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Noemi Schmitt & Frank Westerhoff

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Herding behaviour and volatility clustering in financial markets

NOEMI SCHMITT and FRANK WESTERHOFF*

Department of Economics, University of Bamberg, Feldkirchenstrasse 21, 96045 Bamberg, Germany

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We propose a financial market model in which speculators follow a linear mix of technical and fundamental trading rules to determine their orders. Volatility clustering arises in our model due to speculators' herding behaviour. In case of heightened uncertainty, speculators observe other speculators' actions more closely. Since speculators' trading behaviour then becomes less heterogeneous, the market maker faces a less balanced excess demand and consequently adjusts prices more strongly. Estimating our model using the method of simulated moments reveals that it is able to explain a number of stylized facts of financial markets quite well. Various robustness checks with respect to the model setup reveal that our results are quite stable.

Keywords: Stylized facts of financial markets; Technical and fundamental analysis; Heterogeneity; Herding behaviour; Method of simulated moments

JEL Classification: C63, D84, G15

1. Introduction

Our goal is to develop a herding model to explain a number of important stylized facts of financial markets. In particular, we seek to show that a temporal herding-induced coordination of speculators' trading behaviour can lead to a high volatility period. In a nutshell, the key insight offered by our model may be summarized as follows. Speculators usually follow a diverse set of technical and fundamental trading rules to determine their orders. As a result, their buying and selling orders are roughly balanced and the market maker only has to deal with a relatively modest excess demand. However, speculators are subject to herding behaviour. In periods of heightened uncertainty, speculators observe other speculators' actions more closely. Copying the behaviour of others implies a decrease in heterogeneity among speculators. This affects market stability. Since more speculators are then located on either the buy side or the sell side of the market, the market maker faces a less balanced excess demand and therefore adjusts prices more strongly. We estimate our model using the method of simulated moments. Overall, we find that our model is able to produce bubbles and crashes, excess volatility, fat-tailed return distributions, uncorrelated returns and volatility clustering. Several robustness checks in which we turn our simple herding model into a more involved agent-based herding model suggest that our results are quite stable.

Since herding behaviour plays a crucial role in our approach, let us continue by reviewing related work from this research domain. Baddeley (2010) defines herding behaviour as the phenomenon of individuals deciding to follow others and imitating group behaviour rather than deciding independently and atomistically on the basis of their own private information. Even Keynes stressed the importance of herding behaviour and argued that people tend to herd during periods of heightened uncertainty since they are afraid of making mistakes in isolation. For instance, a famous quote by Keynes (1936, p. 158) is that: 'Worldly wisdom teaches that it is better to fail conventionally than to succeed unconventionally'. Keynes' view on herding behaviour inspired a number of theoretical, experimental and empirical studies. For instance, Scharfstein and Stein (1990) theoretically demonstrate that it might be rational for agents to herd if they are concerned about their reputation. Moreover, Palley (1995) develops a model in which herding occurs if agents are risk averse and if their rewards depend on their relative performance. During periods of uncertainty, agents then seek what he calls 'safety in numbers'. Another prominent study revealing that uncertainty may promote rational herding behaviour is by Avery and Zemsky (1998).

Cipriani and Guarino (2009, 2014) confirm experimentally and empirically that rational herding behaviour may increase due to uncertainty. Conducting a survey study to explore the

*Corresponding author. Email: frank.westerhoff@uni-bamberg.de
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reasons for the US stock market crash of October 1987, Shiller (1990, p. 58) concludes—with a much stronger behavioral emphasis—that

> The suggestion we get of the causes of the crash is one of people reacting to each other with heightened attention and emotion, trying to fathom what other investors were likely to do, and falling back on intuitive models like models of price reversal and continuation.

Similarly, Shiller and Pound (1989) find that herding behaviour increases not only in periods of crises but also when prices rise quickly. It seems that institutional and individual investors are more likely to be subject to contagion effects if they experience stress. In addition, Hommes *et al*. (2005) and Heemeijer *et al*. (2009) conduct learning-to-forecast experiments and report that subjects tend to coordinate on common prediction strategies. Subjects' coordination behaviour is particularly strong in positive feedback systems such as financial markets. Chiang and Zheng (2010) detect strong evidence of herding behaviour in global stock markets. They claim that herding behaviour is present in up and down markets and, most importantly for our approach, that it is stronger in turbulent periods than in calm ones. Further psychological evidence provided by Prechter and Parker (2007) and Baddeley (2010) reveals that, under conditions of certainty, people tend to reason consciously, but tend to herd unconsciously under conditions of uncertainty.

Backed by these insights, we assume in our model that speculators' herding behaviour increases with uncertainty. Moreover, we capture uncertainty via a smoothed measure of the market's past volatility. To be precise, we consider a stock market which is populated by a market maker and a fixed number of heterogeneous speculators. The market maker mediates transactions out of equilibrium and adjusts the stock price with respect to speculators' excess demand. Speculators use a linear mix of technical and fundamental trading rules to determine their orders. To capture the diversity of actual trading rules, we add random variables to speculators' demand functions. These random variables are multivariate normal distributed with a mean vector of zeros and a time-varying variance-covariance matrix. We assume that the correlation between the random parts of speculators' trading signals increases with the market's past volatility. Economically, this means that speculators follow more closely what others do in periods of heightened uncertainty. Note that this kind of herding behaviour, which is in line with the aforementioned empirical and experimental literature, influences the heterogeneity of trading rules applied and thereby the market's price dynamics.

Fortunately, our approach allows a convenient aggregation of speculators' trading behaviour and, as it turns out, its dynamics depends on three equations only. Simulations reveal that our model is able to match several salient statistical properties of financial markets. The functioning of our model may be understood as follows. Simply speaking, there are two coexisting regimes—a calm regime and a turbulent regime. In the calm regime, i.e. in periods when volatility is rather low, speculators act more or less independently. Since large parts of their orders cancel out, the market maker's price adjustments are rather modest, and volatility remains low. In the turbulent regime, i.e. in periods when volatility is high, speculators observe other speculators' actions more closely. This kind of herding behaviour naturally implies that speculators' behaviour becomes increasingly aligned. Since speculators' orders cancel out less strongly, the market maker faces a higher excess demand. As a result, the market maker's price adjustments are more pronounced and volatility remains high.

Although the calm and turbulent regimes are persistent, the model's long-run behaviour is characterized by a regime-switching process. Regime changes, which give rise to volatility clustering, occur as follows. Even in a calm period, there may be a sequence of days when speculators receive strong trading signals and their increased trading intensity drives up volatility due to the market maker's price adjustments. A higher volatility can then lead to a herding-induced coordination of speculators' behaviour and thus the onset of a turbulent period. Alternatively, even in a turbulent period there may be a sequence of days when speculators receive weak trading signals, causing volatility to decline. Consequently, herding-induced coordination among speculators dissolves and a calm period emerges. As we will see, our model can also explain other stylized facts. For instance, speculators' trading behaviour gives rise to misalignments and excess volatility. While the technical components of speculators' trading rules can trigger bubbles, the fundamental components of their trading rules ensure that stock prices eventually revert to their fundamental values. Nevertheless, stock price changes are virtually impossible to predict, i.e. the development of stock prices appears as a random walk. The main reason for this outcome is that speculators rely on a myriad of time-varying trading rules. Moreover, extreme returns may emerge if speculators coordinate their behaviour, leading to fat-tailed return distributions.

We use two different data-sets of the S&P500 to estimate our model. To get an idea of the stock market's average misalignment, we compute the stock market's fundamental value as proposed by Shiller (2015) in his Nobel Prize lecture. This data-set, which runs from January 1871 to October 2015, contains 1738 monthly observations. To capture the stock market's return dynamics, we focus on the S&P500's daily behaviour between 1964 and 2014. This data-set contains 12 797 observations. Inspired by recent econometric work on the possible estimation of models with heterogeneous interacting agents using the method of simulated moments (see, e.g. Winker *et al*. 2007, Franke 2009, Franke and Westerhoff 2012, Chen and Lux 2015), we define 12 summary statistics (moments) to capture five stylized facts of financial markets. To the best of our knowledge, we are the first who utilize the data-set of Shiller (2015) to match the S&P500's misalignment in such an endeavour. As in previous studies, the other 11 moments measure the volatility of the market, the fat-tailedness of the distribution of returns, the random walk behaviour of stock prices and the volatility clustering phenomenon. Overall, the model's moment matching may be deemed as quite acceptable. For instance, we use a bootstrap procedure to compute the 95% confidence intervals of the 12 moments. Generating a large number of simulation runs reveals that, on average, about 85% of the simulated moments drop into the 95% confidence intervals of their empirical counterparts.

To check the robustness of our results, we discuss several model extensions. These extensions concern the market maker's price adjustment, speculators' trading rules, the general herding setup, the volatility updating, and the immunization of some speculators to herding effects. Effectively, these modifications turn our simple herding model into a more involved agent-based herding model. As it turns out, the simulation of the enriched model is quite time-consuming and an estimation of this model by the method of simulated moments is precluded. However, we find it very encouraging to report that the estimated parameters of our simple model provide clear guidance on how to select the parameters of the more general model. Put differently, in determining the parameters of agent-based models, which is often a difficult task, it might be a good idea to begin with a simple approximation of the model which can be estimated using the method of simulated moments and then turning to a more general model. Overall, we find that the dynamics of the enriched model is relatively similar to the dynamics of the simple model, i.e. our herding framework might be regarded as robust.

Other models with heterogeneous interacting agents can, of course, also explain the dynamics of financial markets (for an overview, see LeBaron 2006, Chiarella, Dieci, *et al*. 2009, Hommes and Wagener 2009, Lux 2009). A couple of these studies are concerned explicitly with speculators' herding behaviour.† Kirman (1991, 1993) and Alfarano and Lux (2007) propose herding models in which speculators may convince other speculators to follow their behaviour. In Lux (1995) and Lux and Marchesi (1999), speculators' rule selection behaviour is influenced by the relative popularity of the rules, among other things, and thus a rule may gain in popularity if it has many followers. In Cont and Bouchaud (2000) and Stauffer *et al*. (1999), speculators' herding behaviour is of a local nature. In these models, speculators are situated on a lattice, but not all sites of the lattice are occupied. Speculators who form a local neighbourhood, i.e. a cluster of connected occupied sites, either collectively buy or collectively sell assets. In the herding models of Iori (1999, 2002), speculators can buy or sell one unit of a risky asset or remain inactive. This decision is influenced by social interaction, i.e. by the opinions of speculators' neighbours. Moreover, trade frictions imply that past price changes influence the activity level of speculators. Finally, the market maker's price adjustment speed with respect to order imbalances increases with trading volume. Tedeschi *et al*. (2012) develop a herding model in which speculators imitate the behaviour of more successful speculators and, interestingly, show that speculators have an incentive to imitate and a desire to be imitated since herding is profitable.

Note that in Kirman (1991, 1993), Lux (1995), Lux and Marchesi (1999) and Alfarano and Lux (2007), speculators' herding behaviour influences whether they opt for a technical or a fundamental trading rule. In Cont and Bouchaud (2000) and Stauffer *et al*. (1999), speculators' herding behaviour determines whether they are optimists or pessimists. Within our model, speculators' herding behaviour leads to a coordination of their trading activities. Since speculators take into account other speculators' actions, the heterogeneity of the trading rules applied becomes time-varying. Given the empirical and experimental evidence, especially the view of Shiller (1990), we regard our modelling approach as a worthwhile alternative description of speculators' trading behaviour. It seems that out of the existing herding models the models by Iori (1999, 2002) are the ones which are most closely related to our herding model. Despite many differences, her models and ours have in common that social interactions and imitation behaviour depend on the market's past price dynamics.

The rest of our paper is organized as follows. In section 2, we first present a rather simple financial market model in which speculators are subject to herding behaviour. In section 3, we briefly recap the stylized facts of financial markets and introduce a number of summary statistics to quantify them. In section 4, we bring our simple model to the data and discuss its functioning in further detail. In section 5, we carry out several robustness checks, thereby turning our simple model in a more involved agent-based model. Section 6 concludes our paper and points out a number of avenues for future research.

2. A simple herding model

We consider a single stock market which is populated by a market maker and N heterogeneous speculators. While the task of the market maker is to mediate speculators' transactions and to adjust the price with respect to excess demand, speculators rely on a blend of technical and fundamental trading rules to determine their orders. According to technical analysis (Murphy 1999), prices move in trends, and buying (selling) is suggested when the price increases (decreases). Fundamental analysis (Graham and Dodd 1951) presumes that stock prices revert towards their fundamental values and recommends buying (selling) when the stock is undervalued (overvalued). Note that the stock's fundamental value is constant and known by all market participants. To capture the diversity of actual technical and fundamental trading rules and their possible combinations, we add a random term to speculators' demand functions. These random variables are multivariate normal distributed with a mean vector of zeros and a time-varying variance-covariance matrix. Moreover, speculators tend to herd. Since their herding behaviour increases with market uncertainty, we assume that the correlation between speculators' random trading signals depends positively on the market's past volatility. In volatile periods, speculators' trading behaviour thus becomes more aligned.

Let us turn to the details of our simple herding model. The stock price is adjusted by the market maker, who collects all speculators' individual orders and changes the price with respect to the resulting excess demand. Following Day and Huang (1990), we formalize the behaviour of the market maker as

$$P_{t+1} = P_t + a^M \sum_{i=1}^{N} D_t^i, \tag{1}$$

†Clearly, not all financial market models with heterogeneous interacting agents contain an explicit herding component. In Day and Huang (1990), speculators rely on nonlinear trading rules. In Brock and Hommes (1998), speculators switch between technical and fundamental trading rules with respect to the rules' past performance. In De Grauwe *et al*. (1993), speculators' rule selection behaviour depends on the market's current misalignment. In Westerhoff (2004), speculators switch between different markets. Bouchaud and Cont (1998) present a Langevin approach to explain the boom and bust dynamics of stock markets. Challet *et al*. (2001) show that the stylized facts of financial markets may be explained by Minority Game models. The artificial stock market models by LeBaron *et al*. (1999) and Chen and Yeh (2001) generate realistic dynamics due to the interactions of many different types of speculators.

where P_t represents the log stock price at time step t, a^M is a positive price adjustment coefficient, D_t^i stands for the order placed by speculator i and N denotes the total number of speculators. Accordingly, the market maker quotes a higher (lower) stock price if the sum of the speculators' orders is positive (negative).

As in Chiarella and Iori (2002), Chiarella, Iori *et al*. (2009) and Pellizzari and Westerhoff (2009), we assume that speculators' demand depends on three components. Since speculators use both technical and fundamental analysis to determine their orders, their demand entails a chartist and a fundamentalist component. The third component is a random component that is supposed to capture all digressions from the first two components. Hence, the order placed by a single speculator i can be expressed as

$$D_t^i = b^i (P_t - P_{t-1}) + c^i (F - P_t)^3 + \delta_t^i, \tag{2}$$

where b^i and c^i are positive reaction parameters and F represents the log fundamental value. While the first component implies that speculators follow the current price trend, the second shows that speculators also bet on mean reversion. Since Day and Huang (1990) argue that fundamentalists trade increasingly aggressively as the market's mispricing increases, we use a cubic function to model speculators' fundamental demand. The economic motivation offered by Day and Huang (1990) for such a cubic term is twofold: First, speculators are becoming increasingly convinced that a fundamental price correction is about to set in as the misalignment increases. Second, the gain potential of fundamental analysis increases with the magnitude of the possible fundamental price correction.

The random trading signals $\delta_t = \{\delta_t^1, \delta_t^2, \ldots, \delta_t^N\}'$ are multivariate normal distributed with a mean vector $\boldsymbol{\mu} = (0, 0, \ldots, 0)'$ and a time-varying variance-covariance matrix $\boldsymbol{\Sigma}_t$, i.e. $\delta_t \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}_t)$. The variance-covariance matrix is given by

$$\boldsymbol{\Sigma}_t = \sigma^2 \begin{bmatrix} 1 & \rho_t & \ldots & \rho_t \\ \rho_t & 1 & & \vdots \\ \vdots & & \ddots & \rho_t \\ \rho_t & \ldots & \rho_t & 1 \end{bmatrix}. \tag{3}$$

Before we proceed to describe the remaining model parts, let us introduce some simplifying transformations. Note that the sum of δ_t^i is a normal distributed random variable with a mean of zero and a variance of $\sigma^2(N + N(N-1)\rho_t)$. Without loss of generality, we can thus write $\sum_{i=1}^N \delta_t^i = \sigma \sqrt{N + N(N-1)\rho_t}\ \epsilon_t$ with $\epsilon_t \sim \mathcal{N}(0, 1)$.† Accordingly, (1)–(3) can be summarized by

$$P_{t+1} = P_t + a^M \left\{ (P_t - P_{t-1}) \sum_{i=1}^N b^i + (F - P_t)^3 \sum_{i=1}^N c^i + \sigma \sqrt{N + N(N-1)\rho_t}\ \epsilon_t \right\}. \tag{4}$$

To make the dynamics independent of N, we set $a^M = a/N$, $\sigma = d\sqrt{N}$ and $\rho_t = \frac{X_t}{N-1}$ with $0 \leq X_t \leq N-1$. Accordingly, a^M, σ and ρ_t depend on the number of speculators. Such a rescaling is also implicitly done in related models with a continuum of speculators, e.g. in Day and Huang (1990), Lux (1995) and Brock and Hommes (1998).‡ By defining $\frac{1}{N}\sum_{i=1}^N b^i = b$ and $\frac{1}{N}\sum_{i=1}^N c^i = c$, (4) turns into

$$P_{t+1} = P_t + a\,\{b\,(P_t - P_{t-1}) + c\,(F - P_t)^3 + d\sqrt{1 + X_t}\ \epsilon_t\}, \tag{5}$$

where $\epsilon_t \sim \mathcal{N}(0, 1)$. As can be seen, the market maker's price adjustment depends on speculators' average technical and fundamental trading signal and on the correlation between their random trading signals.

Supported by the empirical and experimental studies reported in the introduction (in particular Shiller (1990) and Chiang and Zheng (2010)), we assume that speculators are subject to herding behaviour and that they tend to herd more strongly in periods of heightened uncertainty. We quantify uncertainty by a smoothed measure of the market's past volatility, i.e.

$$V_t = m V_{t-1} + (1 - m)(P_t - P_{t-1})^2, \tag{6}$$

where $0 \leq m < 1$ is a memory parameter. For $m = 0$, speculators have no memory, and the volatility measure depends only on the most recent price movement. For increasing values of m, i.e. if speculators' memories improve, the volatility measure depends more strongly on past price movements.

To capture the greater synchronization of speculators' trading behaviour in periods of great uncertainty, we assume that the correlation between their random trading signals increases with the market's past volatility. Recall that the correlation coefficient is defined as $0 \leq \rho_t = \frac{X_t}{N-1} \leq 1$. This assumption implies that the correlation increases with X_t and decreases with N. However, we model the time-varying value of X_t using a logistic function, i.e.

$$X_t = \frac{x}{1 + exp[-k(V_t - v)/v]}, \tag{7}$$

where $x > 0$ indicates the maximum value of (7), $k > 0$ describes the steepness of (7) and $v > 0$ determines at which volatility level (7) has its midpoint ($= x/2$). The S-shaped (sigmoid) curve (7) ensures that X_t, and thus speculators' coordination behaviour, increases with the market's past volatility.

The dynamics of our simple herding model is driven by equations (5)–(7). Parameter F, i.e. the log fundamental value, only affects the level of the model's log price dynamics but not its statistical properties. Moreover, a is a scaling parameter and since our dynamics does not depend on N, there remain seven parameters to be specified, namely b, c, d, k, v, x and m. To improve our understanding of the model's functioning, let us briefly explore two extreme model cases. For $k = 0$, the

†In general, the variance of the sum of correlated random variables δ_t^i is given by the sum of their covariances, i.e. $Var(\sum_{i=1}^N \delta_t^i) = \sum_{i=1}^N \sum_{j=1}^N Cov(\delta_t^i, \delta_t^j) = \sum_{i=1}^N Var(\delta_t^i) + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^N Cov(\delta_t^i, \delta_t^j)$. In our case, random variables δ_t^i have equal variances $Var(\delta_t^i) = \sigma^2$ and their covariances are given by $Cov(\delta_t^i, \delta_t^j) = \rho_t \sigma^2$. Therefore, we have $Var(\sum_{i=1}^N \delta_t^i) = N\sigma^2 + 2\frac{N(N-1)}{2}\rho_t\sigma^2 = \sigma^2(N + N(N-1)\rho_t)$.

‡In the appendix 1, we discuss economic implications of these assumptions as well as consequences of a partial rescaling of our model in more detail.

sigmoid function (7) becomes a straight line, i.e. $X_t = x/2$. This turns the model's law of motion into

$$P_{t+1} = P_t + a\,\{b\,(P_t - P_{t-1}) + c\,(F - P_t)^3 + d'\,\epsilon_t\} \quad (8)$$

with $d' = d\,\sqrt{1 + 0.5x}$. As can be seen, (8) represents a nonlinear stochastical dynamical system. Since shocks are IID distributed, this model is unable to explain the stylized facts of financial markets.

In the other extreme case, i.e. for $k = \infty$, the logistic function takes either its minimum value or its maximum value. Since $X_t = 0$ for $V_t < v$ and $X_t = x$ for $V_t \geq v$, the model dynamics is due to

$$P_{t+1} = P_t + a\,\{b\,(P_t - P_{t-1}) + c\,(F - P_t)^3 + \begin{cases} d\,\sqrt{1+x}\,\epsilon_t & \text{if } V_t \geq v \\ d\,\epsilon_t & \text{if } V_t < v \end{cases}. \quad (9)$$

A few comments are required. First, formulation (9) allows a neat illustration of the effects of X_t. Assume that $a = 1$, $P_t = P_{t-1} = F$, $d = 1$, $\epsilon = 0.01$ and $x = 8$. For $V_t < v$, we have $X_t = 0$. As a result, we observe a return of $r_{t+1} = P_{t+1} - P_t = \epsilon_t = 0.01$. For $V_t \geq v$, we have $X_t = 8$ and the return is given by $r_{t+1} = P_{t+1} - P_t = \sqrt{1+8}\,\epsilon_t = 0.03$. Hence, the market maker's price adjustment triples if the market switches from a low volatility regime in which speculators act independently to a high volatility regime in which their behaviour is correlated. Second, we will see in the sequel that even a small correlation between random trading signals may suffice to obtain realistic dynamics. For instance, $x = 8$ implies for $N = 101$ a mere correlation of either $\rho_t = 0$ (if $V_t < v$) or $\rho_t = 0.08$ (if $V_t \geq v$). Assuming that there are 201 speculators would imply that the correlation is even lower, given either by $\rho_t = 0$ (if $V_t < v$) or $\rho_t = 0.04$ (if $V_t \geq v$). Third, from a mathematical point of view, model (9) is similar to that of Manzan and Westerhoff (2005). They propose a behavioural exchange rate model in which speculators' overreactions and underreactions to news depend on the market's past volatility. In a low (high) volatility regime, speculators underreact (overreact) to fundamental shocks and thus volatility remains persistently low (high). Note that this model was successfully estimated by Franke (2009). Therefore, the more flexible functional form of model (5) to (7) may be a good candidate for replicating the dynamics of financial markets. Needless to say, the economic story of Manzan and Westerhoff (2005) is quite different to ours.

3. Stylized facts and estimation strategy

The goal of our paper is to develop a herding model which is able to explain bubbles and crashes, excess volatility, fat-tailed return distributions, uncorrelated returns and volatility clustering. We begin this section by briefly reviewing these properties. For general reviews of the statistical properties of financial markets see, for instance, Mantegna and Stanley (2000), Cont (2001) or Lux and Ausloos (2002). Our exposition of these universal features is based on two different data-sets of the S&P500. The first data-set, obtained from Shiller (2015), ranges from January 1871 to October 2015 and contains 1738 monthly observations. The second data-set, downloaded from Thomson Reuters Datastream, runs from 1 January 1964 to 31 December 2014 and contains 12 797 daily observations.

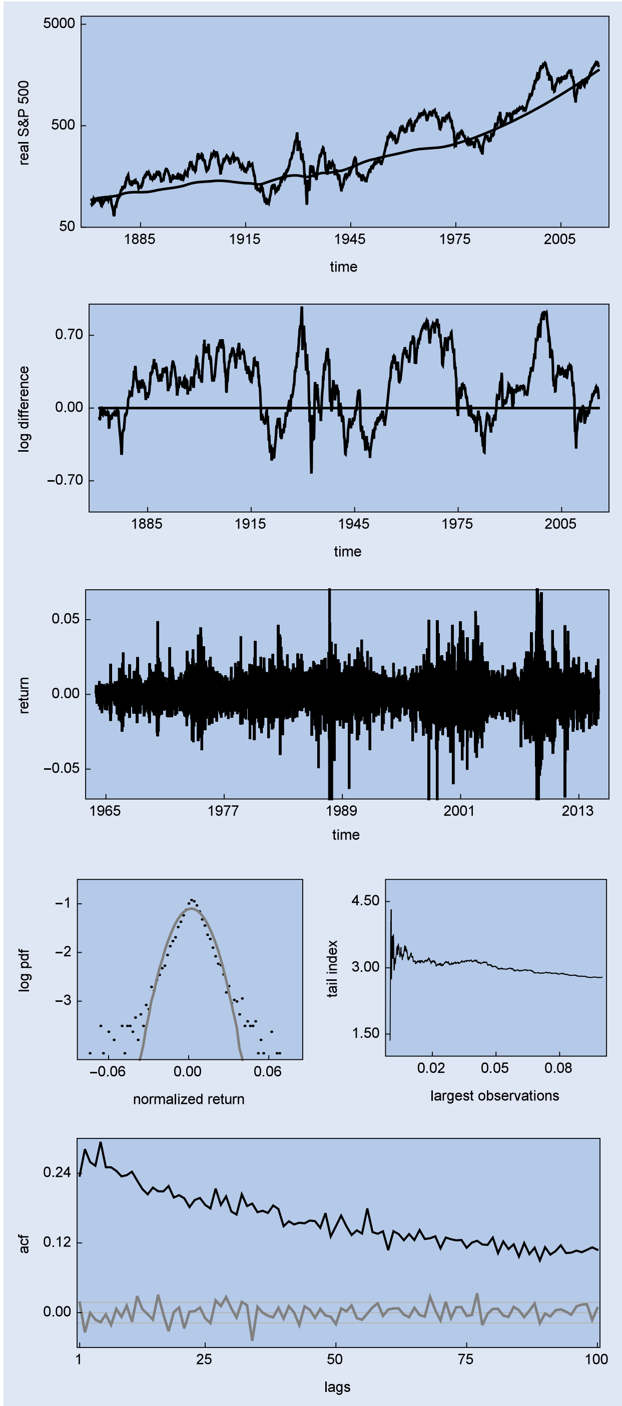

Figure 1. The dynamics of the S&P500. The first panel shows the real S&P500 (black) and its real fundamental value (gray) on a log scale between 1871 to 2015, while their respective log difference is plotted in the second panel. The remaining panels illustrate the S&P500's daily behaviour between 1964 and 2014 and show the returns, the log distribution of normalized returns, the Hill tail index estimator as a function of the largest returns and the autocorrelation function of raw returns (gray) together with the autocorrelation function of absolute returns (black), respectively. The first two panels are based on a time series with 1738 monthly observations; the other panels are based on 12 797 daily observations.

The dynamics of the S&P500 is depicted in figure 1. In the first panel, we illustrate the evolution of the real S&P500

(black) and its real fundamental value (gray) on a log scale between 1871 and 2015. The fundamental value is computed as proposed in Shiller (2015). As can be seen, stock prices display bubbles and crashes. To visualize the S&P500's distortion more clearly, we plot the log difference between the real S&P500 and its real fundamental value in the second panel of figure 1. Obviously, the market's distortion varies over time and there are periods with strong mispricing as well as periods when prices are less distorted.

The third panel of figure 1 displays the daily returns of the S&P500 between 1964 and 2014 and reveals that prices fluctuate strongly. Overall, it seems that the volatility of prices is higher than warranted by fundamentals. The fat tail property of the distribution of returns is exemplified in the fourth line of figure 1. In the left panel, the log-linear plot of the distribution of normalized returns (black) is compared with that of standard normally distributed returns (gray). Recall that returns are normalized by dividing by the standard deviation. The empirical distribution clearly possesses more probability mass in the tails and implies that extreme returns occur more frequently than suggested by a normal distribution. To quantify the fat tail property, we estimate the Hill tail index (Hill 1975) and plot it as a function of the largest returns (in percent) in the right panel of the fourth line. For instance, computing the Hill tail index at the 5% level yields a value of 3.02. Note that the lower the value for the tail index, the fatter the tails. The bottom panel displays the autocorrelation functions of raw (gray) and absolute returns (black) for the first 100 lags. Since raw returns reveal autocorrelation coefficients that are insignificant for almost all lags, the evolution of prices is close to a random walk. In contrast, absolute returns show significant autocorrelations for more than 100 lags, which is clear evidence of temporal persistence in volatility.

To estimate our model using the method of simulated moments, we follow previous studies (e.g. Franke and Westerhoff 2012) and introduce summary statistics (moments) to measure the five stylized facts we seek to match.† Due to the data-set of Shiller (2015), we are fortunately able to capture the S&P500's misalignment and define the distortion D as the mean value of the absolute difference between log stock prices and log fundamentals. By computing the average value of absolute returns, the Hill tail index at the 5% level, the autocorrelation coefficients of raw returns for lags 1, 2 and 3 and the autocorrelation coefficients of absolute returns for lags 3, 6, 12, 25, 50 and 100, we measure the volatility, the fat tail property, the random walk behaviour of stock prices and the volatility clustering phenomenon, respectively. Estimates of these 12 moments are presented in the first lines of table 1.

Additionally, we compute a frequency distribution for each of the moments. For this purpose, we use a bootstrap approach to obtain more empirical samples. To account for the long-range dependence in the time series, we initially follow Winker *et al.* (2007) and choose a block bootstrap. Thus, we subdivide our first empirical time series of the S&P500 into 28 blocks of 60 monthly observations (5 years) and construct a new time series of the same size by randomly drawing (with replacement) from these 28 blocks. We compute the distortion from this bootstrapped time series and repeat it 50 000 times to obtain its distribution. Using block lengths of four and six years produces quite similar results. All other moments' distributions are calculated from the second time series comprising daily data. In order to compute distributions of the mean value of absolute returns and the tail index, we repeat the approach described above but use 51 blocks with 250 data points (1 year) each. However, for the distributions concerning the lagged autocorrelation statistics we apply another bootstrap procedure. Franke and Westerhoff (2016) argue that the long-range dependence in the return series gets interrupted every time two non-adjacent blocks are linked. This may cause the bootstrapped coefficients to show a tendency towards lower values. To avoid this join-point problem, they suggest sampling single days and the associated past data points required to compute the lagged autocorrelation. Therefore, we form new time series by 12 800 random draws with replacement of consecutive data points from the S&P500 time series and compute the moments from them. Repeating this 5000 times yields samples that allow us to compute the moments' distributions. The median, the lower and the upper boundary of the 95% confidence intervals of these 12 bootstrapped distributions are reported in the second, third and fourth lines of table 1, respectively. Note that in all 12 cases, the estimated moments from the original time series are very close to the computed median values, i.e. the moments' distributions are nicely centred around their respective empirical observations.

The basic idea of our estimation strategy is to find the parameter setting for which the model's simulated moments fall most frequently into the 95% confidence intervals of their empirical counterparts. To be more precise, we assign points to cases in which the considered moments match their empirical intervals and set up an objective function that simply adds up these points and divides the total point score by 12. The estimation then searches for the parameter setting that maximizes this average moment matching score (= AMMS).‡

4. The dynamics of the simple herding model

We are now ready to bring our simple herding model to the data. In section 4.1, we report our estimation results and discuss the performance of our model. In section 4.2, we study the impact of single model parameters on the model dynamics. In section 4.3, we present a representative simulation run of our model to explain its functioning in further detail.

†One advantage of the method of simulated moments is its broad flexibility. However, Alfarano *et al.* (2005) estimate their model using maximum likelihood, while Boswijk *et al.* (2007) rely on non-linear least squares.

‡Our objective function is related to the joint moment coverage ratio (= JMCR) used in Franke and Westerhoff (2012). Their aim is to maximize the fraction of simulation runs for which model-generated moments jointly drop into the 95% confidence intervals of their empirical counterparts. Alternatively, one may use an objective function which minimizes the average of the (squared) deviations between the empirical moments and the model-generated moments, using an appropriate weighting scheme. For pioneering contributions in this direction see, for instance, Gilli and Winker (2003) and Franke (2009). Although the method of simulated moments is a powerful tool to estimate models with heterogeneous interacting agents, it requires a number of subjective choices with respect to specifying the objective function and selecting moments. We hope that our objective function stimulates more work in this important line of research.

Table 1. The empirical moments of the S&P500 and their frequency distributions. The first lines contain estimates of the distortion D, the volatility V, the tail index $\alpha_{5.0}$, the autocorrelation function of raw returns $ac\ r_i$ for lags $i \in \{1, 2, 3\}$ and the autocorrelation function of absolute returns $ac\ |r_i|$ for lags $i \in \{3, 6, 12, 25, 30, 100\}$. In the second, third and fourth lines, we report for all 12 moments the median, and the lower and upper boundary of the 95% confidence intervals of their bootstrapped distributions, respectively.

	D	V	$\alpha_{5.0}$	$ac\ r_1$	$ac\ r_2$	$ac\ r_3$
Estimates	0.310	0.697	3.052	0.017	−0.034	0.000
Median	0.316	0.696	3.073	0.017	−0.033	−0.001
Lower bound	0.255	0.625	2.691	−0.022	−0.083	−0.036
Upper bound	0.385	0.776	3.615	0.056	0.010	0.035
	$ac\ \|r_3\|$	$ac\ \|r_6\|$	$ac\ \|r_{12}\|$	$ac\ \|r_{25}\|$	$ac\ \|r_{50}\|$	$ac\ \|r_{100}\|$
Estimates	0.259	0.250	0.227	0.186	0.149	0.108
Median	0.259	0.249	0.227	0.185	0.149	0.109
Lower bound	0.217	0.209	0.191	0.155	0.124	0.087
Upper bound	0.309	0.296	0.268	0.218	0.177	0.132

4.1. The performance of the simple herding model

Recall that the market maker's price adjustment parameter is a scaling parameter and that the statistical properties of model (5)–(7) do not depend on the level of the log fundamental value. Without loss of generality, we thus set $a = 1$ and $F = 0$. For the remaining seven model parameters, the maximization of the average moment matching score yields the following results:

$$b = 0.135,\ c = 0.0012,\ d = 0.005364,$$
$$v = 0.0001475,\ k = 3.950,\ x = 10 \ \text{ and } m = 0.8712.$$

These parameters were identified via a multidimensional grid search. Accordingly, we computed 500 simulation runs for each parameter setting. The length of each time series comprises $T = 12\,800$ observations, reflecting a time span of 51 years with about 250 daily observations per year. The only exception is the calculation of the distortion, which is based on $T = 36\,250$ observations, corresponding to a time span of 145 years. The lengths of the simulated data sets are thus comparable to the lengths of the data sets we have for the S&P500. As it turns out, the average moment matching score for the above parameter setting reaches an astonishing value of $AMMS = 0.855$, i.e. the simulated model moments drop on average in 85.5% of the cases in the 95% confidence intervals of their empirical counterparts. Since any further change of an individual model parameter would result in an AMMS loss, we can conclude that we have found at least a local AMMS maximum.†

Before discussing how the model's parameters affect its performance, we briefly study how well our model matches the 12 individual moments. For instance, table 2 reveals that the distortion, the volatility and the tail index fall in 84.2, 90.6 and 80.4% of the cases in the 95% confidence intervals of their empirical counterparts. We may thus argue that our model is able to produce reasonable levels of distortion and volatility and fat-tailed return distributions. The moment matching of the autocorrelation coefficients of raw returns at lags 1, 2 and 3 scatters between 81.4 and 99.6%, suggesting that the model's price dynamics resembles a random walk. The individual average scores of the autocorrelation coefficients of absolute returns at lags 3, 6 and 12 are above 95%; the individual average scores at lags 25, 50 and 100 are distributed between 83.6 and 40.4%. Hence, the model's ability to generate volatility clustering is exceptionally good for lags 3, 6 and 12, yet appears to be somewhat limited for lags 25, 50 and 100.

However, the model's performance looks even better if we take into account the whole distributions of the 12 simulated moments. Table 3 therefore shows the median and the lower and upper boundaries of the 95% confidence intervals of the 12 simulated moments. Most importantly, we now see that in 97.5% of the simulation runs, the autocorrelation coefficient of absolute returns at lag 100 is larger than 0.03. Since the 95% significance band of raw and absolute returns' autocorrelation coefficients, assuming a white noise process, is about $\pm 2/\sqrt{12\,800} \approx \pm 0.02$, we can conclude that almost all simulation runs display long-range volatility clustering effects. Table 3 also shows that the tail index is below 3.92 in 97.5% of the simulation runs. Since Lux and Ausloos (2002) argue that the tail index of major financial markets hovers between 3 and 4, the matching of the distribution of returns may also be regarded as satisfactory.

Table 2. Overview of the model's moment matching. The table shows the average moment matching score (= AMMS) and the matching of the 12 individual moments. Estimations are based on 500 simulation runs.

AMMS	0.855
D	0.842
V	0.906
$\alpha_{5.0}$	0.804
$ac\ r_1$	0.996
$ac\ r_2$	0.814
$ac\ r_3$	0.994
$ac\ \|r_3\|$	0.974
$ac\ \|r_6\|$	0.972
$ac\ \|r_{12}\|$	0.968
$ac\ \|r_{25}\|$	0.836
$ac\ \|r_{50}\|$	0.754
$ac\ \|r_{100}\|$	0.404

†Since we use a numerical grid-search procedure, it remains unclear whether there are better local maxima or whether we have already arrived at the global maximum. Thus, our AMMS of 85.5% should be regarded as a lower boundary for the performance of our model.

Table 3. The frequency distributions of the model's moments. The first line contains estimates of the median of the distortion D, the volatility V, the tail index $\alpha_{5.0}$, the autocorrelation function of raw returns $ac\ r_i$ for lags $i \in \{1, 2, 3\}$ and the autocorrelation function of absolute returns $ac\ |r_i|$ for lags $i \in \{3, 6, 12, 25, 30, 100\}$. The second and third lines show the lower and upper boundaries of the 95% confidence intervals of the 12 simulated moments. Estimations are based on 500 simulation runs.

	D	V	$\alpha_{5.0}$	$ac\ r_1$	$ac\ r_2$	$ac\ r_3$
Median	0.304	0.720	3.450	0.010	0.000	0.000
Lower bound	0.228	0.650	3.130	−0.010	−0.030	−0.030
Upper bound	0.388	0.800	3.920	0.040	0.020	0.030
	$ac\ \|r_3\|$	$ac\ \|r_6\|$	$ac\ \|r_{12}\|$	$ac\ \|r_{25}\|$	$ac\ \|r_{50}\|$	$ac\ \|r_{100}\|$
Median	0.250	0.250	0.230	0.200	0.150	0.080
Lower bound	0.220	0.210	0.190	0.150	0.100	0.030
Upper bound	0.280	0.270	0.260	0.230	0.190	0.130

To visualize the matching of the 12 individual moments, we plot the smoothed distributions of the simulated moments (black lines) and the smoothed distributions of the empirical moments (shaded in gray) in figure 2. Accordingly, the matching of the volatility, the distortion, the autocorrelation coefficients of raw returns at lags 1 and 3, and the autocorrelation coefficients of absolute returns at lags 3, 6, 12, 25 and 50 immediately appears to be quite acceptable. As already mentioned, the autocorrelation coefficients of absolute returns at lag 100 are, on average, lower than their empirical counterparts but almost all of them are statistically significant. Moreover, almost all tail indices are below 4. The only moment we have not yet discussed is the autocorrelation coefficient of raw returns at lag 2. Note that the distribution of the empirical moment is centred around $ac\ r_2 = -0.033$ (see also table 1) while the distribution of the simulated moment is centred around $ac\ r_2 = 0$ (see also table 3). Since a stock market's return dynamics usually shows no signs of predictability, our results are in line with what one would like to see from a theoretical point of view.

4.2. *The impact of single parameters on the dynamics*

Let us now discuss how individual model parameters affect the performance of our simple herding model. First of all, figure 3 suggests that all of our model parameters are well identified, at least in the sense that a change in a single estimated model parameter immediately decreases the model's performance. The top left panel shows how the AMMS depends on the average reaction parameter of the speculators' technical trading component. Unless stated otherwise, we use 500 simulation runs to compute the AMMS. Moreover, each parameter is increased in 10 discrete steps and for each parameter combination we use the same seeds of random variables. As can be seen, if we deviate from $b = 0.0135$, the average moment matching score decreases. The main reason for this is that changing parameter b affects the matching of the autocorrelation coefficients of raw returns. The economic reason for this is clear: if speculators extrapolate the current price trend more (less) strongly, their orders increase (decrease) the momentum of the price dynamics. The top right panel shows the effects of changes in the average reaction parameter of the speculators' fundamental trading component. Note that parameter c mainly affects the model's ability to produce a realistic level of distortion. If parameter c increases, the distortion shrinks; if parameter c decreases, the distortion increases. For $c = 0.0012$, If parameter c increases, the distortion the model matches the distortion of the S&P500 best and thus maximizes the AMMS. The economic reason for this outcome is as follows. If speculators buy (sell) more strongly in overvalued (undervalued) markets, their orders push prices faster towards fundamental values. Alternatively, if speculators buy (sell) less strongly in overvalued (undervalued) markets, their orders push prices slower towards fundamental values.

The model's performance depends even more strongly on parameters d and v, as indicated in the second line of figure 3. Even small changes in these two parameters can decrease the AMMS substantially. Note that parameter d directly influences the overall volatility of the market. The higher parameter d, the more pronounced is the random trading behaviour of speculators. Parameter v is critical for the model's regime switching behaviour. If parameter v increases, speculators' herding behaviour is tamed (they perceive less often periods with heightened uncertainty). To obtain realistic dynamics, parameters d and v must stand in a certain relation to each other. For instance, if one seeks to calibrate the model to a higher volatility level, parameters d and v must be increased simultaneously.

Parameters k, x and m also influence the model's regime switching dynamics, and thus have an impact on a number of individual moments. Changing one of these parameters from its current position leads to a marked reduction of the AMMS. From an economic perspective, we can conclude that an increase in parameter x increases the maximal possible correlation between speculators' random trading behaviour, thereby elevating price fluctuations. Relatedly, parameter k determines how sensitive speculators' herding behaviour depends on volatility changes. The higher k, the more often periods with rather weak or rather strong herding behaviour emerge. The memory parameter m controls how intense past volatility experiences are for the speculators' perception of the current market uncertainty. Low (high) values of m imply that speculators have a bad (good) memory. Volatility outbursts may thus occur more (less) often but with a shorter (longer) duration.

To get an idea whether our estimated parameters may be viewed as significant, we carry out the following experiment. In the bottom right panel of figure 3, we repeat the computation of the AMMS, using our estimated parameter setting, for 100 different sets of seeds of random variables. As mentioned above,

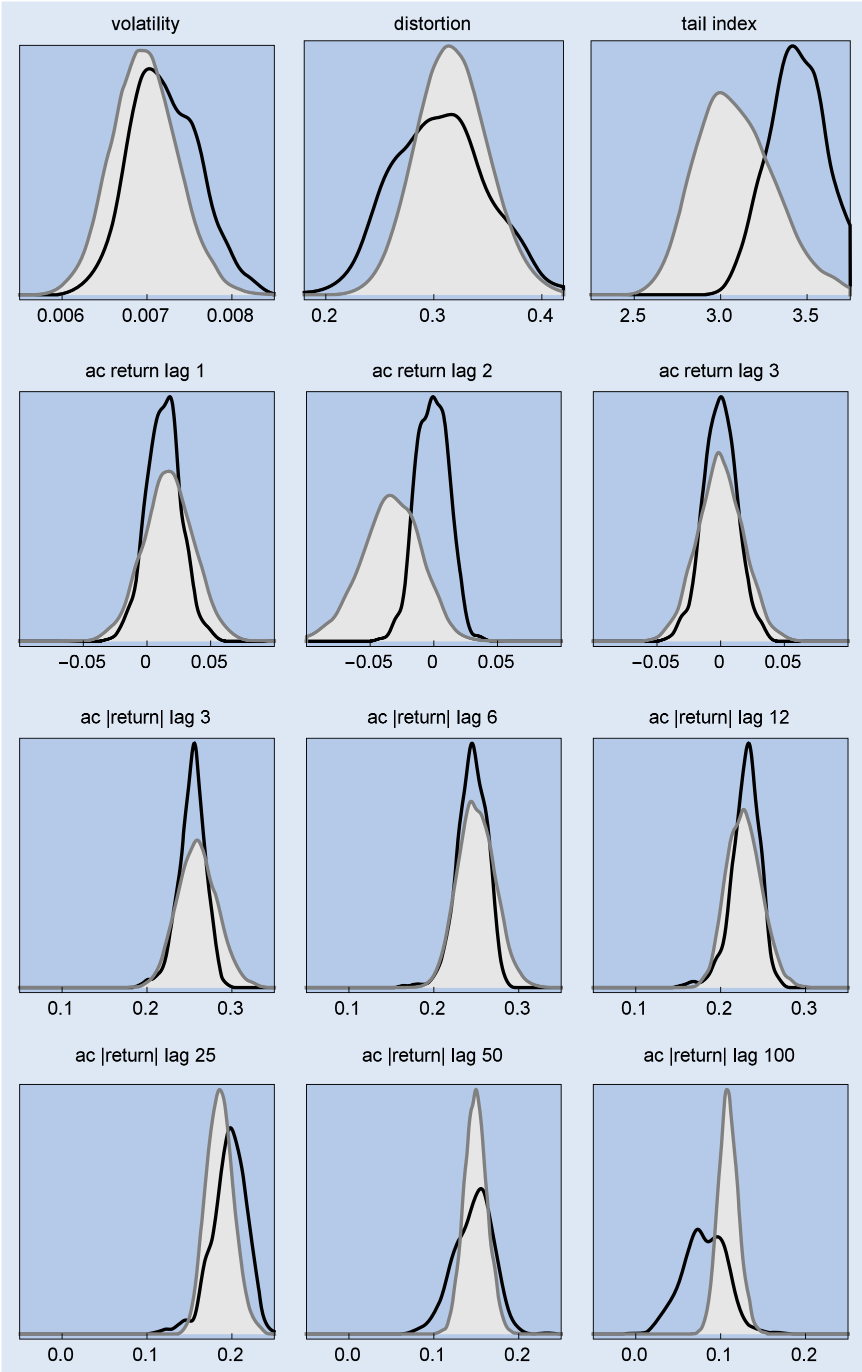

Figure 2. Distributions of simulated and empirical moments. The figure compares the smoothed distributions of the 12 simulated moments (black) with the bootstrapped distributions of their respective empirical counterparts (shaded in gray). Estimations of simulated moments are based on 500 simulation runs, while the bootstrapped distributions of empirical moments are obtained as described in section 3.

each computation of the AMMS requires 500 different seeds of random variables, which we now regard as one set of seeds of random variables. Note that the computed AMMSs fluctuate in a narrow band around 85%. Out of these 100 estimates, we can calculate the AMMS's 90% confidence interval. This confidence interval is represented in all eight panels of figure 3 by the two horizontal gray lines. Note that 95% of the estimated AMMSs are above the lower gray line. As can be seen, if we vary a single model parameter in a certain neighbourhood of its estimated value, the AMMS based on the original set of random seeds, indicated by the black line, stays above the lower gray line. However, if we vary the model parameters sufficiently strong, we obtain an AMMS score below the lower gray lines. On the one hand, this indicates that each of the model parameters actually matters (otherwise the AMMS would not significantly react to a parameter change) and on the other hand, we obtain information about how quickly and in which parameter region the AMMS leaves the significance band when a single parameter is varied. Overall, it seems that reaction parameter b has the weakest impact on the AMMS. Since the critical limits of parameter b are given by -0.01 and 0.05, we may even obtain for $b = 0$ an AMMS within the significance

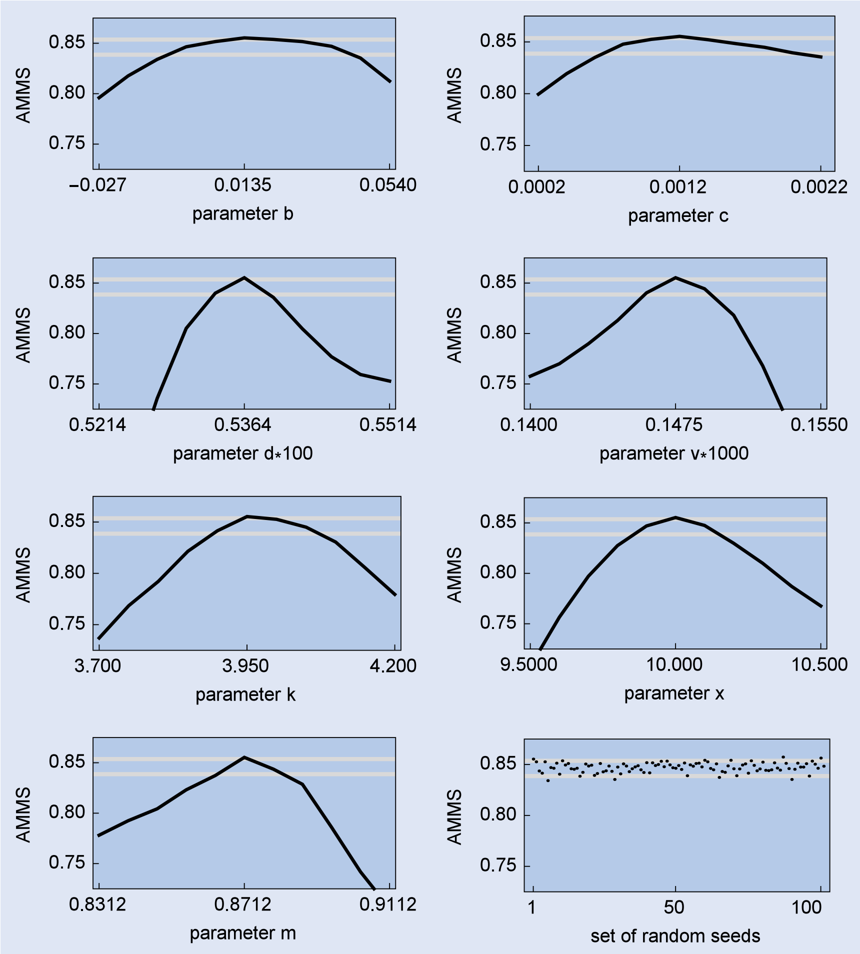

Figure 3. Identification of model parameters. The first seven panels reveal how the AMMS depends on parameters b, c, d, v, k, x and m. The AMMS is computed on the basis of 500 simulation runs. The bottom right panel reports 100 estimates of the AMMS for our estimated parameter setting and different sets of seeds of random variables, along with a 90% confidence band (gray lines, added to all panels).

bands (although not with the best possible performance). Note that in all other cases, these critical parameter ranges are always in the positive parameter domain, and typically rather narrow. For instance, the critical parameters for the memory parameter are given by 0.86 and 0.88, respectively. Behind this background, the effects of the seven model parameters on the model's performance may be deemed to be significant.

4.3. The functioning of the simple herding model

To understand the functioning of our simple herding model, it is helpful to explore one particular simulation run in more detail. We therefore depict in figure 4 a representative simulation run of our model with a sample length of $T = 12\,800$ observations. The top panel shows the evolution of the log price in the time domain. As can be seen, there are boom and bust periods, i.e. the price fluctuates in an intricate manner around its fundamental value. This panel is comparable to the second panel of figure 1 (except that we now look at a time span of 51 years instead of a time span of 145 years). However, the amplitudes of the price swings are on a similar level, and we know already from the statistical analysis provided in section 4.1 that our model matches the average distortion of the S&P500 quite well. The explanation for the model's oscillatory price behaviour is quite simple. While the technical and random demand components tend to drive the price away from its fundamental value, the fundamental demand component ensures that the price eventually reverts to its fundamental value.

The second panel of figure 4 shows the model's return dynamics. Besides a high average volatility, we also notice a number of larger returns and several volatility outbursts. The third panel of figure 4, depicting the development of X_t, explains these phenomena. Suppose that the volatility in the market is low so that speculators trade more or less independently. Technically, this means that X_t is rather low and that the random trading signals are hardly correlated. Large parts of the speculators' orders then cancel out, i.e. the market maker faces a relatively balanced excess demand. As a result, the market maker adjusts prices only weakly and volatility remains low. However, speculators may even receive in a low volatility regime a sequence of stronger trading signals. Their increased trading behaviour then forces the market maker to adjust prices more strongly. As the market's volatility increases, speculators start to display a stronger herding behaviour. Unsure about what is going on, they copy the behaviour of others. Technically, X_t is now rather high, i.e. the speculators' random trading signals are correlated. Since a synchronized trading behaviour

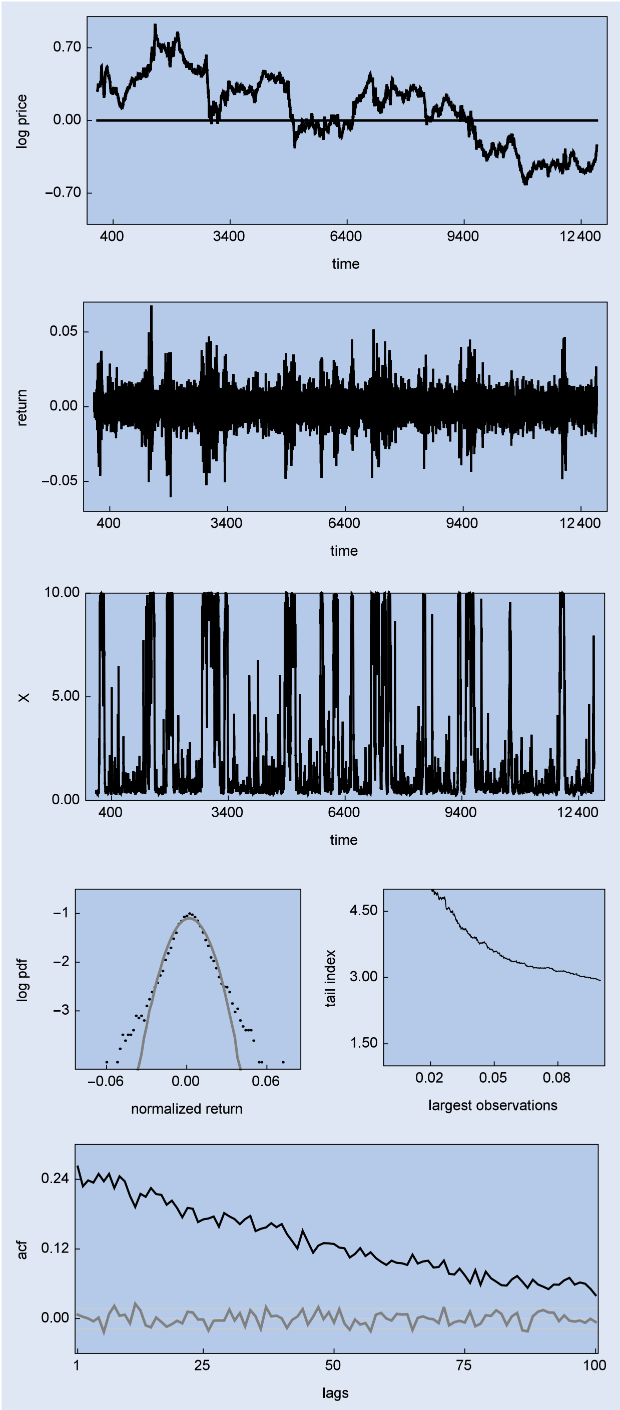

Figure 4. The dynamics of the simple herding model. The panels show, from top to bottom, the evolution of the log price, the returns, the strength of speculators' coordination behaviour, the log distribution of normalized returns, the Hill tail index estimator as a function of the largest returns and the autocorrelation function of raw returns (gray line) together with the autocorrelation function of absolute returns (black line), respectively. The simulation run is based on 12 800 observations.

leads to a less balanced excess demand, the market maker continues to adjust prices more strongly. Clearly, the high volatility regime becomes persistent. Even in a high volatility regime, a sequence of trading days when speculators receive only weak trading signals may eventually occur. Speculators' excess demand then decreases, as does the market maker's price adjustment. Since volatility ebbs away, speculators start to relax. Behaving more independently further weakens speculators' excess demand, and the market maker continues to adjust prices only weakly. Volatility then remains persistently low—until speculators receive again a sequence of stronger trading signals.

Since the volatility outburst may result in larger price changes, the model also gives rise to fat-tailed return distributions. This can be detected from the left panel of the fourth line of figure 4. As in the case of the S&P500 (see left panel of the fourth line of figure 1), extreme returns occur more frequently than warranted by the normal distribution. The right panel of the fourth line shows the tail index as a function of the tail fraction. Taking into account the largest 5% of the observations, we observe a tail index of about 3.59.† The final panel of figure 4 shows the autocorrelation functions of absolute returns and of raw returns. Our observations are quite comparable to what we observe for the S&P500. Raw returns are basically serially uncorrelated while the autocorrelation coefficients of absolute returns are clearly significant, even for lags up to 100 periods. All in all, we find the simulated dynamics to be strikingly similar to the actual behaviour of financial markets.

5. Robustness analysis

The goal of this section is to show that the dynamics of our simple herding model are robust with respect to several cumulative model extensions. In section 5.1, we first turn our simple model into a more involved agent-based model. In section 5.2, we consider additionally that a certain fraction of speculators is immune to herding effects. In section 5.3, we linearize the fundamental demand component of speculators. As we will see, the basic functioning of our simple herding model survives all three model extensions.

5.1. Enriching the simple herding model

In this section, we generalize our simple herding model. In the following, the market maker's price adjustment is given with

$$P_{t+1} = P_t + \frac{a_t}{N}\sum_{i=1}^{N} D_t^i. \tag{10}$$

Accordingly, the market maker's price adjustment speed is now represented by a uniformly distributed random variable, i.e. $a_t \sim \mathcal{U}[a(1-a_0), a(1+a_0)]$, where a and a_0 control the location and length of the interval from which a_t is drawn. For instance, setting $a = 1$ and $a_0 = 0.05$ implies that a_t fluctuates between 0.95 and 1.05. Such fluctuations may represent idiosyncratic shocks to the market maker or, in a stylized way, the presence of heterogeneous market makers.

The order placed by speculator i in period t is formalized as

$$D_t^i = b_t^i(P_t - P_{t-1}) + c_t^i(F - P_t)^3 + d_t^i\sqrt{N}\delta_t^i, \tag{11}$$

†While our estimated model systematically produces tail indices below 4 for a tail fraction of 5%, it does not produce tail indices of comparable size for smaller tail fractions. However, it seems to us that virtually no other model with heterogeneous interacting agents of a comparable size performs significantly better than our model.

where $b_t^i \sim \mathcal{U}[b(1-b_0), b(1+b_0)]$, $c_t^i \sim \mathcal{U}[c(1-c_0), c(1+c_0)]$ and $d_t^i \sim \mathcal{U}[d(1-d_0), d(1+d_0)]$. Parameters b, b_0, c, c_0, d and d_0 control the location and length of the intervals from which the uniformly distributed random variables b_t^i, c_t^i and d_t^i are drawn. Note that speculator i uses in each period a different trading rule and that this trading rule differs from the trading rules of all other speculators.†

Nevertheless, the random trading component in (11) may lead to a certain correlation among speculators' trading behaviour. We assume that the random trading signals $\delta_t = \{\delta_t^1, \delta_t^2, \ldots, \delta_t^N\}'$ are multivariate standard normal distributed, i.e. $\delta_t \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}_t)$, where the mean vector is again given with $\boldsymbol{\mu} = \{0, 0, \ldots, 0\}'$ but the variance-covariance vector now reads

$$\boldsymbol{\Sigma}_t = \left\{\begin{matrix} 1 & \rho_t^{1,2} & \ldots & \rho_t^{1,N} \\ \rho_t^{2,1} & 1 & & \vdots \\ \vdots & & \ddots & \rho_t^{N-1,N} \\ \rho_t^{N,1} & \ldots & \rho_t^{N,N-1} & 1 \end{matrix}\right\}. \tag{12}$$

Moreover, the correlation coefficients in (12) obey

$$\rho_t^{i,j} = \rho_t^{j,i} = \frac{X^{i,j}}{N-1}, \tag{13}$$

where

$$X_t^{i,j} = \frac{x_t^{i,j}}{1 + exp[-k_t^{i,j}(V_t - v)/v]}. \tag{14}$$

According to (13), speculator i's random trading behaviour is in each time step differently influenced by the other speculators and their influence on speculator i may not be identical. For a given trading period, however, we assume that the influence of j on i is the same as of i on j. Equation (14) captures how speculators i and j mutually influence each other's random trading behaviour. As in our simple model, speculators become nervous in periods of heightened uncertainty and thus copy the trading behaviour of others more strongly. However, we now assume that the maximal value of $x_t^{i,j}$ in (14) and the steepness of (14) are uniformly distributed random variables with $x_t^{i,j} \sim \mathcal{U}[x(1-x_0), x(1+x_0)]$ and $k_t^{i,j} \sim \mathcal{U}[k(1-k_0), k(1+k_0)]$, capturing time-varying and speculator-specific influence factors. Parameters x, x_0, k and k_0 determine the locations and lengths of the intervals from which these random variables are drawn.

Finally, market uncertainty is approximated by a smoothed measure of the market's past volatility, i.e.

$$V_t = m_t V_{t-1} + (1-m_t)(P_t - P_{t-1})^2. \tag{15}$$

Note that speculators' memory parameter is now time-varying. As with the other parameters, we assume that m_t is uniformly distributed, i.e. $m_t \sim \mathcal{U}[m(1-m_0), m(1+m_0)]$. Of course, parameters m and m_0 have to be selected such that $0 \leq m_t \leq 1$. Hommes (2013) argues that speculators agree more strongly on a stock market's volatility than on its future direction which is why we assume that market uncertainty has no speculator-specific component.‡

Figure 5 presents a simulation run which has been generated on the basis of the enriched model, i.e. (10)–(15). Since the simulation of the new model is quite time-consuming, it is not possible to estimate its parameters using the method of simulated moments. As it turns out, however, the estimated parameters of our simple herding model provide a very good orientation for the selection of the parameters of the enriched model. To be precise, we use for parameters b, c, d, k, v, m and x exactly the parameter values which we have estimated for our simple model. Moreover, we set $a = 1$, $F = 0$ and $N = 100$. Finally, all variables which are uniformly distributed fluctuate in a $\pm 5\%$ band around their (estimated or assumed) midpoints. Since figure 5 has the same design as figure 4, we can easily compare the dynamics of our enriched model with the dynamics of our simple model. Visual inspection reveals that the core features of our simple model are robust with respect to the modifications presented in this section. Put differently, the main story of our paper—an increased herding in periods of heightened uncertainty causes an increased synchronization of speculators' trading behaviour which may lead to strong price changes and lasting volatility outbursts—seems to be robust.

A few comments are in order. First, to simulate a time series with 12 800 observations of the enriched model with $N = 100$, we need about 870 s.§ In contrast, the same simulation of the simple model only takes 1.5 s. Taking these numbers literally, one may argue that turning from the enriched model to the simple model reduces the computational effort by more than 99%. Second, we checked that our results also hold if we increase the number of speculators to, for instance, $N = 200$ or $N = 400$. Similar as in the case of our simple herding model, the set-up of our enriched model implies that the dynamics is basically independent with respect to the number of speculators. However, the time to generate a simulation run increases with N. Third, due to the time needed to generate a single simulation run, we cannot optimize the parameters of the enriched model. While the dynamics depicted in figure 5 look quite promising, there may be parameter settings for which the model produces even more realistic dynamics. However, we leave this as a challenge for future research.

5.2. *Herding and non-herding of speculators*

In the model presented in section 5.1, all speculators are subject to herding behaviour. How does the model dynamics get affected if a certain fraction of speculators is immune to herding effects? To address this question, we use the model proposed in section 5.1 and assume that half of the speculators does not display any kind of herding behaviour, i.e. their random

†Note that we assume in (12) that the δ_t^i are multivariate standard normal distributed. To increase speculators' heterogeneity, we control the intensity of the random demand component by $d_t^i \sqrt{N}$. Of course, if $d_t^i \sqrt{N} = d\sqrt{N} = \sigma$ we obtain the setting of our simple herding model again.

‡Technically, the variance-covariance matrix has to be symmetric to generate multivariate normal distributed random variables. Assuming speculator-specific volatility functions would imply that $\rho_t^{i,j} \neq \rho_t^{j,i}$.

§We are not claiming that we have found the most efficient way to run the enriched model. However, the simulation of the enriched model requires to generate in each time step a new $N \times N$ variance-covariance matrix and it seems that increasingly more time is needed to compute vectors of multivariate normal distributed random disturbance terms if the number of speculators increases.

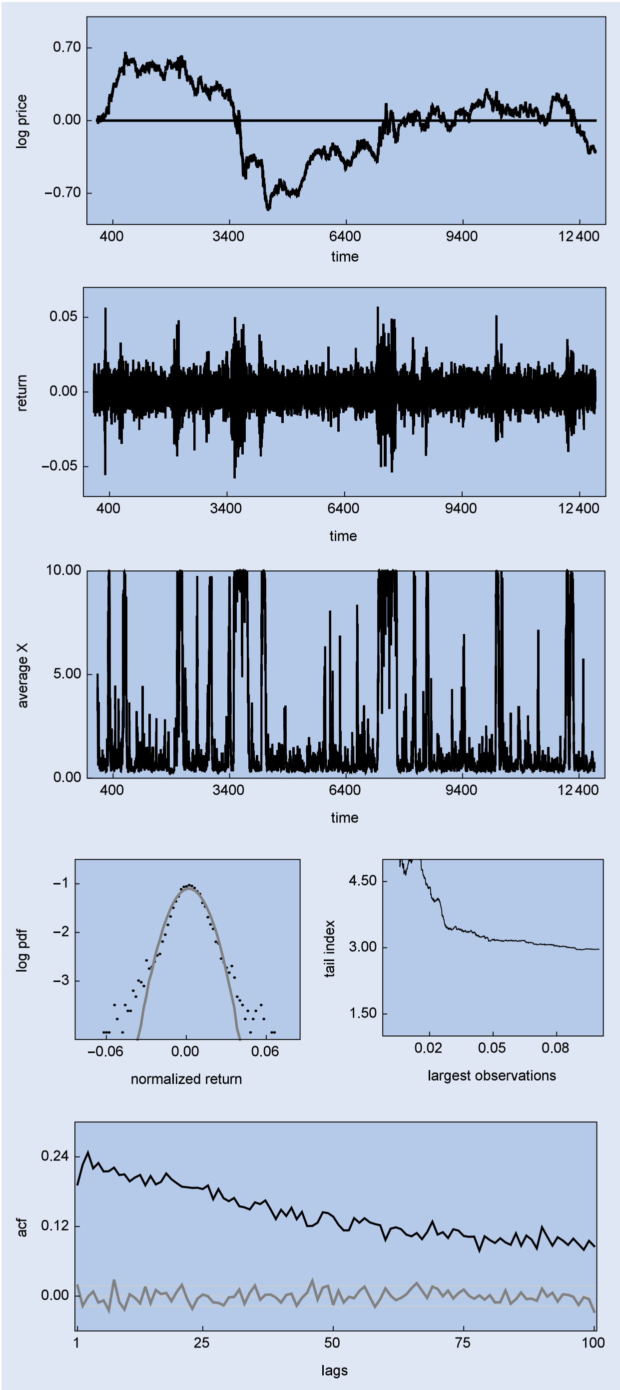

Figure 5. The dynamics of the enriched model. Same design as in figure 4, except that the third panel depicts the evolution of X_t^i averaged over all speculators. Parameters a, b, c, d, k, m and x fluctuate within a band of ±5% around the parameter values of the simple model. Other parameters: $v = 0.0001475$, $F = 0$ and $N = 100$.

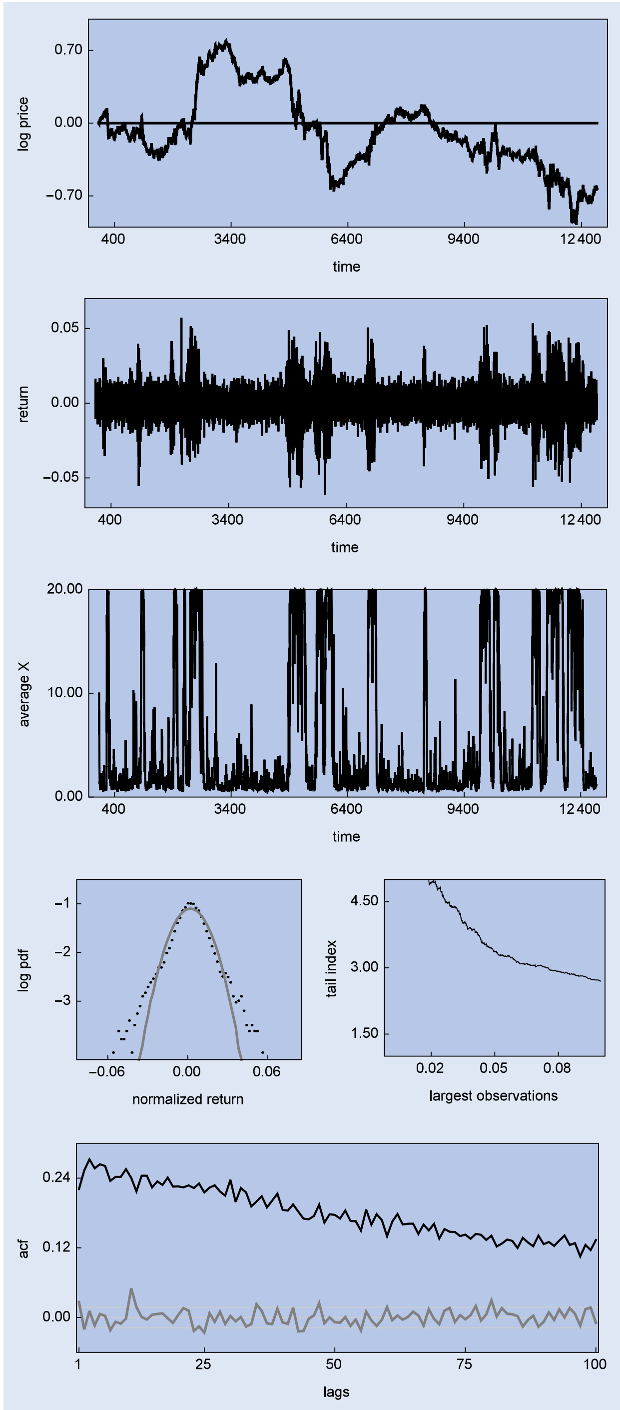

Figure 6. The dynamics of the enriched model with a partial immunization of speculators to herding effects. Same design as in figure 5, except that half of the speculators are not subject to herding effects and that parameter x has been doubled.

demand component is independent from the other speculators' random demand component. If we use the same parameter values as in the simple model, the most interesting features of the model dynamics vanish. However, figure 6 reveals that we can recover all interesting model properties if we adjust the model parameters. To be precise, the simulation design of figures 5 and 6 differs in the sense that in figure 6 only 50 out of the 100 speculators are subject to herding behaviour and that parameter x is increased from 10 to 20. Economically, this means that the correlation between the speculators which are subject to herding behaviour must increase if the fraction of speculators which is subject to herding behaviour decreases.

Further simulations (not depicted) reveal that the model can also produce the stylized facts of financial markets for $x = 20$ and $N = 400$. What do these number imply? If all speculators are subject to herding behaviour and $x = 10$, than the correlation coefficients defined by (13) and (14) can reach values up to 10%. If we consider the scenario depicted in figure 6, i.e. if only half of the $N = 100$ traders are subject to herding behaviour and if $x = 20$, then these correlation

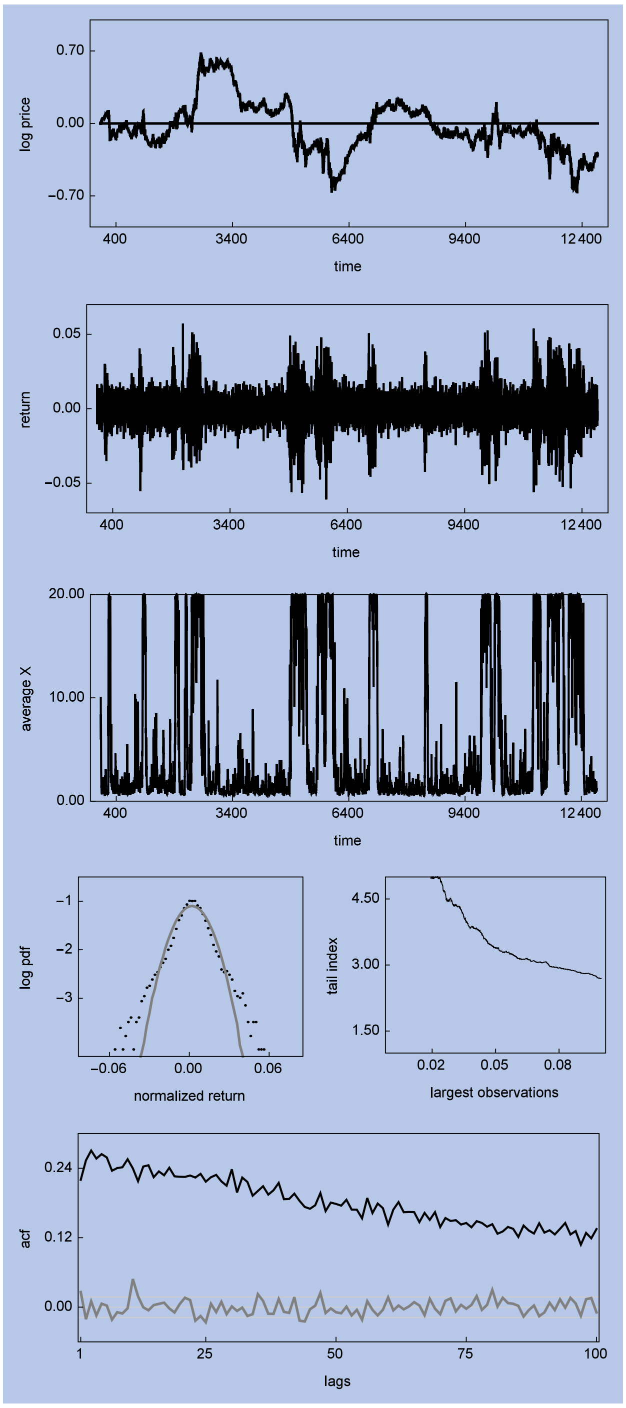

Figure 7. The dynamics of the enriched model with a partial immunization of speculators to herding effects and a linear fundamental demand component. Same design as in figure 6, except that speculators' fundamental demand component has been linearized.

coefficients can reach values up to 40%. However, if there are $N = 400$ speculators of which half of them are subject to herding behaviour and if $x = 20$, the maximal correlation coefficient is again given with 10%. Hence, stock markets which are populated by a large number of speculators of which only a fraction is subject to herding behaviour may even in the presence of a rather weak correlation display extreme price changes and lasting volatility outbursts.†

5.3. *Linear fundamental demand component*

Finally, we show that the model dynamics is also robust if we change speculators' trading rules. Figure 7 is based on the simulation design depicted in figure 6 except that the cubic fundamental demand component in (11) is replaced by a linear fundamental demand component. As can be seen, the dynamics presented in figures 6 and 7 are quite comparable, i.e. the exact specification of the fundamental demand component is not crucial for the model's ability to replicate the stylized facts of financial markets. However, there are some subtle differences. Speculators relying on the cubic fundamental demand component do hardly react to a small mispricing but once the mispricing is more pronounced, they trade rather aggressively. As a result, prices may wander more quickly away from the fundamental value as long as the mispricing is moderate. Comparing, for instance, the bubbles which emerge in figures 6 and 7 around periods 3400, 6400 and 12 400, we see that these bubbles are more pronounced when speculators rely on the cubic fundamental demand component. Overall, this seems to describe the dynamics of actual financial markets better. Recall that we observe for our simple estimated model that the distortion drops in about 84.2% of the cases in the 95% confidence interval of the S&P500's distortion (see table 2). If we re-estimate our simple model with a linear fundamental demand component, this number goes down to 65.8%.‡

6. Conclusions

We propose a novel herding model to explain the dynamics of financial markets. To be precise, we consider a stock market which is populated by a market maker and a fixed number of heterogeneous speculators. The market maker adjusts the stock's price with respect to excess demand while speculators rely on a mix of technical and fundamental trading rules to determine their orders. We model the heterogeneity of the trading rules applied by adding a multivariate normal distributed random variable to speculators' trading rules. Guided by empirical and experimental work, we assume that speculators observe other speculators' actions in periods of heightened uncertainty more strongly. Note that such herding behaviour implies a decrease in the heterogeneity of the trading rules applied. To take this into account, we assume that the correlation between speculators' random trading signals increases with the market's past volatility.

Fortunately, our approach allows a convenient aggregation of speculators' trading behaviour. Since the dynamics of our simple herding model depends on three equations only, we can use the method of simulated moments to estimate its parameters. For this purpose, we define 12 summary statistics (moments) to quantify five important stylized facts of financial markets. In particular, we rely on a distortion measure

†We also checked other scenarios. If one-third of $N = 100$ speculators is immune to herding effects, then x has to be set to 15 to replicate the stylized facts of financial markets. Relatedly, if 60% of $N = 100$ speculators are immune to herding, one needs to set $X = 25$ to obtain realistic dynamics.

‡As pointed out by one of the referees, the fact that the inclusion of a cubic fundamental demand term brings our model closer to the data than a linear fundamental demand term suggests the presence of non-linear mean reversion forces in financial markets.

to capture the stock market's boom-bust dynamics, a volatility measure to capture the stock market's average variability, a tail index to capture the fat-tailedness of the stock market's return distribution, three autocorrelation coefficients of raw returns to capture the random walk property of the stock market and six autocorrelation coefficients of absolute returns to capture the stock market's volatility clustering behaviour. Our empirical analysis focuses on the S&P500. Here we are fortunate to have access to 145 years of monthly data to identify the stock market's distortion and 51 years of daily data to characterize the other phenomena. Based on our 12 summary statistics, we define a goodness-of-fit criterion which indicates the effectiveness of the model's average moment matching. As it turns out, our estimated model achieves an astonishing AMMS of around 85%, i.e. on average, about 85% of the model-implied moments drop into the 95% confidence intervals of their empirical counterparts.

Speculators' herding behaviour is crucial for the model's ability to produce realistic dynamics. The key message delivered by our model may be summarized as follows. Consider a situation in which the stock market's volatility is low. In such a situation, speculators trade more or less independently from each other, i.e. speculators' trading behaviour is relatively heterogeneous. As a result, many orders placed by speculators offset each other and excess demand is rather balanced. Consequently, the market maker's price adjustment is modest and volatility remains low. Eventually, however, there appears a sequence of trading days when speculators happen to receive stronger trading signals. Their increased trading intensity amplifies excess demand so that the market maker adjusts prices more strongly. At this point, the low volatility regime turns into a high volatility regime. Speculators become nervous and start to observe other speculators' actions more closely. The synchronization of speculators' trading behaviour leads to a persistently high excess demand and a sustained period of significant price changes. Eventually, however, there appears a sequence of trading days when speculators receive weak trading signals. Although their trading behaviour remains coordinated, excess demand decreases and the market maker adjusts prices less forcefully. As volatility decreases, speculators start to relax and behave more independently. Now the market enters a period of low volatility until speculators once again receive a sequence of stronger trading signals.

Our robustness analysis reveals that it is possible to extend our simple herding model into a more involved agent-based model. In particular, simulations reveal that the enriched model is also able to replicate the stylized facts of financial markets. Since simulating the enriched model is much more time-consuming than simulating the simple model, it is not possible to estimate the parameters of the enriched model, at least not by the method of simulated moments. However, we show that the estimation results for our simple model may be very useful in determining the parameters of our enriched model. To calibrate agent-based models, it might thus be a good strategy to try to find and estimate a structurally similar but simpler model version. In our case, for instance, the time to generate a simulation run decreases by about 99% if one switches from the enriched model to the small model. We hope that this insight is helpful for other researchers who seek to bring their models to the data. Of course, the simple model is also helpful to understand the functioning of the enriched model.

Our approach may be extended in various ways, and we conclude our paper by pointing out four avenues for future research. First, one may assume that speculators' herding behaviour depends not only on the stock market's volatility but also on its distortion. The more the stock price deviates from its fundamental value, the more risky the stock market may appear to speculators. Second, while our estimated model is able to generate fat-tailed return distributions, it does not produce extreme market changes. In Schmitt and Westerhoff (2016), we assume that exogenously occurring sunspots, such as investment advice by financial gurus, may coordinate the trading behaviour of speculators. If sufficiently many speculators trade in the same direction, extreme market changes may occur. Put differently, one may easily condition the correlation between speculators' random trading signals on additional influence factors. Third, our model may be used to explore which types of policy tools may be used to stabilize financial markets. For instance, our model suggests that a policy which manages to reduce the stock market's volatility automatically relaxes speculators' herding behaviour and should thereby enhance market stability. Fourth, we determined the parameters of our model by maximizing the model's average moment matching score. Of course, different objective functions may be used to estimate our model. In this paper, we opt for an egalitarian treatment of the moments. Alternatively, one could search for a parameter setting which maximizes the worst performing summary statistic. Similarly, one may reconsider the choice of summary statistics. We would like to remark at this point that we know of no other study which also uses Shiller's (2015) data to quantify stock market mispricing. We are under the impression that, in order to make the most out of the method of simulated moments, it would be useful to experiment further with the objective function and the underlying summary statistics. To sum up, our model reveals that herding behaviour can reduce the heterogeneity of the trading rules applied and thereby amplify stock price changes. To make the functioning of our model as clear as possible, we ensure maximum simplicity. However, we hope that our paper will stimulate more work in this exciting research direction.

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References

Alfarano, S. and Lux, T., A noise trader model as a generator of apparent finanical power laws and long memory. *Macroecon. Dyn.*, 2007, **11**, 80–101.

Alfarano, S., Lux, T. and Wagner, F., Estimation of agent-based models: The case of an asymmetric herding model. *Comput. Econ.*, 2005, **26**, 19–49.

Alfi, V., Cristelli, M., Pietronero, L. and Zaccaria, A., Minimal agent based model for financial markets I. *Eur. Phys. J. B*, 2009a, **67**, 385–397.

Alfi, V., Cristelli, M., Pietronero, L. and Zaccaria, A., Minimal agent based model for financial markets II. *Eur. Phys. J. B*, 2009b, **67**, 399–417.

Avery, C. and Zemsky, P., Multidimensional uncertainty and herd behaviour in financial markets. *Amer. Econ. Rev.*, 1998, **80**, 724–748.

Baddeley, M., Herding, social influence and economic decision making: Socio-psychological and neuroscientific analyses. *Philos. Trans. R. Soc. B*, 2010, **365**, 281–290.

Blaurock, I., Schmitt, N. and Westerhoff, F., Market entry waves and volatility outbursts in stock markets. Working Paper, University of Bamberg, 2016.

Boswijk, P., Hommes, C. and Manzan, S., Behavioral Heterogeneity in Stock Prices. *J. Econ. Dyn. Control*, 2007, **31**, 1938–1970.

Bouchaud, J.-P. and Cont, R., A Langevin approach to stock market fluctuations and crashes. *Eur. Phys. J. B*, 1998, **6**, 543–550.

Bouchaud, J.-P., Farmer, D. and Lillo, F., How markets slowly digest changes in supply and demand. In *Handbook of Financial Markets: Dynamics and Evolution*, edited by T. Hens and K.R. Schenk-Hoppé, pp. 57–160, 2009 (North-Holland: Amsterdam).

Brock, W. and Hommes, C., Heterogeneous beliefs and routes to chaos in a simple asset pricing model. *J. Econ. Dyn. Control*, 1998, **22**, 1235–1274.

Challet, D., Marsili, M. and Zhang, Y.-C., Stylized facts of financial markets and market crashes in Minority Games. *Physica A*, 2001, **294**, 514–524.

Chen, Z. and Lux, T., Estimation of sentiment effects in financial markets: A simulated method of moments approach. FinMaP-Working Paper, No. 37, University of Kiel, 2015.

Chen, S.-H. and Yeh, C.-H., Evolving traders and the business school with genetic programming: A new architecture of the agent-based artificial stock market. *J. Econ. Dyn. Control*, 2001, **25**, 363–393.

Chiang, T. and Zheng, D., An empirical analysis of herd behavior in global stock markets. *J. Bank. Finance*, 2010, **34**, 1911–1921.

Chiarella, C., Dieci, R. and He, X.-Z., Heterogeneity, market mechanisms, and asset price dynamics. In *Handbook of Financial Markets: Dynamics and Evolution*, edited by T. Hens and K.R. Schenk-Hoppé, pp. 277–344, 2009 (North-Holland: Amsterdam).

Chiarella, C. and Iori, G., A simulation analysis of the microstructure of double auction markets. *Quant. Finance*, 2002, **2**, 1–8.

Chiarella, C., Iori, G. and Perelló, J., The impact of heterogeneous trading rules on the limit order book and order flows. *J. Econ. Dyn. Control*, 2009, **33**, 525–537.

Cipriani, M. and Guarino, A., Herd behaviour in financial markets: An experiment with financial market professionals. *J. Eur. Econ. Assoc.*, 2009, **7**, 206–233.

Cipriani, M. and Guarino, A., Estimating a structural model of herd behaviour in financial markets. *Amer. Econ. Rev.*, 2014, **104**, 224–251.

Cont, R., Empirical properties of asset returns: Stylized facts and statistical issues. *Quant. Finance*, 2001, **1**, 223–236.

Cont, R. and Bouchaud, J.-P., Herd behaviour and aggregate fluctuations in financial markets. *Macroecon. Dyn.*, 2000, **4**, 170–196.

Day, R. and Huang, W., Bulls, bears and market sheep. *J. Econ. Behav. Organiz.*, 1990, **14**, 299–329.

De Grauwe, P., Dewachter, H. and Embrechts, M., *Exchange Rate Theory: Chaotic Models of Foreign Exchange Markets*, 1993 (Blackwell: Oxford).

Franke, R., Applying the method of simulated moments to estimate a small agent-based asset pricing model. *J. Empir. Finance*, 2009, **16**, 804–815.

Franke, R. and Westerhoff, F., Structural stochastic volatility in asset pricing dynamics: Estimation and model contest. *J. Econ. Dyn. Control*, 2012, **36**, 1193–1211.

Franke, R. and Westerhoff, F., Why a simple herding model may generate the stylized facts of daily returns: Explanation and estimation. *J. Econ. Interac. Coord.*, 2016, **11**, 1–34.

Gilli, M. and Winker, P., A global optimization heuristic for estimating agent based models. *Comput. Stat. Data Anal.*, 2003, **42**, 299–312.

Graham, B. and Dodd, D., *Security Analysis*, 1951 (McGraw-Hill: New York).

Heemeijer, P., Hommes, C., Sonnemans, J. and Tuinstra, J., Price stability and volatility in markets with positive and negative expectations feedback: An experimental investigation. *J. Econ. Dyn. Control*, 2009, **32**, 1052–1072.

Hill, B., A simple general approach to inference about the tail of a distribution. *Ann. Stat.*, 1975, **3**, 1163–1174.

Hommes, C., *Behavioral Rationality and Heterogeneous Expectations in Complex Economic Systems*, 2013 (Cambridge University Press: Cambridge).

Hommes, C., Sonnemans, J., Tuinstra, J. and van de Velden, H., Coordination of expectations in asset pricing experiments. *Rev. Financ. Stud.*, 2005, **18**, 955–980.

Hommes, C. and Wagener, F., Complex evolutionary systems in behavioral finance. In *Handbook of Financial Markets: Dynamics and Evolution*, edited by T. Hens and K.R. Schenk-Hoppé, pp. 217–276, 2009 (North-Holland: Amsterdam).

Iori, G., Avalanche dynamics and trading friction effects on stock market returns. *International J. Mod. Phys. C*, 1999, **10**, 1149–1162.

Iori, G., A microsimulation of traders activity in the stock market: The role of heterogeneity, agents' interactions and trade frictions. *J. Econ. Behav. Organiz.*, 2002, **49**, 269–285.

Keynes, J.M., *The General Theory of Employment, Interest, and Money*, 1936 (Harcourt, Brace and Company: New York).

Kirman, A., Epidemics of opinion and speculative bubbles in financial markets. In *Money and Financial Markets*, edited by M. Taylor, pp. 354–368, 1991 (Blackwell: Oxford).

Kirman, A., Ants, rationality, and recruitment. *Q. J. Econ.*, 1993, **108**, 137–156.

LeBaron, B., Agent-based computational finance. In *Handbook of Computational Economics: Agent-based Computational Economics*, edited by L. Tesfatsion and K. Judd, pp. 1187–1233, 2006 (North-Holland: Amsterdam).

LeBaron, B., Arthur, B. and Palmer, R., Time series properties of an artificial stock market. *J. Econ. Dyn. Control*, 1999, **23**, 1487–1516.

Lux, T., Herd behaviour, bubbles and crashes. *Econ. J.*, 1995, **105**, 881–896.

Lux, T., Stochastic behavioural asset-pricing models and the stylized facts. In *Handbook of Financial Markets: Dynamics and Evolution*, edited by T. Hens and K.R. Schenk-Hoppé, pp. 161–216, 2009 (North-Holland: Amsterdam).

Lux, T. and Ausloos, M., Market fluctuations I: Scaling, multiscaling, and their possible origins. In *Science of Disaster: Climate Disruptions, Heart Attacks, and Market Crashes*, edited by A. Bunde, J. Kropp and H. Schellnhuber, pp. 373–410, 2002 (Berlin: Springer).

Lux, T. and Marchesi, M., Scaling and criticality in a stochastic multi-agent model of a financial market. *Nature*, 1999, **397**, 498–500.

Mantegna, R. and Stanley, E., *An Introduction to Econophysics*, 2000 (Cambridge University Press: Cambridge).

Manzan, S. and Westerhoff, F., Representativeness of news and exchange rate dynamics. *J. Econ. Dyn. Control*, 2005, **29**, 677–689.

Murphy, J., *Technical Analysis of Financial Markets*, 1999 (New York Institute of Finance: New York).

Palley, T., Safety in numbers: A model of managerial herd behavior. *J. Econ. Behav. Organiz.*, 1995, **28**, 443–450.

Pellizzari, P. and Westerhoff, F., Some effects of transaction taxes under different microstructures. *J. Econ. Behav. Organiz.*, 2009, **72**, 850–863.

Prechter, R. and Parker, W., The financial/economic dichotomy in social behavioral dynamics: The socioeconomic perspective. *J. Behav. Finance*, 2007, **8**, 84–108.

Scharfstein, D. and Stein, J., Herd behavior and investment. *Amer. Econ. Rev.*, 1990, **80**, 465–479.

Schmitt, N. and Westerhoff, F., Heterogeneity, spontaneous coordination and extreme events within large-scale and small-scale

agent-based financial market models. BERG Working Paper No. 111, University of Bamberg, 2016.
Shiller, R., Speculative prices and popular models. *J. Econ. Perspect.*, 1990, **4**, 55–65.
Shiller, R., *Irrational Exuberance*, 2015 (Princeton University Press: Princeton).
Shiller, R. and Pound, J., Survey evidence on diffusion of interest and information among investors. *J. Econ. Behav. Organiz.*, 1989, **12**, 47–66.
Stauffer, D., de Oliveira, P. and Bernardes, A., Monte Carlo simulation of volatility clustering in market model with herding. *Int. J. Theor. Appl. Finance*, 1999, **2**, 83–94.
Tedeschi, G., Iori, G. and Gallegati, M., Herding effects in order driven markets: The rise and fall of gurus. *J. Econ. Behav. Organiz.*, 2012, **81**, 82–96.
Westerhoff, F., Multiasset market dynamics. *Macroecon. Dyn.*, 2004, **8**, 596–616.
Winker, P., Gilli, M. and Jeleskovic, V., An objective function for simulation based inference on exchange rate data. *J. Econ. Interac. Coord.*, 2007, **2**, 125–145.

Appendix 1.

In this appendix, we first explain the economic implications behind our rescaling assumptions $a^M = a/N, \sigma = d\sqrt{N}$ and $\rho_t = X_t/(N-1)$, which make our model independent of the number of speculators. Afterwards, we present a number of model versions that are only partially rescaled and thus still depend on the system size N.

Combining (1) and (2) reveals that

$$P_{t+1} = P_t + a^M \sum_{i=1}^{N} \{b^i(P_t - P_{t-1}) + c^i(F - P_t)^3 + \delta_t^i\}. \quad \text{(A1)}$$

Since we define $b = \frac{1}{N}\sum_{i=1}^{N} b^i$ and $c = \frac{1}{N}\sum_{i=1}^{N} c^i$ and since we have $\sum_{i=1}^{N} \delta_t^i = \sigma\sqrt{N + N(N-1)\rho_t}\,\epsilon_t$ with $\epsilon_t \sim \mathcal{N}(0,1)$, (A1) is equivalent to

$$P_{t+1} = P_t + a^M\{Nb(P_t - P_{t-1}) + Nc(F - P_t)^3 + \sigma\sqrt{N + N(N-1)\rho_t}\,\epsilon_t\}. \quad \text{(A2)}$$

To make the model dynamics independent of the number of speculators, we set $a^M = a/N$, $\sigma = d\sqrt{N}$ and $\rho_t = X_t/(N-1)$, which turns (A2) into

$$P_{t+1} = P_t + a\,\{b\,(P_t - P_{t-1}) + c\,(F - P_t)^3 + d\sqrt{1 + X_t}\,\epsilon_t\}. \quad \text{(A3)}$$

Note that (A2) and (A3) represent the dynamics of the same model—the only difference between these two equations is that the model parameters get rescaled in (A3). What does such a rescaling imply?

The rescaling assumption $a^M = a/N$ implies that a given order has a lower price impact in markets with a large number of speculators than in markets with a small number of speculators. Such a model property is in line with empirical evidence (Bouchaud *et al.* 2009) according to which the price impact of a given order decreases with market liquidity. For $a^M = a/N$, (A2) turns into

$$P_{t+1} = P_t + a\{b(P_t - P_{t-1}) + c(F - P_t)^3 + \sigma\sqrt{\frac{1}{N} + \frac{N-1}{N}\rho_t}\,\epsilon_t\}. \quad \text{(A4)}$$

Accordingly, the speculators' aggregate deterministic trading behaviour becomes independent of N. For a large number of speculators we can write that $\sigma\sqrt{\frac{1}{N} + \frac{N-1}{N}\rho_t}\,\epsilon_t \approx \sigma\sqrt{\rho_t}\,\epsilon_t$ which reveals that the speculators' aggregate random trading behaviour increase with ρ_t. However, if $\rho_t = 0$ and N becomes large, the speculators' random trading behaviour washes out.

Rescaling assumption $\rho_t = X_t/(N-1)$ implies that the correlation between the speculators' random trading signals decreases with the number of speculators. This assumption seems reasonable since large markets are more anonymous than small markets and make a coordination less likely. Assuming $a^M = a/N$ and $\rho_t = X_t/(N-1)$ turns (A2) into

$$P_{t+1} = P_t + a\{b(P_t - P_{t-1}) + c(F - P_t)^3 + \frac{\sigma}{\sqrt{N}}\sqrt{1 + X_t}\,\epsilon_t\}. \quad \text{(A5)}$$

Apparently, also specification (A5) suffers from the problem that the speculators' aggregate random trading behaviour becomes less relevant for the price dynamics as N increases.

Obviously, rescaling assumption $\sigma = d\sqrt{N}$ turns (A5) into the fully rescaled model (A3), which is independent of N. What does $\sigma = d\sqrt{N}$ imply? First of all, this assumption implies that an individual speculator's random trading behaviour increases with N, i.e. speculators trade more forcefully in large markets than in small markets. Moreover, this assumption implies that the signal-to-noise ratio of the trading rule of an individual speculator decreases if the number of speculators increases. Put differently, an individual speculator trades more strongly on the random trading signal than on the deterministic trading signal if the market size increases. However, the speculators' aggregate signal-to-noise ratio in the fully rescaled model is independent of N and thus remains constant.

Suppose next that we only rely on rescaling assumption $\rho_t = X_t/(N-1)$. Such a partially rescaled model is given by

$$P_{t+1} = P_t + a^M\{Nb(P_t - P_{t-1}) + Nc(F - P_t)^3 + \sigma\sqrt{N}\sqrt{1 + X_t}\,\epsilon_t\}. \quad \text{(A6)}$$

Assuming further that $\sigma = d\sqrt{N}$ yields

$$P_{t+1} = P_t + a^M\{Nb(P_t - P_{t-1}) + Nc(F - P_t)^3 + Nd\sqrt{1 + X_t}\,\epsilon_t\}. \quad \text{(A7)}$$

If we only assume that $\sigma = d\sqrt{N}$, we obtain

$$P_{t+1} = P_t + a^M\{Nb(P_t - P_{t-1}) + Nc(F - P_t)^3 + Nd\sqrt{1 + (N-1)\rho_t}\,\epsilon_t\}. \quad \text{(A8)}$$

Finally, if we only set $a^M = a/N$ and $\sigma = d\sqrt{N}$, our model becomes

$$P_{t+1} = P_t + a\{b(P_t - P_{t-1}) + c(F - P_t)^3 + d\sqrt{1 + (N-1)\rho_t}\,\epsilon_t\}. \quad \text{(A9)}$$

Specifications (A2), (A6), (A7), (A8) and (A9) may display interesting properties if the number of speculators varies endogenously over time. For instance, if there is a temporary inflow of additional speculators, then these specifications imply a burst of volatility (see Blaurock *et al.* 2016 for a related application). In this respect, we also recall the work of Alfi *et al.* (2009a, 2009b). They first propose an agent-based financial market model which is able to explain the stylized facts of financial markets, but only if the number of speculators is in a certain range. Afterwards they endogenize the number of active speculators and propose a self-organizing mechanism that drives the number of active speculators in the range which produces realistic dynamics. An extension of scenarios (A2), (A6), (A7), (A8) and (A9) in that direction may be interesting. For simplicity, and to keep the speculators' individual signal-to-noise ratio constant, we recommend scenario (A2) for such an endeavor.

A few final comments are in order. One benefit of our system-size independent setup is that it allows us to isolate an empirically relevant volatility clustering mechanism. In fact, our model reveals that even a moderate herding-induced correlation among the trading behaviour of heterogeneous speculators may produce large and persistent price fluctuations. The mechanism responsible for this result works in markets with a small number of speculators as well as in markets with a large number of speculators. While some agent-based models lose their ability to match the stylized facts of financial markets if the number of speculators increases, this is not the case for our model. Moreover, we do not have to vary the number of speculators to obtain realistic dynamics. Within our model, the number of speculators is constant. Our model also generates realistic dynamics if it is only partially rescaled, provided that the other model parameters are suitable adjusted. Clearly, the dynamics of models (A2)–(A9) are due to the same dynamical system. Finally, one may also argue that our rescaling assumptions make at least clear what has to be invested to make our model independent of N.