T\frac{\pi}{\alpha+1}(1+C\delta^2)\theta\left(\frac{\tilde{s}}{1+C\delta^2}\right). \tag{5.17}$$

One can now compute carefully what this is. In fact,

$$\begin{aligned}&\lim_{\delta\to 0}\frac{1}{\delta^2}\left(\theta(\tilde{s}) - T\frac{\pi}{\alpha+1}(1+C\delta^2)\theta\left(\frac{\tilde{s}}{1+C\delta^2}\right)\right)\\ &= \left(\left(\frac{\alpha+1}{\alpha+2} - 1\right)\frac{\alpha+1}{\pi} + C\right)\left(\frac{\pi(\alpha+2)C}{\pi(\alpha+2)C - (\alpha+1)}\right)^{-(\alpha+1)} + \frac{\alpha+1}{\pi} - C.\end{aligned}$$

We only need to show that this is positive for each $C > 1$, and each fixed $\alpha > -1$, and then we are able to conclude that the right-hand side of (5.17) behaves as $\rho_0\delta^2 = \rho_0(1-a_0^2)$ when δ is close to zero, for a positive, absolute constant $\rho_0 > 0$.

In fact, the above limit being positive is the same as the condition

$$\frac{C^{\alpha+2}}{(C - \frac{\alpha+1}{\pi(\alpha+2)})^{\alpha+2} + \frac{\alpha+1}{\pi}C^{\alpha+1}} < 1.$$

The function on the left-hand side tends to 1 as $C \to \infty$ and at $C = 1$ it is smaller than 1. Also, for $C > 1$ it is increasing, hence it always lies beneath 1. This completes the proof of the wavelet analogue of Lemma 4.7. ☐

5.2.3 Stability for the set

Now we complete the result with the stability for the set.

Proof of Theorem 5.3. Let $f \in \mathcal{A}_\alpha$ with $\|f\|_{\mathcal{A}_\alpha} = 1$. We will write $\delta = \delta(f;\Omega,\alpha)$ and $\mu(\Omega) = s$ for simplicity, and we may assume, if necessary, that $\delta \leq \delta_0$ for some arbitrarily small constant δ_0. We may also assume that u attains its maximum at $z = 0$, and we will decompose f as $f = \sqrt{(\alpha+1)/\pi} + \rho g$, as suggested by the stability of the function, where

$$\rho = \|f - \sqrt{(\alpha+1)/\pi}\|_{\mathcal{A}_\alpha} \leq C(s)\delta^{1/2}, \|g\|_{\mathcal{A}_\alpha} = 1.$$

By the uniqueness of the super-level sets of u with respect to the measure μ, we write $A_\Omega = A_{u^*(|\Omega|)}$, and throughout the rest of the proof, we will consider T to be a map transporting $\mathbb{1}_{A_\Omega\setminus\Omega}\,\mathrm{d}\mu$ into $\mathbb{1}_{\Omega\setminus A_\Omega}\,\mathrm{d}\mu$.

Step 1. In this first step, we are concerned with finding a bound for the difference $|T(z)|-|z|$ whenever z lies in the set

$$B\coloneqq\{z\in A_\Omega\setminus\Omega\colon d(0,T(z))-d(0,z)>K_\Omega\gamma\}\,,$$

where $d(0,y)$ is the hyperbolic distance between y and the origin, $d(0,y)=\log(1+|y|)-\log(1-|y|)$. Rewrite the condition defining B as

$$\frac{1-|T(z)|}{1+|T(z)|}\cdot\frac{1+|z|}{1-|z|}<e^{-K_\Omega\gamma}.$$

Now set $d=1-e^{-K_\Omega\gamma}$. This gives

$$2\frac{|z|-|T(z)|}{1-|z|^2}<-d\frac{1+|T(z)|}{1+|z|}.$$

Rearranging,

$$|T(z)|-|z|>\frac{d}{2}\left(1+|T(z)|\right)\left(1-|z|\right).$$

We can bound $|T(z)|\geq 0$ and use the fact that $z\in B\subset A_\Omega\setminus\Omega$ to bound $1-|z|>1-r$, that is, r being the radius of the smallest ball containing A_Ω: in particular, thanks to the analogue of (4.44) in the current setting, found via the same argument, which reads

$$\{v>u^*(s)+2\rho\}\subset\{u>u^*(s)\}\subset\{v>u^*(s)-2\rho\}\,, \tag{5.18}$$

we find that r can be made to depend only on $|\Omega|$. Then

$$|T(z)|-|z|>\frac{1-r}{2}(1-e^{-K_\Omega\gamma}).$$

In particular, by concavity of $x\mapsto 1-e^{-x}$, if $K_\Omega\gamma$ is small enough, we have

$$|T(z)|-|z|>\frac{1-r}{2}(1-e^{-K_\Omega\gamma})\geq\frac{1-r}{4}K_\Omega\gamma.$$

Step 2. We now aim to bound the measure of the set B. To this end, we will need to use the bound we found in Step 1.

$$\begin{aligned}u(z)-u(T(z))&\geq\frac{\alpha+1}{\pi}(1-|z|^2)^{\alpha+2}\left(1-\left(\frac{1-|T(z)|^2}{1-|z|^2}\right)^{\alpha+2}\right)-\frac{\alpha+1}{\pi}\rho^2-4\sqrt{\frac{\alpha+1}{\pi}}\rho\\&\geq C(s,\alpha)\left(1-\left(1-\frac{|T(z)|^2-|z|^2}{1-|z|^2}\right)^{\alpha+2}\right)-\frac{\alpha+1}{\pi}\rho^2-4\sqrt{\frac{\alpha+1}{\pi}}\rho.\end{aligned}$$

Given that $|T(z)|<1$, the quantity $(|T(z)|^2-|z|^2)/(1-|z|^2)$ is at most 1, and if $|T(z)|>|z|$, then it is also positive. Since the function $x\mapsto 1-(1-x)^{\alpha+2}$ is concave on $x\in[0,1]$, and maps $0\mapsto 0$ and $1\mapsto 1$, whenever $|T(z)|>|z|$ we have

$$u(z)-u(T(z))\geq C(s,\alpha)\frac{|T(z)|^2-|z|^2}{1-|z|^2}-\frac{\alpha+1}{\pi}\rho^2-4\sqrt{\frac{\alpha+1}{\pi}}\rho.$$

Knowing this, we can estimate

$$u(z)-u(T(z))\geq C(s,\alpha)\frac{r(1-r)}{4}K_\Omega\gamma-\frac{\alpha+1}{\pi}\rho^2-4\sqrt{\frac{\alpha+1}{\pi}}\rho$$

whenever $z \in B$. With the appropriate choice of K_Ω and as long as $\gamma \geq \rho$ we have $u(z) - u(T(z)) \geq C(\alpha)\gamma$, which provides a bound for the measure of B,

$$\mu(B) \leq C(\alpha)\frac{d(\Omega)}{\gamma} = C(\alpha)\frac{\int_{A_\Omega} u \,\mathrm{d}\mu - \int_\Omega u \,\mathrm{d}\mu}{\gamma} < C(\alpha)\frac{\theta(|\Omega|)}{\gamma}\delta.$$

Here, $C(\alpha)$ can be set to, for example, $(\alpha + 1)/\pi$.

STEP 3. Finally, (5.9) follows from the same argument used in steps 2 and 3 of the proof of (4.4) in Section 4.2. Indeed, (4.50) holds true still, with the volume of the sets replaced by their μ-measure, and it can be bounded using the same technique, since (4.52) holds true as well, in the form of (5.18). $\square$

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