Junior High Math Contest Problems

GBN and GBS Math Teams

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Questions tested by Joshua Kuruvilla (GBS junior) and GBN math team members.

1 Individual Problems (No calculator)

The Individual portion is a 60 minute, 15 question test with an additional tiebreaker question at the end.
All answers are integers or common/improper fractions.
Figures are not necessarily drawn to scale.
NO calculators or any other external aids/communication are allowed.
You MUST press "Submit" for each question, or we will not receive your answers.

Problem 1. (SaM) Find the smallest positive integer k for which the expression

$$\sqrt{20+\sqrt{19+\sqrt{18+k}}}$$

is an integer.

Problem 2. (AZ) Consider the following target where the radius of the smallest circle is 1, the radius of the second smallest circle is 2, and the radius of the outside circle is 3. The area of the shaded region is $k\pi$, for some number k. Find k.

Problem 3. (HD) Queen Elizabeth is selling chocolates throughout the school week, starting from Monday and ending on Friday. Each day, she sells 5 more than the day before. If she sold a total of 165 chocolates over the week, how many did she sell on Tuesday?

Problem 4. (AZ) Rank the following numbers from least to greatest: $a = 2^{(3^5)}, b = 5^{(3^2)}, c = 3^{(2^5)}, d = 3^{(5^2)}$. For your answer, submit the letters with no spaces (i.e. *abcd*).

Problem 5. (BB) Compute the smallest prime p such that $p + 6$ is prime but $p + 1, p + 2, \ldots, p + 5$ are all composite.

Problem 6. (AZ) Three sticks of positive integer lengths form a triangle, and two of these sticks have lengths 10 and 11. If m is the smallest possible value of the third stick, and n is the greatest possible value of the third stick. What is mn?

Problem 7. (IT) Fred takes 6 hours to clean a car. Charles takes 2 hours to clean a car. Amelia takes 1 hour to clean a car. Working together, how many <u>minutes</u> does it take Fred, Charles, and Amelia together to clean a car?

Problem 8. (HD) 100 people sit in a circle, and are each given a label $1, 2, 3, \ldots, 100$. A random integer between 1 and 100, inclusive, is chosen, and all people with labels divisible by this number are removed from the circle (so if 49 was chosen, the people labelled "49" and "98" would leave). What is the probability that the person labelled "75" leaves, but the person labelled "25" stays?

Problem 9. (HD) How many ways are there to place one X and one O on a 3×3 tic-tac-toe board such that they are adjacent over an edge (so diagonals don't count)?

Problem 10. (LL, RM, RY) The three lines below enclose a triangular region. Find the area of this region.

$$y = -\tfrac{2}{7}x + 3$$
$$y = x - 6$$
$$y = -3.$$

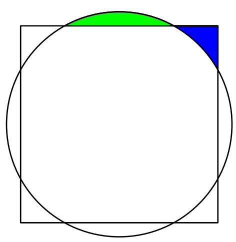
Problem 11. (HD) What is the maximum number of chess pieces that can be placed on an 8×8 chess board such that no two pieces occupy the same diagonal?

Problem 12. (SN, LG, OZ) Joe has a square pool in his backyard. His parents want to build a walkway to surround the pool. They want this walkway to have rounded corners so that it is 5 feet wide at all points. They measured the pool and discovered its diagonal length to be 18 feet. The area of the walkway is in the form $a\sqrt{2} + b\pi$ square feet, for positive integers a, b. What is $a + b$?

Problem 13. (AA) We say a number is *happy* if all of its individual digits are perfect squares (for example: 994 is happy, but 117 is not). How many positive integers less than or equal to 100,000 are happy?

Problem 14. (AZ) What is the area of the region enclosed by the graph of $|2x| + |y - 2021| = 4$?

Problem 15. (BB) A square with side length 7 and and a circle with radius 4 have the same center. The exact value of the green area minus the blue area can be written in the form $a\pi + b$ for rational numbers a and b. Find $a + b$.



Problem 16 (Tiebreaker)**.** (TM) For each integer N between 1 and $1,000,000$, inclusive, Sam writes down the largest factor of N that is a perfect cube. Find the sum of all of the numbers he wrote down. (tiebreaker points are awarded based on how close you are.)

2 Team Problems (Calculator)

The team portion is a 45 minute, 10 question test with an additional tiebreaker question at the end.
All answers are integers or common/improper fractions.
Figures are not necessarily drawn to scale.
Calculators ARE allowed, and collaboration with team members is allowed (and encouraged!)
You MUST press "Submit" for each question, or we will not receive your answers.

Problem 1. (HD) Consider the first 30,000 terms of the Fibonacci sequence: $1, 1, 2, 3, 5, 8, \ldots$ where each number is the sum of the previous two terms. How many of these terms will be divisible by 3?

Problem 2. (AZ) Given that a, b are real numbers such that $a - b = ab = 6$, find $(a + b)^2$.

Problem 3. (SM) Dua Lipa surveys 138 people on their tastes of music. 65 people answered that they liked both kpop and pop, 60 people in total said they didn't like pop, and 99 people in total said they liked kpop. Assuming kpop and pop were the only choices of music provided, how many people surveyed liked pop but not kpop?

Problem 4. (JWJ) Every day, Martin runs directly north from his house and returns on the same path traveling south, and his goal is to run twice as far as the previous day. On day 1, Martin runs a total $\frac{5}{4}$th of a mile (so he runs $\frac{5}{8}$th of a mile north and $\frac{5}{8}$th of a mile south). On day 8, he collapses of exhaustion after running $\frac{7}{8}$ths of his total route for the day. A hospital is located 21 miles directly east of Martin's house. A helicopter flies the fastest route from the hospital to pick up Martin from where he fell (assuming no obstacles in the sky), and then flies the same route back to the hospital. How far, in miles, did the helicopter travel?

Problem 5. (HD) Josh is playing a game. He starts with the number $-\frac{1}{2}$, and he has four "operation tokens" to perform mathematical operations on his number. For each token, he can either add 2, multiply by 2, or square his current number (for example, $\times 2, +2, +2,$ ^2, would produce 9). John wants to end with an integer– what is the largest possible integer that John can end up with after using all four tokens?

Problem 6. (HD) Frankenstein is thinking of a number. He gives you the following clues:

- The number is a four digit positive integer (no leading zeros).
- When rounded to the nearest hundred, the number increases.
- The thousands place is the product of the units place and the tens place.
- When the number is flipped (i.e. $1234 \to 4321$), it is divisible by 66 (hint: focus on the individual prime factors of 66).

What is Frankenstein's number?

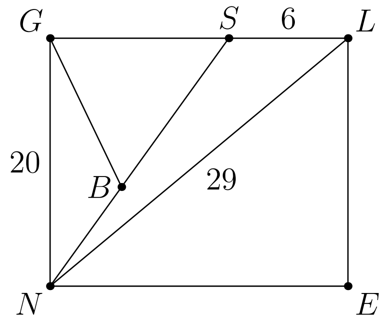
Problem 7. (BB) A certain math competition is scored as follows: Before the competition, the coach selects 5 students from the team as *contestants*, and the rest are considered *alternates*. The sum of the top 3 scores out of the contestants forms the team score (the other 2 scores are ignored).

At the competition, 7 students from the team took the contest, and suppose the coach chose the 5 contestants from these 7 at random. If exactly 4 students (not necessarily contestants) got perfect scores, what is the probability the team did not get a perfect score?

Problem 8. (BB) Find the minimum value of $4^{x+1} - 2^{x-1} + 1$ over all real numbers x.

Problem 9. (HD) A deck with 8 cards, numbered 1, 2, 3 . . . 8, is shuffled. Abby and Brandon each randomly draw and keep two of the cards, without any replacement. What is the probability that Abby's lowest card is higher than Brandon's highest card?

Problem 10. (HD) Let $GLEN$ be a rectangle, where side $GN = 20$ and diagonal $LN = 29$. Let S be on side $\overline{GL}$ such that $LS = 6$, and let B be a point on segment $\overline{SN}$. If the area of $\triangle GBN$ is 60, what is the length of segment $\overline{SB}$?



Problem 11 (Tiebreaker)**.** (BB, BW) Submit a number between 0 and 1, inclusive. If your number is less than or equal to the mean of all (valid) answers submitted, you will receive that many points.

3 Sprint Problems (No calculator)

Problem 0 (Warmup)**.** (BB) What is the median of the set

$$\left\{2, \sqrt{7}, 2.8, \tfrac{17}{7}, 8^{1/\pi}\right\}?$$

Problem 1. (BB) A square $\mathcal{S}$ has two vertices at $(5, 3)$ and $(10, 6)$. Find the sum of all (distinct) possible areas of $\mathcal{S}$.

Problem 2. (HD) Maghan rolls three standard six-sided dice and writes down the results. What is the probability that the product of the three numbers is 2?

Problem 3. (BB) What is 10 times the answer to this question minus 3?

Problem 4. (BB) A regular hexagon and equilateral triangle have the same perimeter. Find the ratio of the area of the hexagon to the area of the triangle.

Problem 5. (TM) What is the value of

$$10^2 - 9^2 + 8^2 - 7^2 ... + 4^2 - 3^2 + 2^2 - 1^2?$$

Problem 6. (BB) An equiangular octagon has side lengths alternating between 3 and $\sqrt{2}$. Find the area of this octagon.

Problem 7. (BB) Find the greatest integer less than or equal to

$$\sqrt{98} + \sqrt{99} + \sqrt{100} + \sqrt{101} + \sqrt{102}.$$

Problem 8. (HD) Lebron took 15 tests, all worth an equal number of points. If his test average over these 15 tests is 59%, what is the lowest percentage he can get on his 16th test to get a final average of at least 60%?

Problem 9. (MW, MS) Find all points (x, y) that satisfy the two equations

$$\tfrac{(x-2)^2}{2} + \tfrac{(y-3)^2}{2} = 2$$
$$y = x + 3.$$

Express your answer as a list of ordered pairs.

Problem 10. (HD, BB) Genghis Khan chooses a random integer between 1 and

$$\frac{7}{9}\left(10^{50} - 1\right),$$

inclusive. What is the probability that this integer is divisible by 7?

Problem 11 (Extra)**.** (IT) If the number $0.\overline{151515} = 0.151515\ldots$ is written as a simplified fraction, find the positive difference between the numerator and denominator.

Problem 12 (Extra)**.** (BB) For some positive integer n, the number 0.385 is the decimal representation of the fraction $\frac{5}{n}$, rounded to the nearest thousandth. Find n.