Quantitative Finance
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/rquf20

# Empirical distributions of stock returns: between the stretched exponential and the power law?

Y. Malevergne [a b], V. Pisarenko [c] & D. Sornette [d e]

[a] ISFA Graduate School of Actuarial Science, University of Lyon 1, 50 avenue Tony Garnier, 69366 Lyon Cedex 07, France

[b] EM-Lyon Graduate School of Management, 23 avenue Guy de Collongue, 69134 Ecully Cedex, France

[c] International Institute of Earthquake Prediction Theory and Mathematical Geophysics, Russian Academy of Sciences, Warshavskoye sh. 79, kor. 2, 79, kor. 2, Moscow 113556, Russia

[d] Laboratoire de Physique de la Matière Condensée, CNRS UMR 6622 and Université des Sciences, Parc Valrose, 06108 Nice Cedex 2, France

[e] Institute of Geophysics and Planetary Physics and Department of Earth and Space Science, University of California, Los Angeles, CA 90095, USA

[f] ISFA Graduate School of Actuarial Science, University of Lyon 1, 50 avenue Tony Garnier, 69366 Lyon Cedex 07, France E-mail:
Published online: 18 Feb 2007.

To cite this article: Y. Malevergne, V. Pisarenko & D. Sornette (2005) Empirical distributions of stock returns: between the stretched exponential and the power law?, Quantitative Finance, 5:4, 379-401, DOI: 10.1080/14697680500151343

To link to this article: http://dx.doi.org/10.1080/14697680500151343

*Quantitative Finance*, Vol. 5, No. 4, August 2005, 379–401



# Empirical distributions of stock returns: between the stretched exponential and the power law?

Y. MALEVERGNE*†‡, V. PISARENKO§ and D. SORNETTE¶||

†ISFA Graduate School of Actuarial Science, University of Lyon 1, 50 avenue Tony Garnier, 69366 Lyon Cedex 07, France
‡EM-Lyon Graduate School of Management, 23 avenue Guy de Collongue, 69134 Ecully Cedex, France
§International Institute of Earthquake Prediction Theory and Mathematical Geophysics, Russian Academy of Sciences, Warshavskoye sh. 79, kor. 2, 79, kor. 2, Moscow 113556, Russia
¶Laboratoire de Physique de la Matière Condensée, CNRS UMR 6622 and Université des Sciences, Parc Valrose, 06108 Nice Cedex 2, France
||Institute of Geophysics and Planetary Physics and Department of Earth and Space Science, University of California, Los Angeles, CA 90095, USA

(*Received 16 April 2004; in final form 29 March 2005*)

A large consensus now seems to take for granted that the distributions of empirical returns of financial time series are regularly varying, with a tail exponent $b$ close to 3. We develop a battery of new non-parametric and parametric tests to characterize the distributions of empirical returns of moderately large financial time series, with application to 100 years of daily returns of the Dow Jones Industrial Average, to 1 year of 5-min returns of the Nasdaq Composite index and to 17 years of 1-min returns of the Standard & Poor's 500. We propose a parametric representation of the tail of the distributions of returns encompassing both a regularly varying distribution in one limit of the parameters and rapidly varying distributions of the class of the stretched-exponential (SE) and the log-Weibull or Stretched Log-Exponential (SLE) distributions in other limits. Using the method of nested hypothesis testing (Wilks' theorem), we conclude that both the SE distributions and Pareto distributions provide reliable descriptions of the data but are hardly distinguishable for sufficiently high thresholds. Based on the discovery that the SE distribution tends to the Pareto distribution in a certain limit, we demonstrate that Wilks' test of nested hypothesis still works for the non-exactly nested comparison between the SE and Pareto distributions. The SE distribution is found to be significantly better over the whole quantile range but becomes unnecessary beyond the 95% quantiles compared with the Pareto law. Similar conclusions hold for the log-Weibull model with respect to the Pareto distribution, with a noticeable exception concerning the very-high-frequency data. Summing up all the evidence provided by our tests, it seems that the tails ultimately decay slower than any SE but probably faster than power laws with reasonable exponents. Thus, from a practical viewpoint, the log-Weibull model, which provides a smooth interpolation between SE and PD, can be considered as an appropriate approximation of the sample distributions.

*Keywords*: Non-nested hypothesis testing; Pareto distribution; Weibull distribution

## 1. Motivation of the study and main results

From a practical as well as from an academic point of view, the shape of the tail of the distribution of returns has important implications. From a fundamental point of view, economic and financial theories often rely on a specific parameterization of the distributions whose parameters are intended to represent the 'macroscopic' variables the agents are sensitive to. For practitioners, and more specifically for practical market risk management purposes, one typically needs to assess tail risks associated with the distribution of returns or profit and losses. Following the recommendations of the Bank for International Settlements, one has to focus on risks associated with ten days' holding positions. Therefore, this

*Corresponding author. Email: malevergne@univ-lyon1.fr

*Quantitative Finance*
ISSN 1469–7688 print/ISSN 1469–7696 online © 2005 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/14697680500151343

requires the consideration of the distributions of ten day returns. However, on such a large time scale, the number of (non-overlapping) observations dramatically decreases. Even over a century, one can only collect 2500 data points or so. Therefore, the assessment of risks associated with high quantiles is particularly unreliable.

Recently, the use of high-frequency data has allowed for an accurate estimation of the very far tails of the distributions of returns. Indeed, using samples of one to ten million points allows one to efficiently calibrate probability distributions up to probability levels of order 99.9995%. Then, one can hope to reconstruct the distribution of returns on a larger time scale by convolution. It is the stance taken by many researchers advocating the use of Lévy processes to model the dynamics of asset returns. The recent study by Eberlein and Özkan (2003) shows the relevance of this approach, at least for fluctuations of moderate size. However, for large fluctuations, this approach does not work, as shown in figure 1 where we compare the pdf of raw 60-min returns of the S&P 500 index with the hypothetical pdf obtained by 60 convolutions of the pdf of returns at the 1-min time scale. Since the return at the time scale of 60 min is the sum of 60 returns at the 1-min time scale, if the returns were iid, we would indeed expect to predict the pdf of the 60-min returns by performing 60 convolutions of the pdf of returns at the 1-min time scale. It is clear that the former exhibits significantly fatter tails than the latter, illustrating the existence of dependence and of its impact on the tails of pdfs.

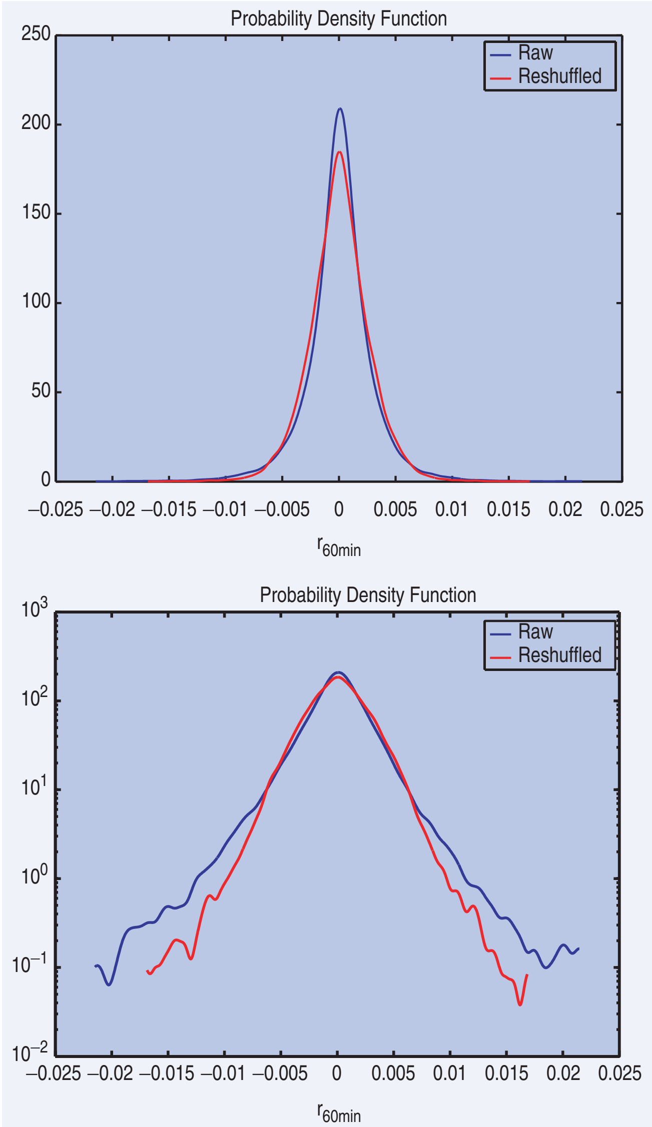


Figure 1. Kernel density estimates of the raw 60-min returns (blue line) and the density (red line) obtained by 60 times convolution of the raw 1-min returns kernel density.

This phenomenon derives naturally from the fact that asset returns cannot be described by independent random variables. Volatility clustering, also called the ARCH effect (Engle 1982), is a clear manifestation of the existence of nonlinear dependencies between returns observed at different lags. These dependencies prevent the use of convolution for estimating tail risks with sufficient accuracy. Figure 1 illustrates the well-known stylized fact according to which fat tails of asset return distributions owe their origin, at least in part, to the existence of volatility correlations. When reshuffling the data to remove these correlations, one observes a faster decay of the pdf in the tails. Thus, assessing extreme risks at large time scales (1 or 10 days) by simple convolution of the distribution of returns at time scales of 1 or 5 min leads to crude approximations and to dramatic under-estimations of the amount of risk actually incurred.

The only way to reliably aggregate high-frequency data is to have a consistent model at one's disposal. By *consistent* model we mean a model that accounts for the complex time structure of asset returns. (FI)-GARCH, $\alpha$-ARCH or MRW (multi-fractal random walks) processes can be used for this purpose, but none of them is yet universally recognized since they do not rely on well-established founding economic principles. As a consequence, one is exposed to model error: for instance, a simple GARCH model (Bollerslev 1986, Bollerslev *et al.* 1994) still underestimates the tail risks since it underestimates the long range dependence of the volatility.

In the absence of such a reliable dynamic model, the most relevant approach is perhaps to let the data speak for themselves and first to focus on the shape of the stationary distribution of returns. This was the stance taken, in the early 1960s, by Mandelbrot (1963) and Fama (1965), who presented evidence that distributions of returns can be well approximated by a symmetric Lévy stable law with tail index $b$ about 1.7. These estimates of the power tail index have recently been confirmed by Mittnik *et al.* (1988), and slightly different indices of the stable law ($b = 1.4$) were suggested by Mantegna and Stanley (1995, 2000). On the other hand, there is much evidence of a larger value of the tail index $b \cong 3$ (De Vries 1994, Longin 1996, Guillaume *et al.* 1997, Gopikrishnan *et al.* 1998, 1999, Müller *et al.* 1998, Farmer 1999, Plerou *et al.* 1999, Lux 2000). See also the various alternative parameterizations in terms of the Student distribution (Blattberg and Gonnedes 1974, Kon 1984), or Pearson type-VII distributions (Nagahara and Kitagawa 1999), which all have an asymptotic power law tail and are regularly varying. Thus, a general conclusion of this group of authors concerning tail fatness can be formulated as follows: the tails of the distribution of returns are heavier than a Gaussian tail and heavier than an exponential tail; they certainly admit the existence of a finite variance ($b > 2$), whereas the existence of the third (skewness) and the fourth (kurtosis) moments is questionable. The two values $b = 1.4$–$1.7$ and $b \cong 3$ can be reconciled by the fact that they do not apply to the same quantiles of the distributions of returns (Mantegna and Stanley 1995, Plerou *et al.* 1999, Bouchaud and Potters 2000). More recently, other works have advocated the so-called inverse-cubic law ($b = 3$) based on the analysis of distributions of returns of high-frequency data aggregated over hundreds to thousands of stocks. But this aggregating procedure could lead to novel problems of interpretation.

However, the class of regularly varying distributions is not the only one able to account for the large kurtosis and fat-tailness of the distributions of returns. For instance, Gouriéroux and Jasiak (1998) claims that the distribution of returns on the French stock market decays faster than any power law. Cont *et al.* (1997) have proposed the use of exponentially truncated stable distributions, Barndorff-Nielsen (1997), Eberlein *et al.* (1998) and Prause (1998) have respectively considered normal inverse Gaussian and (generalized) hyperbolic distributions, which asymptotically decay as $x^{\alpha} \exp(-\beta x)$, while Laherrère and Sornette (1999) suggest the fitting of the distributions of stock returns by a stretched-exponential law. These results, challenging the traditional hypothesis of a power-like tail, offer a new representation of the returns distributions and need to be confronted.

Due to the complicated dependencies of financial time series, such as GARCH effects leading to heteroscedasticity and to clustering of high threshold exceedances resulting from the long memory of the volatility, it has been shown (Malevergne *et al.* 2003) that the standard generalized extreme value (GEV) and the Generalized Pareto distribution (GPD) estimators can be quite inefficient and thus cannot distinguish reliably between some rapidly—namely stretched-exponential distributions—and regularly varying classes of distributions for data similar to those studied here. This is why we propose to examine once more the delicate problem of the tail behaviour of distributions of returns in order to shed new light on this problem†. To this end, we investigate three time series: the daily returns of the Dow Jones Industrial Average (DJ) Index over a century (kindly provided by Professor H.-C.G. Bothmer), the 5-min returns of the Nasdaq Composite index (ND) over 1 year from April 1997 to May 1998 obtained from Bloomberg, and the 1-min returns of the Standard & Poor's 500 over the time interval from April 21, 1982 to February 26, 1999 obtained from Tick Data (www.tickdata.com/). These three sets of data have been chosen since they are typical of the data sets used in most previous studies.

We use a parametric representation of the tail of the distributions of returns of our three time series, encompassing both a regularly varying distribution in one limit of the parameters and rapidly varying distributions of the class of the stretched-exponential (SE) and the log-Weibull or Stretched Log-Exponential (SLE) distributions in other limits. Using the method of nested hypothesis testing, we conclude that none of the standard parametric family distributions that we consider satisfactorily fits the Dow Jones, Nasdaq and S&P 500 data over the whole range of either positive or negative returns. While this is also true for the family of stretched-exponential and the log-Weibull distributions, these families appear to be the best among the five considered parametric families, in so far as they are able to fit the data over the largest interval. For the high quantiles (far in the tails), both the SE distributions and Pareto distributions provide reliable descriptions of the data and are hardly distinguishable for sufficiently high thresholds.

Based on a test developed for non-nested hypotheses, the SE distribution is found to be significantly better than the Pareto distribution over the whole quantile range, but becomes *unnecessary* beyond the 95% quantiles compared with the Pareto law. The log-Weibull (SLE) model, which provides a smooth interpolation between the SE and PD models, is found to be at least as good as the stretched-exponential model, on a large range of data, but, again, the Pareto distribution is ultimately the most parsimonious beyond the 95% quantiles. For the shortest time scale investigated here (1 min), the tail of the distribution of returns is, over a large range, well described by a SLE distribution with an exponent $c$ less than 1, i.e. is fatter than any power law. However, a change of behaviour is ultimately observed and the very extreme tail decays faster than any power law, for which both a SE and a SLE with exponent $c > 1$ provides a reasonable description. Collectively, these results suggest that the extreme tails of the true distribution of returns of our three data sets are fatter than any stretched-exponential, strictly speaking, i.e. with a strictly positive fractional exponent, but should be thinner than any power law.

This paper is organized as follows. Section 2 briefly presents our three data sets and the precautions taken to address their most flagrant non-stationarity properties. Section 3 proposes two general parametric representations of the distribution of returns encompassing both a regularly varying distribution in one limit of the parameters and rapidly varying distributions of the class of stretched exponential and log-Weibull distributions in another limit. Calibration of the various models is then discussed. Next, based upon tests of nested and non-nested hypotheses, the descriptive power of these different models is compared in section 4. Section 5 summarizes our results and concludes.

†Picoli *et al.* (2003) have also presented fits comparing the relative merits of SE and so-called $q$-exponentials (which are similar to a Student distribution with power law tails) for the description of the frequency distributions of basketball baskets, cyclone victims, brand-name drugs by retail sales, and highway length.

## 2. Our data and its basic statistical features

Our first data sample consists of the daily returns (throughout the paper we will use compound returns, i.e. log-returns) of the Dow Jones Industrial Average Index (DJ) over the time interval from May 27, 1896 to May 31, 2000, which represents a sample size of $n = 28\,415$. The second data set contains the high-frequency (5 min) returns of the Nasdaq Composite (ND) index for the period from April 8, 1997 to May 29, 1998, which represents $n = 22\,123$ data points. The third data set contains the high-frequency (1 min) returns of the Standard & Poor's 500 (SP 500) index for the period from April 21, 1982 to February 26, 1999, which represents $n = 1\,548\,972$ non-zero data points. The choice of these data sets is justified by their similarity to (1) the data set of daily returns used by Longin (1996), particularly, (2) the high-frequency data used by Guillaume *et al.* (1997), Lux (2000) and Müller *et al.* (1998), and (3) the large sample of high-frequency data used by Gopikrishnan *et al.* (1998, 1999), Matia *et al.* (2002) and Mizuno *et al.* (2002), among others.

For the intra-day Nasdaq and S&P 500 data, we address two problems. First, we have removed from our sample the returns calculated from the close of each day to the opening of the next day. This corresponds to focusing only on the intra-day returns. We also take into account the systematic change of the intra-day volatility with the time of day. Indeed, the volatility of intra-day data is known to exhibit a U-shape, also called the 'lunch-effect'. In order to test whether this systematic effect has an influence on our results, we construct a 'corrected' Nasdaq and S&P 500 return time series by re-normalizing the returns at a given moment of the trading day by the corresponding average absolute return at the same moment. We examine both the raw and corrected Nasdaq and S&P 500 returns for comparison. Below, we only report the results for the raw returns since they are not qualitatively different from the results obtained from the corrected returns.

The Dow Jones daily returns also exhibit some non-stationarity. Indeed, one can observe a clear excess volatility roughly covering the time of the bubble ending in the October 1929 crash following the Great Depression. To investigate the influence of such non-stationarity, the statistical study presented below has been performed twice: first with the entire sample, and after having removed the period from 1927 to 1936 from the sample. The results are somewhat different, but, on the whole, the conclusions concerning the nature of the tail are the same. Thus, only the results concerning the whole sample will be detailed in the paper.

Although the distributions of positive and negative returns are known to be very similar (Jondeau and Rockinger 2001, for instance), we have chosen to treat them separately. For the Dow Jones, this gives us 14 949 positive and 13 464 negative data points, while, for the Nasdaq, we have 11 241 positive and 10 751 negative data points and for S&P 500 it yields 783 164 positive and 765 808 negative data points, at the 1 min time scale.

Table 1 summarizes the main statistical properties of these time series in terms of the average returns, their standard deviations, the skewness and the excess kurtosis for several time scales from 1 min to 1 month.

## 3. Fitting distributions of returns with parametric densities

As discussed in the introductory section, Power-like distributions and stretched-exponential distributions seems to be two natural models for the empirical distributions of returns. In order to decide which of these models is the best, we propose to pit a parametric champion for this first class of distribution functions against the Pareto champion of regularly varying functions. Recall that stretched-exponential distributions are parsimonious examples of sub-exponential distributions endowed with fat tails in the sense of the asymptotic probability weight of the maximum compared with the sum of large samples (Feller 1971), for instance. Notwithstanding their fat-tailness, stretched-exponential distributions have all their moments finite (however, they do not admit an exponential moment, which leads to problems in the reconstruction of the distribution from the knowledge of their moments (Stuart and Ord 1994)), in contrast to

Table 1. Descriptive statistics for the Dow Jones returns calculated over 1 day and 1 month, for the Nasdaq returns calculated over 5 min and 1 h, and for the S&P500 returns calculated over 1 min, 5 min, 30 min and 1 h. The numbers within parentheses represent the $p$ value of Jarque–Bera's normality test.

| | Mean | SD | Skewness | Ex. kurtosis | Jarque–Bera |
|---|---|---|---|---|---|
| Dow Jones (1 day) | $8.96 \times 10^{-5}$ | $4.70 \times 10^{-3}$ | −0.6101 | 22.5443 | $6.03 \times 10^{5}$ (0.00) |
| Dow Jones (1 month) | $1.80 \times 10^{-3}$ | $2.54 \times 10^{-2}$ | −0.6998 | 5.3619 | $1.28 \times 10^{3}$ (0.00) |
| Nasdaq (5 min)[a] | $1.80 \times 10^{-6}$ | $6.61 \times 10^{-4}$ | 0.0326 | 11.8535 | $1.30 \times 10^{5}$ (0.00) |
| Nasdaq (1 h)[a] | $2.40 \times 10^{-5}$ | $3.30 \times 10^{-3}$ | 1.3396 | 23.7946 | $4.40 \times 10^{4}$ (0.00) |
| Nasdaq (5 min)[b] | $-6.33 \times 10^{-9}$ | $3.85 \times 10^{-4}$ | −0.0562 | 6.9641 | $4.50 \times 10^{4}$ (0.00) |
| Nasdaq (1 h)[b] | $1.05 \times 10^{-6}$ | $1.90 \times 10^{-3}$ | −0.0374 | 4.5250 | $1.58 \times 10^{3}$ (0.00) |
| S&P 500 (1 min) | $1.11 \times 10^{-6}$ | $4.29 \times 10^{-4}$ | 5.58 | $1.36 \times 10^{3}$ | $1.19 \times 10^{11}$ (0.00) |
| S&P 500 (5 min) | $4.86 \times 10^{-6}$ | $1.06 \times 10^{-3}$ | 6.46 | $1.37 \times 10^{3}$ | $1.58 \times 10^{10}$ (0.00) |
| S&P 500 (30 min) | $2.64 \times 10^{-5}$ | $2.57 \times 10^{-3}$ | 3.88 | $4.47 \times 10^{2}$ | $5.38 \times 10^{8}$ (0.00) |
| S&P 500 (1 h) | $4.93 \times 10^{-5}$ | $3.58 \times 10^{-3}$ | 0.73 | $3.02 \times 10^{2}$ | $1.30 \times 10^{8}$ (0.00) |

[a]The raw data.
[b]The data corrected for the U-shape of the intra-day volatility due to the opening, lunch and closing effects.

regularly varying distributions for which moments of order equal to or larger than the tail index $b$ are not defined. This property may provide a substantial advantage to exploit in generalizations of the mean-variance portfolio theory using higher-order moments (Rubinstein 1973, Fang and Lai 1997, Adcock *et al.* 2005, for instance). Moreover, the existence of all moments is an important property allowing for an efficient estimation of any high-order moment, since it ensures that the estimators are asymptotically Gaussian. In particular, for stretched-exponentially distributed random variables, the variance, skewness and kurtosis can be well estimated, in contrast to random variables with a regularly varying distribution with tail index $b$ in the range 3–4.

### *3.1. Definition of two parametric families*

**3.1.1. A general three-parameter family of distributions.** We thus consider a general three-parameter family of distributions and its particular restrictions corresponding to some fixed value(s) of two (one) parameters. This family is defined by its density function, given by

$$f_u(x|b,c,d) = \begin{cases} A(b,c,d,u)x^{-(b+1)}\exp[-(x/d)^c], & \text{if } x \geqslant u > 0, \\ 0, & \text{if } x < u. \end{cases} \tag{1}$$

Here, $b$, $c$ and $d$ are unknown parameters, $u$ is a known lower threshold that will be varied for the purposes of our analysis, and $A(b,c,d,u)$ is a normalizing constant given by the expression

$$A(b,c,d,u) = \frac{d^b c}{\Gamma(-b/c,(u/d)^c)}, \tag{2}$$

where $\Gamma(a,x)$ denotes the (non-normalized) incomplete Gamma function. The parameter $b$ ranges from minus infinity to infinity, while $c$ and $d$ range from zero to infinity. In the particular case where $c = 0$, the parameter $b$ also needs to be positive to ensure the normalization of the probability density function (pdf). The interval of definition of this family is the positive semi-axis. Negative log-returns will be studied by taking their absolute values. The family (1) includes several well-known pdfs often used in different applications. We will enumerate them.

1. The Pareto distribution:

$$F_u(x) = 1-(u/x)^b, \tag{3}$$

   which corresponds to the set of parameters $(b>0, c=0)$ with $A(b,c,d,u) = b\cdot u^b$. Several works have attempted to derive or justify the existence of a power tail of the distribution of returns from agent-based models (Challet and Marsili 2002), from optimal trading of large funds with sizes distributed according to the Zipf law (Gabaix *et al.* 2002) or from stochastic processes (Sobehart and Farengo 2002, Biham *et al.* 1998, 2002).

2. The Weibull distribution:

$$F_u(x) = 1-\exp\Big[-\Big(\frac{x}{d}\Big)^c + \Big(\frac{u}{d}\Big)^c\Big], \tag{4}$$

   with parameter set $(b=-c, c>0, d>0)$ and normalization constant

$$A(b,c,d,u) = \frac{c}{d^c}\exp\Big[\Big(\frac{u}{d}\Big)^c\Big].$$

   This distribution is said to be a 'stretched-exponential' distribution when the exponent $c$ is smaller than 1, namely when the distribution decays more slowly than an exponential distribution. Stationary distributions exhibiting this kind of tail arise, for instance, from the so-called $\alpha$-ARCH processes introduced by Diebolt and Guegan (1991).

3. The exponential distribution:

$$F_u(x) = 1-\exp\Big(-\frac{x}{d}+\frac{u}{d}\Big), \tag{5}$$

   with parameter set $(b=-1, c=1, d>0)$ and normalization constant

$$A(b,c,d,u) = \frac{1}{d}\exp\Big(-\frac{u}{d}\Big).$$

   For sufficiently high quantiles, the exponential behaviour can derive, for instance, from the hyperbolic model introduced by Eberlein *et al.* (1998) or from a simple model where stock price dynamics is governed by a geometrical (multiplicative) Brownian motion with stochastic variance. Dragulescu and Yakovenko (2002) have found good fits of the Dow Jones index for time lags from 1 to 250 trading days with a model with an asymptotic exponential tail of the distribution of log-returns.

4. The incomplete Gamma distribution:

$$F_u(x) = 1-\frac{\Gamma(-b,x/d)}{\Gamma(-b,u/d)}, \tag{6}$$

   with parameter set $(b, c=1, d>0)$ and normalization

$$A(b,c,d,u) = \frac{d^b}{\Gamma(-b,u/d)}.$$

   Such asymptotic tail behaviour can be observed, for instance, for the generalized hyperbolic models, whose description can be found in Prause (1998).

Thus, the Pareto distribution (PD) and exponential distribution (ED) are one-parameter families, whereas the stretched-exponential (SE) and the incomplete Gamma distributions (IG) are two-parameter families. The comprehensive distribution (CD) given by equation (1) contains three unknown parameters.

Very interesting for our present study is the behaviour of the (SE) model when $c \to 0$ and $u > 0$. In this limit, and provided that

$$c\cdot\Big(\frac{u}{d}\Big)^c \to \beta, \quad \text{as } c \to 0, \tag{7}$$

where $\beta$ is a positive constant, the (SE) model goes to the

Pareto model. Indeed, we can write

$$\begin{aligned}
&\frac{c}{d^c}\cdot x^{c-1}\cdot \exp\left(-\frac{x^c-u^c}{d^c}\right)\\
&= c\left(\frac{u}{d}\right)^c\cdot\frac{x^{c-1}}{u^c}\exp\left[-\left(\frac{u}{d}\right)^c\cdot\left(\left(\frac{x}{u}\right)^c-1\right)\right]\\
&\simeq \beta\cdot x^{-1}\exp\left[-c\left(\frac{u}{d}\right)^c\cdot\ln\frac{x}{u}\right],\quad \text{as } c\to 0\\
&\simeq \beta\cdot x^{-1}\exp\left[-\beta\cdot\ln\frac{x}{u}\right]\\
&\simeq \beta\frac{u^\beta}{x^{\beta+1}},
\end{aligned}\tag{8}$$

which is the pdf of the (PD) model with tail index $\beta$. The condition (7) comes naturally from the properties of the maximum-likelihood estimator of the scale parameter $d$ (see appendix 2A of Malevergne and Sornette (2005)). It implies that, as $c\to 0$, the characteristic scale $d$ of the (SE) model must also go to zero as $d\sim c^{1/c}$, i.e. extremely fast with $c\to 0$ to ensure the convergence of the (SE) model towards the (PD) model.

This shows that the Pareto model can be approximated with any desired accuracy on an arbitrary interval $(u>0, U)$ by the (SE) model with parameters $(c,d)$ satisfying equation (7) where the arrow is replaced by an equality. Although the value $c=0$ does not give, strictly speaking, a stretched-exponential distribution, the limit $c\to 0$ provides any desired approximation to the Pareto distribution, uniformly on any finite interval $(u, U)$. This deep relationship between the SE and PD models allows us to understand why it can be very difficult to decide, on a statistical basis, which of these models fits the data best.

**3.1.2. The log-Weibull family of distributions.** Let us also introduce the two-parameter log-Weibull family:

$$1-F(x)=\exp[-b(\ln(x/u))^c],\quad \text{for } x\ge u, \tag{9}$$

whose density is

$$f_u(x|b,c,d)=\begin{cases}\dfrac{b\cdot c}{x}\left(\ln\dfrac{x}{u}\right)^{c-1}\exp\left[-b\left(\ln\dfrac{x}{u}\right)^c\right], & \text{if } x\geqslant u>0,\\ 0, & \text{if } x<u.\end{cases}\tag{10}$$

This family of pdf interpolates smoothly between the stretched-exponential and Pareto classes. It recovers the Pareto family for $c=1$, in which case the parameter $b$ is the tail exponent. For $c$ larger than 1, the tail of the log-Weibull is thinner than any Pareto distribution but heavier than any stretched-exponential†. In particular, when $c$ equals two, the log-normal distribution is retrieved (above threshold $u$). For $c$ smaller than 1, the tails of the SLE are even heavier than any Pareto distribution. It is interesting to remark that, in this case, the SLE do not belong to the domain of attraction of a law of the maximum. Therefore, the standard results of extreme value theory do not apply to such distributions. So, if it would appear that the SLE with index $c$ less than one provides a reasonable description of the tail of the distributions of returns, it would mean that risk management methods based upon standard results of the EVT (see Longin (2000), for instance) are particularly unreliable.

### 3.2. Methodology

We start by fitting our data sets by the six distributions enumerated above (1), (3)–(6) and (10). Our first goal is to show that no single parametric representation among any of the cited pdfs fits the *whole range* of the data sets. Recall that we analyse positive and negative returns separately (the latter being converted to the positive semi-axis). We shall use in our analysis a *movable* lower threshold $u$, restricting by this threshold our sample to observations satisfying $x>u$.

In addition to estimating the parameters involved in each representation ((1), (3)–(6), (10)) by maximum likelihood for each particular threshold $u$ (the expression of the estimators and their asymptotic properties can be found in appendix 2A of Malevergne and Sornette (2005)), we need a characterization of the goodness-of-fit. For this, we have retained the Anderson–Darling distance (Anderson and Darling 1952), which can be estimated from a sample of $N$ realizations as follows:

$$\mathrm{ADS}=N\cdot\int\frac{[F_N(x)-F(x)]^2}{F(x)(1-F(x))}\,\mathrm{d}F(x)\tag{11}$$

$$=-N-2\sum_1^N\{w_k\log(F(y_k))+(1-w_k)\log(1-F(y_k))\},\tag{12}$$

where $w_k=2k/(2N+1)$, $k=1,\dots,N$, $y_1\leqslant\cdots\leqslant y_N$ is its ordered sample, while $F(x)$ and $F_N(x)$ denote, respectively, the theoretical and empirical distribution functions. If the sample is drawn from a population with distribution function $F(x)$, the Anderson–Darling statistics (ADS) has a standard AD distribution: *free of the theoretical df* $F(x)$. It should be noted that the ADS weights the squared difference in equation (11) by $1/F(x)(1-F(x))$, which is nothing but the inverse of the variance of the difference $F_N(x)-F(x)$ in square brackets. Thus the AD distance emphasizes the tails of the distribution more than, say, the Kolmogorov distance, which is determined by the *maximum absolute* deviation of $F_N(x)$ from $F(x)$ or the mean-squared error, which is mostly controlled by the middle range of the distribution. Since we have to insert the estimated parameters into the ADS, this statistic no longer obeys the standard AD distribution: the ADS decreases because the use of the fitting parameters ensures a better fit to the sample distribution. However, we can still use the standard quantiles of the AD distribution as *upper* boundaries of the ADS. If the

†A generalization of the SLE to the following three-parameter family also contains the SE family in some formal limit. Consider indeed $1-F(x)=\exp(-b(\ln(1+x/D))^c)$ for $x>0$, which has the same tail as expression (9). Taking $D\to+\infty$ together with $b=(D/d)^c$ with $d$ finite yields $1-F(x)=\exp(-(x/d))^c)$.

observed ADS is larger than the standard quantile with a high significance level $(1-\varepsilon)$, we can then conclude that the null hypothesis $F(x)$ is rejected with a significance level larger than $(1-\varepsilon)$. If we wish to estimate the real significance level of the ADS in the case where it does not exceed the standard quantile of a high significance level, we are forced to use some other method of estimation of the significance level of the ADS, such as the bootstrap method.

Let us stress that the chosen distance is also useful for estimating the parameters of the considered model. Thus, in the following, the estimates minimizing the Anderson–Darling distance will also be evaluated and referred to as AD estimates. Obviously, the maximum likelihood estimates (ML estimates) are asymptotically more efficient than AD estimates for independent data and under the condition that the null hypothesis (given by one of the four distributions (3)–(6), for instance) corresponds to the true data generating model. When this is not the case, the AD estimates provide a *better practical tool* for approximating sample distributions compared with the ML estimates. We have determined the AD estimates for several standard probability levels $q_1, q_2, \ldots$ given in table 2. The *sample quantiles* corresponding to these probability levels or thresholds $u_1, u_2, \ldots$ for our samples are also shown in table 2.

Despite the fact that the thresholds $u_k$ vary from sample to sample, they always correspond to the same fixed set of probability levels $q_k$ throughout the paper and allow us to compare the goodness-of-fit for samples of different sizes. In the sequel of this study, we will only consider sub-samples with at least 100 data points or so, in order to allow for a sufficiently accurate assessment of the quantile under consideration. Therefore, the study of the SP 500 1-min returns will be restricted to quantiles $q_1$ to $q_{21}$, the study of the SP 500 5-min returns will be restricted to quantiles $q_1$ to $q_{20}$, while the study of the three other cases will be be restricted to quantiles $q_1$ to $q_{18}$.

### *3.3. Empirical results*

The Anderson–Darling statistics (ADS) for four parametric distributions (Weibull or stretched-exponential, exponential, Pareto and log-Weibull) are shown in table 3 for two quantile ranges, the top half of the table corresponding to the 90% lowest thresholds while the bottom half corresponds to the 10% highest thresholds. For the lowest thresholds, the ADS rejects all distributions at the 95% confidence level, except the stretched-exponential for the negative tail of the SP for the 60-min returns and for the Nasdaq. Thus, none of the considered distributions is adequate to model the data over such large ranges. For the 10% highest quantiles, the exponential model is rejected at the 95% confidence level except for the negative tails of the Dow Jones (daily returns) and the Nasdaq. The log-Weibull and the stretched-exponential distributions are the best since they are only rejected at the 1-min time scale (SP). The Pareto distribution provides a reliable description for time scales larger than or equal to 30 min. However, it remains less accurate than the log-Weibull and the stretched-exponential distributions, on average. Overall, it can be seen that the Nasdaq and the 60-min returns of the S&P 500 behave very similarly. We now present an analysis of each case in more detail.

**3.3.1. Pareto distribution.** Figure 2(a) shows the cumulative sample distribution function $1-F(x)$ for the Dow Jones Industrial Average index, and figure 2(b) gives the cumulative sample distribution function for the Standard & Poor's 500 index at the 30-min time scale. The mismatch between the Pareto distribution and the data can be seen with the naked eye: if samples were taken from a Pareto population, the graph in double log-scale should be a straight line. Even in the tails, this is doubtful. To formalize this impression, we calculate the Hill and AD estimators for each threshold $u$. Denoting $y_1 \geqslant \cdots \geqslant y_{n_u}$ the ordered sub-sample of values exceeding $u$ where $N_u$ is the size of this sub-sample, the Hill maximum likelihood estimate of parameter $b$ and its standard deviation are (Hill 1975)

$$\hat{b}_u = \left[\frac{1}{N_u}\sum_{1}^{N_u}\log(y_k/u)\right]^{-1} \quad \text{and} \quad \mathrm{Std}(\hat{b}_u) = b/\sqrt{N_u}. \tag{13}$$

The latter expression is derived under the assumption of iid data, but very severely underestimates the true standard deviation when samples exhibit dependence, as reported by Kearns and Pagan (1997).

Figure 3 shows the Hill estimator $\hat{b}_u$ for all data sets (positive and negative branches of the distribution of returns for the DJ, the ND and the SP) as a function of the index $n = 1, 2, \ldots, 18$ of the quantiles or standard significance levels $q_1, \ldots, q_{18}$ given in table 2. Similar results are obtained with the AD estimates. The three branches of the distribution of returns for the Dow Jones and the negative tail of the Nasdaq suggest a continuous growth of the Hill estimator $\hat{b}_u$ as a function of $n = 1, \ldots, 18$. However, it turns out that this apparent growth may be explained solely on the basis of statistical fluctuations and slow convergence to a moderate $b$ value. Indeed, the two thick lines show the 95% confidence bounds obtained from synthetic time series of 10 000 data points generated with a Student distribution with exponent $b = 3.5$. It is clear that the growth of the upper bound can explain the observed behaviour of the $b$ value obtained for the DJ and ND data. It would thus be incorrect to extrapolate this apparent growth of the $b$ value. However, we cannot conclude with certainty that the growth of the $b$ value has been exhausted and that we have access to the asymptotic exponent value. This is another illustration of the statement by Embrechts *et al.* (1997) that "there is no free lunch when it comes to high quantiles estimation!".

**3.3.2. Weibull distributions.** Let us now fit our data with the Weibull (SE) distribution (4). The Anderson–Darling statistics (ADS) for this case are shown in table 3. The ML

Table 2. Probability levels $q_k$ and their corresponding lower thresholds $u_k$ for the three samples, at different time scales. The number $n_u$ provides the size of the sub-sample beyond the threshold $u_k$.

| | S&P 500 (1 min) | | | | Nasdaq | | | | Dow Jones | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Pos. tail | | Neg. tail | | Pos. tail | | Neg. tail | | Pos. tail | | Neg. tail | |
| | $10^3 \times u$ | $n_u$ | $10^3 \times u$ | $n_u$ | $10^3 \times u$ | $n_u$ | $10^3 \times u$ | $n_u$ | $10^2 \times u$ | $n_u$ | $10^2 \times u$ | $n_u$ |
| $q_1 = 0$ | 0.0135 | 783 164 | 0.0135 | 765 808 | 0.0053 | 11 241 | 0.0053 | 10 751 | 0.0032 | 14 949 | 0.0028 | 13 464 |
| $q_2 = 0.1$ | 0.0566 | 704 848 | 0.0559 | 689 228 | 0.0573 | 10 117 | 0.0571 | 9676 | 0.0976 | 13 454 | 0.0862 | 12 118 |
| $q_3 = 0.2$ | 0.0885 | 626 532 | 0.0879 | 612 647 | 0.1124 | 8993 | 0.1129 | 8601 | 0.1833 | 11 959 | 0.1739 | 10 771 |
| $q_4 = 0.3$ | 0.1229 | 548 215 | 0.1228 | 536 066 | 0.1729 | 7869 | 0.1723 | 7526 | 0.2783 | 10 464 | 0.263 | 9425 |
| $q_5 = 0.4$ | 0.1597 | 469 899 | 0.1598 | 459 485 | 0.238 | 6745 | 0.2365 | 6451 | 0.3872 | 8969 | 0.3697 | 8078 |
| $q_6 = 0.5$ | 0.1988 | 391 582 | 0.1992 | 382 904 | 0.3157 | 5620 | 0.3147 | 5376 | 0.5055 | 7475 | 0.4963 | 6732 |
| $q_7 = 0.6$ | 0.2553 | 313 266 | 0.2569 | 306 324 | 0.406 | 4496 | 0.412 | 4300 | 0.6426 | 5980 | 0.6492 | 5386 |
| $q_8 = 0.7$ | 0.3192 | 234 950 | 0.3224 | 229 743 | 0.5211 | 3372 | 0.5374 | 3225 | 0.8225 | 4485 | 0.8376 | 4039 |
| $q_9 = 0.8$ | 0.4138 | 156 633 | 0.4190 | 153 162 | 0.6901 | 2248 | 0.7188 | 2150 | 1.0545 | 2990 | 1.1057 | 2693 |
| $q_{10} = 0.9$ | 0.5752 | 78 317 | 0.5871 | 76 581 | 0.973 | 1124 | 1.0494 | 1075 | 1.4919 | 1495 | 1.6223 | 1346 |
| $q_{11} = 0.925$ | 0.6461 | 58 738 | 0.6610 | 57 436 | 1.1016 | 843 | 1.1833 | 806 | 1.6956 | 1121 | 1.8637 | 1010 |
| $q_{12} = 0.95$ | 0.7539 | 39 159 | 0.7736 | 38 291 | 1.2926 | 562 | 1.3888 | 538 | 1.9846 | 747 | 2.2285 | 673 |
| $q_{13} = 0.96$ | 0.8152 | 31 327 | 0.8383 | 30 633 | 1.3859 | 450 | 1.4955 | 430 | 2.1734 | 598 | 2.4197 | 539 |
| $q_{14} = 0.97$ | 0.8985 | 23 495 | 0.9237 | 22 975 | 1.53 | 337 | 1.639 | 323 | 2.413 | 448 | 2.7218 | 404 |
| $q_{15} = 0.98$ | 1.0247 | 15 664 | 1.0539 | 15 317 | 1.713 | 225 | 1.8557 | 215 | 2.7949 | 299 | 3.1647 | 269 |
| $q_{16} = 0.99$ | 1.2696 | 7832 | 1.3155 | 7659 | 2.1188 | 112 | 1.8855 | 108 | 3.5704 | 149 | 4.1025 | 135 |
| $q_{17} = 0.9925$ | 1.3889 | 5874 | 1.4366 | 5744 | 2.3176 | 84 | 2.4451 | 81 | 3.9701 | 112 | 4.3781 | 101 |
| $q_{18} = 0.995$ | 1.5796 | 3916 | 1.6261 | 3830 | 3.0508 | 56 | 2.7623 | 54 | 4.5746 | 75 | 5.0944 | 67 |
| $q_{19} = 0.999$ | 2.7979 | 784 | 2.8287 | 766 | | | | | | | | |
| $q_{20} = 0.9995$ | 3.8029 | 392 | 3.8311 | 383 | | | | | | | | |
| $q_{21} = 0.9999$ | 7.9492 | 79 | 8.9869 | 77 | | | | | | | | |

Table 3. Mean Anderson–Darling distances in the range of thresholds $u_1$–$u_9$ and in the range $u_i \geq u_{10}$. The figures within parentheses characterize the goodness of fit: they represent the significance levels with which the considered model can be rejected. Note that these significance levels are only lower bounds since one or two parameters are fitted.

| | Mean AD statistics for $u_1$–$u_9$ | | | | | |
|---|---|---|---|---|---|---|
| | S&P 500 1 min | | S&P 500 5 min | | S&P 500 30 min | |
| | Pos. tail | Neg. tail | Pos. tail | Neg. tail | Pos. tail | Neg. tail |
| Weibull | 292.85 (100%) | 299.46 100%) | 36.62 (100%) | 41.04 (100%) | 7.36 (100%) | 4.84 (100%) |
| Exponential | 771.70 (100%) | 718.56 (100%) | 86.79 (100%) | 108.17 (100%) | 17.47 (100%) | 16.36 (100%) |
| Pareto | 23 998.94 (100%) | 23 337.60 (100%) | 6834.06 (100%) | 6563.26 (100%) | 1847.40 (100%) | 1298.47 (100%) |
| Log-Weibull | 1559.70 (100%) | 1470.11 (100%) | 360.18 (100%) | 331.45 (100%) | 60.03 (100%) | 67.22 (100%) |
| | Mean AD statistics for $u_i \geq u_{10}$ | | | | | |
| Weibull | 6.80 (100%) | 5.80 (100%) | 1.81 (88%) | 1.93 (90%) | 0.67 (42%) | 0.79 (51%) |
| Exponential | 143.97 (100%) | 136.66 (100%) | 28.12 (100%) | 30.88 (100%) | 8.19 (100%) | 9.75 (100%) |
| Pareto | 19.97 (100%) | 19.24 (100%) | 8.10 (100%) | 7.61 (100%) | 1.63 (85%) | 1.77 (88%) |
| Log-Weibull | 3.60 (99%) | 4.10 (99%) | 1.20 (73%) | 1.55 (84%) | 0.64 (39%) | 0.42 (17%) |
| | Mean AD statistics for $u_1$–$u_9$ | | | | | |
| | S&P 500 60 min | | Nasdaq | | Dow Jones | |
| | Pos. tail | Neg. tail | Pos. tail | Neg. tail | Pos. tail | Neg. tail |
| Weibull | 3.58 (99%) | 2.36 (94%) | 1.37 (80%) | 0.85 (55%) | 4.96 (100%) | 3.86 (99%) |
| Exponential | 8.12 (100%) | 12.20 (100%) | 5.41 (100%) | 3.33 (98%) | 16.48 (100%) | 10.30 (100%) |
| Pareto | 1001.68 (100%) | 702.47 (100%) | 475.00 (100%) | 441.40 (100%) | 691.30 (100%) | 607.30 (100%) |
| Log-Weibull | 34.44 (100%) | 36.55 (100%) | 35.90 (100%) | 30.92 (100%) | 32.30 (100%) | 28.27 (100%) |
| | Mean AD statistics for $u_i \geq u_{10}$ | | | | | |
| Weibull | 0.66 (41%) | 0.68 (42%) | 0.67 (42%) | 0.50 (29%) | 0.38 (13%) | 0.35 (10%) |
| Exponential | 4.99 (100%) | 4.89 (100%) | 3.06 (97%) | 1.97 (90%) | 3.06 (97%) | 1.89 (89%) |
| Pareto | 1.12 (70%) | 1.28 (76%) | 1.30 (78%) | 1.33 (78%) | 0.78 (50%) | 1.26 (75%) |
| Log-Weibull | 0.48 (23%) | 0.57 (32%) | 0.46 (29%) | 0.49 (30%) | 0.38 (13%) | 0.69 (43%) |

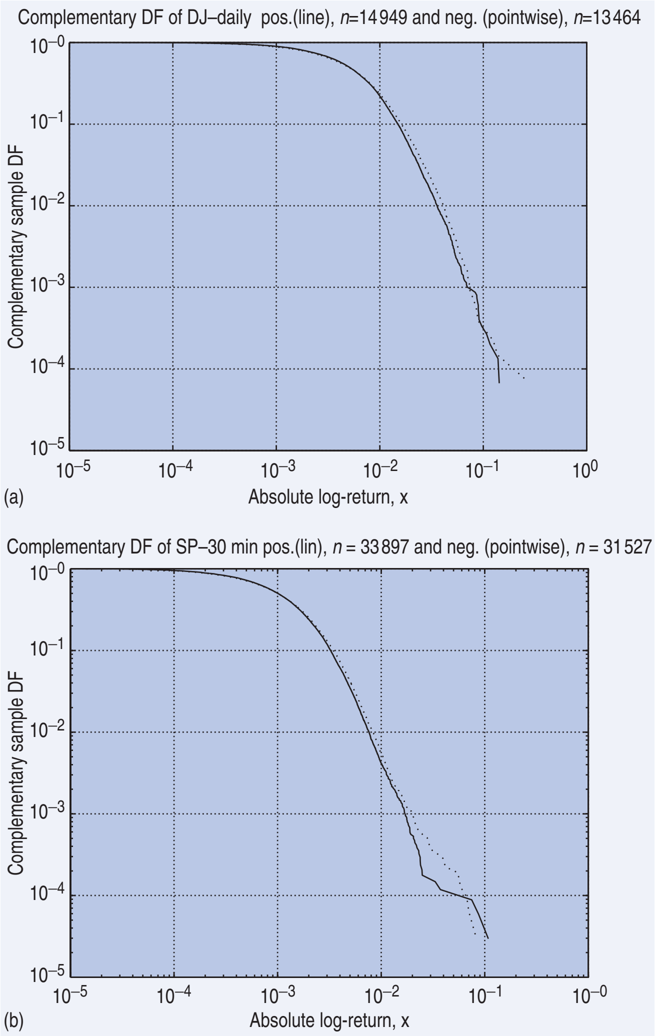


Figure 2. Cumulative sample distributions for the Dow Jones (a) and for the Standard & Poor's 500 30-min (b) data sets.

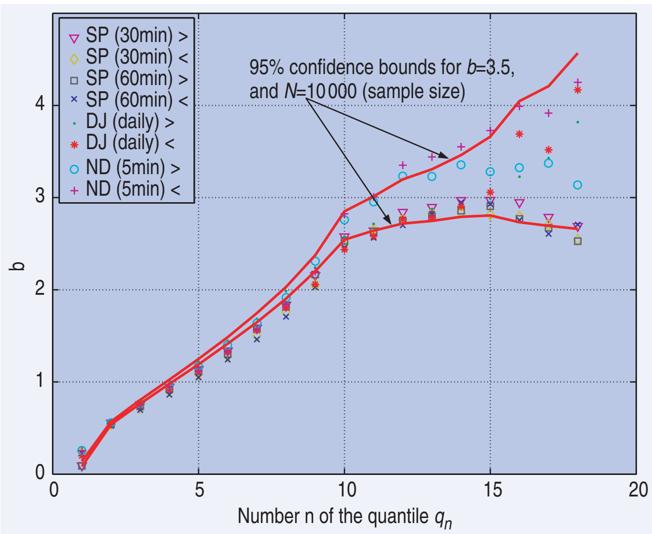


Figure 3. Hill estimator $\hat{b}_u$ for all sets (positive and negative branches of the distribution of returns for the DJ, ND and SP) as a function of the index $n = 1, \ldots, 18$ of the 18 quantiles or standard significance levels $q_1, \ldots, q_{18}$ given in table 2. The two thick lines (in red) show the 95% confidence bounds obtained from synthetic time series of 10 000 data points generated with a Student distribution with exponent $b = 3.5$.

estimates and AD estimates of the form parameter $c$ are presented in table 4. Table 3 shows that, for the highest quantiles, the ADS for the stretched-exponential is the smallest of all ADS, suggesting that the SE is the best model of all. Moreover, for the lowest quantiles, it is the sole model not systematically rejected at the 95% level.

The $c$ estimates are found to decrease when increasing the order $q$ of the threshold $u_q$ beyond which the estimations are performed. In addition, several $c$ estimates are found to be equal to zero. However, this does not automatically imply that the SE model is not the correct model for the data, even for these highest quantiles. Indeed, numerical simulations show that, even for synthetic samples drawn from genuine stretched-exponential distributions with exponent $c$ smaller than 0.5 and whose size is comparable to that of our data, in about one case out of three (depending on the exact value of $c$) the estimated value of $c$ is zero. This *a priori* surprising result comes from a certain condition for the existence of a solution of the maximum likelihood equations given in appendix 2A of Malevergne and Sornette (2005), which is not fulfilled with certainty even for samples drawn for SE distributions.

Notwithstanding this cautionary remark, note that the $c$ estimate of the positive tail of the Nasdaq data equals zero for all quantiles higher than $q_{14} = 0.97\%$. In fact, in every case, the estimated $c$ is not significantly different from zero, at the 95% significance level, for quantiles higher than $q_{12}$–$q_{14}$, except for quantile $q_{21}$ of the negative tail of the SP, but this value is probably doubtful. In addition, table 5 gives the values of the estimated scale parameter $d$, which are found to be very small, particularly for the Nasdaq—beyond $q_{12} = 95\%$—and the S&P 500—beyond $q_{10} = 90\%$. In contrast, the Dow Jones maintains significant scale factors until $q_{16}$–$q_{17}$.

This evidence taken together provides a clear indication of the existence of a change of behaviour of the true pdf of these distributions: while the bulk of the distributions seem rather well approximated by a SE model, a tailed distribution fatter than that of the (SE) model is required for the highest quantiles. Actually, the fact that both $c$ and $d$ are extremely small may be interpreted, according to the asymptotic correspondence given by (7) and (8), as the existence of a possible power law tail.

**3.3.3. Exponential and incomplete Gamma distributions.** Let us now fit our data with the exponential distribution (5). The average ADS for this case are shown in table 3. The maximum likelihood and Anderson–Darling estimates of the scale parameter $d$ for the DJ and ND data are given in table 6. Note that they always decrease as the threshold $u_q$ increases. Comparing the mean ADS values of table 3 with the standard AD quantiles, we can conclude that, on the whole, the exponential distribution (*even with moving scale parameter d*) does not fit our data: this model is systematically rejected at the 95% confidence level for the lowest and highest quantiles,

Table 4. Maximum likelihood and Anderson–Darling estimates of the form parameter $c$ of the Weibull (stretched-exponential) distribution. Figures within parentheses give the standard deviation of the maximum likelihood estimator.

| | S&P 500 (1 min) | | | | Nasdaq | | | | Dow Jones | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Pos. tail | | Neg. tail | | Pos. tail | | Neg. tail | | Pos. tail | | Neg. tail | |
| | MLE | ADE | MLE | ADE | MLE | ADE | MLE | ADE | MLE | ADE | MLE | ADE |
| 1 | 1.065 (0.001) | 1.175 | 1.051 (0.001) | 1.158 | 1.007 (0.008) | 1.053 | 0.987 (0.008) | 1.017 | 1.040 (0.007) | 1.104 | 0.975 (0.007) | 1.026 |
| 2 | 0.927 (0.002) | 1.049 | 0.915 (0.002) | 1.035 | 0.983 (0.011) | 1.051 | 0.953 (0.011) | 0.993 | 0.973 (0.010) | 1.075 | 0.910 (0.010) | 0.989 |
| 3 | 0.8754 (0.002) | 1.0196 | 0.8634 (0.002) | 1.0027 | 0.944 (0.014) | 1.031 | 0.912 (0.014) | 0.955 | 0.931 (0.013) | 1.064 | 0.856 (0.012) | 0.948 |
| 4 | 0.813 (0.002) | 0.970 | 0.799 (0.002) | 0.947 | 0.896 (0.018) | 0.995 | 0.876 (0.018) | 0.916 | 0.878 (0.015) | 1.038 | 0.821 (0.015) | 0.933 |
| 5 | 0.763 (0.003) | 0.952 | 0.752 (0.003) | 0.932 | 0.857 (0.021) | 0.978 | 0.861 (0.021) | 0.912 | 0.792 (0.019) | 0.955 | 0.767 (0.018) | 0.889 |
| 6 | 0.733 (0.003) | 0.985 | 0.727 (0.003) | 0.971 | 0.790 (0.026) | 0.916 | 0.833 (0.026) | 0.891 | 0.708 (0.023) | 0.873 | 0.698 (0.022) | 0.819 |
| 7 | 0.593 (0.004) | 0.799 | 0.590 (0.004) | 0.791 | 0.732 (0.033) | 0.882 | 0.796 (0.033) | 0.859 | 0.622 (0.028) | 0.788 | 0.612 (0.028) | 0.713 |
| 8 | 0.504 (0.005) | 0.740 | 0.502 (0.005) | 0.730 | 0.661 (0.042) | 0.846 | 0.756 (0.042) | 0.834 | 0.480 (0.035) | 0.586 | 0.531 (0.035) | 0.597 |
| 9 | 0.337 (0.007) | 0.537 | 0.342 (0.007) | 0.531 | 0.509 (0.058) | 0.676 | 0.715 (0.059) | 0.865 | 0.394 (0.047) | 0.461 | 0.478 (0.047) | 0.527 |
| 10 | 0.152 (0.010) | 0.394 | 0.159 (0.010) | 0.387 | 0.359 (0.092) | 0.631 | 0.522 (0.099) | 0.688 | 0.304 (0.074) | 0.346 | 0.403 (0.076) | 0.387 |
| 11 | 0.079 (0.012) | 0.327 | 0.091 (0.012) | 0.339 | 0.252 (0.110) | 0.515 | 0.481 (0.120) | 0.697 | 0.231 (0.087) | 0.158 | 0.379 (0.091) | 0.337 |
| 12 | $<10^{-8}$ | 0.151 | $<10^{-8}$ | 0.169 | 0.039 (0.138) | 0.177 | 0.273 (0.155) | 0.275 | 0.269 (0.111) | 0.207 | 0.357 (0.119) | 0.288 |
| 13 | $<10^{-8}$ | 0.0793 | $<10^{-8}$ | 0.084 | 0.057 (0.155) | 0.233 | 0.255 (0.177) | 0.274 | 0.253 (0.127) | 0.147 | 0.428 (0.136) | 0.465 |
| 14 | $<10^{-8}$ | 0.008 | $<10^{-8}$ | 0.020 | $<10^{-8}$ | 0 | 0.215 (0.209) | 0.194 | 0.290 (0.150) | 0.174 | 0.448 (0.164) | 0.641 |
| 15 | $<10^{-8}$ | 0.008 | $<10^{-8}$ | 0.008 | $<10^{-8}$ | 0 | 0.103 (0.260) | 0 | 0.379 (0.192) | 0.407 | 0.451 (0.210) | 0.863 |
| 16 | $<10^{-8}$ | 0.008 | $<10^{-8}$ | 0.008 | $9.6 \times 10^{-8}$ | 0 | 0.064 (0.390) | 0 | 0.398 (0.290) | 0.382 | 0.022 (0.319) | 0.110 |
| 17 | $<10^{-8}$ | 0.008 | $<10^{-8}$ | 0.008 | $<10^{-8}$ | 0 | 0.158 (0.452) | 0.224 | 0.307 (0.346) | 0.255 | 0.178 (0.367) | 0.703 |
| 18 | $<10^{-8}$ | 0.008 | $<10^{-8}$ | 0.008 | $<10^{-8}$ | 0 | $<10^{-8}$ | 0 | $2 \times 10^{-8}$ | 0 | $<10^{-8}$ | 0 |
| 19 | 0.035 (0.082) | 0.007 | 0.009 (0.032) | 0.007 | | | | | | | | |
| 20 | 0.111 (0.119) | 0.075 | 0.316 (0.117) | 0.007 | | | | | | | | |
| 21 | $<10^{-8}$ | 0.008 | 0.827 (0.393) | 0.900 | | | | | | | | |

Table 5. Maximum likelihood and Anderson–Darling estimates of the form parameter $d$ ($\times 10^3$) of the Weibull (stretched-exponential) distribution. Figures within parentheses give the standard deviation of the maximum likelihood estimator.

| | S&P 500 (1 min) | | | | Nasdaq | | | | Dow Jones | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Pos. tail | | Neg. tail | | Pos. tail | | Neg. tail | | Pos. tail | | Neg. tail | |
| | MLE | ADE | MLE | ADE | MLE | ADE | MLE | ADE | MLE | ADE | MLE | ADE |
| 1 | 0.275 (0.000) | 0.274 | 0.277 (0.000) | 0.276 | 0.443 (0.004) | 0.441 | 0.455 (0.005) | 0.452 | 7.137 (0.060) | 7.107 | 7.268 (0.068) | 7.127 |
| 2 | 0.230 (0.001) | 0.250 | 0.232 (0.001) | 0.252 | 0.429 (0.006) | 0.440 | 0.436 (0.006) | 0.443 | 6.639 (0.082) | 6.894 | 6.726 (0.094) | 6.952 |
| 3 | 0.208 (0.001) | 0.242 | 0.209 (0.001) | 0.243 | 0.406 (0.008) | 0.432 | 0.410 (0.009) | 0.424 | 6.236 (0.113) | 6.841 | 6.108 (0.131) | 6.640 |
| 4 | 0.179 (0.001) | 0.226 | 0.178 (0.001) | 0.225 | 0.372 (0.011) | 0.414 | 0.383 (0.012) | 0.402 | 5.621 (0.155) | 6.655 | 5.656 (0.175) | 6.515 |
| 5 | 0.153 (0.001) | 0.220 | 0.153 (0.001) | 0.218 | 0.341 (0.015) | 0.404 | 0.369 (0.016) | 0.399 | 4.515 (0.215) | 5.942 | 4.876 (0.235) | 6.066 |
| 6 | 0.138 (0.001) | 0.234 | 0.140 (0.002) | 0.236 | 0.283 (0.020) | 0.364 | 0.345 (0.021) | 0.383 | 3.358 (0.277) | 5.081 | 3.801 (0.305) | 5.220 |
| 7 | 0.067 (0.002) | 0.154 | 0.069 (0.002) | 0.156 | 0.231 (0.026) | 0.339 | 0.309 (0.028) | 0.358 | 2.192 (0.326) | 4.073 | 2.475 (0.366) | 3.764 |
| 8 | 0.032 (0.002) | 0.126 | 0.033 (0.002) | 0.126 | 0.166 (0.034) | 0.311 | 0.269 (0.039) | 0.336 | 0.682 (0.256) | 1.606 | 1.385 (0.389) | 2.149 |
| 9 | 0.001 (0.000) | 0.037 | 0.002 (0.000) | 0.038 | 0.053 (0.030) | 0.164 | 0.225 (0.057) | 0.365 | 0.195 (0.163) | 0.510 | 0.810 (0.417) | 1.297 |
| 10 | 0.000 (0.000) | 0.005 | 0.000 (0.000) | 0.005 | 0.005 (0.010) | 0.128 | 0.058 (0.057) | 0.184 | 0.019 (0.048) | 0.065 | 0.276 (0.361) | 0.207 |
| 11 | 0.000 (0.000) | 0.001 | 0.000 (0.000) | 0.001 | 0.000 (0.001) | 0.049 | 0.036 (0.053) | 0.194 | 0.001 (0.003) | 0.000 | 0.169 (0.316) | 0.065 |
| 12 | 0.000 (0.000) | 0.000 | 0.000 (0.000) | 0.000 | 0.000 (0.000) | 0.000 | 0.000 (0.001) | 0.000 | 0.005 (0.025) | 0.000 | 0.103 (0.291) | 0.012 |
| 13 | 0.000 (0.000) | 0.000 | 0.000 (0.000) | 0.000 | 0.000 (0.000) | 0.000 | 0.000 (0.001) | 0.000 | 0.001 (0.010) | 0.000 | 0.427 (0.912) | 0.729 |
| 14 | 0.000 (0.000) | 0.000 | 0.000 (0.000) | 0.000 | 0.000 (0.000) | 0.000 | 0.000 (0.000) | 0.000 | 0.009 (0.055) | 0.000 | 0.577 (1.357) | 3.509 |
| 15 | 0.000 (0.000) | 0.000 | 0.000 (0.000) | 0.000 | 0.000 (0.000) | 0.000 | 0.000 (0.000) | 0.000 | 0.149 (0.629) | 0.282 | 0.613 (1.855) | 9.640 |
| 16 | 0.000 (0.000) | 0.000 | 0.000 (0.000) | 0.000 | 0.000 (0.000) | 0.000 | 0.000 (0.000) | 0.000 | 0.145 (0.960) | 0.179 | 0.000 (0.000) | 0.000 |
| 17 | 0.000 (0.000) | 0.000 | 0.000 (0.000) | 0.000 | 0.000 (0.000) | 0.000 | 0.000 (0.000) | 0.000 | 0.007 (0.109) | 0.002 | 0.000 (0.000) | 5.528 |
| 18 | 0.000 (0.000) | 0.000 | 0.000 (0.000) | 0.000 | 0.000 (0.000) | 0.000 | 0.000 (0.000) | 0.000 | 0.000 (0.000) | 0.000 | 0.000 (0.000) | 0.000 |
| 19 | 0.000 (0.000) | 0.000 | 0.000 (0.000) | 0.000 | | | | | | | | |
| 20 | 0.000 (0.000) | 0.177 | 0.017 (0.058) | 0.000 | | | | | | | | |
| 21 | 0.000 (0.000) | 0.000 | 2.853 (3.698) | 3.512 | | | | | | | | |

Table 6. Maximum likelihood and Anderson–Darling estimates of the scale parameter $d = 10^{-3}d'$ of the exponential distribution. Figures within parentheses give the standard deviation of the maximum likelihood estimator.

| | Nasdaq | | | | Nasdaq | | | | Dow Jones | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Pos. tail | | Neg. tail | | Pos. tail | | Neg. tail | | Pos. tail | | Neg. tail | |
| | MLE | ADE | MLE | ADE | MLE | ADE | MLE | ADE | MLE | ADE | MLE | ADE |
| 1 | 0.265 (0.000) | 0.269 | 0.269 (0.000) | 0.272 | 0.441 (0.004) | 0.441 | 0.458 (0.004) | 0.451 | 7.012 (0.057) | 7.055 | 7.358 (0.063) | 7.135 |
| 2 | 0.249 (0.000) | 0.243 | 0.254 (0.000) | 0.247 | 0.435 (0.004) | 0.431 | 0.454 (0.005) | 0.444 | 6.793 (0.059) | 6.701 | 7.292 (0.066) | 6.982 |
| 3 | 0.246 (0.000) | 0.238 | 0.252 (0.000) | 0.242 | 0.431 (0.005) | 0.424 | 0.452 (0.005) | 0.438 | 6.731 (0.062) | 6.575 | 7.275 (0.070) | 6.890 |
| 4 | 0.245 (0.000) | 0.234 | 0.251 (0.000) | 0.238 | 0.428 (0.005) | 0.416 | 0.453 (0.005) | 0.437 | 6.675 (0.065) | 6.444 | 7.358 (0.076) | 6.938 |
| 5 | 0.246 (0.000) | 0.233 | 0.253 (0.000) | 0.238 | 0.429 (0.005) | 0.415 | 0.458 (0.006) | 0.443 | 6.607 (0.070) | 6.264 | 7.429 (0.083) | 6.941 |
| 6 | 0.252 (0.000) | 0.238 | 0.260 (0.000) | 0.245 | 0.429 (0.006) | 0.411 | 0.464 (0.006) | 0.447 | 6.630 (0.077) | 6.186 | 7.529 (0.092) | 6.951 |
| 7 | 0.251 (0.000) | 0.231 | 0.259 (0.000) | 0.238 | 0.436 (0.006) | 0.413 | 0.472 (0.007) | 0.453 | 6.750 (0.087) | 6.207 | 7.700 (0.105) | 7.005 |
| 8 | 0.261 (0.001) | 0.236 | 0.270 (0.001) | 0.244 | 0.447 (0.008) | 0.421 | 0.483 (0.009) | 0.463 | 6.920 (0.103) | 6.199 | 8.071 (0.127) | 7.264 |
| 9 | 0.275 (0.001) | 0.242 | 0.286 (0.001) | 0.251 | 0.462 (0.010) | 0.425 | 0.503 (0.011) | 0.482 | 7.513 (0.137) | 6.662 | 8.797 (0.170) | 7.908 |
| 10 | 0.320 (0.001) | 0.271 | 0.333 (0.001) | 0.282 | 0.517 (0.015) | 0.468 | 0.529 (0.016) | 0.496 | 8.792 (0.227) | 7.745 | 10.205 (0.278) | 9.175 |
| 11 | 0.345 (0.001) | 0.286 | 0.359 (0.002) | 0.299 | 0.540 (0.019) | 0.479 | 0.551 (0.019) | 0.514 | 9.349 (0.279) | 8.148 | 10.835 (0.341) | 9.751 |
| 12 | 0.385 (0.002) | 0.309 | 0.401 (0.002) | 0.322 | 0.574 (0.024) | 0.489 | 0.570 (0.025) | 0.516 | 10.487 (0.383) | 9.265 | 11.796 (0.454) | 10.657 |
| 13 | 0.413 (0.002) | 0.325 | 0.429 (0.002) | 0.339 | 0.615 (0.029) | 0.526 | 0.594 (0.029) | 0.537 | 11.017 (0.451) | 9.722 | 12.598 (0.543) | 11.581 |
| 14 | 0.454 (0.003) | 0.349 | 0.473 (0.003) | 0.366 | 0.653 (0.035) | 0.543 | 0.627 (0.035) | 0.564 | 11.920 (0.563) | 10.626 | 13.349 (0.664) | 12.386 |
| 15 | 0.526 (0.004) | 0.392 | 0.549 (0.004) | 0.413 | 0.750 (0.050) | 0.625 | 0.671 (0.046) | 0.594 | 13.251 (0.766) | 12.062 | 14.462 (0.880) | 13.521 |
| 16 | 0.703 (0.008) | 0.501 | 0.724 (0.008) | 0.515 | 0.917 (0.086) | 0.741 | 0.760 (0.073) | 0.674 | 15.264 (1.246) | 13.943 | 15.294 (1.316) | 13.285 |
| 17 | 0.800 (0.010) | 0.561 | 0.826 (0.011) | 0.579 | 0.991 (0.107) | 0.783 | 0.827 (0.092) | 0.744 | 15.766 (1.483) | 14.210 | 17.140 (1.705) | 15.327 |
| 18 | 0.965 (0.015) | 0.664 | 1.005 (0.016) | 0.700 | 1.178 (0.156) | 0.978 | 0.857 (0.117) | 0.742 | 16.207 (1.871) | 13.697 | 16.883 (2.047) | 13.476 |
| 19 | 2.172 (0.078) | 1.668 | 2.299 (0.083) | 1.775 | | | | | | | | |
| 20 | 2.930 (0.148) | 2.329 | 3.193 (0.163) | 2.760 | | | | | | | | |
| 21 | 5.047 (0.568) | 3.370 | 4.474 (0.510) | 4.382 | | | | | | | | |

except for the negative tail of the Nasdaq. The rejection is even more serious for the SP data at the intra-day time scale and is not shown here.

The results of the fit of our data by the IG distribution (6) have not been reported since they are hardly better than those provided by the exponential model. The mean ADS values again allow us to conclude that, on the whole, the IG distribution does not fit our data. The model is rejected at the 95% confidence level, except for the negative tail of the Nasdaq, for which it is marginally not rejected (significance level 94.13%). However, for the largest quantiles, this model again becomes relevant since it cannot be rejected at the 95% level.

**3.3.4. Log-Weibull distributions.** The parameters $b$ and $c$ of the log-Weibull defined by (9) are estimated with both the Maximum Likelihood and Anderson–Darling methods for the 18 standard significance levels $q_1, \ldots, q_{18}$ given in table 2 for the DJ and ND data and up to $q_{21}$ for the SP data. The results of these estimations are given in the set of three tables 7. For both positive and negative tails of the Dow Jones, we

Table 7. Maximum likelihood and Anderson–Darling estimates of the parameters $b$ and $c$ of the log-Weibull distribution. Numbers in parentheses give the standard deviations of the estimates.

| | S&P 500 (1 min) positive tail} | | | | S&P 500 (1 min) negative tail | | | |
|---|---|---|---|---|---|---|---|---|
| | MLE | | ADE | | MLE | | ADE | |
| | $c$ | $b$ | $c$ | $b$ | $c$ | $b$ | $c$ | $b$ |
| 1 | 3.261 (0.003) | 0.029 (0.000) | 3.298 | 0.027 | 3.232 (0.003) | 0.030 (0.000) | 3.264 | 0.028 |
| 2 | 1.875 (0.002) | 0.433 (0.001) | 1.878 | 0.410 | 1.884 (0.002) | 0.420 (0.001) | 1.881 | 0.399 |
| 3 | 1.645 (0.002) | 0.723 (0.001) | 1.642 | 0.690 | 1.647 (0.002) | 0.707 (0.001) | 1.641 | 0.676 |
| 4 | 1.471 (0.002) | 1.017 (0.001) | 1.477 | 0.970 | 1.465 (0.002) | 1.000 (0.001) | 1.470 | 0.954 |
| 5 | 1.414 (0.002) | 1.277 (0.002) | 1.405 | 1.233 | 1.411 (0.002) | 1.251 (0.002) | 1.401 | 1.208 |
| 6 | 1.382 (0.002) | 1.512 (0.002) | 1.387 | 1.477 | 1.383 (0.002) | 1.477 (0.002) | 1.389 | 1.443 |
| 7 | 1.233 (0.002) | 1.862 (0.003) | 1.234 | 1.811 | 1.232 (0.002) | 1.823 (0.003) | 1.239 | 1.776 |
| 8 | 1.187 (0.002) | 2.155 (0.005) | 1.192 | 2.116 | 1.192 (0.002) | 2.117 (0.005) | 1.196 | 2.079 |
| 9 | 1.112 (0.002) | 2.508 (0.007) | 1.111 | 2.470 | 1.113 (0.002) | 2.455 (0.007) | 1.112 | 2.415 |
| 10 | 1.069 (0.003) | 2.876 (0.011) | 1.078 | 2.896 | 1.062 (0.003) | 2.818 (0.011) | 1.074 | 2.831 |
| 11 | 1.048 (0.003) | 2.961 (0.014) | 1.066 | 3.016 | 1.055 (0.003) | 2.927 (0.014) | 1.069 | 2.972 |
| 12 | 1.016 (0.004) | 3.048 (0.018) | 1.033 | 3.123 | 1.015 (0.004) | 3.006 (0.017) | 1.034 | 3.076 |
| 13 | 1.002 (0.004) | 3.063 (0.020) | 1.021 | 3.151 | 1.001 (0.004) | 3.033 (0.020) | 1.020 | 3.115 |
| 14 | 0.981 (0.005) | 3.054 (0.023) | 1.003 | 3.153 | 0.990 (0.005) | 3.033 (0.023) | 1.012 | 3.134 |
| 15 | 0.961 (0.006) | 3.015 (0.027) | 0.985 | 3.133 | 0.978 (0.006) | 3.004 (0.027) | 1.003 | 3.132 |
| 16 | 0.941 (0.008) | 2.867 (0.036) | 0.961 | 2.980 | 0.937 (0.008) | 2.871 (0.037) | 0.957 | 2.987 |
| 17 | 0.937 (0.010) | 2.798 (0.040) | 0.951 | 2.899 | 0.927 (0.010) | 2.780 (0.041) | 0.947 | 2.887 |
| 18 | 0.902 (0.011) | 2.649 (0.046) | 0.902 | 2.677 | 0.925 (0.012) | 2.644 (0.046) | 0.940 | 2.726 |
| 19 | 0.994 (0.028) | 2.256 (0.084) | 0.971 | 2.201 | 0.962 (0.027) | 2.134 (0.080) | 0.923 | 2.063 |
| 20 | 0.999 (0.039) | 2.245 (0.118) | 0.967 | 2.139 | 1.011 (0.040) | 2.037 (0.107) | 0.933 | 1.879 |
| 21 | 0.949 (0.083) | 2.686 (0.330) | 0.957 | 2.801 | 1.288 (0.115) | 3.387 (0.455) | 1.234 | 3.272 |
| | Nasdaq positive tail | | | | Nasdaq negative tail | | | |
| | MLE | | ADE | | MLE | | ADE | |
| | $c$ | $b$ | $c$ | $b$ | $c$ | $b$ | $c$ | $b$ |
| 1 | 3.835 (0.006) | 0.004 (0.000) | 4.310 | 0.002 | 3.872 (0.006) | 0.003 (0.000) | 4.220 | 0.002 |
| 2 | 2.175 (0.010) | 0.217 (0.002) | 2.280 | 0.198 | 2.126 (0.010) | 0.219 (0.002) | 2.220 | 0.202 |
| 3 | 1.797 (0.012) | 0.508 (0.005) | 1.860 | 0.493 | 1.753 (0.012) | 0.506 (0.005) | 1.790 | 0.495 |
| 4 | 1.590 (0.013) | 0.812 (0.009) | 1.620 | 0.800 | 1.558 (0.013) | 0.785 (0.009) | 1.580 | 0.775 |
| 5 | 1.479 (0.014) | 1.096 (0.013) | 1.500 | 1.092 | 1.472 (0.014) | 1.032 (0.013) | 1.480 | 1.030 |
| 6 | 1.363 (0.015) | 1.412 (0.019) | 1.380 | 1.412 | 1.385 (0.015) | 1.312 (0.018) | 1.390 | 1.311 |
| 7 | 1.301 (0.015) | 1.723 (0.026) | 1.310 | 1.724 | 1.310 (0.016) | 1.622 (0.025) | 1.310 | 1.623 |
| 8 | 1.243 (0.017) | 2.065 (0.036) | 1.250 | 2.070 | 1.250 (0.017) | 1.968 (0.035) | 1.250 | 1.969 |
| 9 | 1.152 (0.018) | 2.479 (0.052) | 1.160 | 2.488 | 1.228 (0.020) | 2.425 (0.052) | 1.230 | 2.427 |
| 10 | 1.124 (0.023) | 2.981 (0.089) | 1.130 | 3.003 | 1.148 (0.024) | 3.113 (0.095) | 1.140 | 3.106 |
| 11 | 1.090 (0.025) | 3.141 (0.108) | 1.100 | 3.175 | 1.148 (0.027) | 3.343 (0.118) | 1.150 | 3.344 |
| 12 | 1.000 (0.028) | 3.226 (0.136) | 1.020 | 3.268 | 1.037 (0.030) | 3.448 (0.149) | 1.040 | 3.460 |
| 13 | 1.042 (0.033) | 3.327 (0.157) | 1.050 | 3.356 | 1.051 (0.033) | 3.582 (0.173) | 1.050 | 3.584 |
| 14 | 1.020 (0.036) | 3.401 (0.185) | 1.020 | 3.390 | 1.064 (0.038) | 3.738 (0.208) | 1.040 | 3.676 |
| 15 | 1.037 (0.046) | 3.359 (0.224) | 1.020 | 3.333 | 0.967 (0.043) | 3.601 (0.245) | 0.941 | 3.521 |
| 16 | 0.961 (0.061) | 3.202 (0.301) | 0.959 | 3.195 | 1.020 (0.061) | 4.030 (0.388) | 0.991 | 3.953 |
| 17 | 0.888 (0.067) | 3.064 (0.332) | 0.861 | 3.003 | 1.015 (0.071) | 3.924 (0.436) | 1.010 | 3.937 |
| 18 | 0.864 (0.083) | 2.807 (0.372) | 0.816 | 2.710 | 0.999 (0.084) | 4.168 (0.567) | 1.010 | 4.255 |

(*continued*)

Table 7. Continued.

| | Dow Jones positive tail | | | | Dow Jones negative tail | | | |
|---|---|---|---|---|---|---|---|---|
| | MLE | | ADE | | MLE | | ADE | |
| | $c$ | $b$ | $c$ | $b$ | $c$ | $b$ | $c$ | $b$ |
| 1 | 5.262 (0.005) | 0.000 (0.000) | 5.55 | 0.000 | 5.085 (0.005) | 0.000 (0.000) | 5.320 | 0.000 |
| 2 | 2.140 (0.009) | 0.241 (0.002) | 2.25 | 0.220 | 2.125 (0.009) | 0.211 (0.002) | 2.240 | 0.191 |
| 3 | 1.790 (0.010) | 0.531 (0.005) | 1.87 | 0.510 | 1.751 (0.010) | 0.495 (0.005) | 1.800 | 0.481 |
| 4 | 1.616 (0.012) | 0.830 (0.008) | 1.65 | 0.820 | 1.593 (0.012) | 0.744 (0.008) | 1.630 | 0.735 |
| 5 | 1.447 (0.012) | 1.165 (0.012) | 1.47 | 1.160 | 1.459 (0.013) | 1.022 (0.011) | 1.480 | 1.015 |
| 6 | 1.339 (0.012) | 1.472 (0.017) | 1.36 | 1.473 | 1.353 (0.013) | 1.311 (0.016) | 1.370 | 1.311 |
| 7 | 1.259 (0.013) | 1.768 (0.023) | 1.28 | 1.773 | 1.269 (0.014) | 1.609 (0.022) | 1.270 | 1.610 |
| 8 | 1.173 (0.013) | 2.097 (0.031) | 1.17 | 2.096 | 1.188 (0.015) | 1.885 (0.030) | 1.190 | 1.887 |
| 9 | 1.125 (0.015) | 2.362 (0.043) | 1.12 | 2.358 | 1.158 (0.017) | 2.178 (0.042) | 1.150 | 2.174 |
| 10 | 1.090 (0.020) | 2.705 (0.070) | 1.08 | 2.695 | 1.087 (0.022) | 2.545 (0.069) | 1.090 | 2.545 |
| 11 | 1.035 (0.022) | 2.771 (0.083) | 1.03 | 2.762 | 1.074 (0.024) | 2.688 (0.085) | 1.070 | 2.681 |
| 12 | 1.047 (0.027) | 2.867 (0.105) | 1.04 | 2.857 | 1.068 (0.029) | 2.880 (0.111) | 1.050 | 2.857 |
| 13 | 1.046 (0.030) | 2.960 (0.121) | 1.03 | 2.933 | 1.067 (0.032) | 2.900 (0.125) | 1.080 | 2.924 |
| 14 | 1.044 (0.034) | 3.000 (0.142) | 1.03 | 2.976 | 1.132 (0.038) | 3.171 (0.158) | 1.120 | 3.155 |
| 15 | 1.090 (0.043) | 3.174 (0.184) | 1.09 | 3.165 | 1.163 (0.047) | 3.439 (0.209) | 1.180 | 3.472 |
| 16 | 1.085 (0.059) | 3.424 (0.280) | 1.09 | 3.425 | 1.025 (0.056) | 3.745 (0.322) | 1.010 | 3.731 |
| 17 | 1.093 (0.066) | 3.666 (0.345) | 1.09 | 3.650 | 1.108 (0.069) | 3.822 (0.380) | 1.120 | 3.891 |
| 18 | 0.935 (0.071) | 3.556 (0.411) | 0.902 | 3.484 | 0.921 (0.071) | 3.804 (0.461) | 0.933 | 3.846 |



find very stable results for all quantiles lower than $q_{10}$: $c = 1.09 \pm 0.02$ and $b = 2.71 \pm 0.07$. These results reject the Pareto distribution degeneracy $c = 1$ at the 95% confidence level. Only for the quantiles higher than or equal to $q_{16}$, do we find an estimated value $c$ compatible with the Pareto distribution. Moreover, both for the positive and negative Dow Jones tails, we find that $c \approx 0.92$ and $b \approx 3.6$–3.8, suggesting a possible change of regime or a sensitivity to 'outliers' or a lack of robustness due to the small sample size. For the positive Nasdaq tail, the exponent $c$ is found to be compatible with $c = 1$ (the Pareto value), at the 95% significance level, above $q_{11}$ while $b$ remains almost stable at $b \simeq 3.2$. For the negative Nasdaq tail, we find that $c$ decreases almost systematically from 1.1 for $q_{10}$ to 1 for $q_{18}$ for both estimators, while $b$ regularly increases from about 3.1 to about 4.2. The Anderson–Darling distances are no worse but not significantly better than for the SE and this statistic cannot be used to conclude either in favour of or against the log-Weibull class.

The situation is different for the S&P 500 (1 min). For the positive tail, the parameter $c$ remains significantly less than one (Pareto case) from $q_{14} = 97\%$ to $q_{21}$, except for $q_{19}$ and $q_{20}$. Therefore, it seems that, for the very small time scales, the tails of the distribution of returns might be even fatter than a power law. As stressed in section 3.1.2, when $c$ is less than one, the SLE does not belong to the domain of attraction of a law of the maximum. As a consequence, standard results of EVT cannot provide reliable results when applied to such data, either from a theoretical point of view or from a practical stance (e.g. extreme risk assessment). The conclusions are the same for a 5-min time scale. For the 30-min and 60-min time scales, $c$ remains systematically less than one for the highest quantiles, but this difference ceases to be significant. In the negative tail, the situation is overall the same.

### *3.4. Summary*

At this stage, two conclusions can be drawn. First, it appears that none of the considered distributions fits the data over the entire range, which is not surprising. Second, for the highest quantiles, four models seem to be able to represent the data, the Gamma model, the Pareto model, the stretched-exponential model and the log-Weibull model. The latter two have the lowest Anderson–Darling statistics and thus seem to be the most reasonable models among the four models compatible with the data. For all the samples, their Anderson–Darling statistics remain so close to each other for the highest quantiles that the descriptive power of these two models cannot be distinguished.

## 4. Comparison of the descriptive power of the different families

As we have seen by comparing the Anderson–Darling statistics corresponding to the five parametric families (3)–(6) and (10), the best models in the sense of minimizing the Anderson–Darling distance are the stretched-exponential and the log-Weibull distributions.

We now compare the four distributions (3)–(6) with the comprehensive distribution (1) using Wilks' theorem (Wilks 1938) of nested hypotheses to check whether or not the four distributions are sufficient to describe the data compared with the comprehensive distribution. It will appear that the Pareto and the stretched-exponential models are the most parsimonious. We then turn to a direct comparison of the best two-parameter models (the SE and log-Weibull models) with the best one-parameter model (the Pareto model), which will require an extension of Wilks' theorem that will allow us to directly test the SE model against the Pareto model.

### *4.1. Comparison between the four parametric families (3)–(6) and the comprehensive distribution (1)*

According to Wilks' theorem, the doubled generalized log-likelihood ratio $\Lambda$,

$$\Lambda = 2\log\frac{\max\mathcal{L}(CD, X, \Theta)}{\max\mathcal{L}(z, X, \theta)}, \tag{14}$$

has asymptotically (as the size $N$ of the sample $X$ tends to infinity) the $\chi^2$ distribution. Here $\mathcal{L}$ denotes the likelihood function, and $\theta\subset\Theta$ are parametric spaces corresponding to hypotheses $z$ and the comprehensive distribution (CD) correspondingly (hypothesis $z$ is one of the four hypotheses (3)–(6) that are particular cases of the CD under some parameter relations). The statement of the theorem is valid under the condition that the *sample $X$ obeys hypothesis $z$ for some particular value of its parameter belonging to the space $\theta$*. The number of degrees of freedom of the $\chi^2$ distribution equals the difference of the dimensions of the two spaces $\Theta$ and $\theta$. We have $\dim(\Theta)=3$, $\dim(\theta)=2$ for the stretched-exponential and for the incomplete Gamma distributions, while $\dim(\theta)=1$ for the Pareto and the exponential distributions. This corresponds to one degree of freedom for the two former cases and two degrees of freedom for the latter pdfs.

The double log-likelihood ratios (14) are shown for the positive and negative branches of the distribution of returns in figure 4 for the Standard & Poor's 500 (1 min), in figure 5 for the Nasdaq and in figure 6 for the Dow Jones. The figures for the SP 500 at time scales of 5, 30 and 60 min are not depicted since they do not provide new information with respect to figure 4. The 95% $\chi^2$ confidence levels for one and two degrees of freedom are given by the horizontal lines.

For the Nasdaq data, figure 5 clearly shows that the exponential distribution is totally insufficient: for all

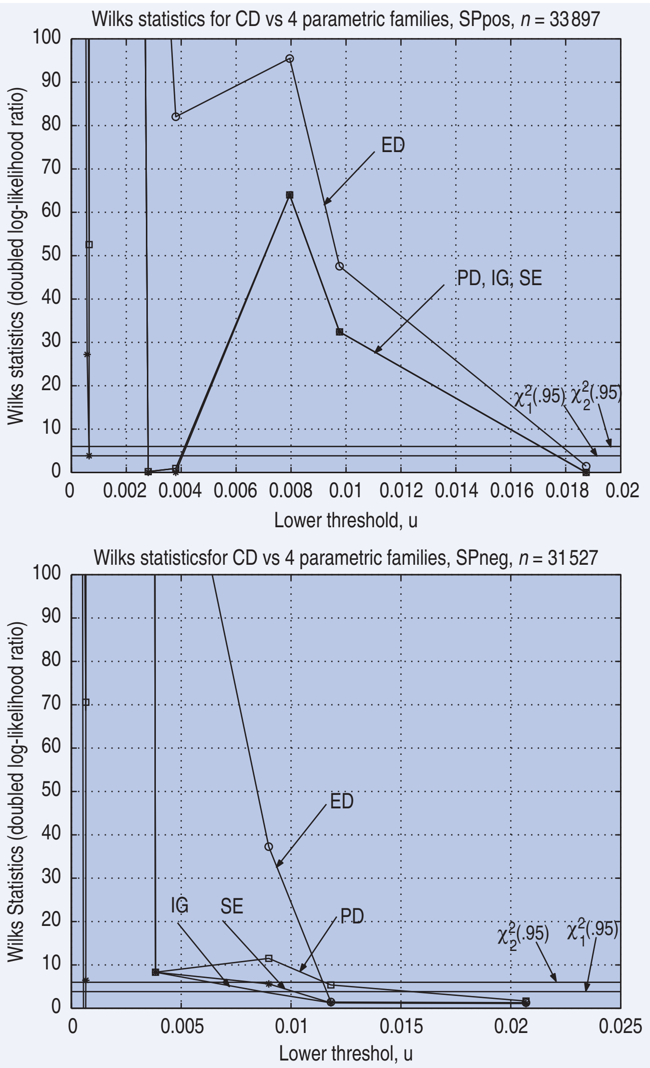


Figure 4. Wilks statistic for the comprehensive distribution versus the four parametric distributions, Pareto (PD), Weibull (SE), exponential (ED) and incomplete Gamma (IG), for the Standard & Poor's 500, 1 min returns. The upper panel refers to the positive returns and the lower panel to the negative returns.

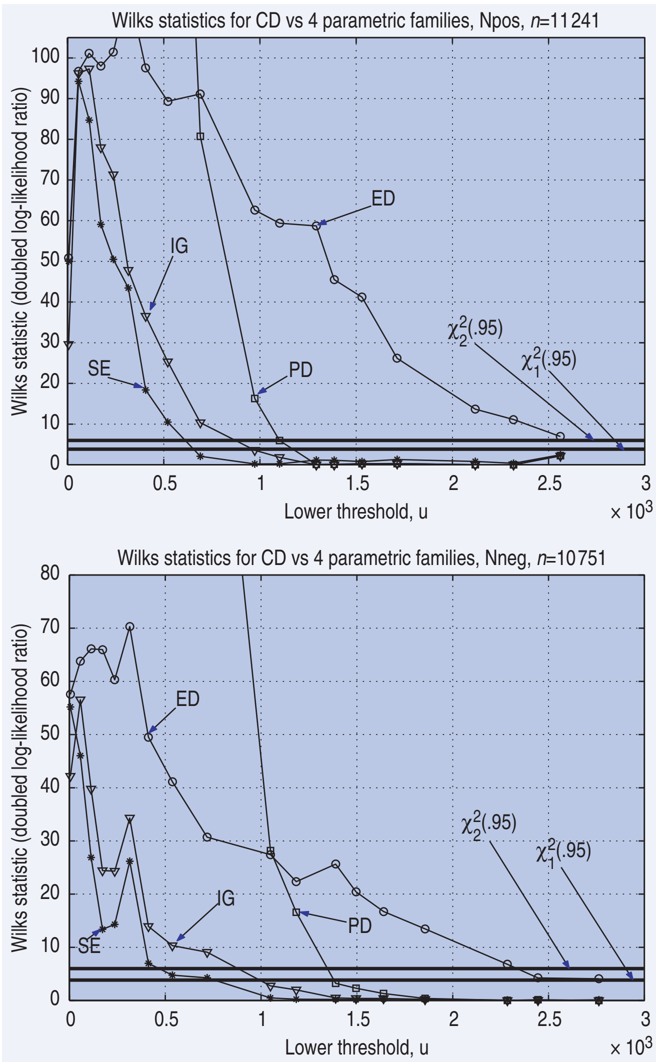


Figure 5. Wilks statistic for the comprehensive distribution versus the four parametric distributions, Pareto (PD), Weibull (SE), exponential (ED) and incomplete Gamma (IG), for the Nasdaq 5-min returns. The upper panel refers to the positive returns and the lower panel to the negative returns.

lower thresholds, the Wilks log-likelihood ratio exceeds the 95% $\chi_1^2$ level of 3.84. The Pareto distribution is insufficient for thresholds $u_1$–$u_{11}$ (92.5% of the ordered sample) and becomes comparable to the comprehensive distribution in the tail $u_{12}$–$u_{18}$ (7.5% of the tail probability). It is natural that the two-parametric families incomplete Gamma and stretched-exponential have higher goodness-of-fit than the one-parametric exponential and Pareto distributions. The incomplete Gamma distribution is comparable to the comprehensive distribution starting with $u_{10}$ (90%), whereas the stretched-exponential is somewhat better ($u_9$ or $u_8$, i.e. 70%). For the tails representing 7.5% of the data, all parametric families, except the exponential distribution, fit the sample distribution with almost the same efficiency. The results obtained for the Dow Jones data shown in figure 6 are similar. The stretched-exponential is comparable to the comprehensive distribution starting with $u_8$ (70%). On the whole, one can say that the stretched-exponential distribution performs better than the three other parametric families. The situation is different for the S&P shown in figure 4. For the positive tail, none of the four distributions is really sufficient to accurately describe the data. The comprehensive distribution is, overall, the best. In the negative tail, we retrieve a behaviour more similar to that observed in the two previous cases, except for the exponential distribution, which also appears to be better than the comprehensive distribution. However, it should be noted that the comprehensive distribution is only rejected in the very far tail. The PD, SE, ED and IG models are better than the comprehensive distribution only for the two highest quantiles ($q_{20}$ and $q_{21}$) of the negative tail. In contrast, the PD, SE, and IG models are better than the comprehensive distribution over the ten highest quantiles (or so) for the Nasdaq and the Dow Jones.

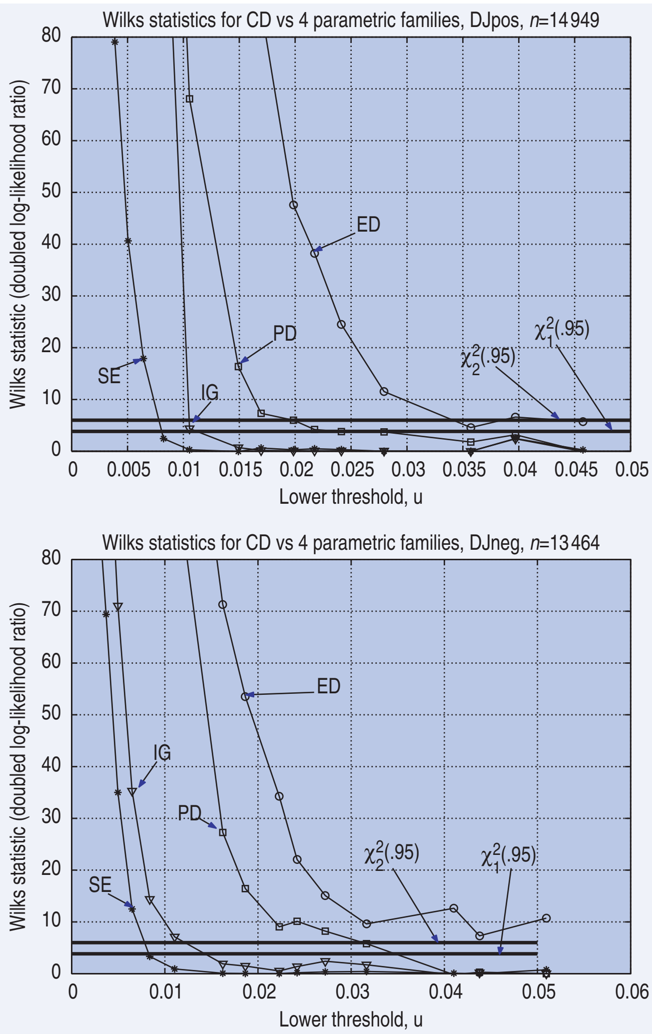


Figure 6. Wilks statistic for the comprehensive distribution versus the four parametric distributions, Pareto (PD), Weibull (SE), exponential (ED) and incomplete Gamma (IG) for the Dow Jones daily returns. The upper panel refers to the positive returns and the lower panel to the negative returns.

### 4.2. Pair-wise comparison of the Pareto model with the stretched-exponential and log-Weibull models

We now want to compare formally the descriptive power of the stretched-exponential distribution and the log-Weibull distribution (the two best two-parameter models) with that of the Pareto distribution (the best one-parameter model). For the comparison of the log-Weibull model versus the Pareto model, Wilks' theorem can still be applied since the log-Weibull distribution encompasses the Pareto distribution. *A contrario*, the comparison of the stretched-exponential versus the Pareto distribution should, in principle, require that we use the methods for testing non-nested hypotheses (Gouriéroux and Monfort 1994), such as the Wald encompassing test or the Bayes factors (Kass and Raftery 1995). Indeed, the Pareto model and the (SE) model are not, strictly speaking, nested. However, as shown in section 3.1.1, the Pareto distribution is a limit case of the stretched-exponential distribution, as the fractional exponent $c$ goes to zero, so that we can show (proof is available upon request and is given in Appendix 2A of Malevergne and Sornette (2005)) that the doubled log-likelihood ratio,

$$W = 2 \log \frac{\max \mathcal{L}_{\mathrm{SE}}}{\max \mathcal{L}_{\mathrm{PD}}}, \tag{15}$$

still follows Wilks' statistic, namely is asymptotically distributed according to a $\chi^2$ distribution, with one degree of freedom in the present case. Thus, even in this case of non-nested hypotheses, Wilks' statistic still allows us to test the null hypothesis $H_0$ according to which the Pareto model is sufficient to describe the data.

The results of these tests for the S&P 500 1-min and 30-min returns, for the Nasdaq and for the Dow Jones, are given in tables 8 and 9. The $p$-value (figures within parentheses) gives the significance with which one can reject the null hypothesis $H_0$ that the Pareto distribution is sufficient to accurately describe the data. Table 8 compares the stretched-exponential with the Pareto distribution. $H_0$ (Pareto) is found to be rejected more often for the Dow Jones than for the Nasdaq and the S&P 500. Indeed, beyond the quantile $q_{12} = 95\%$, $H_0$ cannot be rejected at the 95% confidence level for the Nasdaq and the S&P 500 data. For the Dow Jones, we

Table 8. Wilks' test for the Pareto distribution versus the stretched-exponential distribution. The *p*-value (figures within parentheses) gives the significance with which one can reject the null hypothesis that the Pareto distribution is sufficient to accurately describe the data.

| | SP 500 (1 min) | | SP 500 (30 min) | |
|---|---|---|---|---|
| | Pos. tail | Neg. tail | Pos. tail | Neg. tail |
| 1 | 1 511 182.27 (100%) | 1 454 941.73 (100%) | 335 711.06 (100%) | 90 910.14 (100%) |
| 2 | 456 411.68 (100%) | 447 046.72 (100%) | 27 157.88 (100%) | 24 784.94 (100%) |
| 3 | 259 490.16 (100%) | 253 365.20 (100%) | 14 175.71 (100%) | 13 569.71 (100%) |
| 4 | 149 696.47 (100%) | 144 970.11 (100%) | 8039.11 (100%) | 7717.52 (100%) |
| 5 | 90 548.88 (100%) | 88 382.04 (100%) | 4656.77 (100%) | 4497.13 (100%) |
| 6 | 58 394.16 (100%) | 57 893.97 (100%) | 2615.95 (100%) | 2519.15 (100%) |
| 7 | 24 161.00 (100%) | 24 070.92 (100%) | 1244.70 (100%) | 1269.01 (100%) |
| 8 | 10 961.80 (100%) | 10 924.94 (100%) | 567.54 (100%) | 546.01 (100%) |
| 9 | 2669.63 (100%) | 2775.46 (100%) | 175.45 (100%) | 184.79 (100%) |
| 10 | 231.70 (100%) | 252.97 (100%) | 12.99 (100%) | 15.04 (100%) |
| 11 | 46.18 (100%) | 60.89 (100%) | 7.70 (99%) | 1.27 (74%) |
| 12 | 0.00 (0%) | 0.00 (0%) | 0.00 (0%) | 0.00 (0%) |
| 13 | 0.00 (0%) | 0.00 (0%) | 0.00 (0%) | 0.00 (0%) |
| 14 | 0.00 (0%) | 0.00 (0%) | 0.00 (0%) | 0.00 (0%) |
| 15 | 0.00 (0%) | 0.00 (0%) | 0.00 (0%) | 0.00 (0%) |
| 16 | 0.00 (0%) | 0.00 (0%) | 0.00 (0%) | 0.00 (0%) |
| 17 | 0.00 (0%) | 0.00 (0%) | 0.00 (0%) | 0.00 (0%) |
| 18 | 0.00 (0%) | 0.00 (0%) | 0.00 (0%) | 0.00 (0%) |
| 19 | 0.18 (33%) | 0.01 (9%) | | |
| 20 | 0.90 (66%) | 6.96 (99%) | | |
| 21 | 0.00 (0%) | 4.62 (97%) | | |
| | **Nasdaq** | | **Dow Jones** | |
| | Pos. tail | Neg. tail | Pos. tail | Neg. tail |
| 1 | 19 335.00 (100%) | 18 201.00 (100%) | 28 910.00 (100%) | 24 749.00 (100%) |
| 2 | 7378.00 (100%) | 6815.00 (100%) | 9336.00 (100%) | 8377.00 (100%) |
| 3 | 4162.00 (100%) | 3795.00 (100%) | 5356.00 (100%) | 4536.00 (100%) |
| 4 | 2461.00 (100%) | 2311.00 (100%) | 3172.00 (100%) | 2832.00 (100%) |
| 5 | 1532.00 (100%) | 1520.00 (100%) | 1734.00 (100%) | 1681.00 (100%) |
| 6 | 853.00 (100%) | 933.00 (100%) | 930.00 (100%) | 933.00 (100%) |
| 7 | 491.00 (100%) | 555.00 (100%) | 483.00 (100%) | 466.00 (100%) |
| 8 | 248.00 (100%) | 301.00 (100%) | 177.00 (100%) | 218.00 (100%) |
| 9 | 78.60 (100%) | 141.00 (100%) | 68.00 (100%) | 98.00 (100%) |
| 10 | 16.10 (100%) | 28.00 (100%) | 16.00 (100%) | 27.00 (100%) |
| 11 | 5.70 (98%) | 16.00 (99%) | 6.69 (99%) | 16.00 (100%) |
| 12 | 0.10 (25%) | 3.03 (92%) | 5.71 (98%) | 9.00 (100%) |
| 13 | 0.14 (30%) | 2.17 (86%) | 3.70 (95%) | 9.90 (100%) |
| 14 | 0.00 (0%) | 1.04 (68%) | 3.48 (93%) | 7.90 (100%) |
| 15 | 0.00 (0%) | 0.15 (30%) | 3.73 (95%) | 5.40 (98%) |
| 16 | 0.00 (0%) | 0.03 (14%) | 1.77 (83%) | 0.01 (6%) |
| 17 | 0.00 (0%) | 0.13 (28%) | 0.73 (41%) | 0.30 (42%) |
| 18 | 0.00 (0%) | 0.00 (0%) | 0.00 (0%) | 0.00 (0%) |

must consider quantiles higher than $q_{16} = 99\%$—at least for the negative tail—in order not to reject $H_0$ at the 95% significance level. These results are in qualitative agreement with what we could expect from the action of the central limit theorem: the power-law regime (if it really exists) is pushed back to higher quantiles due to time aggregation (recall that the Dow Jones data are at the daily scale while the Nasdaq data are at the 5-min time scale).

It is more difficult to rationalize the non-rejection (at the 99% confidence level) of the SE model for the two highest quantiles of the negative tail of the S&P 500 1-min returns and for the quantiles $q_{19}$ and $q_{20}$ for its positive tail. This could be ascribed to the lack of power of the test, but recall that we have restricted our investigation to empirical quantiles with more than 100 points (or so). Therefore, such an argument is not very convincing. In addition, for these high quantiles, the fractional exponent $c$ in the SE model becomes significantly different from zero (see table 4). This may be an empirical illustration of the existence of a cut-off beyond which the power-law regime is replaced by an exponential (or stretched-exponential) decay of the distribution function as suggested by Mantegna and Stanley (1994), on theoretical grounds. To strengthen this idea, it can be seen that, in figures 4 and 5, the exponential distribution is found to be sufficient to describe the data for the S&P 500 1-min returns, while it is always rejected (with respect to the comprehensive distribution) for the Nasdaq and the Dow Jones. Thus, this non-rejection could actually be

Table 9. Wilks' test for the Pareto distribution versus the log-Weibull distribution. The $p$-value (figures within parentheses) gives the significance with which one can reject the null hypothesis that the Pareto distribution is sufficient to accurately describe the data.

| | SP 500 (1 min) | | SP 500 (30 min) | |
|---|---|---|---|---|
| | Pos. tail | Neg. tail | Pos. tail | Neg. tail |
| 1 | 1 022 836.93 (100%) | 989 235.67 (100%) | 127 153.62 (100%) | 53 486.54 (100%) |
| 2 | 313 946.54 (100%) | 312 166.92 (100%) | 17 455.52 (100%) | 15 720.42 (100%) |
| 3 | 186 778.42 (100%) | 183 746.08 (100%) | 9993.06 (100%) | 9324.65 (100%) |
| 4 | 103 170.08 (100%) | 98 617.18 (100%) | 5799.81 (100%) | 5560.24 (100%) |
| 5 | 74 200.31 (100%) | 71 636.07 (100%) | 3477.76 (100%) | 3407.46 (100%) |
| 6 | 55 334.83 (100%) | 54 293.16 (100%) | 2187.16 (100%) | 2030.71 (100%) |
| 7 | 19 375.91 (100%) | 18 862.86 (100%) | 1068.58 (100%) | 1055.03 (100%) |
| 8 | 10 036.11 (100%) | 10 386.04 (100%) | 556.30 (100%) | 510.67 (100%) |
| 9 | 2695.60 (100%) | 2664.22 (100%) | 235.08 (100%) | 195.24 (100%) |
| 10 | 555.74 (100%) | 441.28 (100%) | 18.93 (100%) | 38.15 (100%) |
| 11 | 211.77 (100%) | 267.91 (100%) | 15.35 (100%) | 6.32 (99%) |
| 12 | 16.92 (100%) | 14.28 (100%) | 2.68 (90%) | 1.44 (77%) |
| 13 | 0.35 (45%) | 0.17 (32%) | 0.36 (45%) | 0.41 (48%) |
| 14 | 13.51 (100%) | 3.73 (95%) | 0.12 (27%) | 0.21 (35%) |
| 15 | 42.05 (100%) | 12.79 (100%) | 0.57 (55%) | 1.00 (68%) |
| 16 | 49.58 (100%) | 56.52 (100%) | 5.08 (98%) | 1.75 (81%) |
| 17 | 43.61 (100%) | 57.67 (100%) | 0.99 (68%) | 2.84 (91%) |
| 18 | 73.25 (100%) | 40.91 (100%) | 1.51 (78%) | 2.74 (90%) |
| 19 | 0.04 (16%) | 1.86 (83%) | | |
| 20 | 0.00 (1%) | 0.07 (21%) | | |
| 21 | 0.39 (47%) | 7.28 (99%) | | |
| | **Nasdaq** | | **Dow Jones** | |
| | Pos. tail | Neg. tail | Pos. tail | Neg. tail |
| 1 | 17 235.00 (100%) | 16 689.00 (100%) | 27 632.00 (100%) | 23 670.00 (100%) |
| 2 | 6426.00 (100%) | 5834.00 (100%) | 8262.00 (100%) | 7313.00 (100%) |
| 3 | 3533.00 (100%) | 3134.00 (100%) | 4680.00 (100%) | 3933.00 (100%) |
| 4 | 2051.00 (100%) | 1795.00 (100%) | 2959.00 (100%) | 2497.00 (100%) |
| 5 | 1308.00 (100%) | 1209.00 (100%) | 1587.00 (100%) | 1482.00 (100%) |
| 6 | 698.00 (100%) | 730.00 (100%) | 853.00 (100%) | 817.00 (100%) |
| 7 | 426.00 (100%) | 421.00 (100%) | 442.00 (100%) | 414.00 (100%) |
| 8 | 226.00 (100%) | 222.00 (100%) | 164.00 (100%) | 172.00 (100%) |
| 9 | 57.90 (100%) | 127.00 (100%) | 62.40 (100%) | 84.90 (100%) |
| 10 | 22.90 (100%) | 30.50 (100%) | 15.80 (100%) | 14.00 (100%) |
| 11 | 9.77 (100%) | 22.90 (100%) | 2.09 (85%) | 6.91 (99%) |
| 12 | 0.01 (7%) | 0.51 (52%) | 2.48 (89%) | 4.35 (96%) |
| 13 | 0.68 (59%) | 1.56 (79%) | 2.05 (85%) | 2.40 (88%) |
| 14 | 0.19 (33%) | 0.89 (66%) | 1.25 (74%) | 7.88 (100%) |
| 15 | 0.07 (21%) | 0.60 (56%) | 2.89 (91%) | 9.12 (100%) |
| 16 | 0.31 (42%) | 0.10 (25%) | 1.53 (78%) | 0.00 (2%) |
| 17 | 2.21 (86%) | 0.31 (42%) | 1.14 (72%) | 0.91 (66%) |
| 18 | 2.17 (86%) | 0.03 (14%) | 1.03 (69%) | 0.85 (64%) |

the genuine signature of a cut-off beyond which the decay of the distribution is faster than any power law. However, this conclusion is only drawn from the 100 most extreme data points and, therefore, should be considered with caution. Larger samples should be considered to confirm this intuition. Unfortunately, samples with more than ten million (non-zero) data points (for a single asset) are not yet accessible.

Table 9 shows Wilks' test for the Pareto distribution versus the log-Weibull distribution. For quantiles above $q_{12}$, the Pareto distribution cannot be rejected in favour of the log-Weibull for the Dow Jones, the Nasdaq and the S&P 500 30-min returns. This parallels the lack of rejection of the Pareto distribution versus the stretched-exponential beyond the significance level $q_{12}$. The picture is different for the 1-min returns of the S&P 500. The Pareto model is almost always rejected. The most interesting point is the following: in the negative tail (we omit $q_{13}$ and $q_{14}$), the Pareto model is always strongly rejected except for the quantiles $q_{19}$ and $q_{20}$. Comparing with table 7, we clearly see that, between $q_{15}$ and $q_{18}$, the exponent $c$ is significantly (at the 95% significance level) *less than one*, indicating a tail fatter than any power law. On the contrary, for $q_{21}$, the exponent $c$ is found to be significantly *larger than one*, indicating a change of regime and again an ultimate decay of the tail of the distribution faster than any power law.

In summary, the stretched-exponential and log-Weibull models encompass the Pareto model as soon as one considers quantiles higher than $q_6 = 50\%$. The null hypothesis that the true distribution is the Pareto distribution is strongly rejected until quantiles 90–95% or so. Thus, within this range, the SE and SLE models seem to be the best and the Pareto model is insufficient to describe the data. But, for the very highest quantiles (above 95–98%), we can no longer reject the hypothesis that the Pareto model is sufficient compared with the SE and SLE models. These two parameter models can then be seen as a redundant parameterization for the extremes compared with the Pareto distribution, *except for the returns calculated at the smallest time scales*.

## 5. Discussion and conclusions: is there a best model of tails?

Our (re)-investigation of the properties of the stock returns distribution has been motivated by the fact that, for most practical applications, the relevant question is not to determine what is the true asymptotic tail, but what is the best effective description of the tails in the domain of useful applications. Our present work must thus be gauged as an attempt to provide a simple, efficient and effective description of the tails of the distribution of returns covering most of the range of interest for practical applications. We feel that the efforts requested to go deeper in the tails beyond those analysed here, while of great interest from a scientific point of view to potentially help unravel market mechanisms, may be too artificial and unattainable to have significant application.

That is why we have presented a statistical analysis of the tail behaviour of the distributions of the daily log-returns of the Dow Jones Industrial Average, of the 5-min log-returns of the Nasdaq Composite index and of the 1-min to 60-min log-returns of the Standard & Poor's 500 index. We have emphasized the practical aspects of the application of statistical methods to this problem. Although the application of statistical methods to the study of empirical distributions of returns seems to be an obvious approach, it is necessary to keep in mind the existence of the necessary conditions that the empirical data must obey for the conclusions of the statistical study to be valid. Perhaps the most important condition in order to talk meaningfully about distribution functions is the stationarity of the data, a difficult issue that we have barely touched upon here. In particular, the importance of regime switching is now well established (Ramcham and Susmel 1998, Ang and Bekeart 2001) and its possible role should be assessed and accounted for.

Our purpose here has been to revisit the generally accepted fact that the tails of the distributions of returns present power-like behaviour. Although there are some disagreements concerning the exact value of the power indices (the majority of previous workers accept index values between 3 and 3.5, depending on the particular asset and the investigated time interval), the power-like character of the tails of the distributions of returns is not in doubt. Often, the conviction of the existence of a power-like tail is based on the Gnedenko theorem stating the existence of only three possible types of limit distributions of normalized maxima (a finite maximum value, an exponential tail, and a power-like tail), together with the exclusion of the first two types by experimental evidence. The power-like character of the log-return tail $\bar{F}(x)$ then follows simply from the power-like distribution of the maxima. However, in this chain of arguments, the conditions needed for the fulfillment of the corresponding mathematical theorems are often omitted and not discussed properly. In addition, widely used arguments in favour of power-law tails invoke the *self-similarity* of the data, but are often *assumptions* rather than experimental evidence or consequences of economic and financial laws.

The results of a companion paper (Malevergne *et al.* 2005), that standard statistical estimators of heavy tails are much less efficient than often assumed and cannot in general clearly distinguish between a power-law tail and a stretched-exponential tail, can be rationalized by our discovery presented here that, in a certain limit, the stretched exponential pdf tends to the Pareto distribution. Thus, the Pareto (or power-law) distribution can be approximated with any desired accuracy on an arbitrary interval by a suitable adjustment of the parameters of the stretched-exponential model. Our parametric tests indicate that the class of stretched-exponential and log-Weibull distributions provides a significantly better fit to empirical returns than the Pareto, the exponential or the incomplete Gamma distributions. All our tests are consistent with the conclusion that these two models are the best effective apparent and parsimonious models to account for the empirical data on the largest possible range of returns.

However, this does not mean that the stretched-exponential (SE) or the log-Weibull model is the correct description of the tails of empirical distributions of returns. Again, as already mentioned, the strength of these models comes from the fact that they encompass the Pareto model in the tail and offer a better description in the bulk of the distribution. To see where the problem arises, we report in table 10 our best ML estimates for the SE parameters $c$ (form parameter) and $d$ (scale parameter) restricted to the quantile level $q_{12} = 95\%$, which offers a good compromise between a sufficiently large sample size and a restricted tail range, leading to an accurate approximation in this range.

One can see that $c$ is very small (and all the more so for the scale parameter $d$) for the tail of positive returns of the Nasdaq data, suggesting a convergence to a power-law tail. The exponents $c$ for the negative returns of the Nasdaq data and for both positive and negative returns of the Dow Jones data are an order of magnitude larger, but our tests show that they are not incompatible with an asymptotic power-tail. Indeed, we have shown in section 4.2 that, for the very highest quantiles (above 95–98%), we cannot reject the hypothesis that the Pareto model is sufficient compared with the SE model. The values of $c$ and $d$ are even smaller for the S&P 500 data, both at the 1-min and 5-min time scales.

Note also that the exponents $c$ are larger for the daily DJ data than for the 5-min ND data and the 1-min and 5-min S&P 500 data, in agreement with an expected

Table 10. Best parameters $c$ and $d$ of the stretched-exponential model estimated up to quantile $q_{12} = 95\%$. Figures within parentheses give the standard deviation of the estimates. The apparent Pareto exponent $c(u_{12}/d)^c$ (see expression (7)) is also shown. $u_{12}$ are the lower thresholds corresponding to the significance levels $q_{12}$ given in table 2.

| Sample | $c$ | $d$ | $c(u_{12}/d)^c$ |
|---|---|---|---|
| DJ positive returns | 0.274 (0.111) | $4.81 \times 10^{-6}$ ($2.49 \times 10^{-5}$) | 2.68 |
| DJ negative returns | 0.362 (0.119) | $1.02 \times 10^{-4}$ ($2.87 \times 10^{-4}$) | 2.57 |
| ND positive returns | 0.039 (0.138) | $4.54 \times 10^{-52}$ ($2.17 \times 10^{-49}$) | 3.03 |
| ND negative returns | 0.273 (0.155) | $1.90 \times 10^{-7}$ ($1.38 \times 10^{-6}$) | 3.10 |
| SP positive returns (1 min) | | | 3.01 |
| SP negative returns (1 min) | | | 2.97 |
| SP positive returns (5 min) | 0.033 (0.031) | $3.06 \times 10^{-59}$ ($4.82 \times 10^{-57}$) | 2.95 |
| SP negative returns (5 min) | 0.033 (0.031) | $3.26 \times 10^{-56}$ ($4.65 \times 10^{-54}$) | 2.87 |

(slow) convergence to the Gaussian law according to the central limit theory†. However, a $t$-test does not allow us to reject the hypothesis that the exponent $c$ remains the same for a given tail (positive or negative) of the Dow Jones data. Thus, we confirm previous results (Lux 1996, Jondeau and Rockinger 2001, for instance) according to which the extreme tails can be considered as symmetric, at least for the Dow Jones data. In contrast, we find a very strong asymmetry for the 5-min sampled Nasdaq and the S&P 500 data.

This is the evidence in favour of the existence of an asymptotic power-law tail. Balancing this, many of our tests have shown that the power-law model is not as powerful as the SE and SLE models, even arbitrarily far in the tail (as far as the available data allows us to probe). In addition, our results have shown that, for the smallest time scales, the tail of the distribution of returns is, over a large range, well described by a SLE distribution with an exponent $c$ of less than one, i.e. is fatter than any power law. A change of regime is ultimately observed and the very extreme tail decays faster than any power law. Both a SE and a SLE with exponent $c > 1$ provide a reasonable description.

Attempting to wrap up the different results obtained by the battery of tests presented here, we can offer the following conservative conclusion. It seems that the tails of the distributions examined here are decaying faster than any (reasonable) power law, but slower than any stretched exponentials. Perhaps log-normal distributions could offer a better effective description of the distribution of returns‡ as suggested by Serva *et al.* (2002).

In sum, in most practical cases, the PD is sufficient above quantiles $q_{12} = 95\%$, but is not sufficiently stable to ascertain with strong confidence the power-law asymptotic nature of the pdf. We refer to Malevergne and Sornette (2005) for a discussion of the implications of our results for risk assessment.

## References


Adcock, C., Jurczenko, E. and Maillet, B., *Multi-moment Capital Asset Pricing Models and Related Topics*, 2005 (Springer: Berlin), forthcoming.

Anderson, T.W. and Darling, D.A., Asymptotic theory of certain 'goodness of fit' criteria. *Annals of Mathematical Statistics*, 1952, **23**, 193–212.

Ang, A. and Bekeart, G., International asset allocation with regime shifts. *Review of Financial Studies*, 2001, **15**, 1137–1187.

Barndorff-Nielsen, O.E., Normal inverse Gaussian distributions and the modeling of stock returns. *Scandinavian Journal of Statistics*, 1997, **24**, 1–13.

Biham, O., Malcai, O., Levy, M. and Solomon, S., Generic emergence of power law distributions and Levy–Stable intermittent fluctuations in discrete logistic systems. *Physical Review* E, 1998, **58**, 1352–1358.

Blattberg, R. and Gonnedes, N., A comparison of stable and Student distribution as statistical models for stock prices. *Journal of Business*, 1974, **47**, 244–280.

Bollerslev, T., Generalized autoregressive conditional heteroscedasticity. *Journal of Econometrics*, 1986, **31**, 307–327.

Bollerslev, T., Engle, R.F. and Nelson, D.B., ARCH models. In *Handbook of Econometrics*, Vol. IV, edited by R.F. Engle and D.L. McFadden, pp. 2959–3038, 1994 (Amsterdam, North Holland).

Bouchaud, J.-P. and Potters, M., *Theory of Financial Risks: From Statistical Physics to Risk Management*, 2000 (Cambridge University Press: Cambridge).

Challet, D. and Marsili, M., Criticality and market efficiency in a simple realistic model of the stock market. *Phys. Rev. E*, 2003, **68**, 036132.

Cont, R., Potters, M. and Bouchaud, J.-P., Scaling in stock market data: stable laws and beyond. In *Scale Invariance and Beyond*, edited by B. Dubrulle, F. Graner and D. Sornette, 1997 (Springer: Berlin).

De Vries, C.G., Stylized facts of nominal exchange rate returns. In *The Handbook of International Macroeconomics*, edited by F. van der Ploeg, pp. 348–389, 1994 (Blackwell: Oxford).

Diebolt, J. and Guegan, D., Le modèle de série chronologique autorégressive $\beta$-ARCH. *Acad. Sci. Paris*, 1991, **312**, 625–630.


†See Sornette *et al.* (2000) and figures 3.6–3.9, pp. 81–82, of Sornette (2004), where it is shown that SE distributions are approximately stable as a family and the effect of aggregation can be seen to slowly increase the exponent $c$. See also Drozdz *et al.* (2002), which specifically studies this convergence to a Gaussian law as a function of the time scale level.

‡Let us stress that we are speaking of a log-normal distribution of returns, not of price! Indeed, the standard Black and Scholes model of a log-normal distribution of prices is equivalent to a Gaussian distribution of returns. Thus, a log-normal distribution of returns is much more fat tailed, and in fact bracketed by power law tails and stretched-exponential tails.

Dragulescu, A.A. and Yakovenko, V.M., Probability distribution of returns in the Heston model with stochastic volatility. *Quantitative Finance*, 2002, **2**, 443–453.

Drozdz, S., Kwapien, J., Gruemmer, F., Ruf, F. and Speth, J., Are the contemporary financial fluctuations sooner converging to normal? *Physica Polonica* B, 2002, **34**, 4293–4306.

Eberlein, E., Keller, U. and Prause, K., New insights into smile, mispricing and value at risk: the hyperbolic model. *Journal of Business*, 1998, **71**, 371–405.

Eberlein, E. and Özkan, F., Time consistency of Lévy models. *Quantitative Finance*, 2003, **3**, 40–50.

Embrechts P., Klüppelberg, C.P. and Mikosh, T., *Modelling Extremal Events*, 1997 (Springer: Berlin).

Engle, R.F., Autoregressive conditional heteroskedasticity with estimation of the variance of UK inflation. *Econometrica*, 1982, **50**, 987–1008.

Engle, R.F. and Patton, A.J., What good is a volatility model? *Quantitative Finance*, 2001, **1**, 237–245.

Fama E.F., The behavior of stock market prices. *Journal of Business*, 1965, **38**, 34–105.

Fang, H. and Lai, T., Co-kurtosis and capital asset pricing. *Financial Review*, 1997, **32**, 293–307.

Farmer, J.D., Physicists attempt to scale the ivory towers of finance. *Computing in Science and Engineering (IEEE)*, 1999, **1**(6), Nov/Dec, 26–39.

Feller, W., *An Introduction to Probability Theory and its Applications*, Vol. II, 1971 (Wiley: New York).

Frisch, U. and Sornette, D., Extreme deviations and applications. *J. Phys. I France*, 1997, **7**, 1155–1171.

Gabaix, X., Gopikrishnan, P., Plerou, V. and Stanley, H.E., A theory of power-law distributions in financial market fluctuations. *Nature*, 2002, **423**, 267–270.

Gopikrishnan, P., Meyer, M., Amaral, L.A.N. and Stanley, H.E., Inverse cubic law for the distribution of stock price variations. *European Physical Journal* B, 1998, **3**, 139–140.

Gopikrishnan, P., Plerou, V., Amaral, L.A.N., Meyer, M. and Stanley, H.E., Scaling of the distributions of fluctuations of financial market indices. *Physics Review* E, 1999, **60**, 5305–5316.

Gouriéroux, C. and Jasiak, J., Truncated maximum likelihood, goodness of fit tests and tail analysis. Working Paper, CREST, 1998.

Gouriéroux, C. and Monfort, A., Testing non nested hypothesis. *Handbook of Econometrics*, 1994, **4**, 2585–2637.

Guillaume, D.M., Dacorogna, M.M., Davé, R.R., Müller, U.A., Olsen, R.B. and Pictet, O.V., From the bird's eye to the microscope: a survey of new stylized facts of the intra-day foreign exchange markets. *Finance and Stochastics*, 1997, **1**, 95–130.

Hill, B.M., A simple general approach to inference about the tail of a distribution. *Annals of Statistics*, 1975, **3**, 1163–1174.

Hwang, S. and Satchell, S., Modelling emerging market risk premia using higher moments. *International Journal of Finance and Economics*, 1999, **4**, 271–296.

Johansen, A. and Sornette, D., Large stock market price drawdowns are outliers. *Journal of Risk*, 2002, **4**, 69–110.

Jondeau, E. and Rockinger, M., Testing for differences in the tails of stock-market returns. Working Paper, 2001. Available online at: http://papers.ssrn.com/paper.taf?abstract_id=291399

Jurcenko, E. and Maillet, B., The four-moment capital asset pricing model: some basic results. Working Paper, 2002.

Kass, R.E. and Raftery, A.E., Bayes factors. *Journal of the American Statistical Association*, 1995, **90**, 773–795.

Kearns, P. and Pagan, A., Estimating the density tail index for financial time series. *Review of Economics and Statistics*, 1997, **79**, 171–175.

Kon, S., Models of stock returns: a comparison. *Journal of Finance*, 1984, **39**, 147–165.

Laherrère, J. and Sornette, D., Stretched exponential distributions in nature and economy: fat tails with characteristic scales. *European Physical Journal* B, 1999, **2**, 525–539.

Longin, F.M., The asymptotic distribution of extreme stock market returns. *Journal of Business*, 1996, **96**, 383–408.

Longin, F.M., From value at risk to stress testing: the extreme value approach. *Journal of Banking and Finance*, 2000, **24**, 1097–1130.

Lux, T., The stable Paretian hypothesis and the frequency of large returns: an examination of major German stocks. *Applied Financial Economics*, 1996, **6**, 463–475.

Lux, T., The limiting extreme behavior of speculative returns: an analysis of intra-daily data from the Francfurt Stock Exchange. Preprint, March 2000, 29 pp.

Malevergne, Y. and Sornette, D., Extreme financial risks (from dependence to risk management). In *Mathematical Finance Monograph*, 2005 (Springer: Berlin), in press.

Malevergne, Y., Pisarenko, V. and Sornette, D., On the power of Generalized Extreme Value (GEV) and Generalized Pareto Distribution (GPD) estimators for empirical distributions of log-returns. *Appl. Financ. Econ.*, 2005, forthcoming.

Mandelbrot, B., The variation of certain speculative prices. *Journal of Business*, 1963, **36**, 392–417.

Mantegna, R.N. and Stanley, H.E., Stochastic process with ultraslow convergence to a Gaussian: the truncated Lévy flight. *Physical Review Letters*, 1994, **73**, 2946–2949.

Mantegna, R.N. and Stanley, H.E., Scaling behavior of an economic index. *Nature*, 1995, **376**, 46–55.

Mantegna, R.N. and Stanley, H.E., *An Introduction to Econophysics, Correlations and Complexity in Finance*, 2000 (Cambridge University Press: Cambridge, UK).

Matia, K., Amaral, L.A.N., Goodwin, S.P. and Stanley, H.E., Different scaling behaviors of commodity spot and future prices. *Physical Review* E, 2002, **66**, 045103.

McNeil, A. and Frey, R., Estimation of tail-related risk measures for heteroscedastic financial time series: an extreme value approach. *Journal of Empirical Finance*, 2000, **7**, 271–300.

Mittnik, S., Rachev, S.T. and Paolella, M.S., Stable Paretian modeling in finance: some empirical and theoretical aspects. In *A Practical Guide to Heavy Tails*, edited by R.J. Adler, R.E. Feldman and M.S. Taqqu, pp. 79–110, 1998 (Birkhauser: Boston).

Mizuno, T., Kurihara, S., Takayasu, M. and Takayasu, H., Analysis of high-resolution foreign exchange data of USD-JPY for 13 years. *Physica* A, 2002, **324**, 296–302.

Müller, U.A., Dacarogna, M.M. and Picktet, O.V., Heavy tails in high-frequency financial data. In *A Practical Guide to Heavy Tails*, edited by R.J. Adler, R.E. Feldman and M.S. Taqqu, pp. 55–78, 1998 (Birkhauser: Boston).

Muzy, J.-F., Delour, J. and Bacry, E., Modelling fluctuations of financial time series: from cascade process to stochastic volatility model. *European Physical Journal*, 2000, **17**, 537–548.

Muzy, J.-F., Sornette, D., Delour, J. and Arneodo, A., Multifractal returns and Hierarchical Portfolio Theory. *Quantitative Finance*, 2001, **1**, 131–148.

Nagahara, Y. and Kitagawa, G., A non-Gaussian stochastic volatility model. *Journal of Computational Finance*, 1999, **2**, 33–47.

Picoli, Jr., S., Mendes, R.S. and Malacarne, L.C., $q$-Exponential, Weibull, and $q$-Weibull distributions: an empirical analysis. *Physica* A, 2003, **324**, 678–688.

Plerou, V., Gopikrishnan, P., Amaral, L.A.N., Meyer, M. and Stanley, H.E., Scaling of the distribution of price fluctuations of individual companies. *Physics Review* E, 1999, **60**, 6519–6529.

Prause, K., The generalized hyperbolic model. PhD dissertation, University of Freiburg, 1998.

Ramshand, L. and Susmel, R., Volatility and cross correlation across major stock markets. *Journal of Empirical Finance*, 1998, **5**, 397–416.

Rubinstein, M., The fundamental theorem of parameter-preference security valuation. *Journal of Financial and Quantitative Analysis*, 1973, **8**, 61–69.

Serva, M., Fulco, U.L., Lyra, M.L. and Viswanathan, G.M., Kinematics of stock prices. Preprint, 2002. Available online at: http://arXiv.org/abs/cond-mat/0209103

Sobehart, J. and Farengo, R., Fat-tailed bulls and bears. *Risk Investors*, 2002, **15**(12), December, 20–24.

Sornette, D., *Critical Phenomena in Natural Sciences, Chaos, Fractals, Self-organization and Disorder: Concepts and Tools*, 2nd ed., 2004 (Springer Series in Synergetics) (Springer: Heidelberg).

Sornette, D., Simonetti, P. and Andersen, J.V., $\phi^q$-Field theory for portfolio optimization: 'fat tails' and non-linear correlations. *Physics Report*, 2000, **335**, 19–92.

Starica, C. and Pictet, O., The tales the tails of GARCH(1,1) process tell. Working Paper, University of Pennsylvania, 1999.

Stuart, A. and Ord, K., *Kendall's Advances Theory of Statistics*, 1994 (Wiley: New York).

Wilks, S.S., The large sample distribution of the likelihood ratio for testing composite hypotheses. *Annals of Mathematical Statistics*, 1938, **9**, 60–62.