Lecture 1

Part I: Vector calculus

- Motivation: Doing calculations with vectors in mechanics, fluid dynamics, description and calculation of curves and surfaces.

1 Vector analysis

1.1 Vectors and scalars

- *Definition*: A *scalar* a is completely described by its value. A scalar is simply a number. Examples are the temperature, the pressure and the length of an object.

- *Definition*: A *vector* $\vec{a}$ is completely described by its length and its direction. Examples are quantities like the velocity, the acceleration, the position in a coordinate system and forces in general. We indicate vectors by arrows.

- *Components of vectors*: Consider a vector $\vec{a}$ in a 3 dimensional cartesian coordinate system connecting the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ (see Figure 1). The *components* of $\vec{a}$ are then given as

$$\vec{a} = (x_2 - x_1, y_2 - y_1, z_2 - z_1) = (a_1, a_2, a_3)$$

and the length of $\vec{a}$ is

$$|\vec{a}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

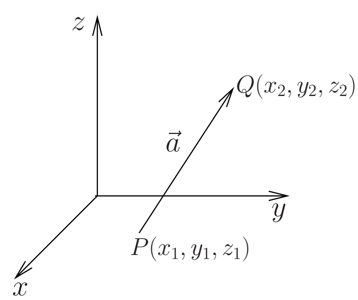

Figure 1: The vector $\vec{a}$ with initial point $P(x_1, y_1, z_1)$ and final point $Q(x_2, y_2, z_2)$ in a 3 dimensional cartesian coordinate system.

- *Equality of vectors*: The vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ are *equal*

$$\vec{a} = \vec{b}$$

if and only if the corresponding components are the same

$$a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3.$$

- *Vector summation* (see Figure 2): We can add two vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ by adding the corresponding components

$$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

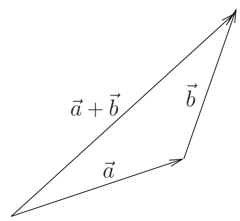

Figure 2: Addition $\vec{a} + \vec{b}$ of the vectors $\vec{a}$ and $\vec{b}$.

- *Scalar multiplication* (see Figure 3): Let $\vec{a} = (a_1, a_2, a_3)$ be a vector and c be a scalar. The multiplication of $\vec{a}$ by c is defined as

$$c\vec{a} = (ca_1, ca_2, ca_3)$$

and the resulting length is

$$|c\vec{a}| = \sqrt{(ca_1)^2 + (ca_2)^2 + (ca_3)^2} = |c||\vec{a}|.$$

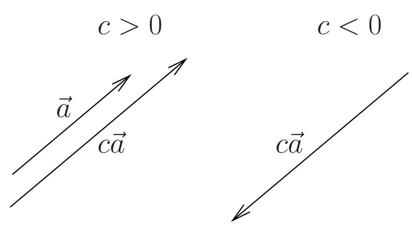

Figure 3: Scalar multiplication of the vector $\vec{a}$.

- *Unit normal vectors* (see Figure 4): The unit normal vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$ have length 1 and are defined as

$$\vec{i} = (1, 0, 0), \quad \vec{j} = (0, 1, 0), \quad \vec{k} = (0, 0, 1).$$

Any vector $\vec{a} = (a_1, a_2, a_3)$ can be rewritten as (using the addition of vectors and the scalar multiplication)

$$\vec{a} = (a_1, a_2, a_3) = (a_1, 0, 0) + (0, a_2, 0) + (0, 0, a_3) = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}.$$

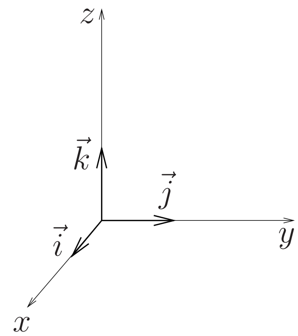

Figure 4: Illustration of the unit normal vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$.

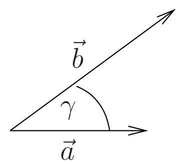

Figure 5: The angle $\gamma = \angle(\vec{a}, \vec{b})$ between the vectors $\vec{a}$ and $\vec{b}$.

1.2 Scalar product, vector product

- *Scalar product*: The *scalar product* between the vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ is defined as

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\vec{a}||\vec{b}| \cos \gamma, \quad \gamma = \angle(\vec{a}, \vec{b})$$

where $0 \leq \gamma \leq \pi$ is the angle between the vectors $\vec{a}$ and $\vec{b}$ (see Figure 5).

- *Orthogonal vectors* (see Figure 6): Two vectors $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$ are called *orthogonal*

$$\vec{a} \perp \vec{b}$$

if and only if

$$\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \cos \gamma = 0 \Leftrightarrow \gamma = \pi/2.$$

Here $\vec{0} = (0, 0, 0)$ denotes the *zero vector*.

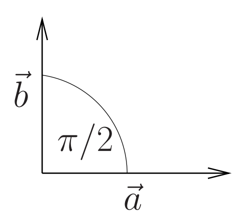

Figure 6: Orthogonal vectors $\vec{a}$ and $\vec{b}$.

- *Length and direction*: Let $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$. We can calculate the length and the angle γ between $\vec{a}$ and $\vec{b}$ using the scalar product as

$$\vec{a} \cdot \vec{a} = |\vec{a}||\vec{a}| \cos 0 = |\vec{a}|^2 \Rightarrow |\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}$$
$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \gamma = \sqrt{\vec{a} \cdot \vec{a}}\sqrt{\vec{b} \cdot \vec{b}} \cos \gamma \Rightarrow \cos \gamma = \frac{\vec{a} \cdot \vec{b}}{\sqrt{\vec{a} \cdot \vec{a}}\sqrt{\vec{b} \cdot \vec{b}}}.$$

- *Example*: The *work* W done by a constant force $\vec{f}$ when moving a particle from point $P(p_1, p_2, p_3)$ to the point $Q(q_1, q_2, q_3)$ along a straight line (see Figure 7) is given as
$$W = \vec{f} \cdot \vec{d} = \vec{f} \cdot (q_1 - p_1, q_2 - p_2, q_3 - p_3).$$

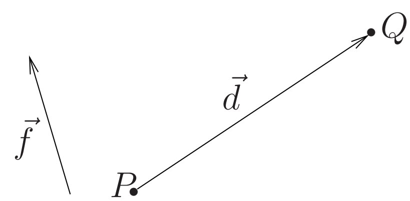

Figure 7: Displacement of a particle from point P to the point Q under the influence of a force $\vec{f}$.

- *Projection*: The *projection* p of $\vec{a} \neq \vec{0}$ in direction of $\vec{b} \neq \vec{0}$ (see Figure 8) is

$$p = |\vec{a}| \cos \gamma = |\vec{a}| \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}.$$

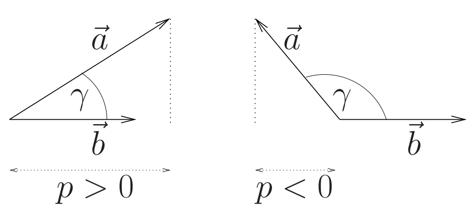

Figure 8: Illustration of the projection p of the vector $\vec{a}$ in direction $\vec{b}$.

- *Example*: Let $\vec{a} = (a_1, a_2, a_3)$ be a vector. The projections in direction of the unit normal vectors are the components of $\vec{a}$

  projection in direction $\vec{i}$: $p = (\vec{a} \cdot \vec{i})/|\vec{i}| = a_1$

  projection in direction $\vec{j}$: $p = (\vec{a} \cdot \vec{j})/|\vec{j}| = a_2$

  projection in direction $\vec{k}$: $p = (\vec{a} \cdot \vec{k})/|\vec{k}| = a_3$.

- *Vector product*: Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ and $\gamma = \angle(\vec{a}, \vec{b})$. We define the *vector product* $\vec{v} = \vec{a} \times \vec{b}$ as

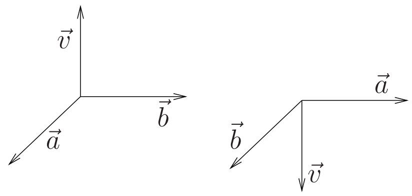

Figure 9: Illustration of the direction of $\vec{v} = \vec{a} \times \vec{b}$.

$\vec{v} = \vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$
length of $\vec{v}$: $|\vec{v}| = |\vec{a}||\vec{b}| \sin \gamma$
direction of $\vec{v}$: right hand rule (see Figure 9).

- *Alternative calculation of the vector product*: Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$. Then

$$\begin{aligned}\vec{v} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} &= \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= \vec{i}(a_2b_3 - a_3b_2) - \vec{j}(a_1b_3 - a_3b_1) + \vec{k}(a_1b_2 - a_2b_1) \\ &= (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).\end{aligned}$$

- *Rules*: For any vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and any scalar q we have

$$\begin{aligned}(q\vec{a}) \times \vec{b} &= q(\vec{a} \times \vec{b}) = \vec{a} \times (q\vec{b}) \\ (\vec{a} + \vec{b}) \times \vec{c} &= (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c}) \\ \vec{a} \times \vec{b} &= -\vec{b} \times \vec{a}.\end{aligned}$$

- *Geometric interpretation of the vector product* $\vec{v} = \vec{a} \times \vec{b}$: $|\vec{v}|$ equals the area of the parallelogram spanned by the vectors $\vec{a}$ and $\vec{b}$ (see Figure 10).

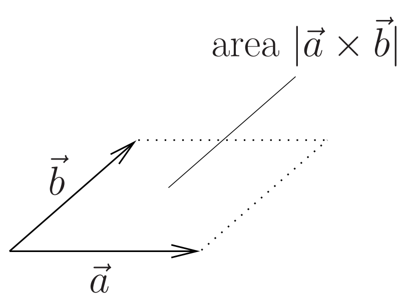

Figure 10: The parallelogram spanned by the vectors $\vec{a}$ and $\vec{b}$.

- *Example*: We consider a rotating body with angular velocity $\vec{w}$ (see Figure 11). The velocity $\vec{v}$ of the point $P(x, y, z)$ on the rotating body is given by the vector product

$$\vec{v} = \vec{w} \times \vec{r}$$

where

$$\vec{r} = (x, y, z).$$

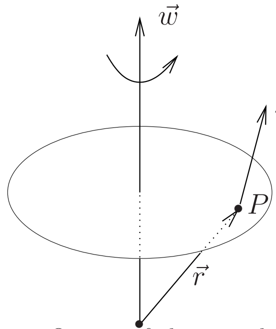

Figure 11: Illustration of a rotating body.

1.3 Vector functions and scalar functions

- *Definition*: We denote the points in the three dimensional space, using a cartesian coordinate system, by $P(x, y, z)$ and by $\vec{r} = (x, y, z)$ the corresponding *position vectors*.

- *Definition*: A *vector field* $\vec{v}(P)$ is a function that associates a vector $\vec{v}$ to each point $P(x, y, z)$

$$\vec{v}(P) = \vec{v}(x, y, z) = \vec{v}(\vec{r}) = (v_1(\vec{r}), v_2(\vec{r}), v_3(\vec{r}))$$

  where v_1, v_2 and v_3 are the components of $\vec{v}$.

- *Example*: The gravitational field is described by the gravitational force

$$\vec{p}(x, y, z) = \vec{p}(\vec{r}) = -c\frac{\vec{r}}{|\vec{r}|^3}$$

  where c is a constant.

- *Example*: The velocity $\vec{v}(\vec{r})$ of a point $P(x, y, z)$ at position $\vec{r} = (x, y, z)$ on a rotating body (see Figure 11) is given as

$$\vec{v}(x, y, z) = \vec{v}(\vec{r}) = \vec{w} \times \vec{r}.$$

- *Definition*: A *scalar field* is a function that associates a scalar f to each point $P(x, y, z)$ with position vector $\vec{r} = (x, y, z)$

$$f(P) = f(x, y, z) = f(\vec{r}).$$

- *Example*: The distance $f(P)$ between the point $P(x, y, z)$ and a fixed point $P_0(x_0, y_0, z_0)$ (see Figure 12) is a scalar field

$$f(P) = f(\vec{r}) = f(x, y, z) = |\vec{r} - \vec{r}_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}.$$

- *Definition*: A *vector function* $\vec{v}(t)$ associates a vector $\vec{v}$ to each scalar t

$$\vec{v}(t) = (v_1(t), v_2(t), v_3(t)).$$

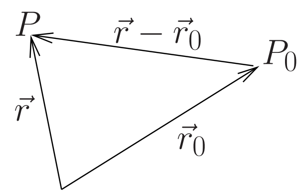

Figure 12: The distance $|\vec{r} - \vec{r}_0|$ between the points P and P_0.

- *Example*: The velocity $\vec{v}(t)$ of the wind at a fixed position and as a function of time t is a vector function.

- *Definition*: A *scalar function* $f(t)$ associates a scalar f to each scalar t (the usual definition of a function).

- *Example*: The temperature $T(t)$ at a fixed position as a function of time t is a scalar function.

- *Derivatives of vector functions and scalar functions*: The *derivative of a scalar function* $f^{'}(t)$ is the usual derivative. The *derivative of a vector function* is defined as
$$\vec{v}\,'(t) = (v_1^{'}(t), v_2^{'}(t), v_3^{'}(t))$$
(the usual derivatives of the components of $\vec{v}$).

- *Example*: Let $\vec{r}(t)$ be the position vector of a particle at time t. The velocity vector is $\vec{v}(t) = \vec{r}\,'(t)$ and the acceleration vector is $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$.

- *Partial derivatives of a scalar field* $f(x, y, z)$: Let $f(x, y, z)$ be a scalar field. The *partial derivatives* are defined as
$$f_x(x, y, z) = \frac{\partial}{\partial x} f(x, y, z) \ \ \text{(keeping } y \text{ and } z \text{ fixed)}$$
$$f_y(x, y, z) = \frac{\partial}{\partial y} f(x, y, z) \ \ \text{(keeping } x \text{ and } z \text{ fixed)}$$
$$f_z(x, y, z) = \frac{\partial}{\partial z} f(x, y, z) \ \ \text{(keeping } x \text{ and } y \text{ fixed).}$$

- *Example*: We consider the scalar field
$$f(x, y, z) = f(\vec{r}) = x^2 + y^2 + z^2 = |\vec{r}|^2.$$
Then
$$f_x(x, y, z) = 2x, \ \ f_y(x, y, z) = 2y, \ \ f_z(x, y, z) = 2z.$$

- *Partial derivatives of a vector field* $\vec{v}(x, y, z)$: Let $\vec{v}(x, y, z)$ be a vector field. The *partial derivatives* are defined as the vectors of the partial derivatives of the components of $\vec{v}(x,y,z)$

$$\vec{v}_x(x,y,z) = \left(\frac{\partial}{\partial x}v_1(x,y,z), \frac{\partial}{\partial x}v_2(x,y,z), \frac{\partial}{\partial x}v_3(x,y,z)\right)$$
$$\vec{v}_y(x,y,z) = \left(\frac{\partial}{\partial y}v_1(x,y,z), \frac{\partial}{\partial y}v_2(x,y,z), \frac{\partial}{\partial y}v_3(x,y,z)\right)$$
$$\vec{v}_z(x,y,z) = \left(\frac{\partial}{\partial z}v_1(x,y,z), \frac{\partial}{\partial z}v_2(x,y,z), \frac{\partial}{\partial z}v_3(x,y,z)\right).$$

- *Example*: Let
$$\vec{v}(x,y,z) = \frac{1}{2}(x^2,y^2,z^2).$$
Then
$$\vec{v}_x(x,y,z) = (x,0,0), \quad \vec{v}_y(x,y,z) = (0,y,0) \quad \vec{v}_z(x,y,z) = (0,0,z).$$

1.4 Curves

- A *curve* C in space (see Figure 13) can be described by a *parametric representation* $\vec{r}(t)$, where $\vec{r}(t)$ denotes the position vector of all points on the curve C. Here t is a convenient variable that identifies the points on C.

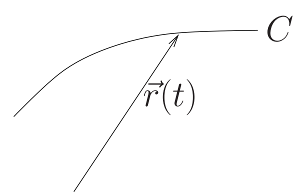

Figure 13: Parametric representation $\vec{r}(t)$ of a curve C.

- *Example*: Let C be a circle in the xy–plane with centre at the origin and radius R. Then
$$\vec{r}(t) = (R\cos t, R\sin t, 0) = (x(t), y(t), 0), \quad 0 \le t \le 2\pi$$
and
$$|\vec{r}(t)|^2 = x(t)^2 + y(t)^2 = R^2\cos^2 t + R^2\sin^2 t = R^2.$$

- *Tangent of a curve C at the point $\vec{r}(t)$*: If the parametric representation $\vec{r}(t)$ of a given curve C is differentiable, then $\vec{r}\,'(t)$ is called the *tangent* at the point $\vec{r}(t)$. The tangent defines the direction of a line L that touches the curve C at the point $\vec{r}(t)$ (see Figure 14).

- *Example*: Consider a circle in the xy–plane given by $\vec{r}(t) = (R\cos t, R\sin t, 0)$, $0 \le t \le 2\pi$ (see Figure 15). Then
$$\vec{r}\,'(t) = (-R\sin t, R\cos t, 0)$$
$$\vec{r}\,'(0) = (0, R, 0)$$
$$\vec{r}\,'(\pi/2) = (-R, 0, 0).$$

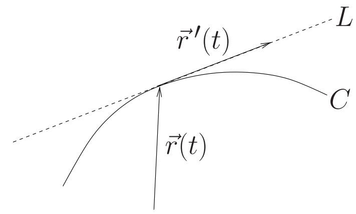

Figure 14: The line L touches the curve C at the point $\vec{r}(t)$ and has direction $\vec{r}'(t)$.

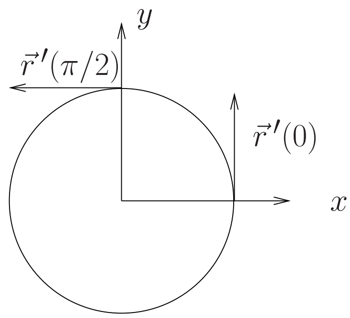

Figure 15: The tangent $\vec{r}'(t)$ on the circle for $t = 0$ and $t = \pi/2$.

Exercises

1. Let $\vec{a} = (1, 1, 1)$ and $\vec{b} = (1, 2, 1)$. Find $\vec{a} \times \vec{b}$, $\vec{a} \times (\vec{a} \times \vec{b})$ and $(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{b})$.

2. Let

$$\vec{v}(x, y, z) = (v_1(x, y, z), v_2(x, y, z), v_3(x, y, z)) = (xyz, x\cos(yz), ze^{xy}) .$$

Calculate

$$\vec{v}_x(x, y, z), \quad \vec{v}_y(x, y, z), \quad \vec{v}_z(x, y, z), \quad \vec{v}_{xy}(x, y, z)$$

where

$$\vec{v}_{xy}(x, y, z) = \left(\frac{\partial^2}{\partial x \partial y} v_1(x, y, z), \frac{\partial^2}{\partial x \partial y} v_2(x, y, z), \frac{\partial^2}{\partial x \partial y} v_3(x, y, z) \right) .$$

3. Consider the position vector $\vec{r}(t) = \vec{c} + t^2\vec{b}$ at time $t \geq 0$ of a moving particle where $\vec{c} = (1, 1, 0)$ and $\vec{b} = (1, 0, 0)$.

   (a) Sketch the path of the particle.

   (b) Calculate the velocity $\vec{v}(t)$ and the acceleration $\vec{a}(t)$.

   (c) Calculate the projection p of $\vec{a}(t)$ in direction of $\vec{v}(t)$.

   (d) The acceleration can be decomposed into a tangential acceleration $\vec{a}_{tan}(t)$ (in direction of $\vec{v}(t)$) and a normal accelaration $\vec{a}_{norm}(t)$ (orthogonal to $\vec{v}(t)$)

$$\vec{a}(t) = \vec{a}_{tan}(t) + \vec{a}_{norm}(t)$$

   where

$$\vec{a}_{tan}(t) = p\frac{\vec{v}(t)}{|\vec{v}(t)|}$$

and

$$\vec{a}_{norm}(t) = \vec{a}(t) - \vec{a}_{tan}(t).$$

Calculate $\vec{a}_{tan}(t)$ and $\vec{a}_{norm}(t)$.

4. Consider the position vector $\vec{r}(t) = (\cos t, \sin t, 0)$ at time $t \geq 0$ of a moving particle.

   (a) Calculate the velocity $\vec{v}(t)$ and the acceleration $\vec{a}(t)$.

   (b) Calculate the projection p of $\vec{a}(t)$ in direction of $\vec{v}(t)$.

   (c) Calculate $\vec{a}_{tan}(t)$ and $\vec{a}_{norm}(t)$.

5. Consider the curve C consisting of all points $P(x, y, z)$ such that

$$(x-1)^2 + (y-1)^2 = 1, \quad z = 0.$$

   (a) Sketch the curve C.

   (b) Find a parametric representation of the curve C.