Lecture 2

- *Length l of a curve C*: Let C be a curve given by the parametric representation $\vec{r}(t)$ and let $\vec{r}(a)$ and $\vec{r}(b)$ denote the position vectors of two points on C. The length of C between these points is then given as

$$l = l(a,b) = \int_a^b \sqrt{\vec{r}\,'(t) \cdot \vec{r}\,'(t)}\mathrm{d}t.$$

- *Example*: We consider a circle with radius R in the xy–plane and centre at the origin with the parametric representation

$$\vec{r}(t) = (R\cos t, R\sin t, 0), \quad 0 \leq t \leq 2\pi.$$

  The tangent vectors are

$$\vec{r}\,'(t) = (-R\sin t, R\cos t, 0)$$

  and the scalar product gives

$$\vec{r}\,'(t) \cdot \vec{r}\,'(t) = R^2\sin^2(t) + R^2\cos^2(t) = R^2.$$

  We therefore have the length of the circle given as

$$l = l(0, 2\pi) = \int_0^{2\pi} \sqrt{\vec{r}\,'(t) \cdot \vec{r}\,'(t)}\mathrm{d}t = \int_0^{2\pi} R\mathrm{d}t = 2R\pi.$$

- *Example*: A helix with radius a and slope c along the z–axis with 3 rotations (see Figure 1) is given by the parametric representation

$$\vec{r}(t) = (a\cos t, a\sin t, ct), \quad 0 \leq t \leq 6\pi.$$

  The tangent vectors are

$$\vec{r}\,'(t) = (-a\sin t, a\cos t, c)$$

  and the scalar product is

$$\vec{r}\,'(t) \cdot \vec{r}\,'(t) = a^2\sin^2(t) + a^2\cos^2(t) + c^2 = a^2 + c^2.$$

  We thus calculate the length of the helix as

$$l = l(0, 6\pi) = \int_0^{6\pi} \sqrt{\vec{r}\,'(t) \cdot \vec{r}\,'(t)}\mathrm{d}t = \int_0^{6\pi} \sqrt{a^2 + c^2}\mathrm{d}t = 6\pi\sqrt{a^2 + c^2}.$$

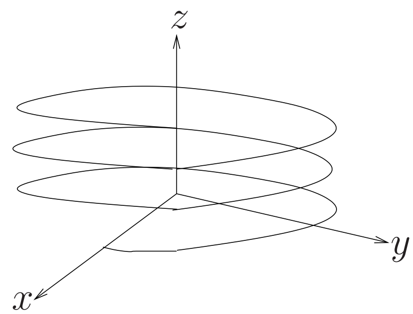

Figure 1: Illustration of a helix along the z–axis with 3 rotations.

1.5 The gradient of a scalar field

- *Taylor expansion*: Consider a scalar field $f(x,y,z)$ and two points $P(x,y,z)$, $P_0(x+\mathrm{d}x, y+\mathrm{d}y, z+\mathrm{d}z)$ that are very close to each other, i.e. $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ are very small. Let $\vec{r}=(x,y,z)$ and $\vec{r}_0=(x+\mathrm{d}x, y+\mathrm{d}y, z+\mathrm{d}z)$ be the corresponding position vectors. The *Taylor expansion* then gives

$$\begin{aligned} f(x+\mathrm{d}x, y+\mathrm{d}y, z+\mathrm{d}z) =& f(x,y,z) + \frac{\partial f(x,y,z)}{\partial x}\mathrm{d}x + \frac{\partial f(x,y,z)}{\partial y}\mathrm{d}y + \frac{\partial f(x,y,z)}{\partial z}\mathrm{d}z \\ & + \text{higher order terms in } \mathrm{d}x, \mathrm{d}y, \mathrm{d}z. \end{aligned}$$

  The higher order terms can be neglected for $\mathrm{d}x \to 0$, $\mathrm{d}y \to 0$ and $\mathrm{d}z \to 0$. Thus

$$\begin{aligned} f(x+\mathrm{d}x, y+\mathrm{d}y, z+\mathrm{d}z) - f(x,y,z) = & \\ &= \left(\frac{\partial f(x,y,z)}{\partial x}, \frac{\partial f(x,y,z)}{\partial y}, \frac{\partial f(x,y,z)}{\partial z}\right) \cdot (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z) \\ &= \left(\frac{\partial f(x,y,z)}{\partial x}, \frac{\partial f(x,y,z)}{\partial y}, \frac{\partial f(x,y,z)}{\partial z}\right) \cdot (\vec{r}_0 - \vec{r}) \,. \end{aligned}$$

- *Definition*: Consider a scalar field $f(x,y,z)$ and its partial derivatives

$$\frac{\partial f(x,y,z)}{\partial x}, \quad \frac{\partial f(x,y,z)}{\partial y}, \quad \frac{\partial f(x,y,z)}{\partial z}.$$

  The *gradient* of $f(x,y,z)$ is defined as

$$\begin{aligned} \mathrm{grad} f(x,y,z) = \nabla f(x,y,z) &= \frac{\partial f(x,y,z)}{\partial x}\vec{i} + \frac{\partial f(x,y,z)}{\partial y}\vec{j} + \frac{\partial f(x,y,z)}{\partial z}\vec{k} \\ &= \left(\frac{\partial f(x,y,z)}{\partial x}, \frac{\partial f(x,y,z)}{\partial y}, \frac{\partial f(x,y,z)}{\partial z}\right) \end{aligned}$$

  where ∇ denotes the so–called Nabla operator

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

  Note that the gradient of the scalar field f is a vector field.

- *Example*: Let $\vec{r}_0 = (x_0, y_0, z_0)$ be a fixed position vector. We consider the scalar field
$$f(x, y, z) = f(\vec{r}) = \vec{r} \cdot \vec{r}_0 = xx_0 + yy_0 + zz_0.$$
The partial derivatives are
$$\frac{\partial f(\vec{r})}{\partial x} = x_0, \quad \frac{\partial f(\vec{r})}{\partial y} = y_0, \quad \frac{\partial f(\vec{r})}{\partial z} = z_0$$
and the gradient is
$$\nabla f(\vec{r}) = x_0\vec{i} + y_0\vec{j} + z_0\vec{k} = (x_0, y_0, z_0) = \vec{r}_0.$$

- *Example*: Consider the gravitational force
$$\vec{p}(x, y, z) = \vec{p}(\vec{r}) = -c\frac{\vec{r}}{r^3}, \quad r = |\vec{r}|$$
where c is a constant and let
$$f(x, y, z) = f(\vec{r}) = \frac{c}{r} = \frac{c}{\sqrt{x^2 + y^2 + z^2}}.$$
Then
$$\frac{\partial f(x, y, z)}{\partial x} = c\left(-\frac{1}{2}\right)(x^2 + y^2 + z^2)^{-3/2}2x = -c\frac{x}{r^3}$$
and
$$\frac{\partial f(x, y, z)}{\partial y} = -c\frac{y}{r^3}, \quad \frac{\partial f(x, y, z)}{\partial z} = -c\frac{z}{r^3}.$$
Therefore
$$\nabla f(x, y, z) = \left(-c\frac{x}{r^3}, -c\frac{y}{r^3}, -c\frac{z}{r^3}\right) = -c\frac{\vec{r}}{r^3} = \vec{p}(\vec{r})$$
i.e. the gravitational force $\vec{p}$ is the gradient of the scalar field f.

- *Directional derivative*: We are interested in the rate of change of a scalar field $f(x, y, z)$ in direction $\vec{b} = (b_1, b_2, b_3)$ at the point $P(p_1, p_2, p_3)$. The rate of change at the point P is the relative difference between $f(Q)$ and $f(P)$ where $Q(q_1, q_2, q_3)$ is a point in direction $\vec{b}$ that gets closer and closer to P (see Figure 2). Let $\vec{q} = (q_1, q_2, q_3)$ be the position vector of Q. We have that
$$\vec{q} = (p_1, p_2, p_3) + s\frac{\vec{b}}{|\vec{b}|} = \left(p_1 + s\frac{b_1}{|\vec{b}|}, p_2 + s\frac{b_2}{|\vec{b}|}, p_3 + s\frac{b_3}{|\vec{b}|}\right).$$
Using the Taylor expansion, we get for $s \to 0$
$$f(Q) - f(P) = \nabla f(P) \cdot s\left(\frac{b_1}{|\vec{b}|}, \frac{b_2}{|\vec{b}|}, \frac{b_3}{|\vec{b}|}\right) = \nabla f(P) \cdot s\frac{\vec{b}}{|\vec{b}|}.$$
Thus
$$\text{rate of change} = \lim_{s\to 0}\frac{f(Q) - f(P)}{s} = \frac{\vec{b} \cdot \nabla f(P)}{|\vec{b}|} = D_{\vec{b}}f(P)$$
where $D_{\vec{b}}f(P)$ is also called the *directional derivative*.

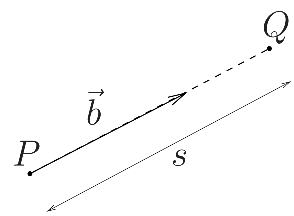

Figure 2: The point Q in direction $\vec{b}$ is approaching P when the distance s is approaching zero.

- *Example*: Consider the scalar field
$$f(x, y, z) = 2x^2 + 3y^2 + z^2$$
and let $\vec{b} = (1, 0, -2)$ and $P(2, 1, 3)$. We then get
$$\nabla f(x, y, z) = (4x, 6y, 2z), \quad \nabla f(P) = (8, 6, 6)$$
and
$$D_{\vec{b}} f(P) = \frac{(1, 0, -2) \cdot (8, 6, 6)}{\sqrt{5}} = -\frac{4}{\sqrt{5}}.$$
The $-$ sign tells us that f is decreasing at P in direction $\vec{b}$.

- *Direction of maximum increase*: Consider the directional derivative of $f(\vec{r})$ in direction $\vec{b}$ at a given point P. We are interested in finding the direction of maximum increase of f. For that, let γ denote the angle between $\nabla f(P)$ and $\vec{b}$. Then
$$D_{\vec{b}} f(P) = \frac{\vec{b} \cdot \nabla f(P)}{|\vec{b}|} = \frac{|\vec{b}||\nabla f(P)| \cos\gamma}{|\vec{b}|} = |\nabla f(P)| \cos\gamma$$
which has a maximum value if $\gamma = 0$, i.e. $\vec{b}$ points in direction of $\nabla f(P)$. The gradient $\nabla f(P)$ always has the *direction of the maximum increase of f at P*.

- *Surfaces*: A surface S consisting of points $P(x, y, z)$ can, in many cases, be described by an equation of the form
$$f(x, y, z) = c$$
where c is a constant and f is a scalar field. Each point $P(x, y, z)$ whose coordinates fulfil this equation belongs to the surface S.

- *Example*: A sphere with radius R is described by
$$f(x, y, z) = x^2 + y^2 + z^2 = R^2.$$

- *Tangent plane and normal vector*: Consider a surface S given by $f(x, y, z) = c$ and let $P(x, y, z)$ be a point on S such that the gradient $\nabla f(P)$ exists. Then there is a unique plane T that touches the surface S at P and $\nabla f(P)$ is orthogonal to T (see Figure 3). T is called the *tangent plane* of S at P and $\nabla f(P)$ is called the *normal vector* of S at P.

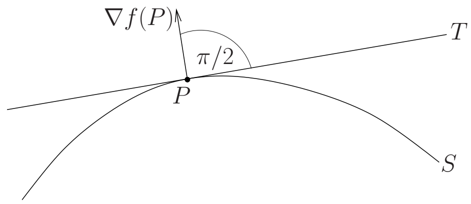

Figure 3: The geometry of a surface S, its tangent plane T and the normal vector $\nabla f(P)$.

- *Example*: Let S be a sphere of radius R given by

$$f(x, y, z) = x^2 + y^2 + z^2 = R^2.$$

The normal vector at $P(x, y, z)$ is always along the position vector $\vec{r} = (x, y, z)$ of P (see Figure 4)

$$\nabla f(x, y, z) = (2x, 2y, 2z) = 2\vec{r}.$$

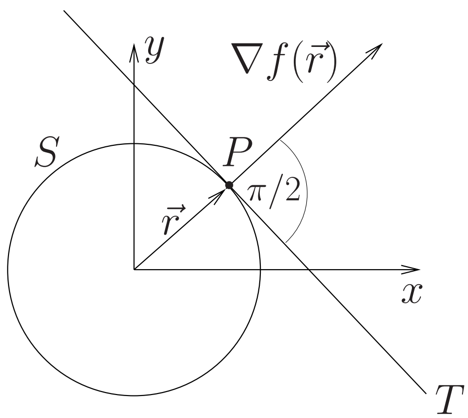

Figure 4: The normal vector of a sphere.

- *Example*: Let S be a cone along the z–axis given by

$$f(x, y, z) = 4(x^2 + y^2) - z^2 = 0, \quad z \geq 0.$$

The normal vector at $P(x, y, z)$ is

$$\nabla f(x, y, z) = (8x, 8y, -2z).$$

We define the *unit surface normal vector* by

$$\vec{n} = \frac{\nabla f(x, y, z)}{|\nabla f(x, y, z)|} = \frac{(8x, 8y, -2z)}{\sqrt{64(x^2 + y^2) + 4z^2}} = \frac{(8x, 8y, -2z)}{\sqrt{20z^2}} = \frac{(4x, 4y, -z)}{\sqrt{5}z}.$$

Figure 5 illustrates the cone and its unit surface normal vector.

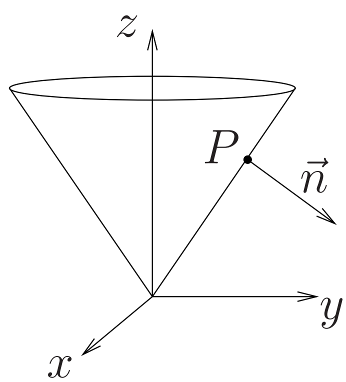

Figure 5: The unit surface normal vector $\vec{n}$ of a cone.

- *Example*: Let S be a cylinder with radius R and height H along the z–axis given by

$$f(x,y,z) = x^2 + y^2 = R^2, \quad 0 \leq z \leq H.$$

The normal vector at $P(x,y,z)$ is

$$\nabla f(x,y,z) = (2x, 2y, 0)$$

and the unit surface normal vector is given as

$$\vec{n} = \frac{\nabla f(x,y,z)}{|\nabla f(x,y,z)|} = \frac{(x,y,0)}{\sqrt{x^2+y^2}} = \frac{(x,y,0)}{R}.$$

Figure 6 illustrates the cylinder and its unit surface normal vector.

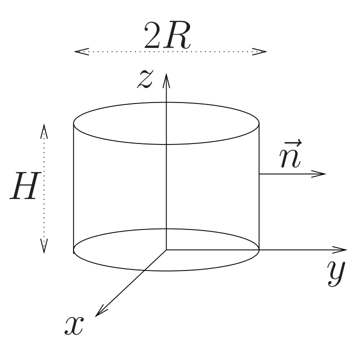

Figure 6: The unit surface normal vector of a cylinder.

1.6 The divergence of a vector field

- *Definition of the divergence*: Let $\vec{v}(x,y,z) = (v_1(x,y,z), v_2(x,y,z), v_3(x,y,z))$ be a differentiable vector field, i.e. all partial derivatives of the components of $\vec{v}$ exist. Then we define the *divergence* of $\vec{v}$ as

$$\text{div}\vec{v}(x,y,z) = \frac{\partial v_1(x,y,z)}{\partial x} + \frac{\partial v_2(x,y,z)}{\partial y} + \frac{\partial v_3(x,y,z)}{\partial z}$$

which is a scalar field. We can also rewrite the divergence in terms of the Nabla operator as

$$\begin{aligned}\mathrm{div}\vec{v}(x,y,z) &= \nabla\cdot\vec{v}(x,y,z) = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot(v_1(x,y,z),v_2(x,y,z),v_3(x,y,z))\\ &= \frac{\partial v_1(x,y,z)}{\partial x}+\frac{\partial v_2(x,y,z)}{\partial y}+\frac{\partial v_3(x,y,z)}{\partial z}.\end{aligned}$$

- *Example*: Let $\vec{v}(x,y,z)=(x,y,z)$. Then

  $$\mathrm{div}\vec{v}(x,y,z)=1+1+1=3.$$

- *Example*: Let $\vec{\omega}=(\omega_1,\omega_2,\omega_3)$ be a fixed vector and consider

  $$\vec{v}(x,y,z)=\vec{v}(\vec{r})=\vec{\omega}\times\vec{r}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \omega_1 & \omega_2 & \omega_3\\ x & y & z\end{vmatrix}=(\omega_2 z-\omega_3 y,\omega_3 x-\omega_1 z,\omega_1 y-\omega_2 x).$$

  Then

  $$\mathrm{div}\vec{v}(\vec{r})=0+0+0=0.$$

- *Example*: Let $f(x,y,z)$ be a scalar field and consider the vector field

  $$\vec{v}(x,y,z)=\nabla f(x,y,z)=\left(\frac{\partial f(x,y,z)}{\partial x},\frac{\partial f(x,y,z)}{\partial y},\frac{\partial f(x,y,z)}{\partial z}\right).$$

  Then

  $$\mathrm{div}\vec{v}(x,y,z)=\nabla\cdot\nabla f(x,y,z)=\frac{\partial^2 f(x,y,z)}{\partial x^2}+\frac{\partial^2 f(x,y,z)}{\partial y^2}+\frac{\partial^2 f(x,y,z)}{\partial z^2}=\nabla^2 f(x,y,z)$$

  where

  $$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

  is the so–called *Laplace operator*.

- *Flow of a fluid*: Consider a fluid in a region R with density $\rho(x,y,z,t)$ and velocity $\vec{v}(x,y,z,t)$ and let $\vec{u}(x,y,z,t)=\rho(x,y,z,t)\vec{v}(x,y,z,t)$. Here x,y,z denote the spatial coordinates and t denotes time. We assume that there are no sources (points where fluid is produced) and no sinks (points where fluid dissapears). In this case we have the so–called continuity equation

  $$\frac{\partial\rho(x,y,z,t)}{\partial t}+\mathrm{div}\vec{u}(x,y,z,t)=0.$$

  To illustrate the physics behind the divergence consider the small box around the point (x,y,z) in Figure 7. If there is more inflow than outflow, then we get

  $$\frac{\partial\rho(x,y,z,t)}{\partial t}>0$$

and

$$\mathrm{div}\vec{u}(x,y,z,t) < 0$$

from the continuity equation.

On the other hand, if there is more outflow than inflow, then we get

$$\frac{\partial \rho(x,y,z,t)}{\partial t} < 0$$

and

$$\mathrm{div}\vec{u}(x,y,z,t) > 0$$

from the continuity equation.

In summary, the divergence of $\vec{u}(x,y,z,t)$ measures outflow minus inflow.

In case the flow is incompressible, i.e. $\rho(x,y,z,t) = \rho_0 =$ constant, then

$$\mathrm{div}(\rho_0\vec{v}(x,y,z,t)) = \rho_0\mathrm{div}\vec{v}(x,y,z,t) = 0$$

and we get the condition of incompressibility

$$\mathrm{div}\vec{v}(x,y,z,t) = 0.$$

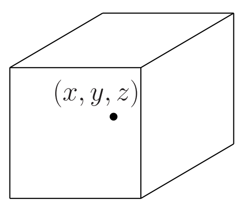

Figure 7: A small box around the point (x,y,z).

Exercises

1. Consider the curve C given by the parametric representation

$$\vec{r}(t) = (\cos(t^2), 0, \sin(t^2)), \quad 0 \leq t \leq \sqrt{2\pi}.$$

   Calculate the length of C.

2. Consider the surface S consisting of all points $P(x, y, z)$ such that

$$y + z = c.$$

   (a) Sketch the surface S for $c = 0$. Calculate the unit surface normal vector. Sketch some of the unit surface normal vectors.

   (b) Sketch the surface S for $c = 1$. Calculate the unit surface normal vector. Sketch some of the unit surface normal vectors.

3. Consider the surface S consisting of all points $P(x, y, z)$ such that

$$x^2 + y^2 = 1, \quad x > 0, \quad y > 0, \quad 0 \leq z \leq 1.$$

   Sketch the surface S. Calculate the unit surface normal vector. Sketch some of the unit surface normal vectors.

4. Calculate the directional derivative of

   (a) $f(x, y, z) = xyz$ at $P(1, 1, 1)$ in direction $\vec{b} = (1, 1, 1)$

   (b) $f(x, y, z) = \cos(xyz)$ at $P(1, \pi/2, 1)$ in direction $\vec{b} = (1, 1, 1)$

5. Calculate the divergence of

   (a)

$$\vec{v}(x, y, z) = \left(x^y, yz, (xy + zx)^3\right)$$

   (b)

$$\vec{v}(x, y, z) = \left(\frac{x + y}{x}, \frac{y}{x + y}, \frac{x + z}{y + z}\right).$$