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The Information Content of a Nonlinear Macro-Finance Model for Commodity Prices

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ABSTRACT

State-of-the-art term structure models of commodity prices have serious difficulties extrapolating the prices of long-maturity futures contracts from short-dated contracts. This situation is problematic for valuing real commodity-linked assets. We estimate a nonlinear four-factor continuous time model of commodity price dynamics. The model nests many previous specifications. To estimate the model, we use crude oil prices and inventories. The inventory data and nonlinear price dynamics have a large impact on oil price forecasts. The additional factor in our model compared with current three-factor models has a significant impact on model-implied long-maturity futures prices. (*JEL* G13)

Long-maturity commodity futures prices are important for valuing real assets like oil and gas reserves. Because reliable market data at these long horizons is often unavailable, investors typically infer these futures prices from short-dated futures contracts using models of commodity price dynamics. Recent research, however, suggests that these models do not fully reflect the information contained in short-dated futures contracts for

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long-maturity futures prices.

We characterize a theoretically motivated contingent claims model of commodity price dynamics. The model builds on several important insights by extending the specifications of Schwartz (1997) and Casassus and Collin-Dufresne (CCD 2005) to allow prices to be nonlinear in a four-factor state vector and by including inventory data in the model's estimation. While our estimation uses inventory data, this setting can readily accommodate other macroeconomic variables. To our knowledge, this paper represents the first attempt to include observable real information in a continuous-time quadratic term structure model of commodity price dynamics.

The key contribution of this paper is to show that our model contains significant incremental information about the prices of long-maturity crude oil futures contracts relative to current models. The additional factor in our model compared with current three-factor models has a significant impact on model-implied long-maturity futures prices. To demonstrate this point, we first estimate our model parameters using short-dated crude oil futures contracts. We then compare futures prices generated by our model to futures prices generated by current models. Long-maturity futures prices implied by our model differ significantly from those implied by current three-factor models. We also find that the fit of futures prices generated by our model to the market prices of long-maturity futures contracts is significantly better than the fit of current models. For instance, at the five-year maturity, the mean pricing error for our model is 50% lower than the error for an existing state-of-the-art model.

Furthermore, our model contains important additional information about expected spot prices. Our model adds nonlinear price dynamics and information on inventory levels to current models of commodity price dynamics. Likelihood ratio tests decisively reject restrictions on both inventory and nonlinearity effects. In addition, both inventories and nonlinearity have a large impact upon spot price forecasts. Each effect contains significant incremental information about oil price dynamics.

Our empirical analysis reveals several implications for crude oil markets. The global nature of oil markets has sparked recent debate on whether U.S. stocks of crude oil reflect global scarcity of crude oil. The model estimates confirm that globally set oil prices feed back into U.S. stocks to induce a component linked to the international economy. We then tease from the data new evidence consistent with Singleton's (2010) prediction that the relation between spot price drift and the level of spot prices can be positive. Next, we quantify the extent to which oil price volatility is spanned by futures contracts. Using a novel dataset of crude oil options, we estimate that about 25% of oil price volatility is unspanned.

This paper adds to the literature on contingent claims models of commodity price dynamics. Schwartz (1997) and CCD (2005) estimate affine three-factor models using price data. We estimate a continuous-time model of commodity price dynamics that nests both of these specifications. Like Schwartz (1997) and CCD (2005), our model specifies prices to be functions of Gaussian state variables. However, unlike Schwartz (1997) and CCD (2005), we allow prices to be quadratic in a four-factor state vector and include information from the real economy on crude oil inventory levels in the model estimation.[1]

Section 1 develops the model and demonstrates how it relates to current reduced-

[1] Our study is also related to the macro-finance literature that estimates discrete-time models of the joint dynamics of yield and macroeconomic variables, the work of Ang and Piazzesi (2003), for example. We independently explore a nonlinear macro-finance model for commodities in continuous time.

form models. Section 2 describes the data, details the model calibration, and presents the model implications for long-maturity futures prices. Section 3 presents empirical evidence and examines further model implications. Section 4 concludes the paper.

1. The Model

Schwartz (1997) and CCD (2005) specify commodity prices as affine in a three-dimensional Gaussian state vector. Our specification extends Schwartz (1997) and CCD (2005), by estimating a four-factor quadratic term structure model with price and inventory data. The additional factor in our model is consistent with Duffee (2009). He argues that building flexibility into term structure models by increasing the dimension of their state vectors can uncover additional information from the time-series dynamics of prices. In a production economy framework, Carlson, Khokher, and Titman (2007) show that time-varying supply inelasticity can in theory induce nonlinearity into spot price dynamics.[2] Including inventory data in our model estimation is motivated by Routledge, Seppi, and Spatt (2000). They suggest that inventory levels should be informative about future demand and supply in the market and, related, to expected spot prices.

1.1. Risk-neutral dynamics

We first develop a nonlinear model (QTSM) that describes the joint dynamics of the commodity's spot price and inventory level under the risk-neutral measure *Q*. We define the model as follows:

[2] Very little work studying commodity prices under the quadratic class exists. See Leippold and Wu (2002) for a theoretical development of asset pricing in the quadratic class and Ahn, Dittmar, and Gallant (2002) for such models applied to interest rate modeling.

- The logarithm of the spot price is a function of a latent state vector *Y(t)*,

$$X(t) = \log S(t) \equiv \phi_0 + \phi_Y^T Y(t) + Y(t)^T \Phi_Y Y(t). \tag{1.1}$$

- The inventory factor, *q(t)*, is also linear in *Y(t)*:

$$q(t) \equiv \pi_0 + \pi_Y^T Y(t). \tag{1.2}$$

- The latent state vector $Y(t) = \left[Y_1(t), Y_2(t), Y_3(t), Y_4(t)\right]^T$ is four dimensional and

follows a Gaussian process under the risk-neutral measure Q:

$$dY(t) = -\kappa^Q Y(t)\,dt + dZ^Q(t) \tag{1.3}$$

Here, $Z^Q(t)$ is a 4×1 vector of independent Brownian motions. The parameters ϕ_0 and π_0 are scalars; ϕ_Y and π_Y are 4×1 vectors. The matrices κ^Q and Φ_Y are *4x4*; the first is lower triangular and the latter is symmetric.

The fact that the (log) spot price exhibits state-dependent volatility is evident in the dynamics of the spot price process (Appendix A):

$$dX(t) = \left[tr(\Phi_Y) - \phi^T\kappa^Q Y - 2Y^T\left(\Phi_Y\kappa^Q\right)Y\right]dt + [\phi^T + 2Y'\Phi_Y]\,dZ(t)^Q \tag{1.4}$$

The quadratic form enters the spot price drift (μ^Q) to induce local nonlinearity in spot price dynamics. The product of the quadratic form and the state variables ($Y'\Phi$) enters the diffusion coefficient; this fact implies state-dependent volatility in spot prices.[3]

We obtain the futures pricing function in this setting in quasi-closed form, using standard results on the quadratic class of models (Leippold and Wu 2002) as:

$$F^T(t) = E_t^Q[e^{X(T)}] = e^{[A_F(\tau) + B_F(\tau)'Y_t + Y_t'C_F(\tau)Y_t]}. \tag{1.5}$$

[3] Schwartz (1997) and CCD (2005) model constant volatility price processes; Richter and Sorensen (2002) and Nielsen and Schwartz (2004) explicitly allow for stochastic volatility. In a recent paper, Trolle and Schwartz (2009) develop a Heath, Jarrow, and Morton–type model with three factors driving futures prices and two volatility factors. These authors do not address the nonlinearities identified in Carlson, Khokher, and Titman (2007).

Here, A_F, B_F, and C_F are solutions to the following system of ordinary differential equations (ODEs):

$$\begin{aligned}
\frac{\partial A_F}{\partial \tau} &= \frac{1}{2} B_F(\tau)' B_F(\tau) \\
\frac{\partial B_F}{\partial \tau} &= -\kappa^{Q}{}' B_F(\tau) + 2C_F(\tau) B_F(\tau) \\
\frac{\partial C_F}{\partial \tau} &= -C_F(\tau)\kappa^{Q} - \kappa^{Q}{}' C_F(\tau) + 2C_F(\tau)^2
\end{aligned} \quad .$$

The boundary conditions for this system are $A_F(0) = \phi_0$, $B_F(0) = \phi$ and $C_F(0) = \Phi$. The resulting system is solved numerically.

We specify a short interest rate, $r(t)$, using the scalars ψ_0, and ψ_1, to be linear in a univariate Gaussian process:

$$r(t) \equiv \psi_0 + \psi_1 Y_1(t). \tag{1.6}$$

This specification is standard in the empirical commodities literature. While several empirical studies document the weak direct impact of interest rates on spot prices (for example, Schwartz 1997; CCD 2005; Trolle and Schwartz 2009), one plausible possibility is that interest rates might affect expected spot prices and, hence, affect the futures curve. For instance, interest rates might serve as a partial proxy for economic activity, which may, in turn, be correlated with oil prices (see Fama and French 1988 for a detailed discussion).

We can then derive a convenience yield from Equations 1.1–1.6. Specifically, CCD show that a no-arbitrage assumption combined with an interest rate specification implies a convenience yield. This approach is distinct from that of Schwartz, who exogenously specifies a convenience yield process just as he specifies a spot price and interest rate process. Consistent with the CCD approach, we derive a convenience yield factor, $\delta(t)$, such that:

$$\delta(t) = \eta_0 + \eta_Y^T Y(t) + Y(t)^T \mathrm{N}_Y \ Y(t). \tag{1.7}$$

Here η_0 is scalar, η_Y^T is a 4×1 vector, and the matrix N_F is 4×4. Appendix A provides expressions for η_0, η_Y, and N_F in terms of the latent parameters. Upon deriving the dynamics of the convenience yield process we see that, in contrast to many current models, the volatility of the convenience yield process in our model will also be time varying.

1.2. Specification of risk premia

We observe the data under the historical or physical probability distribution (measure), while the model dynamics characterized above are under the risk-neutral measure. Therefore, calibrating the model to the data requires that we specify a relationship between these two probability measures.

While the classic empirical models of commodity futures use a constant market price of risk, as in Duffee (2002), CCD (2005) document that allowing risk premia to be linear in the time-varying latent state variables is empirically relevant. We also allow the market price of risk corresponding to the Brownian shock vector $Z^Q(t)$ to be affine in the latent state vector so that:

$$dZ^Q(t) = dZ(t) + (\beta_{0Y} + \beta_{1Y} Y(t))dt\,. \tag{1.8}$$

Here, β_{0Y} is a 4×1 vector of constants, β_{1Y} is a 4×4 matrix of constants, and $Z(t)$ is a 4×1 vector of Brownian motions under the physical (P) measure. Equation 1.18 specifies risk premia in the model, so that the difference in price drift under the two measures $\left(\mu - \mu^Q\right)$ is given by $\left(\beta_{0Y} + \beta_{1Y} Y(t)\right)$.

1.3. Relation to extant models

In the preceding subsections, we have characterized a nonlinear model of price and inventory dynamics, a linear model with inventory would restrict the quadratic form to zero $(\Phi_Y = 0)$. When we restrict the quadratic term in this fashion, we can develop an economic representation that relates the four economic factors $\left[r(t), q(t), \delta(t), X(t)\right]$ to one another.[4] Such an economic representation is informative in clarifying the relationship to extant models. Although this economic representation does not accommodate nonlinear price dynamics, it does still allow for an explicit inventory state variable.

The rotation to the economic representation is standard (see, for example, Schwartz and Smith 2000 and CCD 2005). Equations 1.1–1.3 and 1.6–1.8 specify the parameters that relate the four economic factors $\left[r(t), q(t), \delta(t), X(t)\right]$ to the latent state vector $Y(t)$. The following proposition records the joint evolution of the four economic factors in terms of each other without relying upon the latent state vector.

Proposition 1: Assuming $\Phi_Y = 0$, the economic affine representation of the model described in Equations 1.1–1.3 and 1.6–1.8 is:

[4] In a quadratic setting, the latent state variables cannot be inverted to yield the economic variables; as a result, the transformation to the economic representation where the economic factors can be directly related to each other is problematic.

$$dr(t) = \kappa_r^Q \left(\theta_r^Q - r(t)\right) dt + \sigma_r dZ_r^Q(t)$$

$$dq(t) = \left(\kappa_{q0}^Q + \kappa_{qr}^Q r(t) + \kappa_q^Q q(t) + \boxed{\kappa_{q\delta}^Q} \delta(t) + \kappa_{qX}^Q X(t)\right) dt + \sigma_q dZ_q^Q(t)$$

$$d\delta(t) = \left(\kappa_{\delta 0}^Q + \kappa_{\delta r}^Q r(t) + \boxed{\kappa_{\delta q}^Q} q(t) + \kappa_\delta^Q \delta(t) + \kappa_{\delta X}^Q X(t)\right) dt + \sigma_\delta dZ_\delta^Q(t)$$

$$dX(t) = \left(r(t) - \delta(t) - \frac{1}{2}\sigma_X^2 \right) dt + \sigma_X dZ_X^Q(t). \tag{1.9}$$

The standard Brownian motions of the economic factors are correlated with each other such that:

$$\begin{aligned} &dZ_r^Q(t)\, dZ_X^Q(t) = \rho_{rX} dt, \quad dZ_r^Q(t)\, dZ_\delta^Q(t) = \rho_{r\delta} dt, \\ &dZ_\delta^Q(t)\, dZ_X^Q(t) = \rho_{\delta X} dt, \quad dZ_q^Q(t)\, dZ_\delta^Q(t) = \boxed{\rho_{q\delta}} dt, \\ &dZ_q^Q(t)\, dZ_X^Q(t) = \boxed{\rho_{qX}} dt \quad and \quad dZ_q^Q(t)\, dZ_r^Q(t) = \boxed{\rho_{qr}} dt. \end{aligned} \tag{1.10}$$

Proof: The proof follows by applying Ito's lemma to the equations relating the economic factors $\left[r(t),\ q(t),\ \delta(t),\ X(t)\right]$ to the latent state vector $\left[Y(t)\right]$. In this affine setting, we can invert the latent state variables to yield the economic factors, so we can express the dynamics of the economic factors in this setting in terms of one another (Appendix C).

The drift of the convenience yield process in this economic representation is related to the current level of the inventory variable. The highlighted drift coefficients of the inventory and convenience yield processes show that the expected change in these variables can simultaneously depend on convenience yield and inventory levels. Unlike our model, Schwartz (1997) assumes that the expected instantaneous change or drift of the convenience yield process does not depend on the remaining economic factors; in other words, the convenience yield process in his model is autonomous. In our model, unexpected shocks to inventory and convenience yields are potentially correlated, as reflected in the instantaneous correlation coefficient $\rho_{q\delta}$. The following proposition further clarifies the joint dynamics of convenience yields and inventories.

Proposition 2: We can decompose the convenience yield and inventory variables in Proposition 1 such that:

$$
\begin{aligned}
q(t) &= \tilde{q}(t) + \alpha_r^q r(t) + \boxed{\alpha_\delta^q}\,\delta(t) + +\alpha_X^q\, X(t) \\
\delta(t) &= \tilde{\delta}(t) + \alpha_r^\delta r(t) + \boxed{\alpha_q^\delta}\, q(t) + \alpha_X^\delta\, X(t).
\end{aligned} \tag{1.11}
$$

Here, $\tilde{q}(t)$ and $\tilde{\delta}(t)$ follow autonomous processes such that their expected instantaneous changes (drifts) are unrelated to the other economic factors:

$$
\begin{aligned}
d\tilde{q}(t) &= \kappa_{\hat{q}}^Q\left(\theta_{\hat{q}}^Q - \tilde{q}(t)\right)dt + \sigma_{\hat{q}}dZ_{\hat{q}}^Q(t) \\
d\tilde{\delta}(t) &= \kappa_{\hat{\delta}}^Q\left(\theta_{\hat{\delta}}^Q - \tilde{\delta}(t)\right)dt + \sigma_{\hat{\delta}}dZ_{\hat{\delta}}^Q(t).
\end{aligned} \tag{1.12}
$$

The (log) spot price $X(t)$ dynamics in this decomposition are:

$$
\begin{aligned}
dX(t) &= \left(r(t) - \delta(t) - \frac{1}{2}\sigma_X^2\right)dt + \sigma_X dZ_X^Q(t) \\
&= \left[\kappa_{Xr}^Q\left(\theta_r^Q - r(t)\right) + \kappa_{X\hat{q}}^Q\left(\theta_{\hat{q}}^Q - \tilde{q}(t)\right) + \kappa_{X\hat{\delta}}^Q\left(\theta_{\hat{\delta}}^Q - \tilde{\delta}(t)\right) + \kappa_X^Q\left(\theta_X^Q - X(t)\right)\right]dt + \sigma_X dZ_X^Q(t).
\end{aligned} \tag{1.13}
$$

In the representation of Proposition 2, spot commodity prices revert to a long-term level θ_X^Q under the risk-neutral measure at a rate κ_X^Q for:

$$
\theta_X^Q = {-\left(\frac{\sigma_X^2}{2} + \kappa_{Xr}^Q\theta_r^Q + \kappa_{X\hat{q}}^Q\theta_{\hat{q}}^Q + \kappa_{X\hat{\delta}}^Q\theta_{\hat{\delta}}^Q\right)} \Big/ {\kappa_X^Q} \tag{1.14}
$$

and

$$
\kappa_{X\hat{r}}^Q = \left(\frac{\alpha_r^\delta + \alpha_q^\delta\alpha_r^q}{1-\alpha_q^\delta\alpha_\delta^q} - 1\right) \quad \kappa_{X\hat{q}}^Q = \left(\frac{\alpha_q^\delta}{1-\alpha_q^\delta\alpha_\delta^q}\right) \quad \kappa_{X\hat{\delta}}^Q = \left(\frac{1}{1-\alpha_q^\delta\alpha_\delta^q}\right) \quad \kappa_X^Q = \left(\frac{\alpha_X^\delta + \alpha_q^\delta\alpha_X^q}{1-\alpha_q^\delta\alpha_\delta^q}\right).
$$

We denote the correlations of the standard Brownian motions $\left[Z_r^Q(t), Z_{\hat{q}}^Q, Z_{\hat{\delta}}^Q, Z_X^Q(t)\right]$ as

$\rho_{r\hat{q}}$, $\rho_{r\hat{\delta}}$, ρ_{rX}, $\rho_{\hat{q}\hat{\delta}}$, $\rho_{\hat{q}X}$, and $\rho_{\hat{\delta}X}$.

Proof: The result follows similarly to Proposition 1 after we apply Ito's lemma to the representations for $\hat{q}$ and $\hat{\delta}$ (Appendix D).

The representation outlined in Proposition 2 facilitates the interpretation of the relationship between the economics factors. In this representation, each element of $\left[r(t),\ \hat{q}(t),\ \hat{\delta}(t)\right]$ represents autonomous processes, characterized by their own speeds of mean reversion, long-term levels to which they revert and their volatilities. These three economic factors can also be potentially correlated with the spot price process and one another as reflected in the coefficients $\rho_{r\hat{q}}$, $\rho_{r\hat{\delta}}$, ρ_{rX}, $\rho_{\hat{q}\hat{\delta}}$, $\rho_{\hat{q}X}$, and $\rho_{\hat{\delta}X}$. The spot price process need not be autonomous; its drift can depend on each of the other economic factors.

The simultaneous determination and co-dependence of convenience yields and inventories is clear in Proposition 2. The highlighted parameters α_q^{δ} and α_{δ}^{q} relate convenience yield and inventory levels to each other. Several theoretical models, for example that of Routledge, Seppi, and Spatt (2000), show how convenience yields arise endogenously as storage operators optimally respond to supply and demand shocks. Such theoretical models predict that during periods of scarcity, stocks of the commodity held in storage will be drawn down and convenience yields will rise. A negative estimate for the parameters α_q^{δ} and α_{δ}^{q} would be evidence in support of the Theory of Storage.

This economic representation explicitly shows that the model addresses the joint determination of spot commodity prices and inventory levels. The model describes a system of simultaneous equations that jointly determines inventories and prices, and we can estimate such models consistently (e.g., Greene 1997). Accordingly, the parameter estimates $\left(\alpha_q^{\delta} \text{ and } \alpha_{\delta}^{q}\right)$ of the underlying structural relation are less likely to exhibit the simultaneous equations biases that can result from empirical least-squares estimates from models that do not consider joint determination. Therefore, this setting identifies the relationship between inventories and convenience yields.[5]

In the economic representation, we can write the risk premia specification as:

$$\mathrm{d}\begin{pmatrix} Z_r^Q \\ Z_{\hat{q}}^Q \\ Z_{\hat{\delta}}^Q \\ Z_X^Q \end{pmatrix} = \mathrm{d}\begin{pmatrix} Z_r \\ Z_{\hat{q}} \\ Z_{\hat{\delta}} \\ Z_X \end{pmatrix} + \Sigma^{-1}\left\{\begin{pmatrix} \beta_{0r} \\ \beta_{0\hat{q}} \\ \beta_{0\hat{\delta}} \\ \beta_{0X} \end{pmatrix} + \begin{pmatrix} \beta_{rr} & \beta_{r\hat{q}} & \beta_{r\hat{\delta}} & \beta_{rX} \\ \beta_{\hat{q}r} & \beta_{\hat{q}\hat{q}} & \beta_{\hat{q}\hat{\delta}} & \beta_{\hat{q}X} \\ \beta_{\hat{\delta}r} & \beta_{\hat{\delta}\hat{q}} & \beta_{\hat{\delta}\hat{\delta}} & \beta_{\hat{\delta}X} \\ \beta_{Xr} & \beta_{X\hat{q}} & \beta_{X\hat{\delta}} & \beta_{XX} \end{pmatrix}\begin{pmatrix} r(t) \\ \hat{q}(t) \\ \hat{\delta}(t) \\ X(t) \end{pmatrix}\right\}; \Sigma = diag\begin{pmatrix} \sigma_r \\ \sigma_{\hat{q}} \\ \sigma_{\hat{\tilde{\delta}}} \\ \sigma_X \end{pmatrix}. \tag{1.15}$$

The autonomous component of inventory and convenience yield will remain independent of the remaining factors under the physical measure once we restrict all cross risk premia terms to zero, except those involving the spot price. This assumption is similar to the restriction imposed in CCD. With this specification of risk premia, the dynamics of the economic factors under the historical measure are immediate.

Proposition 3: For the above specification of risk premia, the dynamics of the economic factors $\left[r(t), \tilde{q}(t), \tilde{\hat{\delta}}(t), X(t)\right]$ under the historical measure are:

[5] The identification of this relation happens subject to this model's assumptions, though. To be explicit, the model assumes that the state variables are Gaussian and the price and inventory variables are Markov processes. The Markov property assumes that predictions about the future rely only upon the current state; in this way our model is distinct from a full-vector autoregression where identification would still be a problem. Furthermore, this identification assumes that the quadratic term in our model is restricted to zero.

$$
\begin{aligned}
dr(t) &= \kappa_r^P\left(\theta_r^P - r(t)\right)dt + \sigma_r dZ_r(t)\\
d\tilde{q}(t) &= \kappa_{\hat{q}}^P\left(\theta_{\hat{q}}^P - \tilde{q}(t)\right)dt + \sigma_{\hat{q}} dZ_{\hat{q}}(t)\\
d\tilde{\delta}(t) &= \kappa_{\hat{\delta}}^P\left(\theta_{\hat{\delta}}^P - \tilde{\delta}(t)\right)dt + \sigma_{\hat{\delta}} dZ_{\hat{\delta}}(t)\\
dX(t) &= \left(\mu(t) - \delta(t) - \frac{1}{2}\sigma_X^2\right)dt + \sigma_X dZ_X(t)\\
&= \left[\kappa_{Xr}^P\left(\theta_r^P - r(t)\right) + \kappa_{X\hat{q}}^P\left(\theta_{\hat{q}}^P - \tilde{q}(t)\right) + \kappa_{X\hat{\delta}}^P\left(\theta_{\hat{\delta}}^P - \tilde{\delta}(t)\right) + \kappa_X^P\left(\theta_X^P - X(t)\right)\right]dt + \sigma_X dZ_X(t).
\end{aligned}
\tag{1.16}
$$

The mean-reversion matrix under the two measures differs by the time-varying risk premia terms, so that $\kappa_r^P = \kappa_r^Q - \beta_{rr}$, $\kappa_{\hat{q}}^P = \kappa_{\hat{q}}^Q - \beta_{\hat{q}\hat{q}}$, $\kappa_{\hat{\delta}}^P = \kappa_{\hat{\delta}}^Q - \beta_{\hat{\delta}\hat{\delta}}$, $\kappa_{Xr}^P = \kappa_{Xr}^Q - \beta_{Xr}$, $\kappa_{X\hat{q}}^P = \kappa_{X\hat{q}}^Q - \beta_{X\hat{q}}$, $\kappa_{X\hat{\delta}}^P = \kappa_{X\hat{\delta}}^Q - \beta_{X\hat{\delta}}$, $\kappa_X^P = \kappa_X^Q - \beta_{XX}$ (Appendix E). The economic factors $\left[r(t), \tilde{q}(t), \tilde{\delta}(t), X(t)\right]$ revert to long-term mean levels given by:

$$
\begin{aligned}
&\theta_r^P = \frac{\beta_{0r} + \kappa_r^Q\theta_r^Q}{\kappa_r^P} \qquad \theta_{\hat{q}}^P = \frac{\beta_{0\hat{q}} + \kappa_{\hat{q}}^Q\theta_{\hat{q}}^Q}{\kappa_{\hat{q}}^P} \qquad \theta_{\hat{\delta}}^P = \frac{\beta_{0\hat{\delta}} + \kappa_{\hat{\delta}}^Q\theta_{\hat{\delta}}^Q}{\kappa_{\hat{\delta}}^P}\\
&\theta_X^P = \frac{\beta_{0X} + \kappa_X^Q\theta_X^Q - \left(\kappa_{Xr}^P\theta_r^P - \kappa_{Xr}^Q\theta_r^Q\right) - \left(\kappa_{X\hat{q}}^P\theta_{\hat{q}}^P - \kappa_{X\hat{q}}^Q\theta_{\hat{q}}^Q\right) - \left(\kappa_{X\hat{\delta}}^P\theta_{\hat{\delta}}^P - \kappa_{X\hat{\delta}}^Q\theta_{\hat{\delta}}^Q\right)}{\kappa_X^P}\\
&\mu(t) = r(t) + \beta_{0X} + \beta_{Xr}r(t) + \beta_{X\hat{q}}\tilde{q}(t) + \beta_{X\hat{\delta}}\tilde{\delta}(t) + \beta_{XX}X(t).
\end{aligned}
\tag{1.17}
$$

Schwartz finds that models of commodity prices that can accommodate mean reversion in spot prices are empirically useful. Mean-reverting representations characterize spot prices by the speed at which they return to a long-term level. Spot prices in this model revert under the actual or historical probability distribution (measure) at rate κ_X^P to a long-term level θ_X^P. Mean reversion under the actual or historical measure $\left(\kappa_X^P\right)$ is related to the mean reversion under the risk-neutral measure $\left(\kappa_X^Q\right)$ and time-varying risk premia $\left(\beta_{XX}\right)$

through a linear transformation $\left(\kappa_X^P = \kappa_X^Q - \beta_{XX}\right)$.[6] The parameter β_{XX} reflects covariation between spot prices and risk premia; when this covariation is negative, it adds to price reversion under the physical measure.

The economic representation of Proposition 3 is comparable to the classic reduced-form empirical models of commodity price dynamics. While the Theory of Storage motivates recent extensions of the classic commodity pricing models suggested by Black (1976), Brennan and Schwartz (1985), and others, these empirical models omit inventory as an explicit state variable. In contrast to current models, our model directly relates spot prices to inventories. Specifically, the spot price system in Proposition 3 is related to inventories through the eight parameters α_r^q, α_δ^q, α_X^q, α_q^δ, $\beta_{X\hat{q}}$, $\rho_{r\hat{q}}$, $\rho_{\hat{q}\hat{\delta}}$, and $\rho_{\hat{q}X}$. We refer to this affine model with inventories as EXT. Once we restrict these eight parameters to zero, the representation of Proposition 3 reduces to the model of CCD (2005); we refer to this model as the standard market model (STD). As CCD (2005) show, this standard model nests those of Gibson and Schwartz (1990) and Schwartz (1997), Schwartz, and Smith (2000).

2. Empirical Analysis

2.1. Data and methods

We focus our empirical analysis on the recent period in the crude oil markets spanning January 2004 to June 2015. Our reason for doing so is that reliable market prices

[6] More broadly, comparing Propositions 2 and 3 clarifies that the structure of the model under either probability measure is invariant; model parameters under the physical measure are linear transformations of their counterparts under risk neutrality.

for crude oil futures contracts with maturities of five years or more have become available only since 2004. We use this data to explore whether our model contains significant incremental information about the prices of long-maturity crude oil futures contracts relative to current models.[7]

We obtain crude oil futures prices from Bloomberg, which provides historical settlement prices for contracts traded on the New York Mercantile Exchange (NYMEX). We use futures prices for the first through the eighth nearest-to-maturity contracts to estimate the model; these prices are available at monthly maturities for the first eight months. These contracts reflect the continental U.S. market and are among the most liquid commodity futures contracts traded. We use prices for the nearest-to-maturity contract to proxy for the spot price. In cases where a particular futures price is missing, we use the closest maturity available.

To capture the relative scarcity of the commodity, we require a proxy for economic inventory; we use data on inventory levels from the Energy Information Administration. The inventory data are based on reports filed by energy companies operating in the United States, and they are announced each Wednesday. We use the demeaned natural logarithm of the inventory series. Treasury yields are also necessary to calibrate the model. We use constant-maturity Treasury yields as measures of zero-coupon risk-free rates, and we focus on bonds with maturities of 6 months and 1, 2, 3, 5, 7, and 10 years. We obtain this data from the Federal Reserve.

[7] Futures prices for maturities greater than three years become continuously available in our data in February of 2006; our goal is to compare this data with model-generated prices. Ideally, at every period, model parameters would be re-estimated using historical data, over the last two years, for example, on short-maturity futures; model-generated long-maturity prices can then be compared with the data. For this reason, we started our sample in 2004. This exercise, however, proved to be computationally problematic and was modified as described in Section 3. Given this modification, starting our sample in 2006 would also be appropriate.

Table 1 reports crude oil summary statistics for the eight closest-to-maturity futures prices and for the futures basis. We compute the futures slope or basis variable, b_T, as:

$$b_{t,T} = \frac{1}{(T-t)}\left[X_t - \tilde{F}_{t,T}\right]. \tag{2.1}$$

Here, X_t is the (log) spot price of the commodity and $\tilde{F}_{t,T}$ denotes the log futures price at time t maturing at time T. For each maturity date T, the basis variable can be decomposed into an implied cash-flow yield less the corresponding maturity risk-free yield. The implied cash-flow yield $\left(c_{t,T}\right)$ is then equal to an interest-adjusted futures basis $\left(r_{t,T} + b_{t,T}\right)$, and its instantaneous analog in continuous time is the convenience yield $\left(\delta_t\right)$. We refer to the interest-adjusted futures basis as simply the basis unless otherwise specified.

The crude oil basis is, on average, negative and quite variable. When the (simple) basis is positive, the futures term structure slopes downward and is said to be backwardated. Such positive basis periods were more common in earlier samples; for example, see Litzenberger and Rabinowitz (1995). Table 1 documents the R^2 statistic from regressing the basis on the interest rate, spot price, and inventory variables. The R^2 statistic from this regression and in this sample is about 20%.

The model implicitly assumes integration between the financial and real markets. In particular, the model is calibrated to crude oil futures prices, bond data and crude oil inventory levels, under the assumption that a few latent state variables can capture variation in the data to a significant degree. In fact, we find that the first three principal components capture over 99% of the variance in the short interest rate, inventory, and futures dataset, so a few factors do seem to drive covariation in these variables.

2.2. Model estimation methodology

We estimate the model parameters in the spirit of Schwartz, using the extended Kalman filter (details in Appendix F). Duffee and Stanton (2004) compare several estimation procedures and report the superior finite sample properties of the extended Kalman filter. We use financial data on crude oil futures and bond yields, as well as data on crude oil inventory levels, to estimate the model. Because the same four latent state variables drive both the financial and real factors, the calibrated latent state vector will capture the common components in the joint dynamics of the financial and real factors.

We estimate the model parameters using weekly data over the first ten years of the sample (2004–13). This leaves the last 78 weeks for out-of-sample analysis. Specifically, we use the second through the eighth nearest-to-maturity futures and zero coupon bonds with maturities 6 months and 1, 2, 3, 5, 7, and 10 years, and the inventory series. We assume this data is measured with error that is independently and identically distributed Gaussian with a diagonal covariance matrix. Following Trolle and Schwartz (2009), we assume one variance parameter (ω_f) for each futures measurement error, a second (ω_r) for each interest rate, and a third (ω_q) for the inventory data series.[8]

2.3. Statistical significance of inventory and nonlinear effects

We first estimate the parameters of the affine economic representation (EXT). When we set to zero the eight parameters that relate the spot price system to inventories, a

[8] For example, on a given day each of the seven futures prices is observed with error and this error will vary with maturity. The error at a particular maturity will also vary over time. As noted, the errors are modeled as Gaussian random variables. The variance of these errors (ω_f) is the same at all maturities; this ω_f parameter and its standard error are estimated as part of the model calibration.

restricted model corresponding to standard market model (STD) results. A likelihood ratio test rejects the standard market model (STD) relative to the extended affine model with inventories (EXT) at the 1% level.[9]

Next, we estimate the nonlinear (QTSM) model parameters and standard errors. Most of these parameters are significant, which indicates that four factors are empirically relevant for oil price dynamics. In this quadratic setting, four parameters $(\pi_1, \pi_2, \pi_3, \pi_4)$ relate the spot price system to inventories; setting these parameters to zero yields a restricted nonlinear specification (QTSM-NOINV) without inventory effects. A likelihood ratio test rejects this restricted nonlinear model without inventories (QTSM-NOINV) relative to the full nonlinear model with inventories (QTSM) at the 1% level. The results of likelihood ratio tests that compare the nonlinear models with and without inventories are consistent with the results that compare the affine models with and without inventories. These likelihood ratio tests suggest that the information content of the inventory data is economically important for oil price dynamics.

The nonlinear drift term Φ is also statistically significant. Restricting this nonlinear term to zero yields the extended affine model with inventories (EXT). Many of the diagonal elements of Φ are individually significant. Moreover, likelihood ratios reject joint restrictions on the nonlinear term Φ at the 1% level. In particular, a likelihood ratio test formally rejects the linear model with inventories (EXT) relative to the full nonlinear model with inventories (QTSM) at the 1% level.

As discussed, nonzero elements in the Φ matrix induce nonlinearity in the drift of

[9] CCD estimate that the relationship between spot prices and convenience yields is positive in their sample (1987–2003). Interestingly, in our more recent oil data (2004–13) the parameter estimates for the standard market model (STD) do not reflect a positive spot price–convenience yield relation.

the spot price process (Equation 1.4). We find strong evidence of such nonlinearity in oil spot price dynamics. Carlson, Khokher, and Titman (2007) predict that time-varying supply inelasticity can induce important nonlinearity into price dynamics. This nonlinearity can, in turn, induce state-dependent volatility. These results establish nonlinearity in the oil price drift tied to state-dependent volatility as a robust stylized fact.

2.4. Model pricing performance

The model estimation assumes that futures prices are measured with error; this assumption avoids the problem of exactly fitting several futures prices with just a few state variables. These measurement errors are Gaussian variables, and the variance of these errors is a measure of the model pricing performance. The estimated variance of the futures measurement error (ω_f) for our model is small, and this small variance indicates that the model matches the futures price data well. The variance of the measurement errors for inventory data (ω_q) are larger. This finding suggests that inventories have multiple components. The model isolates a discretionary component of inventories that is correlated with prices (Equation 1.2). The larger variance on the inventory measurement error is consistent with a separate nondiscretionary component.

The futures contract prices used to estimate our model parameters are matched with very little error. Figure 1 plots weekly pricing errors between the crude oil futures data and the model-implied futures prices over the sample period. We compute these errors as differences between the market prices and model-generated (log) futures prices. We compute the model-generated futures prices using the filtered latent state variables and the estimated model parameters. The data include the first eight futures maturities, of which we use Nearby 2 through 8 in the model estimation; the top left and bottom right plots of Figure 1 report pricing errors for the second and eighth nearest-to-maturity futures, respectively. These pricing errors do not significantly differ from zero; these plots reveal that these short maturity futures are matched very well.[10]

2.5. Model-implied long-maturity futures prices

Table 2 compares futures prices generated by the nonlinear model (QTSM) and the standard model (STD). This table's first column shows that one-year futures prices generated by the standard market model (STD) are very similar to those implied by our model (QTSM). Futures prices for less than a year generated by the standard market model (STD) are, on average, within 1% of those implied by the nonlinear (QTSM) model. This is not surprising, because we calibrate both models using short maturity futures prices and both models match these prices well.

The difference in model-implied futures prices grows significantly larger at longer maturities. For example, at the five-year maturity, the mean absolute difference in prices increases to over 20%. The standard deviation of the difference is over 25%, so the difference in these model-implied futures prices often exceeds 30%. This exercise definitively establishes that the QTSM has a substantial impact on long-maturity futures prices relative to extant models.

The difference in model-generated long-maturity futures prices results from the

[10] The fit between realized nearest-to-maturity prices and the model-implied spot prices provides some insight into the fit of the model to a contract that is not used to estimate the model parameters. Recall that nearest-to-maturity futures are not directly matched in our model estimation. The median measurement error with respect to the near-term contract does not reveal a significant nonzero bias. The mean absolute error, at less than one-half of 1% of the mean price of the nearest-to-maturity futures, is also quite small.

additional factor in our model compared with extant three-factor models. This becomes apparent by comparing futures prices generated by the full nonlinear specification (QTSM), the restricted specification without inventories (QTSM-NOINV), and the affine representation with inventories (EXT). The term structure of futures prices implied by these four-factor specifications are quite similar, only the three-factor standard (STD) model implies significantly different futures term structures. [11]

Duffee (2009) argues that building flexibility into term structure models by increasing the dimension of their state vector can uncover additional information from the time-series dynamics of prices. Current state-of-the-art models of commodity price dynamics choose a three-dimensional state vector; this choice is motivated by a principal component analysis that finds that three factors suffice to accommodate the observed variation in the futures term structure. Duffee (2009) argues that when there is information that is hidden from the current term structure such models will be misspecified. In particular, he notes that increasing the size of the state vector enables filtering-based estimation to tease additional information from price dynamics. Consistent with this insight, the additional factor in our model relative to current three-factor models uncovers significant incremental information about the prices of long-maturity crude oil futures contracts.

We also note that the nonlinearity and inventory effects contain significant incremental information about future spot prices. We can see this result by comparing the

[11] Furthermore, the long run futures prices implied by the QTSM are estimated with greater precision. Lower long-run price volatility estimates in the QTSM relative to the restricted standard specification reflect this precision. The greater precision also stems from the extra factor underlying the price dynamics of the QTSM relative to the STD model. This added precision increases confidence in the QTSM-implied estimates of long-run futures prices.

full QTSM model with the restricted QTSM-NOINV and EXT specifications. These restricted specifications imply significantly different long-horizon spot price forecasts relative to the QTSM.[12] The top panel of Table 3 shows that the mean absolute difference between the five-year forecast generated by the nonlinear (QTSM) model and the affine model (EXT) is over 50%.[13] As noted, long-maturity futures curves implied by these three models are similar. We conclude that the impact of inventories and nonlinearity upon spot price forecasts in our model operates largely through the risk-premia channel.

3. Empirical Evidence and Model Implications

In this section, we compare our model-generated long-maturity futures prices to the market prices of long-maturity futures contracts. In the previous section, we performed this comparison at short maturities. We also study model implications for the impact of globalization on U.S. stocks, spot price reversion, and unspanned stochastic volatility.

3.1. Long-maturity futures pricing errors

Given the difference in long-maturity futures prices between our model and the standard market model (STD), we now compare these model-generated prices to market prices. For this exercise, we collect long-maturity crude oil futures price data for maturities up to six years. The exercise we run has become possible because reliable market prices

[12] In comparing the full nonlinear model with the restricted specification without inventories, we also note that inventories contain some information about short-run price volatility. However, we find the improvement in the model's performance in matching the prices of option prices is small compared with that for long-maturity futures prices.

[13] Tables 2 and 3 report the results for the last 78 weeks following the period over which the model parameters are estimated (2004–13). The results of comparing model-generated futures prices and price forecasts over the full sample (2004–June 2015) are qualitatively similar.

for crude oil futures contracts with maturities of five years or more have become available since 2004. Previously, Schwartz was restricted to using simulated data in an analogous exercise.

Table 4 records that the fit of futures prices generated by our model (QTSM) to the market prices of long-maturity futures contracts is significantly better than the fit of a current state-of-the-art model (STD). The reported results are for an out-of-sample exercise over the 78 weeks (2014–June 2015) immediately following the period over which the model parameters are estimated (2004–13). Panels A and B record the mean pricing errors for the nonlinear and standard market models, respectively.[14] The difference in mean pricing errors between the models is statistically significant at any reasonable significance level for contracts that will mature five and six years hence. In addition, the mean pricing error is lower at these maturities for the nonlinear model than for the standard model. For example, at the five-year maturity, the mean pricing error for the QTSM is about 50% lower than the mean pricing error for the standard model (STD). Moreover, the QTSM pricing errors are less variable and lead to less extreme valuation errors.[15]

3.2. Globalization and domestic crude oil stocks

As in Gorton, Hayashi, and Rouwenhorst (2012), we use U.S. oil inventory data.

[14] We compute these errors as differences between the market and model-generated (log) futures prices.

[15] The improved performance of our model in fitting the market prices of long-maturity futures contracts results from the additional factor in our model relative to current three-factor models. This result is consistent with the large impact of the additional factor on model-generated long-maturity futures prices. The source of the improvement becomes apparent by first noting that the nonlinear model (QTSM) and the four-factor specification without inventories (QTSM-NOINV) generate comparable mean pricing errors. However, restricting the nonlinear specification without inventories to three factors significantly increases the mean pricing error.

However, the global nature of the crude oil market and the fact that oil prices are set internationally has sparked a recent debate on whether domestic crude oil stocks from any single nation can adequately capture economic or discretionary crude inventories.

Gorton, Hayashi, and Rouwenhorst (2012) note that because most commodity futures contracts call for physical delivery at a particular location, futures prices should reflect the perceived relative scarcity for delivery at that location. Singleton (2010) raises the issue of globalization and notes that U.S. reserves might give only a partial picture of the relevant stocks. However, Singleton (2010), as well as Fama and French (1988), express serious concerns about the accuracy of international data.

U.S. stocks might plausibly be an adequate measure of economic inventory because inventories and prices are jointly determined. Routledge, Seppi, and Spatt (2009) emphasize that storage operators will react to spot and futures prices, even as prices will reflect inventory levels. Global market conditions, reflected in internationally set oil prices, will likely feed back into domestic inventories and induce a component of domestic inventories that is closely tied to the international economy.

Examining the data through the lens of the affine economic representation explicitly highlights novel and interesting feedback effects between the real economy (inventories) and the financial economy (convenience yields). The joint determination of inventory levels and convenience yields is reflected in the affine economic representation by the parameters α_q^{δ} and α_{δ}^{q} of Equation 1.11. Several theoretical models, for example Routledge, Seppi, and Spatt (2000), characterize endogenous convenience yields when production is exogenous and storage operators optimally respond to supply and demand shocks. These theoretical models predict that during periods of scarcity stocks of the commodity held in storage will be drawn down and convenience yields will rise.

We find the estimates for α_q^δ and α_δ^q are both negative and significant. These negative estimates are consistent with the theory of storage hypothesis. Further, these estimates of the underlying structural relation are obtained within a system of simultaneous equations that addresses the joint determination of inventories and prices. Hence, they are robust to the simultaneous equations biases that might result from empirical least squares estimates derived from models that do not consider their joint determination.

The empirical estimates for α_q^δ and α_δ^q document novel and informative feedback effects between the real economy (inventories) and the financial economy (convenience yields). Convenience yields are related to inventories $\left(\alpha_q^\delta\right)$, and we identify additional marginal co-movement between inventories and convenience yields $\left(\alpha_\delta^q\right)$.[16] This evidence confirms that internationally-set oil prices do feed back into domestic inventories; this feedback induces a component of U.S. stocks that is tied to the global economy. This is consistent with our finding that the U.S. inventory data are sufficiently correlated with economic inventory to have a significant impact on price dynamics.[17]

Feedback effects are also apparent in the nonlinear model. For instance, inventories and spot prices share at least two state variables to which they are significantly related. We can also note the relationship between the nonlinear model's economic factors by recording their response to a shock. Figure 2 illustrates the results of this exercise. The dashed (solid)

[16] These parameters also impact the spot price-convenience yield relation; together with this fact, the evidence is consistent with these parameters reflecting feedback effects between convenience yields and inventories (Appendix D).

[17] However, we can reasonably expect that more refined empirical measures of economic inventory are likely to yield even stronger results on the relevance of inventories for oil price dynamics.

line corresponds to a small (large) shock. The top two panels of Figure 2 confirm that in response to a shock, interest rates and inventory levels increase together. Panels C and D reveal that the shock decreases both the futures basis and the term structure of volatility.

3.3. Mean reversion in oil prices

Documenting mean reversion in spot prices has been the focus of several studies of oil markets, see, for example, Bessembinder et al. (1995). We repeat Bessembinder et al.'s regression of the futures slope on spot prices in our crude oil data and confirm price reversion. We briefly summarize these results here but, for brevity, do not tabulate them. We find that, assuming no bias in futures prices relative to the expected spot, market participants expect that 7% of a typical crude oil price shock will decay in two months, but they expect 25% of a typical shock to decay eight months hence. The autocorrelations of crude oil spot price changes are significantly negative at long holding periods, and oil spot price volatility declines monotonically from 33% to 28% at the eighth nearest maturity. This evidence is consistent with oil price reversion.

Schwartz (1997) emphasizes the importance of models that accommodate oil price reversion. Spot price reversion implies a negative relation between price drift and the level of spot prices. Singleton (2010) argues that informational frictions could cause this relation to be negative. Figure 3 presents the time series of the spot price drift (μ) implied by the model. The spot price drift includes a convenience yield (δ) component, and the bottom panel of Figure 3 depicts the convenience yield. The convenience yield fluctuates over quite a large range, and Figure 4 confirms the convenience yield is quite volatile. Figure 4 also presents term structures of price volatility. Consistent with the mean-reverting specifications of the convenience yield and the spot price process in the literature, the model-implied volatility curves for spot prices and convenience yields slope downward.

Table 5 directly examines the relation between the price drift and the spot price. The left-hand panel of this table presents slope coefficients from a regression of the model-implied price drift on the interest rate, inventory, and spot price factors.[18] The spot price coefficient is significant and negative, which directly confirms oil price reversion in the historical data (physical measure). The combined price drift and futures slope evidence sheds light on aspects of the same underlying spot price reversion. The former induces spot price reversion under the physical measure, while the latter reflects price reversion under the risk-neutral measure. The leftmost panel of Table 5 is consistent with the prevailing view that oil prices exhibit mean reversion.

Oil prices are subject to many different types of shocks that can give rise to distinct price components. For example, Schwartz and Smith (2000) propose a decomposition of crude oil spot prices into short- and long-term levels (Figure 5). Accordingly, price drift could plausibly react to distinct price shocks differently. We decompose the spot price factor by regressing it on the other model factors. Spot prices have a component that co-moves with interest rates, inventories, and convenience yields, and this co-movement is reflected in the fitted component from this regression ($\hat{x}$). The autonomous component of spot prices is the regression residual (x^{resid}).

The middle panel of Table 5 repeats the regression in the left panel of that table, but replaces the spot price factor (X) with the price components $\hat{x}$ and x^{resid}. This panel

[18] The table estimates the first-order relation between model factors as overall linear. This does not necessarily preclude a locally nonlinear relation between them, nor does it preclude a nonlinear relation between the spot price and the latent state variables.

reports that price drift is positively related to X^{resid}. Consistent with Singleton's prediction, this exercise isolates a price component that is positively related to price drift.

The rightmost panel of Table 5 explores time variation in risk premia for crude oil prices. Here, we compute risk premia as the difference in price drift under the physical and risk-neutral measure $(\mu - \mu^{Q})$. This panel documents significant variation in risk premia with each of the model's economic factors. It also documents that the slope coefficient on X^{resid} is positive. Comparing the magnitude of the slope coefficients on X^{resid} in the middle and rightmost panels reveals that the positive price drift effect is largely due to time variation in risk premia, just as Singleton (2010) predicts.[19]

3.4. Unspanned stochastic volatility

In this section, we quantify the extent to which oil spot price volatility is spanned by futures contracts. Trolle and Schwartz (2009) convincingly demonstrate that adding factors to a pricing model can help to simultaneously match crude oil futures and options. Li and Zhao (2006) examine the extent to which a model calibrated to bond yields can match interest rate derivatives. Following their approach, we compute the difference between the model-fitted and observed volatility and, hence, quantify the extent to which volatility in the crude oil markets is spanned by futures.

We collect daily settlement prices for all European crude oil options traded on the

[19] Because risk premia are linear in the state variables, the slope coefficients in the middle panel of Table 5 are the sum of the slope coefficients in the right panel (risk premia) and those from an analogous regression that uses the spot price drift under the risk-neutral measure (μ^{Q}).

NYMEX.[20] The raw daily data spans March 2004–June 2015 and consists of over a million option prices. From this raw data, we sort 63,187 call and put option prices for 564 weeks (Wednesdays). In sorting the data, we closely follow Trolle and Schwarz (2009), who focus their analysis on out-of-the-money options because of their greater liquidity. This focus implies puts (calls) with moneyness less (greater) than one, with moneyness defined as the strike price divided by the matching maturity futures. We include options with moneyness between 0.79 and 1.21. Consistent with the second through eighth nearby futures contracts to which the model is calibrated, the option maturities range between 45 and 255 days.

We use each option price to invert the Black (1976) formula for an implied volatility; we then compute a volatility for each date as the average of all of the options traded on that date. Volatility increased to about 80% during the 2008–09 crisis. Volatility rose again when oil prices declined in 2014, although this increase was less dramatic (50%) than the increase in 2008–09. In the full period, oil price volatility exhibited substantial variation around a mean of 30%. The term structure of volatility, in general, decreases with maturity. Short-term out-of-the-money put options tend to be expensive relative to at-the-money options, which is consistent with jump risk being priced. Out-of-the-money put option volatility remains high at all maturities.

Table 6 records the mean pricing error for this options data at various moneyness-maturity categories. Here, we compute pricing error as the difference between the model-generated and market-implied volatility. To quantify the extent to which oil spot price

[20] Data for European crude oil options have become available relatively recently. Trolle and Schwarz (2009) study American option prices; analysts must estimate an early exercise premium for these options. This requirement has the potential to bias estimated magnitudes. Following their lead, we only consider options with open interest greater than 100 contracts and prices above $0.01.

volatility is spanned by futures contracts, we focus on at-the-money options for the Nearby 5 through Nearby 8 maturities. These relatively long-maturity at-the-money option contracts are most likely to reflect pure unspanned volatility components. The mean pricing errors and observed volatilities for these contracts average 0.08 and 0.31, respectively. We conclude that about 26% of crude oil spot price volatility is not spanned by crude oil futures prices.

4. Conclusion

Reliable estimates of long-horizon futures contract prices are useful for valuing real commodity investments. However, as Carlson, Khokher, and Titman (2007, 1663) predict, "when these long dated contracts are illiquid there are problems associated with using existing pricing models to extrapolate their values from the observable prices of more liquid shorter-term contracts." We confirm that calibrating existing models to a cross-section of liquid short-term futures does not detect all of the factors that affect long-maturity futures. We estimate a nonlinear four-factor continuous-time model of commodity price dynamics that significantly improves our ability to infer long-maturity futures prices from short-dated contracts. Thus, the model we develop contains significant incremental information on a factor that affects the prices of long-dated futures contracts.

This study also highlights several intriguing puzzles for future research. The dramatic fluctuation in oil prices over the past decade compared with previous decades has fueled speculation that a fundamental shift in oil prices has occurred. While the statistical evidence is consistent with a structural change in oil price dynamics, our model is reduced form and does not make any predictions as to the cause of this shift.[21] A clear understanding of the cause of this shift would clarify a puzzling question. While econometricians try to accommodate for such breaks using subsample analysis (Baele et al. 2015), we suggest another intriguing question for further research. It would be interesting to investigate stochastic models of commodity prices that can accommodate time-varying parameters or multiple regimes that span many decades. The shift in oil prices does not pose a concern for our study, however, because our focus is on long-maturity futures contracts that have only traded for a relatively short period.

We also leave for future research a thorough empirical analysis of whether contingent claims models with their no-arbitrage restrictions can help out-of-sample forecasts of commodity price changes. However, our model's implication in the first quarter of 2014 is intriguing in this regard. Oil prices fell dramatically in the fall of 2014. We can use our model estimates for 2004–13 to generate out-of-sample long-maturity forecasts of expected spot prices. Figure 6 plots these out-of-sample (five-year) predictions as of the first quarter of 2014. The model-implied oil price forecasts in the first quarter of 2014 were steeply downward sloping. These forecasts should have struck a cautionary note for market participants.

[21] Using the approach of Rudebusch and Wu (2006), we do find evidence of such a shift. In particular, we estimate the models for two subsamples pre/post–2004, as well as for the entire period. A likelihood ratio comparison of the two subsamples with the full period statistically documents a shift at any reasonable significance level. We are not aware of any other study that documents convincing econometric evidence of such a shift.

Appendix A

Given the log (spot) price specification,

$X(t) = F(t,y) = \phi_0 + \phi Y(t) + Y(t)'\Phi_Y Y(t),$ then $\partial F(t,y)/\partial Y = \phi^T + 2Y(t)\Phi_Y{}^T$ and

$\partial F(t,y)/\partial Y = \phi^T + 2Y(t)\Phi_Y{}^T$ and $\partial^2 F(t,y)/\partial Y^2 = 2\mathrm{tr}(\Phi_Y)$. Here, *tr(.)* denotes the trace of a matrix. Applying Ito's lemma results in $dX(t) = \phi^T dY + 2Y^T \Phi_Y dY + tr(\Phi_Y)dt$,

which we can simplify to:

$$
\begin{aligned}
dX(t) &= \phi^T dY + \left[tr(\Phi_Y) - 2Y^T \Phi_Y \kappa^Q Y]dt + 2Y^T \Phi_Y dZ^Q \right] \\
dX(t) &= \phi^T \left[-\kappa^Q Y dt + dZ^Q \right] + \left[tr(\Phi_Y) - 2Y^T \Phi \kappa^Q Y]dt + 2Y^T \Phi_Y dZ^Q \right] \\
dX(t) &= \left[tr(\Phi_Y) - \phi^T \kappa^Q Y - 2Y^T \left(\Phi_Y \kappa^Q \right) Y]dt + [\phi^T + 2Y'\Phi_Y]dZ^Q \right]
\end{aligned}
$$

For completeness, we note that (log) spot price dynamics under the physical measure are:

$$dX(t) = \left[\beta_0 + tr(\Phi_Y) + \left(\beta_1 - \phi^T \kappa^Q \right) Y - 2Y^T \left(\Phi_Y \kappa^Q \right) Y]dt + [\phi^T + 2Y'\Phi_Y]dZ \right]$$

Casassus and Collin-Dufresne (2005) rearrange the no-arbitrage restriction that $E_t^Q\left[dS(t)\right] = [r(t) - \delta(t)]S(t)dt$ so that a convenience yield is implied such that

$\delta(t) = r(t) - \frac{1}{dt} E_t^Q \left[\frac{dS(t)}{S} \right]$. They then set $E_t^Q \left[\frac{dS(t)}{S} \right] = E_t^Q\left[dX(t)\right] + \frac{1}{2} V_t^Q\left[dX(t)\right]$, where

$V_t^Q\left[\bullet\right]$ denotes variance under the risk-neutral measure, to arrive at the implied

convenience yield: $\delta(t) = r(t) - \frac{1}{dt}\{E_t^Q\left[dX(t)\right] + \frac{1}{2}E_t^Q\left[dX(t)\right]^2\}$. We use this

specification of the convenience yield in our setting, so that:

$$\delta(t) = \psi_0 + \psi_1 Y_1(t) - \frac{1}{dt}\{E_t^Q\left[dX(t)\right] + \frac{1}{2}E_t^Q\left[dX(t)\right]^2\}.$$

From the above, $E_t^Q\left[dX(t)\right]=[tr(\Phi_Y)-\phi^T\kappa^Q Y-2Y^T\left(\Phi_Y\kappa^Q\right)Y]dt$ and

$E_t^Q\left[dX(t)^2\right]=[\phi^T+2Y^T\Phi_Y][\phi^T+2Y^T\Phi_Y]^T dt$. Substituting these two expectations into the convenience yield expression and rearranging results in:

$$
\begin{aligned}
&\delta(t)=\psi_0+\psi_1 Y_1(t)\\
&-\left[tr(\Phi_Y)-\phi^T\kappa^Q Y(t)-2Y(t)^T\,\Phi_Y\kappa^Q Y(t)\frac{\left(\phi^T\phi\right)}{2}\right]\\
&-\frac{1}{2}\left[\phi^T+2Y(t)^T\,\Phi_Y\right]\left[\phi^T+2Y(t)^T\,\Phi_Y\right]^T.
\end{aligned}
$$

$$
\begin{aligned}
&\delta(t)=\psi_0-tr(\Phi_Y)-\frac{1}{2}\phi^T\phi\\
&+\left[\psi_1 Y_1(t)+\phi^T\kappa^Q Y(t)-\phi^T\Phi_Y{}^T Y(t)-Y(t)^T\,\Phi_Y\phi\right]\\
&+2Y(t)^T\,\Phi_Y\kappa^Q Y(t)-2Y(t)^T\,\Phi_Y\Phi_Y{}^T Y(t).
\end{aligned}
$$

Collecting terms and recalling that the matrix Φ_Y is symmetric results in:

$$
\delta(t)=\psi_0-tr(\Phi_Y)-\frac{1}{2}\phi^T\phi+\psi_1 Y_1(t)+\phi^T\left(\kappa^Q-2\Phi_Y\right)Y(t)+Y(t)^T\,[2\Phi_Y(\kappa^Q-\Phi_Y)]Y(t).
$$

The dynamics of this convenience yield factor follow from Ito's lemma:

$$
\begin{aligned}
&d\delta(t)=+2tr\left(\Phi(\kappa^Q-\Phi_Y)\right)dt\\
&+\psi_1 dY_1(t)+\phi^T\left(\kappa^Q-2\Phi_Y\right)dY(t)\\
&+2Y(t)^T\left[2\Phi_Y(\kappa^Q-\Phi_Y)\right]dY(t)
\end{aligned}
$$

$$
\begin{aligned}
&d\delta(t)=+\left[2tr\left(\Phi_Y(\kappa^Q-\Phi_Y)\right)+\left[(\psi_1\ 0\ 0\ 0)-\left(\phi^T\kappa^Q-2\phi^T\Phi\right)\kappa^Q\right]Y(t)-2Y(t)^T\left[\Phi(\kappa^Q-\Phi_Y)\kappa^Q\right]Y\right]dt\\
&+\left[(\psi_1\ 0\ 0\ 0)+\phi^T\left(\kappa^Q-2\Phi_Y\right)+4Y(t)^T\,\Phi(\kappa^Q-\Phi_Y)\right]dZ^Q
\end{aligned}
$$

Appendix B

Rearranging the no-arbitrage restriction that $E_t^Q[dS(t)] = [r(t) - \delta(t)]S(t)dt$ gives $\delta(t) = r(t) - \frac{1}{dt}E_t^Q\left[\frac{dS(t)}{S}\right]$. With the current identifying restriction on the short interest rate, $\delta(t) = \psi_0 + \psi_1 Y_1(t) - \frac{1}{dt}E_t^Q\left[\frac{dS(t)}{S}\right]$. From the properties of the log-normal distribution, $E_t^Q\left[\frac{dS(t)}{S}\right] = E_t^Q[dX(t)] + \frac{1}{2}V_t^Q[dX(t)]$, where $V_t^Q[\bullet]$ denotes variance under the risk neutral measure. So, $\delta(t) = \psi_0 + \psi_1 Y_1(t) - \frac{1}{dt}\{E_t^Q[dX(t)] + \frac{1}{2}E_t^Q[dX(t)]^2\}$. Applying Ito's lemma to the definition of the (log) spot price, $X(t) = \phi_0 + \phi_Y^T Y(t)$, yields $dX(t) = \phi_Y^T dY(t) = \phi_Y^T\left[-\kappa^Q Y(t)dt + dZ(t)^Q\right] = -\phi_Y^T \kappa^Q Y(t)dt + \phi_Y^T dZ(t)^Q$. In this case, $E_t[dX(t)] = [-\phi_Y^T \kappa^Q Y(t)]dt$ and $E_t^Q[dX(t)]^2 = \phi_Y^T \phi_Y dt$. Substituting and rearranging yields $\delta(t) = \psi_0 - \frac{1}{2}\phi_Y^T \phi_Y + \psi_1 Y_1(t) + \phi_Y^T \kappa^Q Y(t)$. This result implies $\delta(t) = \eta_0 + \eta_Y^T Y(t)$ with

$$\eta_0 = \psi_0 - \frac{1}{2}\phi_Y^T \phi_Y$$

$$\eta_Y^T = \left[\psi_1 + \kappa_{11}^Q\phi_1 + \kappa_{21}^Q\phi_2 + \kappa_{31}^Q\phi_3 + \kappa_{41}^Q\phi_4 \quad \kappa_{22}^Q\phi_2 + \kappa_{32}^Q\phi_3 + \kappa_{42}^Q\phi_4 \quad \kappa_{33}^Q\phi_3 + \kappa_{34}^Q\phi_4 \quad \kappa_{44}^Q\phi_4\right]^T.$$

Appendix C

When the economic factor vector $W(t) = [r(t), q(t), \delta(t), X(t)]$ is linear in the state vector, we can derive its dynamics without reference to the latent state vector $Y(t)$.

In this case, $W = \vartheta + LY(t)$ with $\vartheta = (\psi_0, \pi_0, \eta_0, \phi_0)$ and $L = \begin{bmatrix} \psi_1 & 0 & 0 & 0 \\ \pi_1 & \pi_2 & \pi_3 & \pi_4 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{bmatrix}$. From

Ito's lemma $dW(t) = LdY = L\kappa^Q L^{-1}(\vartheta - W(t))dt + LdZ^Q(t)$. Matching the coefficients of the economic state variables and the constants in Proposition 1 with $L\kappa^Q L^{-1}$ and $L\kappa^Q L^{-1}\vartheta$, respectively, yields the economic parameters in terms of the latent parameters. For example, $\kappa_{qr}^{Q} = -\left(L\kappa^{Q}L^{-1}\right)_{21}$, $\kappa_{q}^{Q} = -\left(L\kappa^{Q}L^{-1}\right)_{22}$, $\kappa_{q0}^{Q} = L\kappa^{Q}L^{-1}\vartheta_{21}$, and so on.

To obtain the covariance matrix of the economic representation matrix we can similarly match coefficients with LL^{T}. For example, $\sigma_{r} = \sqrt{LL^{T}_{11}}$, $\sigma_{q} = \sqrt{LL^{T}_{22}}$ and

$\rho_{rq} = \frac{LL^{T}_{12}}{\sigma_{r}\sigma_{q}}$ QED.

Appendix D

Here, we use the transformed economic factors $\hat{\bar{W}}(t) = \left[r(t), \bar{q}(t), \hat{\bar{\delta}}(t), X(t)\right]$ with:

$$\begin{aligned}\bar{q}(t) &= q(t) - \alpha_{r}^{q} r(t) - \alpha_{\delta}^{q}\,\delta(t) - \alpha_{X}^{q}\,X(t)\\ \hat{\bar{\delta}}(t) &= \delta(t) - \boxed{\alpha_{r}^{\delta}}r(t) - \boxed{\alpha_{q}^{\delta}}q(t) - \boxed{\alpha_{X}^{\delta}}X(t).\end{aligned}$$

It follows that $\bar{q}(t) = \bar{\pi}_{0} + \bar{\pi}_{Y}^{T}Y(t)$ and $\hat{\bar{\delta}}(t) = \bar{\eta}_{0} + \bar{\eta}_{Y}^{T}Y(t)$ where

$$\bar{\pi}_{0} = \pi_{0} - \alpha_{r}^{q}\psi_{0} - \alpha_{\delta}^{q}\eta_{0} - \alpha_{X}^{q}\phi_{0}\ ;\ \bar{\pi}_{Y} = \begin{bmatrix}\bar{\pi}_{1}\\ \bar{\pi}_{2}\\ \bar{\pi}_{3}\\ \bar{\pi}_{4}\end{bmatrix} = \begin{bmatrix}\bar{\pi}_{1} - \alpha_{\delta}^{q}\eta_{1} - \alpha_{X}^{q}\phi_{1} - \alpha_{r}^{q}\psi_{1}\\ \bar{\pi}_{2} - \alpha_{\delta}^{q}\eta_{2} - \alpha_{X}^{q}\phi_{2}\\ \bar{\pi}_{3} - \alpha_{\delta}^{q}\eta_{3} - \alpha_{X}^{q}\phi_{3}\\ \bar{\pi}_{4} - \alpha_{\delta}^{q}\eta_{4} - \alpha_{X}^{q}\phi_{4}\end{bmatrix}\ ;\ \text{and}$$

$$\bar{\eta}_{0} = \eta_{0} - \alpha_{r}^{\delta}\psi_{0} - \alpha_{\delta}^{\delta}\eta_{0} - \alpha_{X}^{\delta}\phi_{0}\ ;\ \bar{\eta}_{Y} = \begin{bmatrix}\bar{\eta}_{1}\\ \bar{\eta}_{2}\\ \bar{\eta}_{3}\\ \bar{\eta}_{4}\end{bmatrix} = \begin{bmatrix}\eta_{1} - \alpha_{q}^{\delta}\pi_{1} - \alpha_{X}^{\delta}\phi_{1} - \alpha_{r}^{\delta}\psi_{1}\\ \eta_{2} - \alpha_{q}^{\delta}\pi_{2} - \alpha_{X}^{\delta}\phi_{2}\\ \eta_{3} - \alpha_{q}^{\delta}\pi_{3} - \alpha_{X}^{\delta}\phi_{3}\\ \eta_{4} - \alpha_{q}^{\delta}\pi_{4} - \alpha_{X}^{\delta}\phi_{4}\end{bmatrix}.$$

In this case, $\hat{\bar{W}} = \hat{\bar{\vartheta}} + \hat{\bar{L}}Y(t)$ with $\hat{\bar{\vartheta}} = \left(\psi_{0}, \bar{\pi}_{0}, \bar{\eta}_{0}, \phi_{0}\right)$ and $\hat{\bar{L}} = \begin{bmatrix}\psi_{1} & 0 & 0 & 0\\ \bar{\pi}_{1} & \bar{\pi}_{2} & \bar{\pi}_{3} & \bar{\pi}_{4}\\ \bar{\eta}_{1} & \bar{\eta}_{2} & \bar{\eta}_{3} & \bar{\eta}_{4}\\ \phi_{1} & \phi_{2} & \phi_{3} & \phi_{4}\end{bmatrix}$

Proceeding as in Appendix B by applying Ito's lemma, $d\hat{\bar{W}}(t) = \hat{\bar{L}}\kappa^{Q}\hat{\bar{L}}^{-1}\left(\hat{\bar{\vartheta}} - \hat{\bar{W}}(t)\right)dt + \hat{\bar{L}}dZ^{Q}(t)$. Matching the coefficients of the economic state variables and the constants in Proposition 2 with $\tilde{L}\kappa^Q\tilde{L}^{-1}$ and $\tilde{L}\kappa^Q\tilde{L}^{-1}\vartheta$, respectively, yields the economic parameters in terms of the latent parameters. To obtain the covariance matrix of the economic representation matrix we similarly match coefficients with $\tilde{L}\tilde{L}^T$.

The economic factors $\tilde{q}(t)$ and $\tilde{\delta}(t)$ are autonomous once we choose the six parameters $\alpha_r^\delta, \alpha_q^\delta, \alpha_X^\delta, \alpha_r^q, \alpha_\delta^q, \alpha_X^q$, and α_X^q so that the corresponding six non-diagonal elements in the mean-reversion matrix $\tilde{L}\kappa^Q\tilde{L}^{-1}$ are zero. We document the expressions for these six parameters when κ^Q is diagonal (analogous expressions follow for κ^Q lower triangular).

$$\alpha_r^\delta = \frac{\pi_1(\kappa_{44}-\kappa_{22})\phi_2\phi_4 + \pi_2(\phi_1\kappa_{11}-\phi_1\kappa_{44}+\psi_1)\phi_4 - \pi_4(\phi_1\kappa_{11}-\phi_1\kappa_{22}+\psi_1)\phi_2}{(\pi_2\phi_4-\pi_4\phi_2)\psi_1}$$

$$\alpha_r^q = \frac{\pi_1(\kappa_{33}-\kappa_{44})\phi_3\phi_4 + \pi_4(\psi_1-\phi_1\kappa_{33}+\phi_1\kappa_{11})\phi_3\phi_4 - \pi_3\phi_4(\phi_1\kappa_{11}-\phi_1\kappa_{44}+\psi_1)}{\phi_3\phi_4(\kappa_{33}-\kappa_{44})\psi_1}$$

$$\alpha_q^\delta = \frac{(\kappa_{22}-\kappa_{44})\phi_2\phi_4}{(\pi_2\phi_4-\pi_4\phi_2)} \qquad \alpha_X^\delta = \frac{\phi_4\kappa_{44}\pi_2-\phi_2\kappa_{22}\pi_4}{(\pi_2\phi_4-\pi_4\phi_2)}$$

$$\alpha_\delta^q = \frac{\pi_3\phi_4-\pi_4\phi_3}{\phi_3\phi_4(\kappa_{33}-\kappa_{44})} \qquad \alpha_X^q = \frac{\pi_4\phi_3\kappa_{33}-\pi_3\phi_4\kappa_{44}}{\phi_3\phi_4(\kappa_{33}-\kappa_{44})}$$

To see spot price dynamics in this representation, note that rearranging $\tilde{q}(t)$ and $\tilde{\delta}(t)$ yield

$$\delta(t) = \tilde{\delta}(t) + \alpha_r^\delta r(t) + \alpha_q^\delta\left[\tilde{q}(t) + \alpha_r^q r(t) + \alpha_\delta^q\,\delta(t) + \alpha_X^q\,X(t)\right] + \alpha_X^\delta X(t)$$

$$\delta(t) = \frac{(\alpha_r^\delta + \alpha_q^\delta\alpha_r^q)}{(1-\alpha_\delta^q\alpha_q^\delta)} r(t) + \frac{\alpha_q^\delta}{(1-\alpha_\delta^q\alpha_q^\delta)}\tilde{q}(t) + \frac{1}{(1-\alpha_\delta^q\alpha_q^\delta)}\tilde{\delta}(t) + \frac{(\alpha_X^\delta + \alpha_q^\delta\alpha_X^q)}{(1-\alpha_\delta^q\alpha_q^\delta)} X(t)$$

Similarly,

$$q(t) = \frac{(\alpha_r^q + \alpha_\delta^q\alpha_r^\delta)}{(1-\alpha_\delta^q\alpha_q^\delta)} r(t) + \frac{\tilde{q}(t)}{(1-\alpha_\delta^q\alpha_q^\delta)} + \frac{\alpha_\delta^q}{(1-\alpha_\delta^q\alpha_q^\delta)}\tilde{\delta}(t) + \frac{(\alpha_X^q + \alpha_\delta^q\alpha_X^\delta)}{(1-\alpha_\delta^q\alpha_q^\delta)} X(t).$$

Substituting this expression for $\delta(t)$ into the spot price dynamics from Proposition 1 $dX(t)=\left(r(t)-\delta(t)-\frac{1}{2}\sigma_X^2\right)dt+\sigma_X dZ_X^Q(t)$ and simplifying immediately yields :

$$dX(t)=\left[\kappa_{X\hat{r}}^Q\left(\theta_r^Q-r(t)\right)+\kappa_{X\hat{q}}^Q\left(\theta_{\hat{q}}^Q-\hat{q}(t)\right)+\kappa_{X\hat{\delta}}^Q\left(\theta_{\hat{\delta}}^Q-\hat{\delta}(t)\right)+\kappa_X^Q\left(\theta_X^Q-X(t)\right)\right]dt$$
$$+\sigma_X dZ_X^Q(t)$$

with

$$\kappa_{X\hat{r}}^Q=\left(\frac{\alpha_r^\delta+\alpha_q^\delta\alpha_r^q}{1-\alpha_q^\delta\alpha_\delta^q}-1\right)\quad \kappa_{X\hat{q}}^Q=\left(\frac{\alpha_q^\delta}{1-\alpha_q^\delta\alpha_\delta^q}\right)\quad \kappa_{X\hat{\delta}}^Q=\left(\frac{1}{1-\alpha_q^\delta\alpha_\delta^q}\right)\quad \kappa_X^Q=\left(\frac{\alpha_X^\delta+\alpha_q^\delta\alpha_X^q}{1-\alpha_q^\delta\alpha_\delta^q}\right)$$

and

$$\theta_X^Q=\left.-\left(\frac{\sigma_X^2}{2}+\kappa_{X\hat{r}}^Q\theta_r^Q+\kappa_{X\hat{q}}^Q\theta_{\hat{q}}^Q+\kappa_{X\hat{\delta}}^Q\theta_{\hat{\delta}}^Q\right)\right/\kappa_X^Q\quad .$$

Appendix E

Given the specification of the market price of risk in the latent state variable representation, $dZ^Q(t)=dZ(t)+(\beta_{0Y}+\beta_{1Y}Y(t))dt$, the dynamics under the historical measure become $d\hat{W}(t)=\hat{L}\kappa^Q\hat{L}^{-1}\left(\hat{\vartheta}-\hat{W}(t)\right)dt+\hat{L}dZ(t)+\hat{L}(\beta_{0Y}+\beta_{1Y}Y(t))dt$. Substituting $Y(t)=-\hat{L}^{-1}\left(\hat{\vartheta}-\hat{W}\right)$ yields

$$d\hat{W}(t)=\hat{L}\kappa^Q\hat{L}^{-1}\left(\hat{\vartheta}-\hat{W}(t)\right)dt+\hat{L}dZ(t)+(\hat{L}\beta_{0Y}-\hat{L}\beta_{1Y}\hat{L}^{-1}\hat{\vartheta}+\hat{L}\beta_{1Y}\hat{L}^{-1}\hat{W}(t))dt.$$

The risk premia parameters for the economic representation are, therefore:

$$\beta_0=\begin{bmatrix}\beta_{0r}\\ \beta_{0\hat{q}}\\ \beta_{0\hat{\delta}}\\ \beta_{0X}\end{bmatrix}=\hat{L}\beta_{0Y}-\hat{L}\beta_{1Y}\hat{L}^{-1}\hat{\vartheta},\beta_1=\begin{bmatrix}\beta_{rr}&\beta_{r\hat{q}}&\beta_{r\hat{\delta}}&\beta_{rX}\\ \beta_{\hat{q}r}&\beta_{\hat{q}\hat{q}}&\beta_{\hat{q}\hat{\delta}}&\beta_{\hat{q}X}\\ \beta_{\hat{\delta}r}&\beta_{\hat{\delta}\hat{q}}&\beta_{\hat{\delta}\hat{\delta}}&\beta_{\hat{\delta}X}\\ \beta_{Xr}&\beta_{X\hat{q}}&\beta_{X\hat{\delta}}&\beta_{XX}\end{bmatrix}=\hat{L}\beta_{1Y}\hat{L}^{-1}.$$

The market price of risk $dZ_{\hat{W}}^Q(t)=dZ_{\hat{W}}(t)+\Sigma^{-1}\left(\beta_0+\beta_1\hat{W}\right)dt$ in the economic representation follows immediately from $dZ^Q(t) = dZ(t) + (\beta_{0Y} + \beta_{1Y} Y(t))dt$, and recalling $\hat{\tilde{L}} dZ^Q(t) = \Sigma dZ^Q_{\hat{W}}(t)$ and $\hat{\tilde{L}} dZ(t) = \Sigma dZ_{\hat{W}}(t)$.

QED.

Appendix F

Following Schwartz (1997), we estimate our model parameters using the extended Kalman filter. Kalman filter estimation proceeds by maximizing a likelihood function using a state equation and a measurement equation. The state equation describes the dynamics of the state vector, while the measurement equation defines the relationship between the state variables and the data. Several authors, for example Trolle and Schwartz (2009) as well as Li and Zhao (2006), use the Kalman filter to estimate term structure models and document details of this algorithm. Hamilton (1994) is an excellent reference. We follow this standard procedure exactly and summarize this algorithm here for completeness.

We construct the Kalman filter–based log-likelihood function as

$$\log L = -\frac{1}{2}\log 2\pi \sum_{i=1}^{T} N - \frac{1}{2}\sum_{i=1}^{T}\log|V_t| - \frac{1}{2}\sum_{i=1}^{T}\varepsilon_t^{'} V_t^{-1}\varepsilon_t. \tag{F1}$$

Here, T is the number of periods and N is the number of observations each period. The prediction error, ε_t, conditional on the information at t–1, is defined as

$$\varepsilon_t = z_t - h\left(\hat{\bar{Y}}_{t/t-1}\right) \tag{F2}$$

Here, z_t is the data vector, and this consists of seven commodity/yield maturities each and the inventory variable, $z_t = \left[\log F_{t,1} ... \log F_{t,m}, y_{t,1} ... y_{t,m}, q_t\right]$. The model-implied pricing function is denoted h. For example, futures prices are related to the state variables by Equation 1.5.[22] The expected value of Y_t conditional on time $t-1$ is denoted $\hat{Y}_{t/t-1} = E_{t-1}\left(Y_t\right)$; the expectation conditional on the current time t is denoted $\hat{Y}_t = E_t\left(Y_t\right)$. $\hat{P}_t$ and $\hat{P}_{t/t-1}$ are the analogously defined conditional covariance matrices. The derivative of the pricing function (h) with respect to the state variables evaluated at $\hat{Y}_{t/t-1}$ is denoted H_t.

The conditional covariance matrix of the prediction errors V_t is computed as

$$V_t = H_t P_{t/t-1} H_t^T + \Omega \tag{F3}$$

The Kalman filter algorithm requires a measurement equation that defines the relationship between the state variables and the data. This measurement equation is given by

$$z_t = h\left(Y_t\right) + u_t, \qquad u_t \sim N\left(0, \Omega\right) \tag{F4}$$

where, u_t is a vector of iid. Gaussian measurement errors. As in Trolle and Schwartz (2009), we assume Ω is diagonal and has one variance parameter (ω_f) for each futures measurement error, a second (ω_r) for each interest rate maturity, and a third (ω_q) corresponding to the inventory data.

We start the Kalman recursions at the state variable's unconditional means and variances. We then compute $\hat{Y}_{t/t-1}$ and $\hat{P}_{t/t-1}$ from the state equation, which is defined as

$$Y_t = \Theta_0 + \Theta_Y Y_{t-1} + \omega_{t-1}, \qquad \omega_{t-1} \sim N\left(0, \Omega_{\varpi}\right) \tag{F5}$$

where Θ_0 and Θ_Y are immediate from standard results on the transition density of

[22] The closed-form pricing function of zero coupon bonds across different maturities is immediate (as in CCD) from standard results on pricing in the affine framework (e.g., Duffie, Pan and Singleton, 2000). We use Equation 1.2 to obtain the theoretical value of the inventory variable.

multivariate Gaussian variables (for example, Li and Zhao 2006). It follows that $\hat{\bar{Y}}_{t/t-1} = \Theta_0 + \Theta_Y Y_{t-1}$ and $\hat{\bar{P}}_{t/t-1} = \Theta_Y P_{t-1} \Theta_Y^T + \Omega_W$. With these estimates in hand, the log-likelihood function is then updated. The estimates of the mean and variance of the state vector Y_t is also updated to $\hat{\bar{Y}}_t = \hat{\bar{Y}}_{t/t-1} + \hat{\bar{P}}_{t/t-1} H_t^T V_t^{-1} \varepsilon_t$ and $P_t = P_{t/t-1} - P_{t/t-1} H_t^T V_t^{-1} H_t P_{t/t-1}$ and the recursion continues. We then numerically maximize the likelihood function that we compute in this manner to obtain the estimates of the model parameters.

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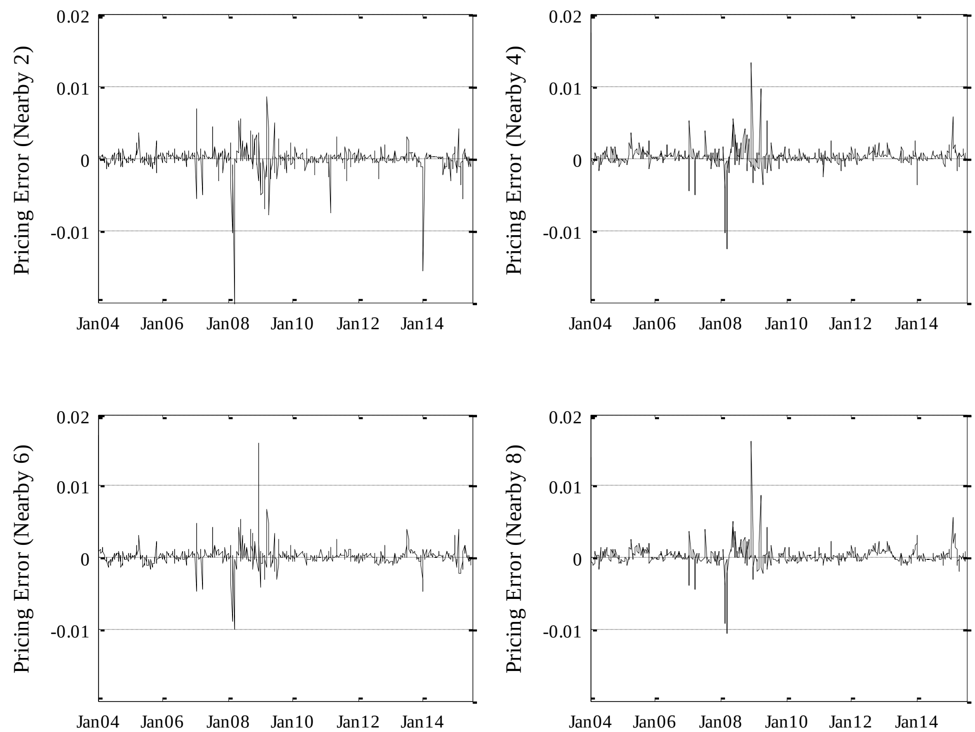

Figure 1
Crude oil futures measurement errors
This figure shows weekly crude oil futures measurement errors over the sample period (January 2004–June 2015) for four futures maturities. These errrors are computed as differences between the actual and model implied (log) futures prices. *Nearby 2* denotes errors computed with the second nearest-to-maturity futures, *Nearby 4* is computed with the fourth nearest-to-maturity price, and so on. The last 78 observations in the figure are over 2014–2015; these errors are out-of-sample because the model is estimated with data up to December 2013.

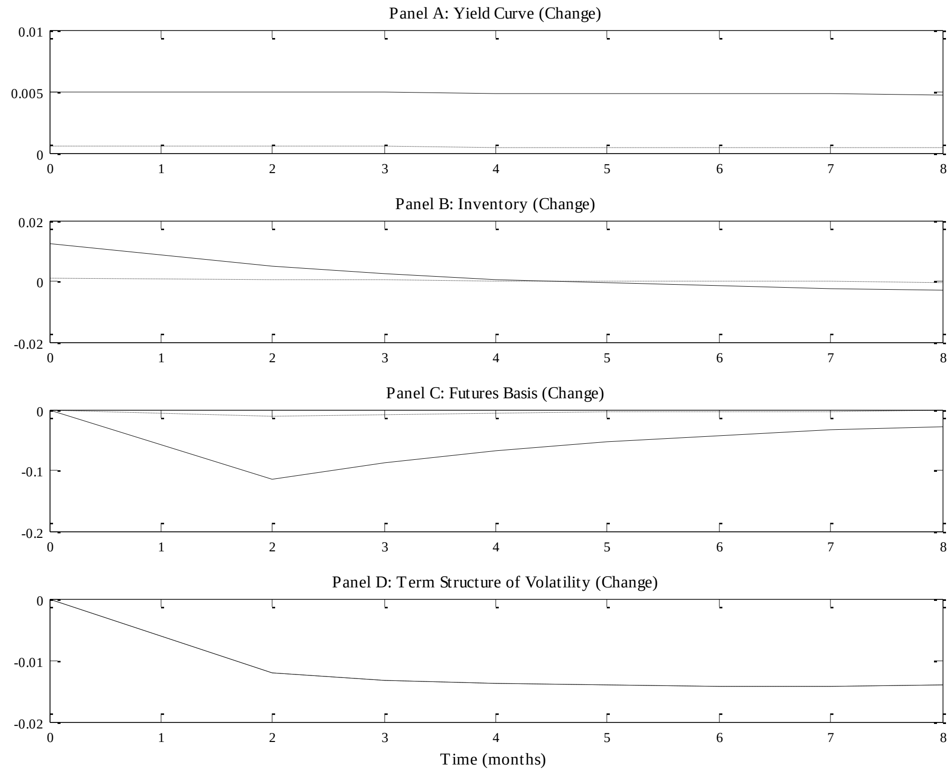

Figure 2
Impulse response of interest rates, inventory, and oil prices to a Y_1 shock
The top panel illustrates the impact of an exogenous Y_1 shock upon model interest rates. Panel B illustrates the differential between the time paths of expected future inventory levels computed at the base and post-shock *Y* vector. Panel C illustrates the effect of this shock upon the interest-adjusted futures basis (by maturity). Panel D presents the response of the term structure of volatility. In this exercise, the state vector *Y* is first set to its unconditional level. From this base case, a small (large) shock is represented as a change to Y_1 that yields a 5- (50–) basis-point interest rate change. The dashed (solid) line corresponds to a small (large) shock. Y_1 is fairly stable over the examined horizon. At the base and the post-shock levels of the *Y* vector, the current futures curve, and the expected time path of inventories is computed. Using the base and post shock futures curve, the interest-adjusted basis is immediate as the futures basis (*b*) plus the matching maturity risk-free yield. For brevity, the figure labels the interest-adjusted basis as the futures basis.

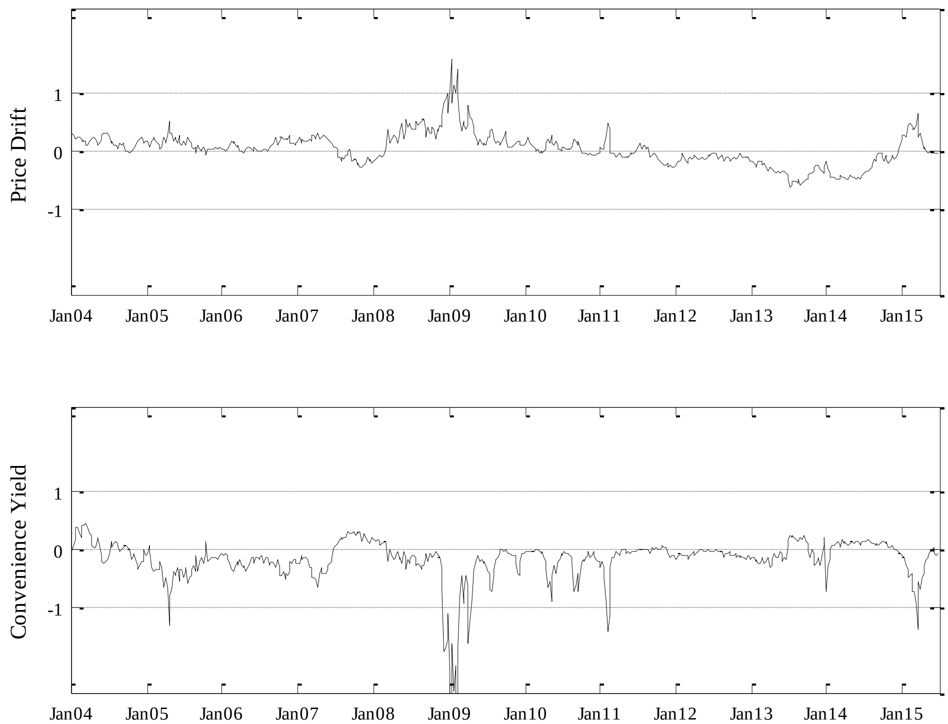

Figure 3
Spot price drift and convenience yield
This figure presents the model-implied spot price drift and convenience yield. The observations are plotted in time series over the sample period (January 2004–June 2015). The last 78 observations in the figure are for January 2014–June 2015; these observations are out-of-sample because the model parameters are estimated with data up to 2013.

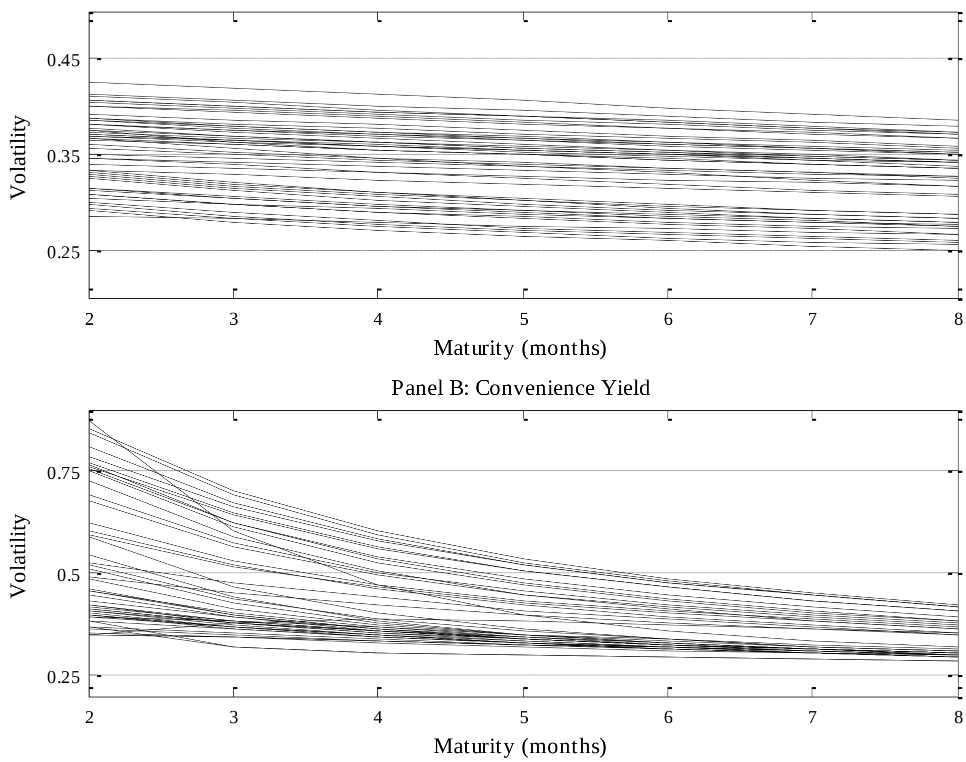

Figure 4
Term structure of volatilty
This figure presents the term structure of volatility for crude oil implied by the nonlinear model. The top row presents spot price volatility, whereas the convenience yield volatility is in the bottom row. These are quarterly volatility curves over the sample period (January 2004–June 2015).

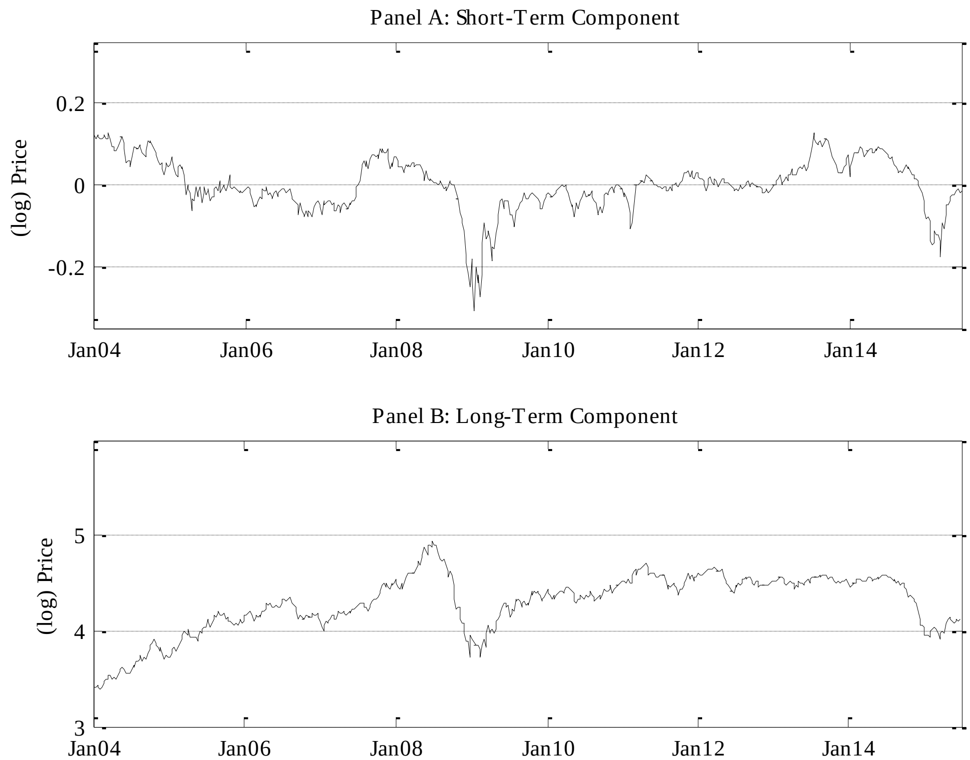

Figure 5
Crude oil short- and long-term price components
This figure shows the Schwartz and Smith decompostion of oil (log) spot prices into short- and long-term components. When the short-term component is positive, spot prices are above the long-run level and vice versa. Schwartz and Smith formally demonstrate the equivalence of their model with short-term temporary price deviations around a long-run level, to a model with stochastic convenience yields (δ_t) that revert to a mean level α at rate $\boldsymbol{\kappa}$. They show their temporary price component (χ_t) is related to the convenience yield, $\chi_t = (\delta_t - \alpha)/\kappa$, and the equilibrium price component is a residual $(\xi_t = X_t - \chi_t)$ relative to the (log) spot. The convenience yield is analogous to a short maturity basis (interest adjusted); we use the model-implied basis at the one-year horizon. The estimate of $\boldsymbol{\kappa}$ is the slope coefficient from a regression of weekly changes in the basis on its level (scaled by the weekly time interval), and α is the mean basis. The last 78 observations in the figure are for January 2014–June 2015; these observations are out-of-sample because the model parameters are estimated with data up to 2013.

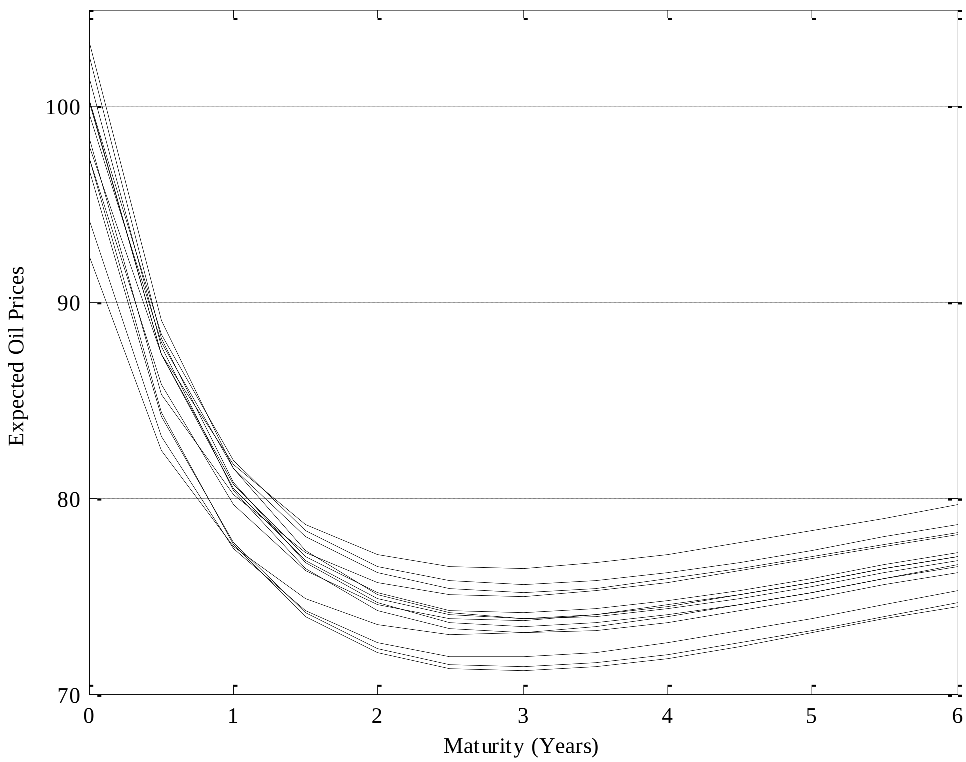

Figure 6
Crude oil expected spot prices
This figure shows weekly forecasts of expected oil spot prices in dollars per barrel for up to five years implied by the nonlinear model (QTSM) in the first quarter of 2014.

Table 1

Statistics for crude oil futures contracts

This table reports summary statistics for crude oil futures contracts. Oil prices are reported in dollars per barrel. A Basis variable is calculated as the (log) spot price minus the contemporaneous (log) futures price; this difference is divided by the maturity of the futures contract. The interest-adjusted basis (*IntAdj*) is the basis plus the matching maturity yield; below we refer to this variable as just the *basis*. Here, we use the six-month yield. The (+) column denotes percentage of positive observations. *Spot* denotes the closest-to-maturity contract, *Nearby 2* the second closest, and so on. Futures contracts are traded for the first eight months. The right-hand columns present results of a regression of the interest-adjusted Basis (8) on the noted dependent variables. The crude oil data starts on January 2004 and ends on June 2015; 599 weekly observations result.

	Futures price		**Basis**			**Basis decomposition**			
Nearby	*Mean*	*(SE)*	*Mean*	*(SE)*	+		β	*(SE)*	*AdjR*^{2}
Spot	76.985	(22.487)				*constant*	–0.520	(0.328)	
2	77.476	(22.171)	–0.059	(0.170)	0.272	*Interest rate*	0.292	(0.949)	
3	77.806	(21.952)	–0.055	(0.158)	0.292	*Inventory*	–0.433	(0.148)	
4	78.016	(21.799)	–0.046	(0.144)	0.311	*Spot*	0.119	(0.072)	0.199
5	78.154	(21.675)	–0.038	(0.133)	0.331				
6	78.239	(21.568)	–0.032	(0.125)	0.337				
7	78.277	(21.47)	–0.026	(0.118)	0.342				
8	78.287	(21.382)	–0.021	(0.112)	0.361				

Table 2

Model-implied futures

This table presents statistics that compare model-implied long-maturity futures contract prices. The table records statistics for the difference between (log) futures price implied by the nonlinear model (QTSM) with inventories and the corresponding maturity-implied futures price from a model without inventories (STD). The mean and maximum absolute differences are reported below. The median and standard deviation of differences are also reported. The recorded statistics are for the 78 weeks immediately following the period over which the model parameters are estimated (2004–13); they are computed using weekly observations from January 2014 to June 2015.

Maturity (years)	1	2	3	4	5	6
Mean	0.021	0.078	0.129	0.171	0.207	0.239
Median	0.008	0.018	0.014	0.007	0.001	–0.005
SD	0.023	0.087	0.144	0.191	0.231	0.267
Max	0.067	0.224	0.356	0.459	0.543	0.617

Table 3

Spot price forecasts

This table presents statistics that record the impact of inventories and nonlinearity on model-generated spot price forecasts. Panel A documents statistics for the difference between the (log) expected spot price implied by the nonlinear model (QTSM) and the corresponding maturity forecast from an affine model (EXT). The mean and maximum absolute differences are reported below. The median and standard deviation of differences are also reported. Panel B similarly compares the full nonlinear model (QTSM) to a nonlinear model without inventories (QTSM-NOINV). The recorded statistics are for the 78 weeks immediately following the period over which the model parameters are estimated (2004–13); they are computed using weekly observations from January 2014 to June 2015.

Maturity (years)	1	2	3	4	5	6
Panel A: Forecast difference (EXT)						
Mean	0.211	0.312	0.366	0.404	0.441	0.484
Median	–0.038	–0.076	–0.118	–0.174	–0.242	–0.322
SD	0.253	0.372	0.430	0.457	0.468	0.469
Max	0.563	0.819	0.922	0.938	0.900	0.896
Panel B: Forecast difference (QTSM-NOINV)						
Mean	0.055	0.056	0.096	0.135	0.169	0.198
Median	0.054	0.051	0.089	0.132	0.169	0.197
SD	0.055	0.029	0.036	0.043	0.045	0.045
Max	0.081	0.122	0.202	0.262	0.301	0.329

Table 4

Long-maturity futures errors

This table presents statistics for model performance in pricing long-maturity futures contracts. The mean and maximum absolute errors are recorded below, where error is calculated as the log of the model-implied futures price minus the log of the market-traded futures price for the corresponding maturities. The median and standard deviation of differences are also reported. Panel A reports nonlinear (QTSM) model pricing errors, whereas Panel B reports errors for the standard model without inventories (STD). The results for the other model specifications are omitted for brevity but available from the authors upon request. The mean pricing error for the nonlinear model is significantly different, at the 1% level, from the mean pricing error for the standard model for all reported maturities over one year. These statistics are out-of-sample errors computed with the 78 weekly observations over the period January 2014 to June 2015.

Maturity (years)	1	2	3	4	5	6
			Panel A: Futures pricing errors (QTSM)			
Mean	0.010	0.028	0.049	0.072	0.098	0.105
Median	–0.004	–0.024	–0.051	–0.084	–0.118	–0.106
SD	0.012	0.030	0.045	0.056	0.064	0.058
Max	0.037	0.074	0.088	0.138	0.184	0.226
			Panel B: Futures pricing errors (STD)			
Mean	0.013	0.063	0.111	0.158	0.211	0.298
Median	–0.008	–0.040	–0.071	–0.109	–0.151	–0.415
SD	0.014	0.065	0.110	0.146	0.180	0.239
Max	0.042	0.183	0.317	0.434	0.541	0.649

Table 5

Spot price drift and model factors

This table presents estimates of relations between the spot price drift and model economic factors in the nonlinear model (QTSM). The leftmost panel records point estimates from a regression that relates the model-implied spot price drift *(μ)* to the interest rate *(r)*, inventory *(q)*, and spot price *(X)* factors. The middle panel repeats this regression, but replaces the aggregate spot price (X) with the price components $\hat{X}$ and X^{resid}. These components are the fitted values and residuals from a regression of the spot price level on the interest rate, inventory, and convenience yield factors. The rightmost panel regresses risk premia, computed as the difference in spot price drift under the historical and risk-neutral measures, on the factors. The crude oil data starts in January 2004 and ends in June 2015; 599 weekly observations result.

	Price drift (μ)				***Price drift (μ)***			***Risk premia (μ- μ^Q)***		
	β	*(SE)*	$AdjR^2$		β	*(SE)*	$AdjR^2$	β	*(SE)*	$AdjR^2$
const	2.231	(0.927)		*const*	6.025	(0.266)		–2.832	(0.284)	
r	–1.465	(1.307)		r	–3.244	(0.696)		–6.045	(0.755)	
q	–0.309	(0.986)		q	2.079	(0.227)		–5.559	(0.239)	
X	–0.507	(0.218)	0.349	$\hat{X}$	–1.390	(0.062)		0.665	(0.066)	
				X^{resid}	0.295	(0.046)	0.904	0.301	(0.050)	0.874

Table 6

Crude oil options

This table reports statistics for model fit to options on crude oil futures. The table reports mean absolute error in pricing European crude oil options traded on the NYMEX. The pricing error is the difference between fitted and observed implied volatility. Pricing errors are reported by moneyness and maturity categories. The options data spans September 2004–June 2015, and the sorted data yields 63,187 observations for 582 weeks.

Nearby	*2*	*3*	*4*	*5*	*6*	*7*	*8*
0.79–0.85	0.312	0.260	0.223	0.189	0.176	0.155	0.135
0.85–0.91	0.251	0.201	0.174	0.143	0.128	0.108	0.101
0.91–0.97	0.194	0.151	0.135	0.109	0.100	0.076	0.074
0.97–1.03	**0.145**	**0.110**	**0.100**	**0.075**	**0.082**	**0.080**	**0.072**
1.03–1.09	0.124	0.099	0.088	0.071	0.073	0.075	0.071
1.09–1.15	0.104	0.086	0.085	0.066	0.065	0.071	0.064
1.15–1.21	0.084	0.077	0.079	0.056	0.059	0.059	0.066