Replica Approach for Minimal Investment Risk with Cost

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In the present work, the optimal portfolio minimizing the investment risk with cost is discussed analytically, where this objective function is constructed in terms of two negative aspects of investment, the risk and cost. We note the mathematical similarity between the Hamiltonian in the mean-variance model and the Hamiltonians in the Hopfield model and the Sherrington–Kirkpatrick model and show that we can analyze this portfolio optimization problem by using replica analysis, and derive the minimal investment risk with cost and the investment concentration of the optimal portfolio. Furthermore, we validate our proposed method through numerical simulations.

1. Introduction

The portfolio optimization problem is one of the most actively researched topics in mathematical finance, coming from the theory of diversification investment management put forth by Markowitz in his pioneer works in 1952 and 1959.[1,2] In mathematical finance, (especially operations research), investment optimality in some practical situations has been discussed,[3–5] but only analysis of an annealed disordered system in the literature of spin glass has been discussed for the portfolio optimization problem, whereas analysis of the quenching system which is desired by rational investors has been given little attention. Recently, however, such analysis of the quenched disordered system desired by rational investors in the context of diversified investment has started to be investigated using the analytical approaches developed in statistical mechanical informatics and econophysics.[6–23] For instance, Ciliberti *et al.* examined the investment risk of the absolute deviation model and the expected shortfall model in the portfolio

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optimization problem with a budget constraint by using replica analysis. Specifically, they analyzed the typical behavior of the ground state in the limit of absolute zero temperature (the optimal solution of the portfolio optimization problem).[6,7] Pafka *et al.* compared the eigenvalue distribution of the variance-covariance matrix derived from practical data with the eigenvalue distribution of the variance-covariance matrix defined by novel variables mapped by Cholesky decomposition and discussed three types of investment risks in diversification investment.[8] Kondor *et al.* evaluated the relationship between noise and estimated error of each optimal portfolio with respect to several risk models: the mean-variance model, the absolute deviation model, the expected shortfall model, and the max-loss model.[9] Caccioli *et al.* used replica analysis to determine whether the optimal solution of the expected shortfall model with ridge regression is stable.[10] Furthermore, Shinzato *et al.* replaced the portfolio optimization problem including a budget constraint with an inference problem using the Boltzmann distribution and derived analytically the trial distribution which can approximate the Boltzmann distribution based on the Kullback–Leibler information criterion using a belief propagation method. They also derived the faster solver algorithm for the optimal solution using the trial distribution.[11]

As described above, various investment models have been examined using replica analysis and a belief propagation method in these previous studies,[12–23] but in recent years, attention has been given to the mathematical similarity between the Hopfield model and the most representative investment models, that is mean-variance model. For instance, Shinzato showed with the Chernoff inequality and replica analysis that the investment risk of the mean-variance model and the investment concentration of the optimal portfolio satisfy the self-averaging property.[12] In addition, Shinzato analyzed the minimization problem of investment risk with constraints of budget and investment concentration by using replica analysis, comparing the results with those of a previous work,[12] as well as analyzing the influence of the investment concentration constraint on the optimal portfolio.[13] Moreover, Shinzato further investigated the maximization problem of investment concentration with constraints of budget and investment risk in a previous work[13] and the corresponding minimization problem as a counterpart, and derived the mathematical structures of the two optimal portfolios of the primal–dual optimization problems.[14] Further, Tada *et al.* resolved the primal–dual optimization problems by using Stieltjes transformation of the asymptotical eigenvalue distribution of the Wishart matrix in order to validate the findings in previous works[13,14] where

the analysis used replica analysis.[15] That is, they reexamined the minimization problem of investment risk with constraints of budget and investment concentration (and the corresponding maximization problem) and the maximization problem of investment concentration with constraints of budget and investment risk (and the corresponding minimization problem) without using replica analysis or the replica symmetry ansatz. In addition, Shinzato considered the minimization problem of investment risk with constraints of budget and expected return, and the maximization problem of expected return with constraints of budget and investment risk as a primal–dual optimization problem, analyzing them by using replica analysis and reexamining the relationship between the two optimal portfolios.[16] Varga-Haszonits et al. generalized the minimization problem of investment risk with constraints of budget and expected return that was considered in the work by Shinzato[16] and analyzed the stability of the replica symmetry solution.[17] Shinzato examined the minimization problem of investment risk with constraints of budget and expected return by using replica analysis and derived a macroscopic theory like the Pythagorean theorem of the Sharpe ratio and opportunity loss.[18] In addition, Shinzato analyzed the minimization problem of investment risk with a budget constraint when the variance of the asset return is not unique using replica analysis and a belief propagation method and calculated the minimum investment risk per asset and the investment concentration of the optimal portfolio.[19] Furthermore, using the asymptotic eigenvalue distribution of the Wishart matrix, Shinzato in a previous work[20] reexamined the minimization problem of investment risk per asset with constraints of budget and investment concentration of the optimal portfolio handled in the earlier work.[19] As a related case, Shinzato examined the minimization problem of investment risk with a budget constraint when the return is characterized by a single-factor model by using replica analysis and succeeded in quantifying the influence of common factors included in the minimal investment risk.[21] Moreover, Shinzato examined the minimization problem of investment risk with constraints of budget and short-selling by using replica analysis when the asset returns are independently and identically distributed, and confirmed that the minimal investment risk per asset based on the replica symmetric ansatz has a first-order phase transition.[22] Following Shinzato's results, Kondor et al. examined the problem of the minimization of a specific type of risk function with constraints of budget and short-selling by using replica analysis for the case that each asset return is not necessarily distributed identically for all assets, and clarified that their minimal risk function has a first-order phase transition.[23]

As described above, at various investment opportunities, objective criteria (such as investment risk, purchase cost, expected return, and investment concentration) that rational investors hope to know have been examined using the approaches of a quenched disordered system (e.g., replica analysis and a belief propagation method). However, it is also known that rational investors do not directly use only these objective criteria, but rather investment activities are carried out based on each investor's utility function.[24, 25] Such a utility function is based on investment preferences (namely, risk averse/risk neutral/risk loving) of each investor, and furthermore, the utility function involves a combination of investment risk, purchase cost, and expected return. Among the previous cross-disciplinary research, few studies discussed the utility function, so it has been difficult to build a theory that appropriately supports investment decisions by rational investors.

Therefore, in order to provide a seamless connection between the analytical approach discussed in previous works[6–23] and the analysis of utility functions, that is, as a first step of an analysis of utility functions, we examine the minimization problem of a loss function defined by two objective criteria under a budget constraint by using replica analysis. In particular, we assume the utility function of the rational investors whose hope is to reduce two negative aspects of investment, the investment risk (fluctuation risk of the held asset occurring during the investment period) and purchasing (or selling) cost (cost incurred in investing).

The remainder of the paper is organized as follow. In the next section, the portfolio optimization problem with a budget constraint for minimizing the loss function defined by the investment risk and purchasing cost (which we refer to hereafter as the investment risk with cost) is formulated. Section 3 demonstrates that the computation complexity for finding the optimal portfolio minimizing the investment risk with cost is increasing with the number of assets by an analysis of this portfolio optimization problem with the Lagrange multiplier method, and therefore that it is difficult to evaluate this problem in practical situations. In section 4, with the aim of avoiding this computational difficulty when using the Lagrange multiplier method, we assess the minimal investment risk with cost per asset and its investment concentration by using replica analysis. Further, we compare the findings obtained by our proposed method with the minimal expected investment risk with cost and its investment concentration derived from the analytical procedure in previous work. In section 5, the effectiveness of our proposed method is verified by numerical simulations. The final section is devoted to summarizing the

present study and discussing future research.

2. Model Setting

In the present work, we consider the portfolio optimization problem with a budget constraint in which one invests in N assets at each of p periods in a stable investment market with no restrictions on short-selling and show the properties of the optimal portfolio minimizing the objective function defined by the two loss functions capturing the negative aspects of investment, investment risk and purchasing cost. First, the portfolio of asset $i(= 1, 2, \cdots, N)$ is $w_i \in \mathbf{R}$, and the portfolio of all N assets is $\vec{w} = (w_1, w_2, \cdots, w_N)^{\mathrm{T}} \in \mathbf{R}^N$. The notation T indicates the transpose of a vector or matrix and, using the same setting as in previous works[6–23] , the budget constraint of the portfolio $\vec{w}$ is defined as

$$\sum_{i=1}^{N} w_i = N. \tag{1}$$

In addition, the return of asset i at period $\mu(= 1, 2, \cdots, p)$ is represented by $\bar{x}_{i\mu}$, and is independently distributed according to some distribution with mean $E[\bar{x}_{i\mu}] = r_i$ and variance $V[\bar{x}_{i\mu}] = v_i$. Moreover, purchasing cost per portfolio of asset i at the first period of investment is c_i. Using this notation, the investment risk and total purchasing cost are giving by

$$\begin{aligned}
Risk &= \frac{1}{2N} \sum_{\mu=1}^{p} \left(\sum_{i=1}^{N} w_i \bar{x}_{i\mu} - \sum_{i=1}^{N} w_i r_i \right)^2 \\
&= \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \left(\frac{1}{N} \sum_{\mu=1}^{p} x_{i\mu} x_{j\mu} \right),
\end{aligned} \tag{2}$$

$$Cost = \sum_{i=1}^{N} w_i c_i. \tag{3}$$

Since the first term of the first line in Eq. (2), $\sum_{i=1}^{N} w_i \bar{x}_{i\mu}$, describes the total return at period μ and the second term $\sum_{i=1}^{N} w_i r_i$ represents its expectation, the investment risk is defined by the sum of the squared of differences between the total return at each period, $\sum_{i=1}^{N} w_i \bar{x}_{i\mu}$, and the expected total return $\sum_{i=1}^{N} w_i r_i$. Further, for the sake of simplicity, here the modified return $x_{i\mu} = \bar{x}_{i\mu} - r_i$ is used; note that the mean and the variance of the modified return $x_{i\mu}$ are $E[x_{i\mu}] = 0$ and $V[x_{i\mu}] = v_i$, respectively. Eq. (3) represents the total purchasing cost.

Based on the above model setting, as the objective function, using the cost toler-

ance $\eta(> 0)$, the investment risk plus the total purchasing cost at the first period of investment is represented as $\mathcal{H}(\vec{w}|X, \vec{c}) = Risk + \eta \times Cost$ (and is what we are calling the investment risk with cost), and is expressed as

$$\mathcal{H}(\vec{w}|X, \vec{c}) = \frac{1}{2}\vec{w}^{\mathrm{T}}J\vec{w} + \eta\vec{c}^{\mathrm{T}}\vec{w}, \tag{4}$$

where the variance-covariance matrix (that is, the Wishart matrix) defined by the modified return $x_{i\mu}$, $J = \{J_{ij}\} \in \mathbf{R}^{N \times N}$, and cost vector $\vec{c} = (c_1, c_2, \cdots, c_N)^{\mathrm{T}} \in \mathbf{R}^N$ are used in Eq. (4). Specifically, the (i, j)th component of Wishart matrix J is $J_{ij} = \frac{1}{N}\sum_{\mu=1}^{p} x_{i\mu}x_{j\mu}$. Moreover, using return matrix $X = \left\{\frac{x_{i\mu}}{\sqrt{N}}\right\} \in \mathbf{R}^{N \times p}$, $J = XX^{\mathrm{T}}$ is also defined. From the definition of the investment risk with cost in Eq. (4), cost tolerance η is the tolerance degree of the investor with respect to the added cost.

One point should be noticed here. The investment risk with cost discussed in this work, $\mathcal{H}(\vec{w}|X, \vec{c})$, is regarded as the Hamiltonian in this investment system, which allows us to apply several analytical approaches developed in spin glass theory to analyze the typical behaviors of the optimal portfolio of this portfolio optimization problem multi-directionally. The reason for this is that, given N Ising spins $\vec{S} = (S_1, S_2, \cdots, S_N)^{\mathrm{T}} \in \{\pm 1\}^N$ and extremal magnetic field $\vec{h} = (h_1, h_2, \cdots, h_N)^{\mathrm{T}} \in \mathbf{R}^N$, square symmetric matrix J represents Hebb's law in the case of the Hopfield model and/or the RKKY interaction matrix in the case of the Sherrington-Kirkpatrick (SK) model. The Hamiltonian of the Hopfield or SK model $\mathcal{H}(\vec{S})$ is defined by

$$\begin{aligned}\mathcal{H}(\vec{S}) &= -\sum_{i>j}J_{ij}S_iS_j - \sum_{i=1}^{N}h_iS_i \\ &= -\frac{1}{2}\vec{S}^{\mathrm{T}}J\vec{S} - \vec{h}^{\mathrm{T}}\vec{S}, \end{aligned} \tag{5}$$

where the notation $\sum_{i>j}$ means the sum over all pairs (i, j) satisfying $i > j$. Comparing Eqs. (4) and (5), it is easily seen that they are mathematically similar with respect to these two models. Moreover, Wishart matrix $J = XX^{\mathrm{T}} \in \mathrm{R}^{N \times N}$ defined in Eq. (4) is related to Hebb's law in the Hopfield model and the aim of both problems is to minimize the Hamiltonian. Therefore, using replica analysis and belief propagation developed in fields engaged in cross-disciplinary research such as spin glass theory and statistical mechanical informatics, we can analyze the portfolio optimization problem and derive several novel insights for diversification investment theory. That is, the optimal portfolio minimizing the Hamiltonian constructed from RKKY interaction terms only (i.e., $\eta = 0$) by using the analytical approach for a quenched disordered system has been investigated

in previous works, where it has been shown that it is difficult to analyze the quenched disordered system (i.e., the rational investors can be regarded as in spin glass theory) by using the analytical approach developed in operations research (i.e., the approach for an annealed disordered system).[6–23] As the natural extension of previous works,[12–14] we here add terms of external magnetic fields to investment risk, that is, the total cost, in order to attempt to construct and analyze a utility function and thereby create a macroscopic theory, which would enrich the theory of optimal investment risk.

Under the above assumptions, in the limit of a large number of assets N, the minimal investment risk with cost per asset ε is

$$\varepsilon = \lim_{N\to\infty} \frac{1}{N} \min_{\vec{w}\in\mathcal{W}} \mathcal{H}(\vec{w}|X,\vec{c}), \tag{6}$$

where the feasible subset of portfolio $\vec{w}$, $\mathcal{W} = \left\{ \vec{w} \in \mathbf{R}^N \, \middle| \, \vec{w}^{\mathrm{T}}\vec{e} = N \right\}$ and the vector of ones $\vec{e} = (1,1,\cdots,1)^{\mathrm{T}} \in \mathbf{R}^N$ are used. From a previous work,[12] this minimal investment risk with cost satisfies the property of self-averaging. Moreover, from the definition of Eq. (6), the minimal investment risk with cost is related to the analysis of a quenched disordered system. On the other hand, from the literature of operations research, the minimal expected investment risk with cost per asset $\varepsilon^{\mathrm{OR}}$ is

$$\varepsilon^{\mathrm{OR}} = \lim_{N\to\infty} \frac{1}{N} \min_{\vec{w}\in\mathcal{W}} E_X[\mathcal{H}(\vec{w}|X,\vec{c})], \tag{7}$$

where $E_X[g(X)]$ is the configuration average of the function $g(X)$. Equation (7) shows that this description is related to the analysis of an annealed disordered system. Therefore, the goal of the present work is also to derive and examine the optimal investment strategy of the portfolio optimization problem with rational investors, so we will discuss ε in Eq. (6) in detail, but not $\varepsilon^{\mathrm{OR}}$ in Eq. (7).

3. Lagrange Multiplier Method

Here, given return matrix $X = \left\{ \frac{x_{i\mu}}{\sqrt{N}} \right\} \in \mathbf{R}^{N\times p}$ and using the Lagrange multiplier method, the minimal investment risk with cost per asset ε and its investment concentration q_w are analytically evaluated. Lagrange function $\mathcal{L}(\vec{w},k)$ for the minimization problem of the investment risk with cost in Eq. (4), $\mathcal{H}(\vec{w}|X,\vec{c})$ under the budget constraint in Eq. (1), is defined by

$$\mathcal{L}(\vec{w},k) = \frac{1}{2}\vec{w}^{\mathrm{T}}J\vec{w} + \eta\vec{c}^{\mathrm{T}}\vec{w} + k(N - \vec{w}^{\mathrm{T}}\vec{e}), \tag{8}$$

where auxiliary variable k is the Lagrange multiplier variable with respect to the budget constraint in Eq. (1).

The extremum of $\mathcal{L}(\vec{w}, k)$ satisfies $\frac{\partial \mathcal{L}(\vec{w},k)}{\partial \vec{w}} = 0$ and $\frac{\partial \mathcal{L}(\vec{w},k)}{\partial k} = 0$, so the minimal investment risk with cost per asset is

$$\varepsilon = \frac{1}{2} \frac{\left(1 + \frac{\eta}{N} \vec{e}^{\mathrm{T}} J^{-1} \vec{c}\right)^2}{\frac{1}{N} \vec{e}^{\mathrm{T}} J^{-1} \vec{e}} - \frac{\eta^2}{2} \frac{1}{N} \vec{c}^{\mathrm{T}} J^{-1} \vec{c}. \tag{9}$$

Moreover, the investment concentration $q_w = \frac{1}{N} \sum_{i=1}^{N} (w_i^*)^2$ of the optimal portfolio $\vec{w}^* = \arg\min_{\vec{w} \in \mathcal{W}} \mathcal{H}(\vec{w}|X, \vec{c}) = (w_1^*, w_2^*, \cdots, w_N^*)^{\mathrm{T}} \in \mathbf{R}^N$ is

$$q_w = \frac{\vec{e}^{\mathrm{T}} J^{-2} \vec{e}}{N} \left(\frac{N}{\vec{e}^{\mathrm{T}} J^{-1} \vec{e}} + \eta \left(\frac{\vec{e}^{\mathrm{T}} J^{-1} \vec{c}}{\vec{e}^{\mathrm{T}} J^{-1} \vec{e}} - \frac{\vec{e}^{\mathrm{T}} J^{-2} \vec{c}}{\vec{e}^{\mathrm{T}} J^{-2} \vec{e}} \right) \right)^2$$

$$+ \eta^2 \frac{\vec{e}^{\mathrm{T}} J^{-2} \vec{e}}{N} \left(\frac{\vec{c}^{\mathrm{T}} J^{-2} \vec{c}}{\vec{e}^{\mathrm{T}} J^{-2} \vec{e}} - \left(\frac{\vec{e}^{\mathrm{T}} J^{-2} \vec{c}}{\vec{e}^{\mathrm{T}} J^{-2} \vec{e}} \right)^2 \right). \tag{10}$$

In the evaluation of the minimal investment risk with cost per asset ε and the investment concentration of the optimal portfolio q_w, we need to assess six moments, $\frac{1}{N} \vec{e}^{\mathrm{T}} J^{-1} \vec{e}$, $\frac{1}{N} \vec{e}^{\mathrm{T}} J^{-1} \vec{c}$, $\frac{1}{N} \vec{c}^{\mathrm{T}} J^{-1} \vec{c}$, and also $\frac{1}{N} \vec{e}^{\mathrm{T}} J^{-2} \vec{e}$, $\frac{1}{N} \vec{e}^{\mathrm{T}} J^{-2} \vec{c}$, $\frac{1}{N} \vec{c}^{\mathrm{T}} J^{-2} \vec{c}$, and also the inverse matrices J^{-1} and J^{-2}. However, computing these inverse matrices accurately requires an $O(N^3)$ computation. Thus, we have the problem that as the number of assets N becomes larger, of course, so does the computation complexity. As the number of assets is typically $N = 10^3$ to 10^5, it is not easy to assess directly either ε in Eq. (9) or q_w in Eq. (10). In the following section, therefore, we avoid the computation of the inverse of the Wishart matrix and propose a method for effectively analyzing the minimal investment risk with cost and the investment concentration of the optimal solution.

4. Replica Analysis

Here, following previous works,[6–23] we consider the minimal investment risk with cost per asset ε and its investment concentration q_w in terms of replica analysis. First, $\mathcal{H}(\vec{w}|X, \vec{c})$ in Eq. (4) is regarded as the Hamiltonian of this investment system. The partition function of the investment market (at inverse temperature β), $Z(X)$, is defined by

$$Z(X) = \int_{\mathcal{W}} d\vec{w} e^{-\beta \mathcal{H}(\vec{w}|X,\vec{c})}, \tag{11}$$

where $\mathcal{W}$ is the subspace of feasible portfolios in Eq. (1). Furthermore, using this description of the partition, from the identity function

$$\varepsilon = -\lim_{\beta \to \infty} \left\{ \frac{\partial}{\partial \beta} \lim_{N \to \infty} \frac{1}{N} E_X[\log Z(X)] \right\}, \tag{12}$$

it is known that the typical behavior of the minimal investment risk with cost per asset can be evaluated.[23] Similar to in this previous work, in order to assess the configuration

average of the logarithm of the partition function $E_X[\log Z(X)]$, we need to analyze the nth moment $E_X[Z^n(X)]$ at $n \in \mathbf{Z}$. That is,

$$\lim_{N \to \infty} \frac{1}{N} \log E_X[Z^n(X)]$$

$$= \operatorname*{Extr}_{\Theta} \left\{ \frac{1}{2} \mathrm{Tr} Q_w \tilde{Q}_w + \frac{1}{2} \mathrm{Tr} Q_s \tilde{Q}_s - \vec{k}^{\mathrm{T}} \vec{e} \right.$$

$$- \frac{\alpha}{2} \log \det |I + \beta Q_s| - \frac{1}{2} \left\langle \log \det \left| \tilde{Q}_w + v \tilde{Q}_s \right| \right\rangle$$

$$\left. + \frac{1}{2} \left\langle \left(\vec{k} - \beta \eta c \vec{e} \right)^{\mathrm{T}} \left(\tilde{Q}_w + v \tilde{Q}_s \right) \left(\vec{k} - \beta \eta c \vec{e} \right) \right\rangle \right\},$$

$$(13)$$

is expanded, where $Q_w = \{q_{wab}\}$ and $Q_s = \{q_{sab}\}$ are order parameters (with auxiliary parameters $\tilde{Q}_w = \{\tilde{q}_{wab}\}, \tilde{Q}_s = \{\tilde{q}_{sab}\} \in \mathbf{R}^{n \times n}, \vec{k} = (k_1, k_2, \cdots, k_n)^{\mathrm{T}} \in \mathbf{R}^n, (a, b = 1, 2, \cdots, n))$. Then the set of order parameters is $\Theta = \left\{ Q_w, Q_s, \tilde{Q}_w, \tilde{Q}_s, \vec{k} \right\}$. Moreover, the notation $\operatorname{Extr}_m g(m)$ means the extremum of $g(m)$ with respect to m, and the period ratio is $\alpha = p/N \sim O(1)$ and $\vec{e} = (1, 1, \cdots, 1)^{\mathrm{T}} \in \mathbf{R}^n$, as above. Note that the order parameters here are defined by

$$q_{wab} = \frac{1}{N} \sum_{i=1}^{N} w_{ia} w_{ib}, \tag{14}$$

$$q_{sab} = \frac{1}{N} \sum_{i=1}^{N} v_i w_{ia} w_{ib}. \tag{15}$$

In addition,

$$\langle f(c, v) \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} f(c_i, v_i) \tag{16}$$

is used.

In the evaluation of Eq. (13), as the replica symmetry solution,

$$q_{wab} = \begin{cases} \chi_w + q_w & a = b \\ q_w & a \neq b \end{cases}, \tag{17}$$

$$q_{sab} = \begin{cases} \chi_s + q_s & a = b \\ q_s & a \neq b \end{cases}, \tag{18}$$

$$\tilde{q}_{wab} = \begin{cases} \tilde{\chi}_w - \tilde{q}_w & a = b \\ -\tilde{q}_w & a \neq b \end{cases}, \tag{19}$$

$$\tilde{q}_{sab} = \begin{cases} \tilde{\chi}_s - \tilde{q}_s & a = b \\ -\tilde{q}_s & a \neq b \end{cases}, \tag{20}$$

$$k_a = k, \tag{21}$$

are set. From this, using replica trick $\lim_{n\to 0} \frac{Z^n - 1}{n} = \log Z$,

$$\phi = \lim_{N\to\infty} \frac{1}{N} E_X[\log Z(X)]$$

$$= \underset{\theta}{\text{Extr}} \left\{ \frac{1}{2}(\chi_w + q_w)(\tilde{\chi}_w - \tilde{q}_w) + \frac{1}{2} q_w \tilde{q}_w \right.$$

$$+ \frac{1}{2}(\chi_s + q_s)(\tilde{\chi}_s - \tilde{q}_s) + \frac{1}{2} q_s \tilde{q}_s - k$$

$$- \frac{\alpha}{2} \log(1 + \beta \chi_s) - \frac{\alpha \beta q_s}{2(1 + \beta \chi_s)}$$

$$- \frac{1}{2} \left\langle \log(\tilde{\chi}_w + v\tilde{\chi}_s) \right\rangle + \frac{1}{2} \left\langle \frac{\tilde{q}_w + v\tilde{q}_s}{\tilde{\chi}_w + v\tilde{\chi}_s} \right\rangle$$

$$\left. + \frac{1}{2} \left\langle \frac{(k - \beta\eta c)^2}{\tilde{\chi}_w + v\tilde{\chi}_s} \right\rangle \right\} \tag{22}$$

is obtained, where the novel set of order parameters $\theta = \{\chi_w, q_w, \chi_s, q_s, \tilde{\chi}_w, \tilde{q}_w, \tilde{\chi}_s, \tilde{q}_s, k\}$ is used. From these terms in the extremum, the order parameters are

$$\chi_w = \frac{\langle v^{-1} \rangle}{\beta(\alpha - 1)}, \tag{23}$$

$$q_w = \frac{1}{\alpha - 1} + \frac{\langle v^{-2} \rangle}{\langle v^{-1} \rangle^2} + C(\eta), \tag{24}$$

$$\chi_s = \frac{1}{\beta(\alpha - 1)}, \tag{25}$$

$$q_s = \frac{\alpha}{\alpha - 1} \left[\frac{1}{\langle v^{-1} \rangle} + \frac{\eta^2 \langle v^{-1} \rangle V_c}{(\alpha - 1)^2} \right], \tag{26}$$

$$\tilde{\chi}_w = 0, \tag{27}$$

$$\tilde{q}_w = 0, \tag{28}$$

$$\tilde{\chi}_s = \beta(\alpha - 1), \tag{29}$$

$$\tilde{q}_s = \beta^2(\alpha - 1) \left[\frac{1}{\langle v^{-1} \rangle} + \frac{\eta^2 \langle v^{-1} \rangle V_c}{(\alpha - 1)^2} \right], \tag{30}$$

$$k = \frac{\beta(\alpha - 1)}{\langle v^{-1} \rangle} + \beta\eta \frac{\langle v^{-1} c \rangle}{\langle v^{-1} \rangle}, \tag{31}$$

where

$$C(\eta) = \frac{\eta^2 \langle v^{-1} \rangle^2 V_c}{(\alpha - 1)^3} + \frac{2\eta}{\alpha - 1} \frac{\langle v^{-2} \rangle}{\langle v^{-1} \rangle} \delta_c$$

$$+\frac{\eta^2 \langle v^{-2} \rangle}{(\alpha-1)^2}(V_{cc} + \delta_c^2), \tag{32}$$

$$V_c = \frac{\langle v^{-1}c^2 \rangle}{\langle v^{-1} \rangle} - \left(\frac{\langle v^{-1}c \rangle}{\langle v^{-1} \rangle} \right)^2, \tag{33}$$

$$\delta_c = \frac{\langle v^{-1}c \rangle}{\langle v^{-1} \rangle} - \frac{\langle v^{-2}c \rangle}{\langle v^{-2} \rangle}, \tag{34}$$

$$V_{cc} = \frac{\langle v^{-2}c^2 \rangle}{\langle v^{-2} \rangle} - \left(\frac{\langle v^{-2}c \rangle}{\langle v^{-2} \rangle} \right)^2. \tag{35}$$

From these and the identity in Eq. (12), $\varepsilon = -\lim_{\beta \to \infty} \frac{\partial \phi}{\partial \beta}$, the minimal investment risk with cost per asset is

$$\begin{aligned}
\varepsilon &= \lim_{\beta \to \infty} \left\{ \frac{\alpha \chi_s}{2(1 + \beta \chi_s)} + \frac{\alpha q_s}{2(1 + \beta \chi_s)^2} \right. \\
&\quad \left. + \left\langle \frac{k - \beta \eta c}{\tilde{\chi}_w + v \tilde{\chi}_s} \eta c \right\rangle \right\} \\
&= \frac{\alpha - 1}{2 \langle v^{-1} \rangle} + \eta \frac{\langle v^{-1}c \rangle}{\langle v^{-1} \rangle} - \frac{\eta^2 \langle v^{-1} \rangle V_c}{2(\alpha - 1)}.
\end{aligned} \tag{36}$$

Further, Eq. (24) gives the extremal investment concentration q_w.

In the next section, we will discuss numerical experiments conducted in order to validate our proposed method. Before then, we should make some comments. First, a previous work[19] has already discussed the portfolio optimization problem in the situation that cost when investing is ignored, giving the minimal investment risk per asset and its investment concentration as follows:

$$\varepsilon = \frac{\alpha - 1}{2 \langle v^{-1} \rangle}, \tag{37}$$

$$q_w = \frac{1}{\alpha - 1} + \frac{\langle v^{-2} \rangle}{\langle v^{-1} \rangle^2}. \tag{38}$$

This corresponds to the case $\eta \to 0$ of our results. Next, for the portfolio optimization problem which minimizes the purchasing cost when ignoring investment risk, the purchasing cost is defined as

$$\mathcal{H}'(\vec{w}|X, \vec{c}) = \sum_{i=1}^{N} c_i w_i \tag{39}$$

and the minimal cost per asset is

$$\varepsilon' = \lim_{N \to \infty} \frac{1}{N} \min_{\vec{w} \in \mathcal{W}} \mathcal{H}'(\vec{w}|X, \vec{c}). \tag{40}$$

Then, from the relationship $\varepsilon' = \lim_{\eta \to \infty} \varepsilon / \eta$ and using Eq. (36), the minimal cost per asset ε' is obtained as $\varepsilon' \to -\infty$. This result is supported by the fact that there does

not exist, for example, a minimum of the function $f(x, y) = 2x + 3y$ of x, y with the two constraint conditions $x + y = 1, -\infty < x, y < \infty$. These comments indicate that the findings obtained by the proposed method are consistent with the well-known properties of the optimal solution of the portfolio optimization problem.

Lastly, the minimal expected investment risk with cost $\varepsilon^{\mathrm{OR}}$ and its investment concentration q_w^{OR} evaluated using the previous analytical procedure (the approach of an annealed disordered system) of operations research are as follows:

$$\varepsilon^{\mathrm{OR}} = \frac{\alpha}{2\langle v^{-1} \rangle} + \eta \frac{\langle v^{-1} c \rangle}{\langle v^{-1} \rangle} - \frac{\eta^2 \langle v^{-1} \rangle V_c}{2\alpha}, \tag{41}$$

$$q_w^{\mathrm{OR}} = \frac{\langle v^{-2} \rangle}{\langle v^{-1} \rangle^2} + \frac{\eta^2 \langle v^{-2} \rangle}{\alpha^2}(V_{cc} + \delta_c^2) + \frac{2\eta \langle v^{-2} \rangle}{\alpha \langle v^{-1} \rangle} \delta_c. \tag{42}$$

As an interpretation of this finding, since, for example, the function $f(x) = x - \frac{b}{x}, (x, b > 0)$ is monotonically increasing in x, compared with Eqs. (36) and (41),

$$\varepsilon < \varepsilon^{\mathrm{OR}} \tag{43}$$

is obtained. That is, in the literature of minimization of investment risk with cost, it has been verified that the minimal investment risk with cost ε does not correspond to the minimal expected investment risk with cost $\varepsilon^{\mathrm{OR}}$; and similarly, the investment concentration of the optimal q_w is not equal to the investment concentration of the solution derived in operations research q_w^{OR}.

5. Numerical Experiments

In this section, using numerical experiments, a verification of the result based on replica analysis in the preceding section is performed. First, if the purchasing cost c_i and the variance of return v_i do not depend on each other, then the second term of ε in Eq. (36) and V_c in Eq. (33) reduce to $\frac{\langle v^{-1} c \rangle}{\langle v^{-1} \rangle} = \langle c \rangle$ and $V_c = \langle c^2 \rangle - \langle c \rangle^2$. However, since this model setting is similar to that of a previous work[19], in this paper, we consider the case that c_i and v_i are correlated. Here, we assume that the mean of return $\bar{x}_{i\mu}$, r_i, is equal to the purchasing cost c_i, that is, $E[\bar{x}_{i\mu}] = r_i = c_i$. Moreover, we assume that the second moment of return $E[\bar{x}_{i\mu}^2]$ is randomly proportional to the square of mean $E[\bar{x}_{i\mu}]$, that is, $E[\bar{x}_{i\mu}^2] = (h_i + 1)c_i^2$. In this setting, the variance of return is $V[\bar{x}_{i\mu}] = v_i = h_i c_i^2$. Note that $h_i(> 0)$ is the random coefficient and does not depend on c_i.

For the concrete setting of the numerical experiments, we assume that c_i, h_i are in-

dependently distributed with the bounded Pareto distributions whose density functions are denoted by

$$
f_c(c_i) = \begin{cases} \frac{(1-b_c)(c_i)^{-b_c}}{(u_c)^{1-b_c}-(l_c)^{1-b_c}} & l_c \le c_i \le u_c \\ 0 & \text{otherwise} \end{cases}, \tag{44}
$$

$$
f_h(h_i) = \begin{cases} \frac{(1-b_h)(h_i)^{-b_h}}{(u_h)^{1-b_h}-(l_h)^{1-b_h}} & l_h \le h_i \le u_h \\ 0 & \text{otherwise} \end{cases}, \tag{45}
$$

where u_c, l_c, u_h, l_h are the upper and lower bounds of c_i, h_i, and $b_c, b_h (> 0)$ are the powers characterizing the bounded Pareto distributions.

We here do not evaluate analytically the inverse matrix J^{-1} in Eqs. (9) and (10) in order to assess the optimal portfolio. Instead, in the following steps, we derive the optimal portfolio numerically by using the steepest descent method and assess the minimal investment risk with cost ε and its investment concentration q_w.

Step 1. (Initial setting) Assign c_i and h_i randomly according to the density function in Eq. (44), $f_c(c_i)$, and that in Eq. (45), $f_h(h_i)$. In particular, random variables s_c^i, s_h^i are independently and identically distributed according to the uniform distribution on $[0, 1)$, so that $c_i = (s_c^i(u_c)^{1-b_c} + (1-s_c^i)(l_c)^{1-b_c})^{\frac{1}{1-b_c}}$ and $h_i = (s_h^i(u_h)^{1-b_h} + (1-s_h^i)(l_h)^{1-b_h})^{\frac{1}{1-b_h}}$.

Step 2. (Initial setting) For asset i, the returns of assets $\bar{x}_{i\mu}$ are independently and identically distributed with $E[\bar{x}_{i\mu}] = c_i$ and $V[\bar{x}_{i\mu}] = v_i = h_i c_i^2$. Moreover, the modified return is $x_{i\mu} = \bar{x}_{i\mu} - E[\bar{x}_{i\mu}]$. Thus, the return matrix $X = \left\{ \frac{x_{i\mu}}{\sqrt{N}} \right\} \in \mathbf{R}^{N \times p}$ is assigned.

Step 3. (Initial setting) Using the modified return $x_{i\mu}$ in Step 2,

$$
J_{ij} = \frac{1}{N} \sum_{\mu=1}^{p} x_{i\mu} x_{j\mu}. \tag{46}
$$

Step 4. (Initial setting) Set the initial portfolio $\vec{w}$ and Lagrange coefficient k as $\vec{w}_0 = \vec{e} = (1, 1, \cdots, 1)^{\mathrm{T}} \in \mathbf{R}^N$ and $k_0 = 1$ and the initial of cost tolerance η as $\eta_{\min}$.

Step 5. (Optimization) Using the portfolio at iteration step t, $\vec{w}_t = (w_{1,t}, w_{2,t}, \cdots, w_{N,t})^{\mathrm{T}} \in \mathbf{R}^N$, and Lagrange coefficient k_t, update $\vec{w}_{t+1}$ (the portfolio at iteration step $t + 1$) and the Lagrange coefficient k_{t+1} (using the

steepest descent method for $L(\vec{w}, k)$ in Eq. (8)) as follows:

$$\vec{w}_{t+1} = \vec{w}_t - \gamma_w \left(\frac{\partial L(\vec{w}, k)}{\partial \vec{w}} \right)_{\vec{w}=\vec{w}_t, k=k_t}, \tag{47}$$

$$k_{t+1} = k_t + \gamma_k \left(\frac{\partial L(\vec{w}, k)}{\partial k} \right)_{\vec{w}=\vec{w}_t, k=k_t}, \tag{48}$$

where $\gamma_w, \gamma_k (> 0)$ are the learning rates of the steepest descent method.

Step 6. (Optimization) Compute the difference between $\vec{w}_t, k_t$ and $\vec{w}_{t+1}, k_{t+1}$,

$$\Delta = \sum_{i=1}^{N} |w_{i,t} - w_{i,t+1}| + |k_t - k_{t+1}|. \tag{49}$$

Step 7. (Optimization) If $\Delta > \delta$, then update $t \leftarrow t + 1$ and go back to Step 5. If $\Delta < \delta$, then, regarding $\vec{w}_{t+1}$ and k_{t+1} as the approximations of the optimal portfolio $\vec{w}^* = \arg\min_{\vec{w} \in \mathcal{W}} \mathcal{H}(\vec{w}|X, \vec{c})$ and Lagrange coefficient k^*, evaluate the minimal investment risk with cost per asset $\varepsilon(\eta, X)$ and its investment concentration $q_w(\eta, X)$, and go to Step 8.

Step 8. (Optimization) If $\eta + d_\eta < \eta_{\max}$, then update $\eta \leftarrow \eta + d_\eta$ and go back to Step 5. If $\eta + d_\eta > \eta_{\max}$, then stop the steepest descent algorithm.

Note that we do not use either the replica symmetry ansatz and a calculation of an inverse matrix in this algorithm. Moreover, using this steepest descent method algorithm M times, with respect to the return matrix assigned in the initial setting of the $m (= 1, 2, \cdots, M)$th trial, $X^m = \left\{ \frac{x_{i\mu}^m}{\sqrt{N}} \right\} \in \mathbf{R}^{N \times p}$, which is used to assess the minimal investment risk with cost $\varepsilon(\eta, X^m)$ and its investment concentration $q_w(\eta, X^m)$ and the sample averages of the minimal investment risk with cost per asset and the investment concentration of the optimal portfolio are

$$\varepsilon(\eta) = \frac{1}{M} \sum_{m=1}^{M} \varepsilon(\eta, X^m), \tag{50}$$

$$q_w(\eta) = \frac{1}{M} \sum_{m=1}^{M} q_w(\eta, X^m), \tag{51}$$

where $\varepsilon(\eta, X^m)$ and $q_w(\eta, X^m)$ are the results of mth trial.

For the numerical simulations, $N = 1000, p = 3000, (\alpha = p/N = 3)$, and the parameters of the bounded Pareto distribution are $(b_c, u_c, l_c) = (b_h, u_h, l_h) = (2, 4, 1)$. Further, $(\eta_{\min}, \eta_{\max}, d_\eta) = (0, 100, 2)$ defines the range of cost tolerance η and its increment, the learning rates of the steepest descent method are $\gamma_w = \gamma_k = 10^{-3}$, and the constant of the stopping condition is $\delta = 10^{-6}$. Finally, the total number of trials is $M = 100$.

The numerical results estimated by this steepest descent method with these numerical settings (orange crosses with error bars) and those based on replica analysis (black solid lines) are shown in Fig. 1. As shown, the results derived by using replica analysis and the numerical results are consistent with each other, which verifies the validity of our proposed method based on replica analysis. In addition, from Eqs. (43), (24), and (42), the analytical approach developed in operations research in previous works is difficult to use to examine the minimization problem of the investment risk with cost under a budget constraint, that is, it is disclosed that the analytical approach developed in operations research cannot examine the properties of the minimal investment risk with cost and the investment concentration of the optimal portfolio.

6. Conclusion

In this work, we have investigated using replica analysis the minimization problem of the investment risk with cost which is defined by two types of loss in investment, the risk and cost. Concretely, based on mathematical similarity, we regarded the investment risk with cost as the Hamiltonian of this investment system, and further, since this system is mathematically analogous to the Hamiltonians of the Hopfield model and the SK model, we recognized that we could analyze the portfolio optimization problem using replica analysis. Similar to in previous works,[6–23] we were able to examine the minimal investment risk with cost and the investment concentration of the optimal portfolio minimizing the investment risk with cost thoroughly based on the replica symmetry ansatz. In addition, we showed that the minimal investment risk with cost and its investment concentration which are evaluated by the approach of a quenched disordered system are in no way consistent with the minimal expected investment risk with cost and the investment concentration minimizing the expected investment risk with cost which are evaluated by the approach developed in operations research (that is, the approach of an annealed disordered system). Using the results of numerical simulations, we verified the validity of our proposed method based on replica analysis. Namely, we showed that the properties of the minimal investment risk with cost and its investment concentration, which are not easily analyzed by the analytical approach developed in operations research, are revealed by the quenched disordered approach.

In this paper, we assumed that, with respect to the cost per unit portfolio, purchasing cost is equal to selling cost; however, as future research, we also need to considered the case that purchasing cost c_i (the cost on $w_i > 0$) and selling cost c_i' (the cost on

$w_i < 0$) are distinct. For this purpose, as a generalization, we need to consider the portfolio optimization problem for the case that the cost needs to be represents as a piecewise linear or nonlinear function; for example, we can change $\sum_{i=1}^{N} c_i w_i$ in Eq. (3) to $\sum_{i=1}^{N}(c_i \max(w_i, 0) - c_i' \max(-w_i, 0))$. Moreover, in order to construct a macroscopic relation of the diversification investment theory, we need to derive a relation between the macroscopic variables like the Pythagorean theorem of the Sharpe ratio and the relation of loss opportunity[18–20]. Further, so as to examine the properties of the utility function of the optimal portfolio, we need to investigate several performance indicators rather than merging risk and cost (see appendix B).

Acknowledgements

Appendix A: Moments

In this appendix, using replica analysis, we will calculate the six moments in the argument of Lagrange multiplier's method, $\frac{1}{N}\vec{e}^{\mathrm{T}}J^{-1}\vec{e}$, $\frac{1}{N}\vec{e}^{\mathrm{T}}J^{-1}\vec{c}$, $\frac{1}{N}\vec{c}^{\mathrm{T}}J^{-1}\vec{c}$, $\frac{1}{N}\vec{e}^{\mathrm{T}}J^{-2}\vec{e}$, $\frac{1}{N}\vec{e}^{\mathrm{T}}J^{-2}\vec{c}$, and $\frac{1}{N}\vec{c}^{\mathrm{T}}J^{-2}\vec{c}$. First, the following partition $Z(y, X)$ is applied:

$$Z(y, X) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{-\infty}^{\infty} d\vec{w}\, e^{-\frac{1}{2}\vec{w}^{\mathrm{T}}(J-yI_N)\vec{w} + k\vec{w}^{\mathrm{T}}\vec{e} + \theta\vec{w}^{\mathrm{T}}\vec{c}},$$

$$(\mathrm{A\cdot 1})$$

where $J = XX^{\mathrm{T}}$. Further, we analyze

$$\begin{aligned}
\log Z(y, X) \\
= -\frac{1}{2}\log\det|J - yI_N| + \frac{k^2}{2}\vec{e}^{\mathrm{T}}(J - yI_N)^{-1}\vec{e} \\
+ \frac{\theta^2}{2}\vec{c}^{\mathrm{T}}(J - yI_N)^{-1}\vec{c} + k\theta\vec{e}^{\mathrm{T}}(J - yI_N)^{-1}\vec{c}.
\end{aligned}$$

$$(\mathrm{A\cdot 2})$$

For this purpose, we define

$$\phi(y) = \lim_{N\to\infty} \frac{1}{N}\log Z(y, X). \qquad (\mathrm{A\cdot 3})$$

Thus, $\phi(0)$ and $\phi'(0)$ are given by

$$
\begin{aligned}
\phi(0) &= -\frac{1}{2} \lim_{N \to \infty} \frac{1}{N} \log \det |J| + \frac{k^2}{2} \lim_{N \to \infty} \frac{\vec{e}^{\mathrm{T}} J^{-1} \vec{e}}{N} \\
&\quad + \frac{\theta^2}{2} \lim_{N \to \infty} \frac{\vec{c}^{\mathrm{T}} J^{-1} \vec{c}}{N} + k\theta \lim_{N \to \infty} \frac{\vec{e}^{\mathrm{T}} J^{-1} \vec{c}}{N},
\end{aligned}
\tag{A·4}
$$

$$
\begin{aligned}
\phi'(0) &= \frac{1}{2} \lim_{N \to \infty} \frac{1}{N} \mathrm{Tr} J^{-1} + \frac{k^2}{2} \lim_{N \to \infty} \frac{\vec{e}^{\mathrm{T}} J^{-2} \vec{e}}{N} \\
&\quad + \frac{\theta^2}{2} \lim_{N \to \infty} \frac{\vec{c}^{\mathrm{T}} J^{-2} \vec{c}}{N} + k\theta \lim_{N \to \infty} \frac{\vec{e}^{\mathrm{T}} J^{-2} \vec{c}}{N}.
\end{aligned}
\tag{A·5}
$$

The second derivatives of $\phi(0)$ and $\phi'(0)$ with respect to k, θ allow the six moments to be analyzed exactly. Moreover, in a similar way to that used in a previous work[16], since the logarithm of the partition function maintains the property of self-averaging, using replica analysis and the replica symmetric ansatz,

$$
\begin{aligned}
\phi(y) &= \lim_{N \to \infty} \frac{1}{N} E_X[\log Z(y, X)] \\
&= \underset{\chi_s, q_s, \tilde{\chi}_s, \tilde{q}_s}{\mathrm{Extr}} \left\{ -\frac{\alpha}{2} \log(1 + \chi_s) - \frac{\alpha q_s}{2(1 + \chi_s)} \right. \\
&\quad + \frac{1}{2}(\chi_s + q_s)(\tilde{\chi}_s - \tilde{q}_s) + \frac{1}{2} q_s \tilde{q}_s \\
&\quad - \frac{1}{2} \langle \log(v\tilde{\chi}_s - y) \rangle + \frac{1}{2} \left\langle \frac{v\tilde{q}_s}{v\tilde{\chi}_s - y} \right\rangle \\
&\quad \left. + \frac{1}{2} \left\langle \frac{(k + c\theta)^2}{v\tilde{\chi}_s - y} \right\rangle \right\},
\end{aligned}
\tag{A·6}
$$

is assessed as follows. From the extremum of the order parameters when $y = 0$, we analytically derive $\chi_s = \frac{1}{\alpha - 1}, \tilde{\chi}_s = \alpha - 1, q_s = \frac{\alpha}{(\alpha-1)^3} \left\langle \frac{(k+c\theta)^2}{v} \right\rangle$, and $\tilde{q}_s = \frac{1}{\alpha-1} \left\langle \frac{(k+c\theta)^2}{v} \right\rangle$. Substituting these into Eq. (A·6),

$$
\begin{aligned}
\phi(0) &= -\frac{\alpha}{2} \log \frac{\alpha}{\alpha - 1} - \frac{1}{2} \log(\alpha - 1) + \frac{1}{2} - \frac{1}{2} \langle \log v \rangle \\
&\quad + \frac{1}{\alpha - 1} \left\langle \frac{(k + c\theta)^2}{v} \right\rangle,
\end{aligned}
\tag{A·7}
$$

and

$$
\begin{aligned}
\phi'(0) &= \frac{\langle v^{-1} \rangle}{2\tilde{\chi}_s} + \frac{\langle v^{-1} \rangle}{2} \frac{\tilde{q}_s}{\tilde{\chi}_s^2} + \frac{1}{2\tilde{\chi}_s^2} \left\langle \frac{(k + c\theta)^2}{v^2} \right\rangle \\
&= \frac{\langle v^{-1} \rangle}{2(\alpha - 1)} + \frac{\langle v^{-1} \rangle}{2(\alpha - 1)^3} \left\langle \frac{(k + c\theta)^2}{v} \right\rangle \\
&\quad + \frac{1}{2(\alpha - 1)^2} \left\langle \frac{(k + c\theta)^2}{v^2} \right\rangle,
\end{aligned}
\tag{A·8}
$$

where $\tilde{\chi}_s = \alpha - 1, \tilde{q}_s = \frac{1}{\alpha-1} \left\langle \frac{(k+c\theta)^2}{v} \right\rangle$ have already been substituted. From this, we can evaluate the second derivatives of $\phi(0)$ and $\phi'(0)$ with respect to k, θ analytically as

$$\lim_{N\to\infty} \frac{1}{N} \vec{e}^{\mathrm{T}} J^{-1} \vec{e} = \frac{\partial^2 \phi(0)}{\partial k^2}$$
$$= \frac{\langle v^{-1} \rangle}{\alpha - 1}, \tag{A.9}$$

$$\lim_{N\to\infty} \frac{1}{N} \vec{e}^{\mathrm{T}} J^{-1} \vec{c} = \frac{\partial^2 \phi(0)}{\partial k \partial \theta}$$
$$= \frac{\langle v^{-1} c \rangle}{\alpha - 1}, \tag{A.10}$$

$$\lim_{N\to\infty} \frac{1}{N} \vec{c}^{\mathrm{T}} J^{-1} \vec{c} = \frac{\partial^2 \phi(0)}{\partial \theta^2}$$
$$= \frac{\langle v^{-1} c^2 \rangle}{\alpha - 1} \tag{A.11}$$

$$\lim_{N\to\infty} \frac{1}{N} \vec{e}^{\mathrm{T}} J^{-2} \vec{e} = \frac{\partial^2 \phi'(0)}{\partial k^2}$$
$$= \frac{\langle v^{-1} \rangle^2}{(\alpha - 1)^3} + \frac{\langle v^{-2} \rangle}{(\alpha - 1)^2}, \tag{A.12}$$

$$\lim_{N\to\infty} \frac{1}{N} \vec{e}^{\mathrm{T}} J^{-2} \vec{c} = \frac{\partial^2 \phi'(0)}{\partial k \partial \theta}$$
$$= \frac{\langle v^{-1} \rangle \langle v^{-1} c \rangle}{(\alpha - 1)^3} + \frac{\langle v^{-2} c \rangle}{(\alpha - 1)^2}, \tag{A.13}$$

$$\lim_{N\to\infty} \frac{1}{N} \vec{c}^{\mathrm{T}} J^{-2} \vec{c} = \frac{\partial^2 \phi'(0)}{\partial \theta^2}$$
$$= \frac{\langle v^{-1} \rangle \langle v^{-1} c^2 \rangle}{(\alpha - 1)^3} + \frac{\langle v^{-2} c^2 \rangle}{(\alpha - 1)^2}. \tag{A.14}$$

Next, using the result of Eq. (9) in the case of a finite number of assets N, in the thermodynamical limit of N, these should maintain the self-averaging property, so we substitute the results in Eqs. (A.9) to (A.11) into (9),

$$\varepsilon = \frac{1}{2} \frac{\left(1 + \eta \frac{\langle v^{-1} c \rangle}{\alpha-1}\right)^2}{\frac{\langle v^{-1} \rangle}{\alpha-1}} - \frac{\eta^2}{2} \frac{\langle v^{-1} c^2 \rangle}{\alpha - 1}$$
$$= \frac{\alpha - 1}{2 \langle v^{-1} \rangle} + \eta \frac{\langle v^{-1} c \rangle}{\langle v^{-1} \rangle} - \frac{\eta^2 \langle v^{-1} \rangle}{2(\alpha - 1)} V_c, \tag{A.15}$$

which is consistent with the result based on replica analysis in Eq. (36). Similarly, if the

results from Eqs. (A·9) to (A·14) are substituted into Eq. (10), then it is also verified that this result corresponds to that based on replica analysis in Eq. (24).

Appendix B: Investment risk with return and cost

Since the model handled in this paper is mathematically analogous to both the Hopfield model and the SK model, we have focused on the minimization problem of the investment risk with cost. Here, however, let us consider the minimization problem of the investment risk with return, which has been widely investigated in operations research. First, the expected return of the portfolio $\vec{w}$ is defined as follows:

$$Return \quad = \quad \sum_{i=1}^{N} w_i r_i, \tag{B·1}$$

where r_i is the mean of return of asset i, that is, $E[\bar{x}_{i\mu}] = r_i$. In this setting, the investment risk with return is

$$\mathcal{H}(\vec{w}|X, \vec{r}) \quad = \quad \frac{1}{2}\vec{w}^{\mathrm{T}} J \vec{w} - g\vec{r}^{\mathrm{T}}\vec{w}, \tag{B·2}$$

where $g(> 0)$ is the mixing degree of return. From this, when ηc_i in the main manuscript is replaced by $-gr_i$, the minimal investment risk with return per asset is $\varepsilon = \lim_{N\to\infty} \frac{1}{N} \min_{\vec{w}\in\mathcal{W}} \mathcal{H}(\vec{w}|X, \vec{r})$ based on Eq. (36). Then,

$$\varepsilon \quad = \quad \frac{\alpha - 1}{2\langle v^{-1}\rangle} - g\frac{\langle v^{-1}r\rangle}{\langle v^{-1}\rangle} - \frac{g^2 \langle v^{-1}\rangle V_r}{2(\alpha - 1)}, \tag{B·3}$$

where

$$V_r \quad = \quad \frac{\langle v^{-1}r^2\rangle}{\langle v^{-1}\rangle} - \left(\frac{\langle v^{-1}r\rangle}{\langle v^{-1}\rangle}\right)^2. \tag{B·4}$$

In addition, we can also consider the minimization problem of the investment risk with both return and cost added, that is, the investment risk with return and cost, as follows:
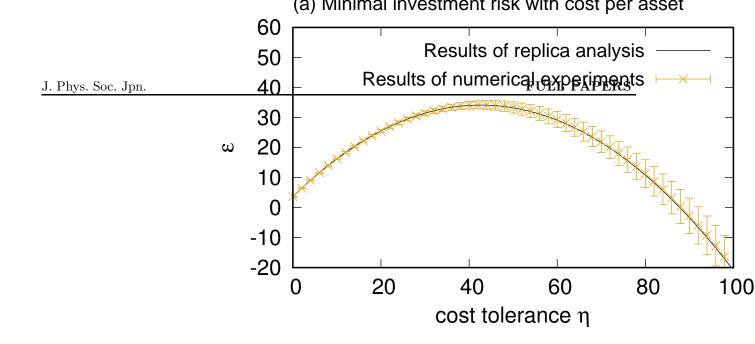
$$\mathcal{H}(\vec{w}|X, \vec{r}, \vec{c}) \quad = \quad \frac{1}{2}\vec{w}^{\mathrm{T}} J \vec{w} - g\vec{r}^{\mathrm{T}}\vec{w} + \eta\vec{c}^{\mathrm{T}}\vec{w}. \tag{B·5}$$

Then the minimal investment risk with return and cost per asset $\varepsilon = \lim_{N\to\infty} \frac{1}{N} \min_{\vec{w}\in\mathcal{W}} \mathcal{H}(\vec{w}|X, \vec{r}, \vec{c})$ can be calculated as

$$\varepsilon \quad = \quad \frac{\alpha - 1}{2\langle v^{-1}\rangle} - g\frac{\langle v^{-1}r\rangle}{\langle v^{-1}\rangle} + \eta\frac{\langle v^{-1}c\rangle}{\langle v^{-1}\rangle} - \frac{\langle v^{-1}\rangle V}{2(\alpha - 1)}, \tag{B·6}$$

$$V \quad = \quad \frac{\langle v^{-1}(\eta c - gr)^2\rangle}{\langle v^{-1}\rangle} - \left(\eta\frac{\langle v^{-1}c\rangle}{\langle v^{-1}\rangle} - g\frac{\langle v^{-1}r\rangle}{\langle v^{-1}\rangle}\right)^2, \tag{B·7}$$

where ηc_i in Eq. (4) is replaced by $-gr_i + \eta c_i$. This shows that we can analyze a utility function which comprises risk, return, and cost. Note that the utility function depends on the preferences of each investor; that is, the utility function is a subjective criterion based on each individual's needs of what the important factors are for the investor to decide to invest. As individual terms in the utility function, it is well known that the utility function may include risk, return, and cost (that is, the mixing degree of return g and cost tolerance η differ between investors). As mentioned in the main manuscript, investment theory should be deepened in order to meet the needs of each investor and an optimal investment strategy should be proposed for the rational investor.
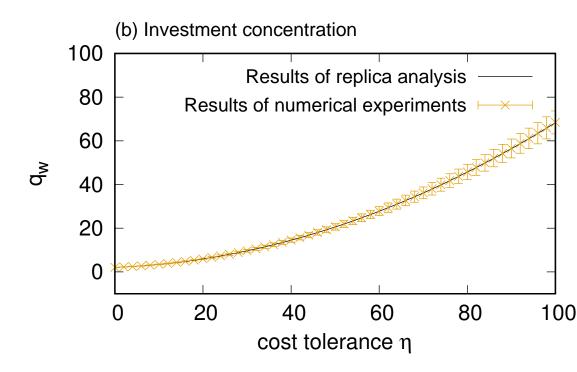
(a) Minimal investment risk with cost per asset

(b) Investment concentration

Fig. 1. Results of the replica analysis and the numerical experiments ($\alpha = p/N = 3$). The horizontal axis indicates the cost tolerance η, and the vertical axes show (a) the minimal investment risk with cost per asset ε, and (b) the investment concentration q_w. The black solid lines indicate the results of the replica analysis for (a) Eq. (36) and (b) Eq. (24). The orange crosses with error bars indicate the results of the numerical simulations.

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