Junior High Math Contest Solutions

GBN and GBS Math Teams

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Individual Round

Answers

1. 18
2. 6
3. 28
4. $bdca$
5. 23
6. 40
7. 36
8. 3/100
9. 24
10. 36
11. 14
12. 205
13. 1024
14. 16
15. $-33/4$
16. (tiebreaker) $83, 336, 576$

Solutions

1. (HD) We can focus on the inner radical first: 7 is the lowest positive integer that gives us a perfect square inside the innermost radical, but it fails immediately after as $\sqrt{19+5}$ is not a perfect square. The next highest number that works for the inside is 18, which gives us a final value of 5 when fully evaluated, which is an integer. Thus, $\boxed{18}$ is the lowest number that works.
2. (HD) The inner shaded circle has an area π, and the outer ring has an area of $9\pi - 4\pi = 5\pi$, giving us a final area of $\pi + 5\pi = \boxed{6\pi}$
3. (HD) Let x be the number of chocolates sold on Tuesday. The sum of the chocolates sold over the five days is then $(x-5)+(x)+(x+5)+(x+10)+(x+15)$, which we can set equal to 165 as given in the problem. Simplifying, we get $5x+25 = 165$ and $x = \boxed{28}$.

4. (HD) We can rearrange the exponents as $(2^4)^{20.5}$, 5^9, $(3^2)^{16}$, $(3^2)^{12.5}$, which simplifies to $16^{20.5}$, 5^9, 9^{16}, $9^{12.5}$. We can see that a has the largest base and exponent, while d has the smallest base and exponent, giving us the ordering $\boxed{bdca}$.

5. (HD) We can just test primes, looking for pairs of primes that differ by 6. We look at 5, 11, which fails because 7 is prime. Our next number to test would be 11 and 17 (11 being prime makes any pair that includes it "in-between" fail). Eventually, we approach the pair 23 and 29, which works because 24, 25, 26, 27, and 28 are all composite numbers. We are looking for the lower number, p, so report $\boxed{23}$.

6. (HD) By Triangle Inequality, the sum of the two smaller side lengths must be strictly greater than the length of the third side. If our third side is not the longest side, we have a lowest possible case of 2 (because $2+10 > 11$). Similarly, if our third side is the longest case, we have a largest possible case of 20 ($11+10 > 20$), giving us a product of $\boxed{40}$.

7. (HD) We can first solve for the number of cars each person can clean in 6 hours. Using the rates given, we find that Fred can clean 1 car, Charles can clean 3 cars, and Amelia can clean 6 cars. Altogether, Fred, Charles, and Amelia can clean a total of 10 cars in 6 hours, giving us 1 car in 0.6 hours or $\boxed{36}$ minutes.

8. (HD) For “25" to stay while “75" leaves, we need to draw a number that is a factor of 75 but not 25. Because $75 = 3 \cdot 25$, we know that 75 has the same prime factorization as 25, only with an extra 3. Thus, we must choose the factors in 75 that include a factor of 3 (so that “25" doesn't leave). The only factors of 75 that are divisible by 3 are 3, 15, and 75, giving us 3 possible numbers out of 100 total, and a $\boxed{\dfrac{3}{100}}$ probability.

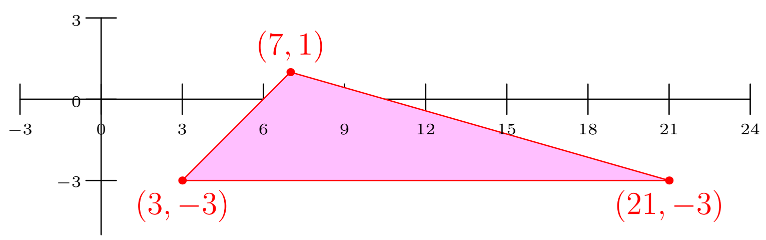
   Alternatively, you could just list out and compare all factors of 25 and 75, and produce the same result.

9. (HD) Because they must be adjacent over an edge, each internal edge will produce a unique pair of positions. Because there are 12 possible edges for the X and O to be adjacent over, and 2 possible orientations for each edge (X and O can be swapped in each case), we have a total of $\boxed{24}$ ways to place the two symbols.

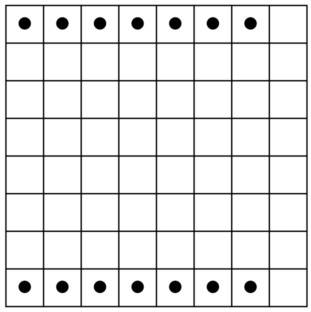
10. (HD, BB) We can find the three intersection points by pairing equations and solving for each. For example, the first two equations can be solved as such:

$$-\frac{2}{7}x + 3 = x - 6$$
$$9 = \frac{9}{7}x$$
$$x = 7$$

   This gives us an intersection point of $(7, 1)$. We can similarly solve for other pairs, getting the points $(21, -3)$ and $(3, -3)$. Using these last two points as our base, we have a base length of 18 and a height of 4, giving us an area of $\boxed{36}$.



11. (HD, BB) Notice that in any direction, there are 15 unique diagonals on the board, giving us a maximum of 15 pieces at the very best. However, the first and last diagonals (the ones at the corners, that are only one unit long) cannot both be occupied, as they are along the same diagonal themselves, so we can only ever have 14 pieces. This can be achieved with the following locations:



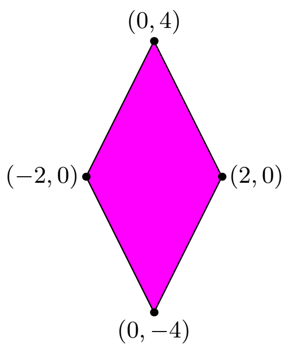
12. (HD) We can divide the path into rectangles (along the edges of the square) and quarter-circles (around the corners). Because the diagonal length of the square pool is 18 feet, we know the side lengths are $9\sqrt{2}$ feet by the Pythagorean Theorem. So, the side length of the rectangular sections are each $9\sqrt{2}$ feet, and they have a width of 5 feet (as given). There are four rectangular sections, giving us a total of $9\sqrt{2} \cdot 5 \cdot 4 = 180\sqrt{2}$ square feet. The quarter-circles around the corners have a radius of 5 feet, and can be combined to form the area of a full circle– this gives us an additional area of 25π square feet. This gives us a total area of $180\sqrt{2} + 25\pi$, so $a + b = \boxed{205}$

13. (HD) Each digit can either be 0, 1, 4, or 9. We have 5 possible digits, including the term $100,000$ but excluding the term 0 (because it must be a positive integer), so we have $4^5 = \boxed{1024}$ possible numbers.

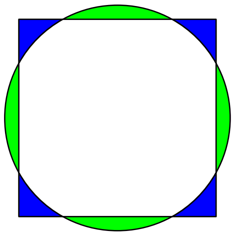
14. (HD, BB) The first thing to recognize is that subtracting all y-values by 2021 is only a translation, and does not affect the area of the region. We can thus rewrite the question as $|2x| + |y| = 4$.

    Because the absolute value is applied to both x and y, we can see that changing the sign values of x and y will not change anything, and our graph will be symmetric along all quadrants of the graph (if we can graph it in one quadrant, we can reflect it into all other quadrants).

    If we start exclusively in the first quadrant, our graph begins as $2x + y = 4$ (because x and y are always positive here), so the portion in the first quadrant is a segment from $(2, 0)$ to $(0, 4)$. We can then reflect this segment into the other three quadrants (using the symmetry), creating a rhombus with diagonal lengths of 8 and 4, and an area of $\frac{8 \cdot 4}{2} = \boxed{16}$. (see it as four right triangles if the rhombus area formula doesn't make sense to you)



15. (BB) *Note: This problem has a relatively simple solution, but is difficult to find.* (Instead, we solve the problem where we color all eight of the regions between the circle and the square similarly.



The trick is to notice that

$$\begin{aligned}\text{Area(green)} - \text{Area(blue)} &= (\text{Area(green)} + \text{Area(white)}) - (\text{Area(blue)} + \text{Area(white)})\\ &= \text{Area(Circle)} - \text{Area(Square)}\\ &= 16\pi - 49.\end{aligned}$$

Therefore the area difference in the original problem is $\frac{1}{4}(16\pi - 49) = 4\pi - \frac{49}{4}$, so $a = 4$ and $b = -49/4$ for a sum of $a + b = \boxed{-33/4}$.

If you're curious, the area of each green region is approximately 1.30805 and the area of each blue region is approximately 0.99168. These were computed by using lots of trigonometry.

16. (tiebreaker) (TM) We can get a rough estimate based on each "highest cube factor". For example, if we only summed the numbers that had 1 as its highest cube divisor or 8 as a divisor and so on, each separate factor would sum relatively close to 1,000,000 (In general, for any factor n, there are about $\frac{1000000}{n}$ numbers that are divisible by n, and we add n to the sum for each of these numbers). Because the cube root of 1,000,000 is 100, we know that we will have at most 100 different cubes to consider. So, we know that roughly below $100 \cdot 1,000,000 = 100,000,000$ is a good estimation. So we can generalize this as $S(T) \leq T \cdot \sqrt[3]{T}$. This tells us the answer is slightly below 100,000,000 which gives us rough estimate of the answer. To calculate the actual answer you need a computer. So ... here is my code in java. Since there are a lot of numbers it take 1-5 seconds to run.

```
import java.util.*;
public class sumCube
{
    public static void main(String[] args)
    {
      long sum = 0;
      long[] cube = new long[100];
      for(int i = 1; i <= 100; i++)
      {
         cube[i-1] = i*i*i;
      }
      int count = 0;
      for(long i = 1; i <= 1000000; i++)
      {
         long val = 1;
         for(int j = 0; j < Math.cbrt(i); j++)
         {
              if(i % cube[j] == 0)
              {
                 val = cube[j];
              }
         }
         sum+=val;
      }
      System.out.println(sum);
    }
}
```

This yields $\boxed{83,336,576}$.

Team Round

Answers

1. 7500
2. 60
3. 13
4. 58
5. 81
6. 8481
7. 2/7
8. 63/64
9. 1/6
10. 15
11. (tiebreaker) 0.514

Solutions

1. (TM) Looking at the first couple terms of the sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 ... we notice that terms are divisible by 3 after every 4th term. So, for the first 30,000 terms we get $\frac{30000}{4} = \boxed{7500}$.

2. (HD) We start with $(a-b)^2 = 36$. Expanding, we get:

$$a^2 - 2ab + b^2 = 36$$

Adding $4ab$ to both sides, we get:

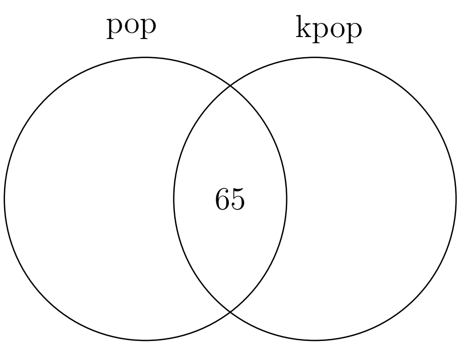
$$a^2 + 2ab + b^2 = 36 + 4ab$$

Because $ab = 6$, we can substitute to get:

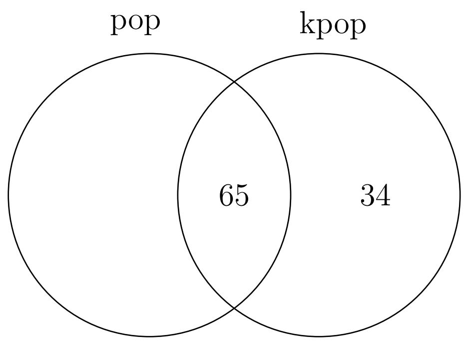
$$a^2 + 2ab + b^2 = 60$$

$$(a+b)^2 = \boxed{60}$$

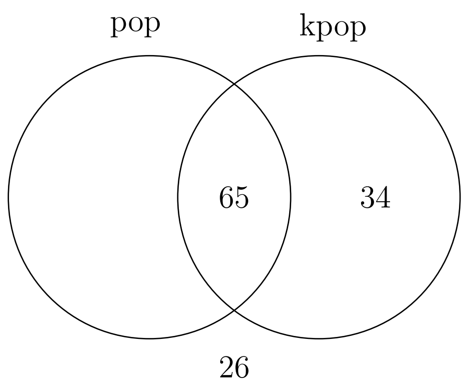
3. (HD, BB, TM) It is helpful to draw a Venn diagram here – we have 65 in the middle who both like pop and kpop.



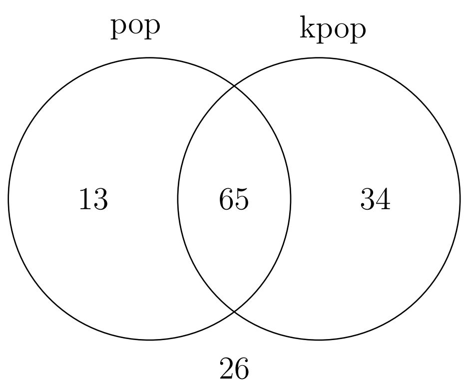
Then we have 99 in total that like kpop, so the ones that only like kpop (and not pop) are $99 - 65 = 34$.



Because we have that 60 people in total do not like pop, and 34 like kpop but not pop, $60-34=26$ people who do not like either.



We now only need to find the final region, which is liking pop but not kpop. Because we have 138 people total, we have $138-65-34-26=\boxed{13}$ people that like pop but not kpop.



4. (TM) Martin runs twice as far every day– so on day 8 he will run $\frac{5}{4}\cdot 2^7=160$ total miles. Then he collapses 7/8 of the way which means he ran a total of 140 miles. So, he would have travelled 80 miles out and reached 60 miles back before collapsing, leaving him 20 miles north of his house. Because the helicopter is exactly 21 miles east of his house, by the Pythagorean theorem we know the distance from the helicopter to his position is 29 miles ($20^2+21^2=29^2$). The helicopter travels there and back, so, this gives us $29\cdot 2=\boxed{58}$.

5. (HD) The general idea is to reach an integer fast enough (so we can square it a bunch at the end, while keeping it an integer) and large enough (to make squaring effective). Messing around with different cases gives us a best case of $+2, \times 2, {}^\wedge 2, {}^\wedge 2$, which gives us $\boxed{81}$.

   Note: after the first $+2$ (currently at 1.5), it may seem better to add two again to reach 3.5, which is larger than 3 when multiplying. However, 3 is an integer, which allows us to square it twice, while 3.5 needs to be multiplied first to make it an integer.

6. (HD, TM) We are given that it is a 4 digit integer: let us write this as $\overline{abcd}$, where each letter represents a digit. Because we are given that the number increases when rounded to the nearest hundred, we know that $c \geq 5$. We are also given that $c \cdot d = a$ (by the third bullet point). If $d = 0$, then a would equal 0, which breaks the "no leading zeros" rule. If $d \geq 2$, then $d \cdot c$ would be too large (because we know $c \geq 5$, so $d \cdot c$ would be larger than 9). So, $d = 1$ and $a = c$. Using this information, we can rewrite the number as $\overline{cbc1}$.

   We are given that the number, when flipped (so $\overline{1cbc}$) is divisible by 66. Focusing on the prime factors of 66, we know that $\overline{1cbc}$ is divisible by 2, 3, and 11, and we can analyze each factor separately.

   Because it is divisible by 2 (so the last digit, c must be even), and we know that $c \geq 5$ (from earlier), we know that c must be either 6 or 8.

   Because it is divisible by 11, we know that $c - b + c - 1$ must be divisible by 11 (look up the rule of 11 if you aren't familiar!). If $c = 6$, then b can only be 0 ($6 - 0 + 6 - 1 = 11$. If $c = 8$, then b can only 4 ($8 - 4 + 8 - 1 = 11$). So, we now have only two possible numbers: 6061 and 8481.

   Because the number is divisible by 3, we know that the sum of the digits must be divisible by 3. $6 + 0 + 6 + 1 = 13$, which is not divisible by 3, while $8 + 4 + 8 + 1 = 21$, which is. Thus, our number can only be $\boxed{8481}$.

7. (HD, TM) For a group of 5 contestants to not have a perfect score, it can have at most 2 perfect scorers (so at least 3 non-perfect). Because there are only 3 non-perfect students in this case, our only way of having a non-perfect score is if all 3 of these non-perfect students are chosen as contestants. For the first of these 3 non-perfect students, there is a $\frac{5}{7}$ chance that he is a contestant (because there are 7 possible positions, 5 of which are contestants). Similarly, for the second and third non-perfect scorers, there are a $\frac{4}{6}$ and $\frac{3}{5}$ chance of them being contestants as well (after the first person is chosen, there are 6 open positions, 4 of which are contestants, same logic for the third person as well). This gives us a $\frac{5}{7} \cdot \frac{4}{6} \cdot \frac{3}{5} = \boxed{\frac{2}{7}}$ probability of having a non-perfect score.

   ALTERNATIVE (if you know choose functions): There are a total of $\binom{7}{5}$ ways to choose a group of 5 contestants from a team of 7. Because we must have all three non-perfect scorers as contestants to have a non-perfect team score, the remaining two contestants can only be chosen in $\binom{4}{2}$ ways (we're choosing who the last 2 contestants are, out of the remaining 4 total people). Thus, our probability is $\frac{\binom{4}{2}}{\binom{7}{5}} = \boxed{\frac{2}{7}}$

8. (HD) Rewrite the expression, using exponent rules, as $4(2^x)^2 - \frac{1}{2}(2^x) + 1$. We can substitute some variable y for 2^x and find the maximum for y first. This gives us the quadratic equation $4y^2 - \frac{1}{2}y + 1$. Here, $-\frac{b}{2a} = \frac{1}{16}$ (the vertex of any quadratic expression $ax^2 + bx + c$ have the x value $-\frac{b}{2a}$). So, our maximum value of this graph occurs when y is $\frac{1}{16}$, which produces $4 \cdot \frac{1}{256} - \frac{1}{2} \cdot \frac{1}{16} + 1 = \boxed{\frac{63}{64}}$.

9. FAST WAY (HD): The key observation here is to focus on only the four cards that Abby and Brandon chose. These cards will always have some kind of "ranking" (highest, second highest, etc.), as they are all unique numbers. To compare Abby's cards to Brandon's cards, we only care about such "ranking" and not the actual numbers on the cards– this lets us narrow our casework to the order of these four cards (in fact, if I wrote that the deck had 1,000,000 cards in it, but they still drew 4 cards, the answer would still be the same!).

   If we label the highest card a, the second highest card b, the third highest card c, and the lowest card d, we can count different combinations of the two cards that Abby has (completely ignore the original 8-card deck at this point). In total, Abby can have 6 possible combinations of two cards: ab, ac, ad, bc, bd, cd (this is also $\binom{4}{2}$ if you are familiar with choose functions). For Abby's lowest card to be higher than Brandon's highest, she must have the top two cards (ab), which is only 1 case. So, there is 1 successful hand out of 6 possible hands, giving us a probability of $\boxed{\frac{1}{6}}$

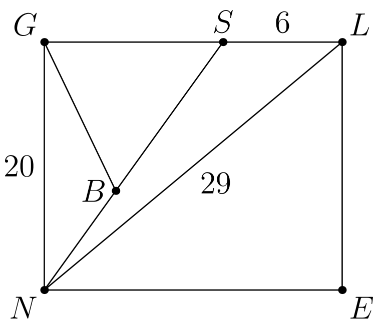
CASEWORK WAY (TM): First, let us find the denominator of this probability. The number of ways they can each choose two cards without replacement is $\binom{8}{2} \cdot \binom{6}{2} = 420$. Next, we can use casework to find the number of ways they can pick cards such Abby's lowest card is higher than Brandon's highest card. We can only care about Brandon's largest card by setting it to 6, 5, 4, 3, 2. Then the number of ways Abby's cards can be larger are respectively $\binom{2}{2}$, $\binom{3}{2}$, $\binom{4}{2}$, $\binom{5}{2}$. Then the smaller card of Brandon's can be switched around the larger number -1 times. So, respectively we get 5, 4, 3, 2, 1. Then we multiply and add our respective values. We have $1 \cdot 5 + 3 \cdot 4 + 6 \cdot 3 + 10 \cdot 2 + 15 \cdot 1 = 70$. So we get $\frac{70}{420} = \boxed{\frac{1}{6}}$.

Note: This expression $\binom{n}{k}$ is called choose it is said as "n choose k". It is the number of ways to choose k things from n when we do not care about order. $\binom{n}{k}$ is denoted as $\frac{n!}{k!(n-k)!}$. Also: For some number $m!$ is denoted as $m \cdot (m-1) \cdot (m-2) \cdot ... \cdot 1$.

10. (TM, HD) By the Pythagorean Theorem, $GL = 21$ because $20^2 + 21^2 = 29^2$. This means $GS = 15$ $(21 - 6 = 15)$. Using GS and GN, we can use Pythagorean Theorem again to find that $SN = 25$ because $15^2 + 20^2 = 25^2$. We can find that $[GNS] = \frac{20 \cdot 15}{2} = 150$. We are given that $[GBN] = 60$, so $[GBS] = [GNS] - [GBN] = 150 - 60 = 90$. Notice that $\triangle GBN$ and $\triangle GBS$ have bases along the same line SN, and share point G, so their heights must be equal. Because their heights must be equal, the ratio of their bases will equal the ratio of their areas (think about how we find triangle areas, and what happens if you set two heights equal).

    Using this, we can set up the proportion $\frac{60}{90} = \frac{BN}{SB}$, and multiply and simplify both sides to get $2(SB) = 3(BN)$. Because we found that $SN = 25$, we know that $BN + SB = 25$. We now have a system of equations, which we can solve to find SB. We can substitute BN for $25 - SB$ in the first equation to get $2(SB) = 75 - 3(SB)$, and solve to get $SB = \boxed{15}$

    Note: [ABC] denotes the area of triangle ABC



11. (tiebreaker) (HD) The answers submitted were 0.3, 0.74, 0.83, 0.000.... (insert two pages of 0's)1, 0.4, and 3/10, which averages to 0.514.

    So 0.3, 0, 0, (a really small number), 0.4, and 0.3 points were awarded to each team respectively.

Sprint Round

Answers

1. 51
2. 1/72
3. 1/3 or 10/3 depending on interpretation
4. 3/2

5. 55
6. 23
7. 49
8. 75 (%)
9. 8
10. 1/7
11. 28 (extra)
12. 13 (extra)

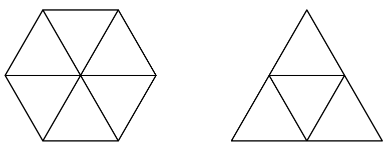
Solutions

1. (TM) There are two possible squares. The first one is the that these two points are a side and the other is that they are vertices of the diagonal. The area of the square when it is a side is $((\sqrt{(10-5)^2 \cdot (6-3)^2})^2 = 34$. The other area is $(\frac{\sqrt{34}}{\sqrt{2}})^2 = 17$. So the total is $17+34 = \boxed{51}$.
2. (TM) The total possible dice rolls are $6^3 = 216$. Then the total rolls where the product is two is 3 ways (1, 1, 2), (1, 2, 1), (2, 1, 1). So our answer is $\frac{3}{216} = \boxed{\frac{1}{72}}$.

   (Note: The dice can be distinguishable or indistinguishable but you must remain consistent with your numerator and denominator)
3. (TM) This question looks tricky but we just use variables to help. Let us set the answer to x. So, $x = 10x - 3$. Solving we get $9x = 3$ so $x = \boxed{\frac{1}{3}}$.

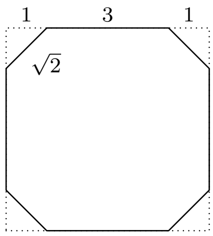
   This can also be interpreted as "ten times (the answer minus 3)", giving $10(x-3) = x$. Solving for x we get $9x = 30$ so x $= \boxed{\frac{10}{3}}$. Both answers are acceptable.
4. (TM, BB) We can notice that the side length of the hexagon will only have side length x while the triangle is going to have side length $2x$. We can then calculate the ratio by knowing that the area of the hexagon will be $\frac{3x^2\sqrt{3}}{2}$ triangle will be $x^2\sqrt{3}$. Thus the ratio is $\frac{\frac{3}{2}}{1} = \boxed{\frac{3}{2}}$.

   Alternatively, dissecting the shapes into congruent triangles should give you the answer immediately:

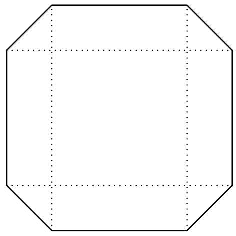
   

   Since the mini triangles all have the same area, we can write the ratio of the areas as $\frac{6}{4} = \boxed{\frac{3}{2}}$.
5. (TM) We can notice that each two consecutive numbers going down is a difference of squares that have 1 times the sum of the two numbers. Such as $10^2 - 9^2$ which can be written as (10+9)(10-9) = 19. As we keep going down we notice that there is a series 19+15+11+7+3. This has $\frac{10}{2} = 5$ terms. So we can write the sum of this series as $\frac{22 \cdot 5}{2} = \boxed{55}$.
6. (TM, BB) There are two ways to do this problem. The first method is more efficient than the second.

   - (Subtract from square) Let us surround the octagon in a square of size 5. The area of the square can be found easily as $5 \cdot 5 = 25$. Then we need to subtract the extract area which is $4 * \frac{1 \cdot 1}{2} = 2$. So we get our answer 25 - 2 $= \boxed{23}$.



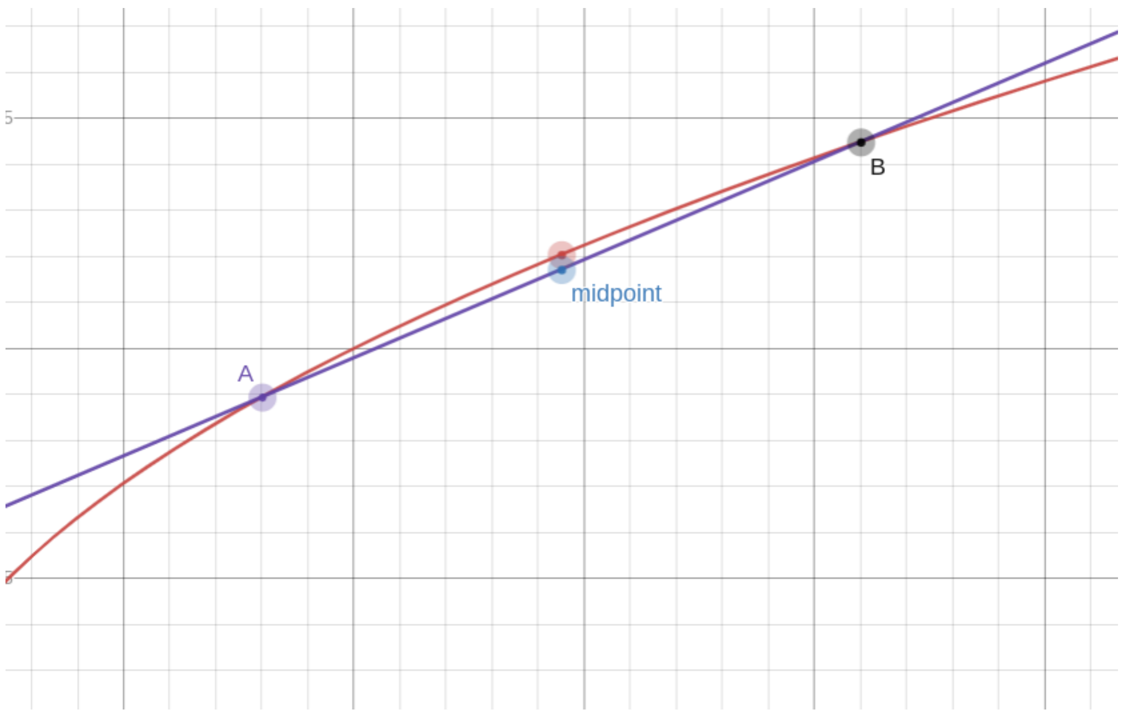
- (Add each area) We can split the octagon into separate pieces. A square, 4 rectangles, and 4, triangles. The area of the square is $3 \cdot 3 = 9$. The area of each rectangle is $3 \cdot 1 = 3$. The area of each triangle is $\frac{1 \cdot 1}{2} = \frac{1}{2}$. So we get $9 + 4 \cdot 3 + 4 \cdot \frac{1}{2} = \boxed{23}$.



7. (BB) We know $\sqrt{100} = 10$, so the value of the expression should be really close to 50. Because the graph of $y = \sqrt{x}$ is curving downward, this means that

$$\sqrt{a} + \sqrt{b} < 2\sqrt{\tfrac{a+b}{2}},$$

for any positive real numbers a, b.



We can see this since the y-coordinate of the midpoint $\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)$ is less than $\sqrt{\frac{a+b}{2}}$. This implies that

$$(\sqrt{98} + \sqrt{102}) + (\sqrt{99} + \sqrt{101}) + \sqrt{100} < 5\sqrt{100} = 50.$$

Since the expression is also at least 49 (since $\sqrt{98} > 9.8$ already), the answer must be $\boxed{49}$.

8. (TM) We can write this as question as $15 \cdot 59 + x = 60 \cdot 16$. Solving for x we get 75 so the answer is $\boxed{75\%}$.

9. (TM) Since we have a system of equations and y is already solved for we can substitute the y in the first equation. So, we can rewrite the first equation as $\frac{(x-2)^2}{2} + \frac{(x+3-3)^2}{2} = 2$. Simplifying, we get $2x^2 - 4x + 4 = 4$. Solving for x we get x = 0, x = 2. Plugging in both values of x in the second equation we get the pairs $(0, 3), (2, 5)$. Finding the square of the distance between these two points we get $(\sqrt{(2-0)^2 + (5-3)^2})^2 = \boxed{8}$.

10. (TM) First, we realize that number
$$\frac{7}{9}\left(10^{50} - 1\right),$$
can be written as 777...777 with 50 total 7's. (This is because $10^{50} - 1 = 9999 \cdots 99$ with 50 9's.) Since this number is divisible by 7 it is the same as the probability as between 1 and 7. So we get $\boxed{\frac{1}{7}}$.

11. (HD) The key here is to find a way to cancel the infinitely repeating decimals. Let $x = 0.\overline{15}$. Because there are two repeating digits in the decimal, we need to multiply by 10^2, giving us $100x = 15.\overline{15}$. Now, because $x = 0.\overline{15}$, we can subtract x from the left side and $0.\overline{15}$ from the right side (we're still subtracting the same thing from both sides, because they're equal). This completely gets rid of the repeating decimal part, giving us $99x = 15$. We divide both sides to get $x = 5/33$ for a difference of $33 - 5 = \boxed{28}$.

12. (BB) Note that $5/15 = 1/3 = 0.333\cdots$ and $5/12 = 5 \cdot 0.08333\cdots = 0.41666\cdots$, so n is either 13 or 14. The approximation 0.385 is closer to $0.41666\cdots$ than $0.333\cdots$ so $n = \boxed{13}$.