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A walk through mathematical physics with disformal transformations and scalar field self-force effects on a particle orbiting a Reissner-Nordström black hole

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Aos meus pais, Bartolomeu e Maria.

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ABSTRACT

This doctorate thesis is divided in two independent parts. The content of the first part is a thorough and original study of the algebraic and geometric properties of what is known as disformal metrics. A new formalism for such metrics is developed and two physical applications are given. The first application concerns the disformal invariance of the Dirac equation. Then, as a second application, the new formalism developed for disfomal metrics is used as the mathematical framework for the description of a phenomenological approach to quantum gravity known as rainbow gravity.

In the second part, scalar field self-force effects on a scalar charge orbiting a Reissner-Nordström black hole are investigated. The scalar wave equation is solved analytically in a post-Newtonian framework, and the solution is used to compute the self-field as well as the components of the self-force at the location of the particle up to 7.5 post-Newtonian order. The energy fluxes radiated to infinity and down the hole are also evaluated. A comparison with previous numerical results in the Schwarzschild case shows a reasonable agreement in both strong-field and weak-field regimes.

Keywords: mathematical physics, disformal metrics, spinor, Dirac equation, rainbow gravity, post-Newtonian approximation, Reissner-Nordström space-time, scalar self-force

PUBLICATIONS

The content of this doctorate thesis puts into context the following published pieces:

1. Eduardo Bittencourt, Iarley P. Lobo and Gabriel G Carvalho, *On the disformal invariance of the Dirac equation*, Class. Quantum Grav. **32** 185016 (2015) [arXiv:1505.03415].

2. Gabriel G. Carvalho, Iarley P. Lobo and Eduardo Bittencourt, *Extended disformal approach in the scenario of rainbow gravity*, Phys. Rev. D **93**, 044005 (2016) [arXiv:1511.00495].

3. Donato Bini, Gabriel G. Carvalho and Andrea Geralico, *Scalar field self-force effects on a particle orbiting a Reissner-Nordström black hole*, Phys. Rev. D **94**, 124028 (2016) [arXiv:1610.02235].

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FOREWORD

> If you don't know, the thing to do is not to get scared, but to learn.
>
> Ayn Rand

Although I have always been interested in physics and the mysterious manner in which nature works and presents itself for us – I credit my father and my brother for that – my academic life was, and still is, mathematically inclined for one reason or another. Long story short, I've chosen to study pure mathematics instead of any natural or applied science, but I always kept myself curious about physical applications of the pure mathematics I was learning (such as differential geometry, analysis and topology). When the opportunity to study physics overseas presented itself, I felt obligated to embrace it or otherwise I would never get in touch with physics in my life.

The present manuscript is the summary of a three years long expedition of a mathematician trying to achieve a Ph.D degree in physics/astrophysics, and I must say it was not an easy task to catch up with the numerous amount of concepts and physical theories, both old and new. Nevertheless, by working with the right people inside the right environment, it was possible to write a thesis using some modern techniques in physics, as well as the mathematical formalism I am used to and that was missing in the literature – at least for the first part of this manuscript. That being said, the present manuscript has a mathematical flavour from cover to cover, with formal definitions and proofs of the nontrivial results presented here. Furthermore, an effort was made to include examples and remarks about the validity of some concepts and definitions. Every example in the text ends with the symbol □ and every proof ends with the symbol ■, both in the right hand side margin, giving a sign to the reader wanting to skip them in a first look.

The bibliography contains all the research material that I needed to write this manuscript and serves the purpose of being the minimum pre-requisite for the missing parts of this text, such as differential geometry, topology, algebra, general relativity, self-force effects, etc. I've done my best to include in the bibliography my favourite books covering the basic topics as well the most up-to-date articles covering the more advanced ones.

As a final word, I would like to thank you, the reader, for reading the present text and I sincerely hope that you have as much fun reading it than I did learning this material and writing it in the present format.

Gabriel Guimarães Carvalho
Roma, February 2017.

PART I

DISFORMAL TRANSFORMATIONS AND APPLICATIONS

Introduction

The role of geometry in physics has always been discussed and studied. With the emergence of the general theory of relativity in 1915, by A. Einstein, two matters became clear: not only the geometry is part of the reality we are immersed in, but this reality is also affected by it. In the case of the general theory of relativity, the "force" we call gravity is nothing more than a manifestation of the space-time geometry, that becomes curved in the presence of matter and energy, and affects the way one measures distances and time. It is because of the important role that geometry plays in this modern era of physics that the interest in geometrical theories, either in gravitational physics or otherwise [**?**], has increased immensely.

In the nineties, the disformal transformations appeared with some notoriety in the literature through Bekenstein's works [**?**, **?**], where the possibility of adding more than one Riemannian geometry to a geometrical theory of gravity is revisited and discussed in the realm of the so-called Finsler geometries. Since then, investigations of new kinds of symmetry – beyond the well-known external (rotational, translations, boosts, etc) and internal (gauge) ones – under which a given dynamical equation could be invariant have increased. As a consequence, disformal transformations have been used with the aim of explaining some of the current open problems in physics, for instance, MOND (modified Newtonian dynamics) [**?**, **?**], modified dispersion relations in quantum gravity phenomenology [**?**], bimetric theories of gravity [**?**], scalar-tensor theories [**?**], disformal inflation [**?**], chiral symmetry breaking [**?**], anomalous magnetic moment for neutrinos [**?**], analogue models of gravity [**?**, **?**], and others.

The study of new kinds of symmetry associated with equations of motion is crucial in modern physics, since it can elucidate hidden features and helps one to find new nontrivial solutions to the equations involved [**?**]. As investigated in what follows, the same reasoning could be applied to disformal transformations. Furthermore, disformal transformations of a metric tensor are shown to have interesting algebraic and geometric features, making them compelling to exploit also from the mathematical viewpoint.

Summary of part I

For convenience, a detailed structure of the subsequent chapters of the first part of this thesis is given below.

Chapter 1: Section 1.1 starts with the definition of disformal metrics that we are going to use extensively throughout the text, showing that they possess an intrinsic (coordinate independent) definition. Then, in section 1.2, the existence of a group acting on the set of metrics defined on the manifold $\mathscr{M}$ giving rise to disformal metrics is explored. Starting in section 1.3, a new formalism, the CLB (Carvalho-Lobo-Bittencourt) formalism, for disformal metrics is developed in terms of linear operators. Then, the algebraic and geometric features of those operators are investigated in section 1.4. Finally, in section 1.5, an easy algebraic criteria that promptly provides the relation between the light cones of the background and disformal metrics is found. Additionally, as suggested by equation (1.25-I), it is provided two interesting and equivalent ways to interpret causal relations in the context of disformal transformations.

Chapter 2: This chapter covers some physical applications of the theory explained and developed in the previous one. In section 2.1, the disformal invariance of the Dirac equation is discussed and a sufficient condition that keeps a solution for the Dirac equation invariant under disformal transformations of the metric tensor is given. All the mathematical ingredients needed were either explained in the previous chapter or they are introduced as demanded, making this section as self-contained as possible. Finally, section 2.2 makes use of the CLB formalism developed in section 1.3 to provide a mathematical framework to a phenomenological approach to quantum gravity known as rainbow gravity.

1
DISFORMAL METRICS: THEN AND NOW

Вдохновение нужно в геометрии
не меньше, чем в поэзии.

Александр С. Пушкин

The purpose of this chapter is twofold. The first goal is to introduce what is known as a disformal transformation of a metric tensor and to show the existence of a group that acts on the background metric giving rise to the disformal one. Subsequently, a new, and more fundamental, formalism for such metrics is developed and scrutinized.

1.1 The definition of a disformal metric

Our starting point is a space-time $(\mathscr{M}, g)$ – a connected, paracompact, Hausdorff, time-orientable, smooth (C^k, for a conveniently large $k \in \mathbb{N}$) and four-dimensional manifold $\mathscr{M}$ equipped with a pseudo-Riemannian smooth metric g with Lorentzian signature $(+---)$ – and a smooth field Φ (scalar, vector, tensor or spinor) defined on $\mathscr{M}$. Traditionally, on the literature, the disformal transformation is explicitly constructed with the components of the tensors involved in the map $(\mathscr{M}, g, \Phi) \longmapsto (\mathscr{M}, \hat{g}(g, \Phi))$[1] according to

$$\hat{g}^{\mu\nu} = \alpha(\Phi, \nabla\Phi) g^{\mu\nu} + \Sigma^{\mu\nu}(\Phi, \nabla\Phi), \tag{1.1-I}$$

where α is a scalar depending on the field Φ and its derivatives and $\Sigma^{\mu\nu}$ is a rank-2 tensor field also depending on Φ and its derivatives. In principle, this could lead to some problems, since in this case the disformal maps might work easier with the contra-variant components of the metric as given in equation (1.1-I) rather than its inverse, that might be expressed as an infinite sum of terms involving g and Σ (see for instance the reference [**?**] for the spin-2 field theory formulation [2]). The main reason to start with contra-variant components expression for disformal metrics is that it is usually defined in terms of a globally defined vector field constructed with the background metric g and the given field Φ. Although it is possible to construct disformal metrics in which the rank-2 tensor $\Sigma^{\mu\nu}$ is more general, the upcoming analysis is going to be restricted to the aforesaid case, in which the disformal metric is constructed with given vector fields or vector fields constructed directly from a

[1] Formally, it should be written $(\mathscr{M}, g, \Phi) \longmapsto (\hat{\mathscr{M}}, \hat{g}(g, \Phi))$. However, the space-time $(\hat{\mathscr{M}}, \hat{g})$ is a subset of the manifold $\mathscr{M}$ endowed with another metric tensor defined on it, hence the abuse of notation. As is the case for conformal transformations, disformal transformations are not, in general, associated with a diffeomorphism of $\mathscr{M}$ [**?**].

[2] As a matter of fact, for two metrics $\hat{g}$ and g related via $\hat{g}^{\mu\nu} = g^{\mu\nu} + \Sigma^{\mu\nu}$, it can be shown that $\hat{g}_{\mu\nu} = g_{\mu\nu} - \Sigma_{\mu\nu} + \Sigma_{\mu}^{\alpha}\Sigma_{\alpha\nu} + \cdots$.

given field Φ. Thus, in this section, one should find the definition of (vector) disformal transformations of a metric tensor in a coordinate independent manner to show that in fact one can choose any representation – either covariant or contra-variant – for the disformal metric without loss of generality. Before starting with all formal definitions we will need throughout the text let us take a look at one

Example 1.1. Let $(\mathscr{M}, g)$ be a space-time, $\{x^\mu\}$ a local coordinate system around a point $p \in \mathscr{M}$ and consider a vector field with components in this local coordinate system given by $V^\mu = \sum_{\nu=0}^{3} g^{\mu\nu} \frac{\partial \Phi}{\partial x^\nu} \equiv g^{\mu\nu} \frac{\partial \Phi}{\partial x^\nu}$ [3] for some scalar field Φ. Let α and β be any scalars. Then

$$\hat{g}^{\mu\nu} = \alpha g^{\mu\nu} + \beta V^\mu V^\nu, \tag{1.2-I}$$

fits the description of the map associated with equation (1.1-I). □

The overt problem we have in hands now is that equation (1.2-I) does not always represent a metric tensor on $\mathscr{M}$. This problem is addressed in sequence and some conditions for the disformal coefficients α and β are found to guarantee that the disformal metrics we are going to deal with are in accordance with the following

Definition 1.2. A pseudo-Riemannian metric tensor field g, on a smooth manifold $\mathscr{M}$, is a smooth, symmetric, bilinear, nondegenerate map which assigns a real function to pairs of vector fields defined on $\mathscr{M}$, that is

$$\begin{aligned} g: \quad \Gamma(T\mathscr{M}) \otimes \Gamma(T\mathscr{M}) &\longrightarrow \mathscr{F}(\mathscr{M}), \\ (X, Y) &\longmapsto g(X, Y), \end{aligned}$$

$\Gamma(T\mathscr{M})$ denoting the set of smooth sections of the tangent bundle, i.e., vector fields over $\mathscr{M}$, and $\mathscr{F}(\mathscr{M})$ denoting the set of smooth functions defined on $\mathscr{M}$. Additionally, it is assumed that the signature convention is $(+---)$ for pseudo-Riemannian metrics.

By nondegenerate it is meant that, once given a coordinate system $\{x^\mu\}$, the determinant of the matrix formed by the metric components is nonzero at each point of the manifold, namely $\det[g(\partial_\mu, \partial_\nu)]|_p \neq 0$, where ∂_μ represent the elements of the vector basis in the tangent space at each $p \in \mathscr{M}$ and from which we define the metric components to be $g_{\mu\nu} \equiv g(\partial_\mu, \partial_\nu)$. Then, for instance, the metric tensor $\hat{g}$ with inverse $\hat{g}^{\mu\nu}$ given in example 1.1 could be degenerate or have a different signature.

Definition 1.3. A single vector disformal transformation of a given metric g is an application that takes scalar functions α and β, the metric tensor g, and a globally defined time-like[4] vector field V in a space-time and associates them to another metric tensor $\hat{g}$ according to

$$\hat{g}(*, \cdot) = \alpha\, g(*, \cdot) + \frac{\beta}{g(V,V)}\, g(V, *) \otimes g(V, \cdot), \tag{1.3-I}$$

where $*$ and $\cdot$ represent the placements of arbitrary vector fields on which the tensors g and $\hat{g}$ act.

[3] This expression introduces the summation convention used in this text, called the Einstein summation convention, in which indices appearing both as subscripts and superscripts are summed over.

[4] It could be extended for light-like vectors as discussed in the subsection 2.1.6 below or even tensor fields as discussed in reference [**?**].

The existence of a nonvanishing time-like vector field on $\mathscr{M}$ is guaranteed when one considers time-orientable space-times, which is reasonable from the physical viewpoint. If $\mathscr{M}$ is not time-orientable, there still exists a time-orientable twofold covering of $\mathscr{M}$, where this formalism may lie (cf. the brief discussion in reference [**?**]). Taking mathematical convenience and physical reasonableness into account, one shall henceforth consider only time-orientable space-times as defined previously. It shall be proven soon that the coefficients of the disformal metric must satisfy $\alpha > 0$ and $\alpha + \beta > 0$ throughout the manifold. This requirement guarantees that the constructed $\hat{g}$ is a pseudo-Riemannian metric. The scalars α and β are not necessarily functions depending only on points of the manifold, but rather arbitrary functions that could also have a functional dependence on V and its derivatives.

One can write down explicitly the components of the disformal metric in a given local coordinate system in its covariant and contra-variant versions respectively as

$$\hat{g}_{\mu\nu} = \alpha g_{\mu\nu} + \frac{\beta}{V^2} V_\mu V_\nu, \tag{1.4-I}$$

$$\hat{g}^{\mu\nu} = \frac{1}{\alpha} g^{\mu\nu} - \frac{\beta}{\alpha(\alpha+\beta)} \frac{V^\mu V^\nu}{V^2}, \tag{1.5-I}$$

where, $V_\mu \equiv g_{\mu\nu} V^\nu$ and $V^2 \equiv g_{\mu\nu} V^\mu V^\nu$. It is straightforward to verify that $\hat{g}_{\mu\nu}\hat{g}^{\nu\sigma} = \delta^\sigma_\mu$, where δ^σ_μ denotes Kronecker's delta. Let us now take a look in a special case of a two vectors disformal transformation that will be needed in the first part of the next chapter. It will be shown that in this case some conditions are necessary to ensure that the new tensor $\hat{g}$ is a metric tensor. These conditions naturally cover the case of a single vector disformal transformations that were mentioned before.

Proposition 1.4. Let $(\mathscr{M}, g)$ be a space-time. Consider also smooth vector fields V and U such that $g(V,V) = -g(U,U) = V^2 > 0$ and $g(V,U) = 0$ and regular scalar fields α, β, γ and δ depending functionally on V and U, such that α is positive and $\Upsilon = (\alpha+\beta)(\alpha+\gamma) + \delta^2$ is never null. Then, the map

$$\begin{aligned} \hat{g}: \quad \Gamma(T\mathscr{M}) \otimes \Gamma(T\mathscr{M}) &\longrightarrow \mathscr{F}(\mathscr{M}), \\ (X,Y) &\longmapsto \hat{g}(X,Y) \end{aligned} \tag{1.6-I}$$

with

$$\begin{aligned} \hat{g}(*,\cdot) &= \alpha\, g(*,\cdot) + \frac{\beta}{V^2} g(V,*) \otimes g(V,\cdot) - \frac{\gamma}{V^2} g(U,*) \otimes g(U,\cdot) \\ &+ \frac{\delta}{V^2} [g(V,*) \otimes g(U,\cdot) + g(U,*) \otimes g(V,\cdot)], \end{aligned} \tag{1.7-I}$$

where $*$ and $\cdot$ represent the placements of arbitrary vector fields on which the metric tensors g and $\hat{g}$ act, is also a pseudo-Riemannian metric for $\mathscr{M}$. Furthermore, if one requires $\alpha + \beta > 0$ and $\alpha + \gamma > 0$, implying that $\Upsilon > 0$, the signatures of $\hat{g}$ and g are the same.

Proof. First, let us find a relation between the determinants of the disformal metric and the background one. Because the determinant of the metric tensor is a *scalar density* [**?**], the value of the determinant of both $\hat{g}$ and g will change if one changes coordinates (in particular, if the basis is changed). Notwithstanding, the relation between the determinants is still preserved since equation (1.7-I) is a tensor identity and the determinant of both sides will be scaled by the same amount when the basis is changed. Fixing a point $p \in \mathscr{M}$ and an orthonormal basis (orthonormal with respect to the metric g) $\mathscr{B} = \{e_0, e_1, e_2, e_3\}$, such that $V = \sqrt{V^2}e_0$ and $U = \sqrt{V^2}e_1$, the disformal metric will have components given, in matrix notation, by

$$\hat{g} = \begin{pmatrix} \alpha+\beta & -\delta & 0 & 0 \\ -\delta & -(\alpha+\gamma) & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha \end{pmatrix}. \tag{1.8-I}$$

Evaluating the determinant yields

$$\det(\hat{g}) = -\alpha^2[(\alpha+\beta)(\alpha+\gamma)+\delta^2] = \alpha^2 \Upsilon \det(g),$$

since $\det(g) = -1$ in the basis $\mathscr{B}$. The authentic relation between the determinants, which holds in any basis, is then given by

$$\det(\hat{g}) = \alpha^2 \Upsilon \det(g). \tag{1.9-I}$$

From this, the determinant of the disformal metric is nondegenerate under the condition $\Upsilon \neq 0$, since $\alpha > 0$. The requirements for preserving the signature could be obtained if one requires that the basis $\mathscr{B}$ has one time-like (e_0) and three space-like vectors ($e_i, i = 1,2,3$). Inspecting the matrix 1.8-I, these conditions are satisfied if $\alpha+\beta > 0$ and $\alpha+\gamma > 0$. ■

Corollary 1.5. The space-time $(\mathscr{M}, \hat{g})$, with a disformal metric as the one constructed above is time-orientable.

Proof. If $\alpha+\beta > 0$, the causal character of V is kept invariant. Since V is initially time-like in the metric g, it will remain time-like in the metric $\hat{g}$. The existence of a globally defined time-like vector fields characterizes $(\mathscr{M}, \hat{g})$ as being time-orientable [**?**]. ■

Consider the operator defined by $\hat{g}^{\mu}{}_{\nu} \equiv \hat{g}_{\nu\alpha} g^{\alpha\mu}$ in the particular case of a single vector disformal transformation given by equation (1.4-I). The reader can verify that the eigenvalue $\lambda = \alpha+\beta$ is associated with the eigenvector V and three eigenvalues equal to α with eigenvectors lying, for every $p \in \mathscr{M}$, on the subspace $V_p^{\perp} \equiv \{W \in T_p\mathscr{M} : g(W,V)|_p = 0\}$, where $T_p\mathscr{M}$ denotes the tangent space of $\mathscr{M}$ at p. This algebraic remark is going to motivate the introduction of the disformal operators defined in section 1.3.

Additionally, from Levi-Civita's theorem [**?**], a given metric tensor on the manifold possesses a unique affine connection, called the Levi-Civita connection (denoted by ∇), satisfying the requirements of symmetry (i.e., $\nabla_X Y - \nabla_Y X = [X,Y]$) and compatibility with the given metric (i.e., $\nabla g = 0$). Therefore, for the metric tensors g and $\hat{g}$ on $\mathscr{M}$ one can relate the Levi-Civita connections ∇ and $\hat{\nabla}$, respectively and unequivocally. Finally, from the proposition above one sees that the attribution of space-time components to the metrics associated with the disformal map as we defined does not lead to contradictions because the map itself has an intrinsic definition. For this reason, one can use in practice either the contra-variant or the covariant components of both tensors, according to convenience.

1.2 On the disformal group structure

Let us recall some algebraic definitions concerning group theory that can be found in [?], for instance. We start with the most fundamental concept, the one of a *monoid*. A monoid is a set G endowed with a binary operation $\cdot : G \times G \longrightarrow G$, denoted by $(G, \cdot)$, satisfying the following properties:

1. Existence of the identity element: there exists an element $e \in G$ such that $e \cdot g = g \cdot e = g$ for all $g \in G$.

2. Associativity : given $g_1, g_2, g_3 \in G$ we have $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$.

If every element of a monoid has an inverse element the set G with the operation $\cdot$ is called a *group*. In the special case that $g_1 \cdot g_2 = g_2 \cdot g_1$ for all $g_1, g_2 \in G$ the group $(G, \cdot)$ is called *commutative* or *Abelian*. The group $(G_1, \cdot)$ is said to be homomorphic to the group $(G_2, *)$ if there exists a function $\phi : (G_1, \cdot) \longrightarrow (G_2, *)$ such that $\phi(a \cdot b) = \phi(a) * \phi(b)$ for all $a, b \in G_1$. In that case, ϕ is called a homomorphism between $(G_1, \cdot)$ and $(G_2, *)$. If ϕ is bijective and its inverse ϕ^{-1} is also a homomorphism, the groups are said to be isomorphic. The main idea of considering isomorphic groups is that they are algebraically indistinguishable.

Definition 1.6. If $(G, \cdot)$ is a group and X is a set, then a (left) *group action* σ of $(G, \cdot)$ on X is a function

$$\begin{aligned} \sigma : G \times X &\longrightarrow X \\ (g, x) &\longmapsto \sigma(g, x) = g.x \end{aligned} \tag{1.10-I}$$

satisfying the following axioms:

1. Identity: $e.x = x$ for all $x \in X$, where $e \in G$ is the identity element.

2. Compatibility: $(g \cdot h).x = g(h.x)$, for all $g, h \in G$.

Example 1.7. The set $\mathbb{K}_4 = \{e, a, b, c\}$ with the binary operation $*$ defined by the table

$$\begin{array}{c|cccc} * & e & a & b & c \\ \hline e & e & a & b & c \\ a & a & e & c & b \\ b & b & c & e & a \\ c & c & b & a & e \end{array} \tag{1.11-I}$$

is an Abelian group called the Klein four-group. It is not difficult to check that the Klein four-group above is isomorphic to the group $(\mathbb{Z}_2 \oplus \mathbb{Z}_2, +)$ defined by the table

$$\begin{array}{c|cccc} + & (0,0) & (1,0) & (0,1) & (1,1) \\ \hline (0,0) & (0,0) & (1,0) & (0,1) & (1,1) \\ (1,0) & (1,0) & (0,0) & (1,1) & (0,1) \\ (0,1) & (0,1) & (1,1) & (0,0) & (1,0) \\ (1,1) & (1,1) & (0,1) & (1,0) & (0,0) \end{array} \tag{1.12-I}$$

once we identify the elements $\{e, a, b, c\}$ and $\{(0,0), (1,0), (0,1), (1,1)\}$, respectively. These are examples of *discrete (or finite) groups*. □

Now let us investigate the existence of a group structure behind the disformal transformations of a metric tensor. Suppose, to begin with, that we have fixed a space-time and a time-like vector field V. The resulting disformal metric has covariant and contra-variant components given, respectively, by equations (1.4-I) and (1.5-I). Let us then start identifying

$$\hat{g}_{\mu\nu} = \alpha g_{\mu\nu} + \frac{\beta}{V^2} V_\mu V_\nu \longleftrightarrow \lfloor \alpha, \beta \rfloor g, \tag{1.13-I}$$

and use this to define the action of $\lfloor \alpha, \beta \rfloor$ on the metric g. We shall denote the set of all such $\lfloor \alpha, \beta \rfloor$ by $\mathscr{G}$. That is,

$$\mathscr{G} = \left\{ \lfloor \alpha, \beta \rfloor : \ \alpha > 0 \text{ and } \alpha + \beta > 0 \right\}. \tag{1.14-I}$$

The set $\mathscr{G}$ with the operation $\star$ defined by

$$(\lfloor \alpha, \beta \rfloor \star \lfloor \alpha', \beta' \rfloor) g \equiv \lfloor \alpha\alpha', \alpha'\beta + \beta'\beta + \alpha\beta' \rfloor g \tag{1.15-I}$$

is a group. Before proving this statement, let us explore the meaning of this composition law: when we first evaluate $\lfloor \alpha', \beta' \rfloor g$, we obtain equation (1.13-I) with primed α and β, i.e., the covariant components of a new metric tensor $\hat{g}$. Then, the composition law is defined in such a way that the compatibility condition

$$(\lfloor \alpha, \beta \rfloor \star \lfloor \alpha', \beta' \rfloor) g = \lfloor \alpha, \beta \rfloor \hat{g} \tag{1.16-I}$$

holds. The proof that $(\mathscr{G}, \star)$ is a group is straightforward and it is shown below:

1. Closure: Given $\lfloor \alpha, \beta \rfloor, \lfloor \alpha', \beta' \rfloor \in \mathscr{G}$, using the composition law given by equation (1.15-I) we have that $\alpha'' = \alpha\alpha' > 0$ and $\alpha'' + \beta'' = \alpha\alpha' + \alpha'\beta + \beta'\beta + \alpha\beta' = (\alpha+\beta)(\alpha' + \beta') > 0$, implying that $\lfloor \alpha\alpha', \alpha'\beta + \beta'\beta + \alpha\beta' \rfloor \in \mathscr{G}$.

2. Existence of the identity element: there exists an element $\lfloor 1, 0 \rfloor \in \mathscr{G}$ such that for all $\lfloor \alpha, \beta \rfloor \in \mathscr{G}$ holds

$$\lfloor 1, 0 \rfloor \star \lfloor \alpha, \beta \rfloor = \lfloor \alpha, \beta \rfloor \star \lfloor 1, 0 \rfloor = \lfloor \alpha, \beta \rfloor. \tag{1.17-I}$$

3. Existence of the inverse element: for each $\lfloor \alpha, \beta \rfloor \in \mathscr{G}$ there exists another element $\left\lfloor \frac{1}{\alpha}, -\frac{\beta}{\alpha(\alpha+\beta)} \right\rfloor \in \mathscr{G}$ such that

$$\left\lfloor \frac{1}{\alpha}, -\frac{\beta}{\alpha(\alpha+\beta)} \right\rfloor \star \lfloor \alpha, \beta \rfloor = \lfloor \alpha, \beta \rfloor \star \left\lfloor \frac{1}{\alpha}, -\frac{\beta}{\alpha(\alpha+\beta)} \right\rfloor = \lfloor 1, 0 \rfloor. \tag{1.18-I}$$

4. Associativity: if we consider $\lfloor \alpha, \beta \rfloor$, $\lfloor \alpha', \beta' \rfloor, \lfloor \alpha'', \beta'' \rfloor \in \mathscr{G}$, we have

$$\begin{aligned} \lfloor \alpha'', \beta'' \rfloor \star \left(\lfloor \alpha', \beta' \rfloor \star \lfloor \alpha, \beta \rfloor \right) &= \lfloor \alpha'', \beta'' \rfloor \star \lfloor \alpha'\alpha, \alpha'\beta + \beta'\beta + \alpha\beta' \rfloor \\ &= \lfloor \alpha''\alpha'\alpha, \alpha''(\alpha'\beta + \beta'\beta + \alpha\beta') \\ &+ \beta''(\alpha'\beta + \beta'\beta + \alpha\beta') + \beta''\alpha\alpha' \rfloor \qquad (1.19\text{-I}) \\ &= \lfloor \alpha''\alpha', \alpha''\beta' + \beta''\beta' + \beta''\alpha' \rfloor \star \lfloor \alpha, \beta \rfloor \\ &= \left(\lfloor \alpha'', \beta'' \rfloor \star \lfloor \alpha', \beta' \rfloor \right) \star \lfloor \alpha, \beta \rfloor. \end{aligned}$$

Besides, it is direct to check that the group $(\mathscr{G},\star)$ is in fact Abelian.

On the other hand, using equation (1.5-I), we find that the inverse of $\lfloor\alpha,\beta\rfloor g$ is written as

$$(\lfloor\alpha,\beta\rfloor g)^{-1} = \frac{1}{\alpha}g^{\mu\nu} - \frac{\beta}{\alpha(\alpha+\beta)}\frac{V^{\mu}V^{\nu}}{V^2}. \tag{1.20-I}$$

Note that the disformal coefficients of the inverse of $\hat{g}$ coincides with the parameters of the inverse element of $\lfloor\alpha,\beta\rfloor$ in $\mathscr{G}$, which motivates the definition of another operation $\lceil\alpha,\beta\rceil$ acting on g according to

$$\lceil\alpha,\beta\rceil g = \frac{1}{\alpha}g^{\mu\nu} - \frac{\beta}{\alpha(\alpha+\beta)}\frac{V^{\mu}V^{\nu}}{V^2}, \tag{1.21-I}$$

and an analogous composition law $\odot$ given by

$$(\lceil\alpha,\beta\rceil \odot \lceil\alpha',\beta'\rceil)g \equiv \lceil\alpha\alpha',\alpha'\beta+\beta'\beta+\alpha\beta'\rceil g. \tag{1.22-I}$$

It should be clear in the notation that $\lceil\alpha,\beta\rceil g$ and $\lfloor\alpha,\beta\rfloor g$ are related to the contravariant (ceil notation) and the covariant (floor notation) representations of the metric $\hat{g}$ in a given coordinate system, respectively. As the reader should check, the set $\mathscr{H} = \{\lceil\alpha,\beta\rceil :\ \alpha > 0 \text{ and } \alpha+\beta > 0\}$ with the operation $\odot$ acting on g according to (1.22-I) is also an Abelian group and $((\lfloor\alpha,\beta\rfloor \star \lfloor\alpha',\beta'\rfloor)g)^{-1} = (\lceil\alpha,\beta\rceil \odot \lceil\alpha',\beta'\rceil)g$. In other words, the inverse of the composition of two disformal transformations is the same as the composition of the inverses of those disformal transformations. This is a crucial fact that allow us, when dealing with disformal transformations, to define with no ambiguity $\hat{g}_{\mu\nu}$ from the given $\hat{g}^{\mu\nu}$, and vice-versa. Furthermore, the application

$$\begin{aligned} \phi : (\mathscr{G},\star) &\longrightarrow (\mathscr{H},\odot) \\ \lfloor\alpha,\beta\rfloor &\longmapsto \lceil\alpha,\beta\rceil, \end{aligned}$$

satisfies

$$\phi(\lfloor\alpha,\beta\rfloor \star \lfloor\alpha',\beta'\rfloor) = \phi(\lfloor\alpha,\beta\rfloor)\odot\phi(\lfloor\alpha',\beta'\rfloor), \tag{1.23-I}$$

hence it is a group homomorphism. In fact, ϕ is a group isomorphism, as expected. The reader should have observed that, given a local coordinate system, the groups $(\mathscr{G},\star)$ and $(\mathscr{H},\odot)$ act on the space of metrics that could be defined on $\mathscr{M}$ and return, respectively, the covariant and contra-variant components of the resulting metric in that coordinate system. Since the groups $(\mathscr{G},\star)$ and $(\mathscr{H},\odot)$ are isomorphic, i.e., algebraically indistinguishable, one could gather all the information above in a single group $(G,\bullet)$, called the disformal group, acting on the set of metrics defined on $\mathscr{M}$ yielding the disformal one according to

$$\begin{aligned} \sigma : G\times \mathrm{Met}(\mathscr{M}) &\longrightarrow \mathrm{Met}(\mathscr{M}) \\ (\llbracket\alpha,\beta\rrbracket, g) &\longmapsto \hat{g}, \end{aligned} \tag{1.24-I}$$

in which $G = \left\{\llbracket\alpha,\beta\rrbracket :\ \alpha > 0 \text{ and } \alpha+\beta > 0\right\}$, the operation $\bullet$ is defined to be equal to $\star$ (or $\odot$) and $\mathrm{Met}(\mathscr{M})$ denotes the set of pseudo-Riemannian metrics defined on

$\mathcal{M}$. Furthermore, in contrast to the groups in example 1.7, the disformal group is an example of a *continuous (or infinite) group*.

The existence of a group acting on the background metric giving rise to the disformal one suggests the occurrence of something in a fundamental level. The new formalism developed and studied in the next sections aims at proving that this is exactly the case.

1.3 A new formalism to describe disformal metrics - The CLB formalism

Because of the group structure, one can consider that α and β are fixed, even though these functions could have been obtained via multiples disformal transformations. Hereafter, let us fix a space-time $(\mathcal{M}, g)$, a nonvanishing time-like vector field $V \in \Gamma(T\mathcal{M})$ and consider also fixed any two scalars α and β satisfying the conditions $\alpha > 0$ and $\alpha + \beta > 0$. Then, the map summarized in equation (1.3-I), henceforth denoted by $(\mathcal{M}, g, \upharpoonleft_V, \alpha, \beta) \mapsto (\mathcal{M}, \hat{g})$, can be equivalently described as an action on the tangent space in each point of the manifold by the following definition:

$$\hat{g}(Y,Z) = g\left(\overrightarrow{D}(Y), \overrightarrow{D}(Z)\right), \tag{1.25-I}$$

where, for any $X \in \Gamma(T\mathcal{M})$, we have

$$\overrightarrow{D}(X) \equiv \sqrt{\alpha+\beta}X_{\parallel} + \sqrt{\alpha}X_{\perp}, \tag{1.26-I}$$

with

$$X_{\parallel} \equiv \frac{g(X,V)}{g(V,V)}V, \quad \text{and} \quad X_{\perp} \equiv X - \frac{g(X,V)}{g(V,V)}V, \tag{1.27-I}$$

where $X_{\parallel}$ is the projection of X onto V and $X_{\perp}$ is the projection onto the orthogonal complement of V, such that $X = X_{\parallel} + X_{\perp}$. So, instead of working with a disformal transformation of the metric, one can define the map

$$\begin{array}{rccl} \overrightarrow{D} : & \Gamma(T\mathcal{M}) & \longrightarrow & \Gamma(T\mathcal{M}) \\ & X & \longmapsto & \hat{X} = \overrightarrow{D}(X) \end{array} \tag{1.28-I}$$

to deform vectors and recover the disformal metric $\hat{g}$. We shall denote by $\hat{X}$ the disformal vector related to X through the action of $\overrightarrow{D}$. Clearly, such a map is $\mathcal{F}(\mathcal{M})$−linear, i.e., $\overrightarrow{D}(\gamma X + Y) = \gamma\overrightarrow{D}(X) + \overrightarrow{D}(Y)$ for any scalar function γ and vector fields X and Y, and from this, $\overrightarrow{D}$ defines a mixed rank-2 tensor field on $\mathcal{M}$ given by

$$\begin{array}{rcl} \overrightarrow{D} : \Gamma(T^*\mathcal{M}) \otimes \Gamma(T\mathcal{M}) & \longrightarrow & \mathcal{F}(\mathcal{M}) \\ (\theta, X) & \longmapsto & \overrightarrow{D}(\theta, X) \equiv \theta\left(\overrightarrow{D}(X)\right), \end{array} \tag{1.29-I}$$

where θ is an arbitrary covector field [5]. Furthermore, $\overrightarrow{D}$ can be seen as a linear transformation on the tangent space $T_p\mathcal{M}$ for each $p \in \mathcal{M}$ and, provided $\alpha > 0$ and $\alpha + \beta > 0$, a linear isomorphism.

[5] It is a common practice, as seen in [**?**], to use the same symbol for the operator defined in equation (1.28-I) and the corresponding mixed tensor 1.29-I.

Evidently, the operator $\overrightarrow{D}$ satisfying equation (1.25-I) is not unique. In fact, all possible operators of the form $\overrightarrow{D}(X) = F_1(\alpha,\beta)X_\parallel + F_2(\alpha,\beta)X_\perp$ satisfying (1.25-I) are fourfold degenerate:

$$\overrightarrow{D}_{\{\pm,\pm\}}(X) = \pm\sqrt{\alpha+\beta}X_\parallel \pm \sqrt{\alpha}X_\perp. \tag{1.30-I}$$

This degeneracy in the choice of $\overrightarrow{D}$ could, in principle, lead to ambiguities in the definition of a disformal metric by a disformal operator according to equation (1.25-I). For reasons that shall subsequently become clear, we shall define the disformal operator as $\overrightarrow{D}(X) = \overrightarrow{D}_{\{+,+\}}(X)$, for every vector field X, and prove the uniqueness of this class afterwards.

1.3.1 The disformal cometric

In chapter 2, concerning the physical applications of our analysis, it will be important to use the disformal cometric instead of the disformal metric. Therefore, we briefly elaborate upon how one can analogously define a disformal operator acting on 1−forms (covectors) to recover the information contained in equation (1.5-I).

For each vector X in a manifold $\mathscr{M}$ endowed with a metric tensor g, there exists its unique metric dual $\widetilde{X} \equiv g(X,*)$. Hence the dual is a linear map $\widetilde{X} : \Gamma(T\mathscr{M}) \to \mathscr{F}(\mathscr{M})$, i.e., $\widetilde{X} \in \Gamma(T^*\mathscr{M})$. In this way, the cometric h is defined by a bilinear, symmetric, and nondegenerate map:

$$\begin{aligned} h : \Gamma(T^*\mathscr{M}) \otimes \Gamma(T^*\mathscr{M}) &\longrightarrow \mathscr{F}(\mathscr{M}) \\ (\widetilde{X},\widetilde{Y}) &\longmapsto h(\widetilde{X},\widetilde{Y}) = g(X,Y). \end{aligned} \tag{1.31-I}$$

In a given local coordinate system $\{x^\mu\}$, the cometric has components $g^{\mu\nu} = h(dx^\mu, dx^\nu)$, i.e., it is the contra-variant components of the metric. From equation (1.5-I), one can define the disformal cometric intrinsically as

$$\hat{h}(*,\cdot) = \frac{1}{\alpha}h(*,\cdot) - \frac{\beta}{\alpha(\alpha+\beta)}\frac{h(\widetilde{V},*)\otimes h(\widetilde{V},\cdot)}{h(\widetilde{V},\widetilde{V})}, \tag{1.32-I}$$

where $*$ and $\cdot$ represent here the placements of arbitrary covector fields on which the tensors h and $\hat{h}$ act. Analogously to what was done in the beginning of section 1.3, one can define the disformal covector $\hat{\omega}$ associated with ω by the application of a linear map $\widetilde{D} : \Gamma(T^*\mathscr{M}) \to \Gamma(T^*\mathscr{M})$, from which one writes the disformal cometric as

$$\hat{h}(\omega,\eta) = h\left(\widetilde{D}(\omega), \widetilde{D}(\eta)\right), \tag{1.33-I}$$

with $\widetilde{D}$ given by

$$\widetilde{D}(\omega) \equiv \frac{1}{\sqrt{\alpha+\beta}}\omega_\parallel + \frac{1}{\sqrt{\alpha}}\omega_\perp \tag{1.34-I}$$

and again using the decomposition

$$\omega_\parallel \equiv \frac{h(\omega,\widetilde{V})}{V^2}\widetilde{V} \quad \text{and} \quad \omega_\perp \equiv \omega - \frac{h(\omega,\widetilde{V})}{V^2}\widetilde{V}. \tag{1.35-I}$$

Similarly to the covariant disformal operator, we have four possible contra-variant disformal operators. We shall set $\widetilde{D}(\omega) = \widetilde{D}_{\{+,+\}}(\omega)$ to be the disformal operator for elements in $\Gamma(T^*\mathscr{M})$ and again prove the uniqueness of this class.

Remark 1.8. By definition, every space-time has a metric tensor defined on it, but, even though we will always have a metric tensor available, it is important to be aware of the logical status of each mathematical objects we introduce. That being said, it is worth mentioning that the introduction of the disformal operators above has both conceptual and technical advantages. In fact, a metric tensor is an additional structure one can endow a manifold with, and the existence of vector fields and 1−form fields relies only on the differentiable structure of the manifold. Additionally, with a prescription to change vector fields and 1−form fields available, such as the disformal operators, one could define more general disformal structures involving any tensor field defined on a manifold $\mathscr{M}$, with or without a metric tensor defined on it.

1.4 Machinery and uniqueness of the disformal operators

In sections 1.1 and 1.2 one can find algebraic properties concerning disformal metrics as the eigenvalue problem for $\hat{g}^{\mu}{}_{\nu} = \hat{g}_{\nu\alpha}g^{\alpha\mu}$ and the existence of a group acting on metric tensors giving rise to a disformal metric. Indeed, we now show that these metrics can be completely characterized by disformal operators, since they share similar properties. For practical purposes, it is useful to provide a coordinate representation for both $\overrightarrow{D}$ and $\widetilde{D}$. We then start with these coordinate expressions and use them to prove some propositions about the disformal operator, and explore their algebraic and geometric features.

1.4.1 Coordinate expressions

To derive a coordinate expression for the disformal operators $\overrightarrow{D}$ and $\widetilde{D}$, let $\{x^{\mu}\}$ be a local coordinate system, $\{\partial_{\mu}\}$ the tangent vectors associated with the coordinate lines and $\{dx^{\mu}\}$ their duals. Thus,

$$\overrightarrow{D}(\partial_{\nu}) \quad = \quad \sqrt{\alpha}\,\partial_{\nu} + \frac{\sqrt{\alpha+\beta}-\sqrt{\alpha}}{V^2}V_{\nu}V.$$

Applying dx^{μ} to this, one gets the desired expression

$$\mathfrak{D}^{\mu}{}_{\nu} \equiv dx^{\mu}\left(\overrightarrow{D}(\partial_{\nu})\right) = \sqrt{\alpha}\,\delta^{\mu}_{\nu} + \frac{\sqrt{\alpha+\beta}-\sqrt{\alpha}}{V^2}V^{\mu}V_{\nu}. \tag{1.36-I}$$

In coordinates, we thus have $\hat{X}^{\mu} = \mathfrak{D}^{\mu}{}_{\nu}X^{\nu}$. Analogously, we get

$$\mathscr{D}_{\nu}{}^{\mu} \equiv \partial_{\nu}\left(\widetilde{D}(dx^{\mu})\right) = \frac{1}{\sqrt{\alpha}}\delta^{\mu}_{\nu} + \left(\frac{1}{\sqrt{\alpha+\beta}} - \frac{1}{\sqrt{\alpha}}\right)\frac{V^{\mu}V_{\nu}}{V^2}, \tag{1.37-I}$$

when defining $\hat{\omega}_{\mu} = \mathscr{D}_{\mu}{}^{\nu}\omega_{\nu}$. It should be remarked that with our definitions, vectors (covectors) transform upon the action of $\overrightarrow{D}$ $(\widetilde{D})$. Although it is possible to transform covectors (vectors) by means of $\mathfrak{D}^{\mu}{}_{\nu}$ $(\mathscr{D}_{\nu}{}^{\mu})$ this shall not be of our general interest here.

With the coordinate expressions for $\overrightarrow{D}$ and $\widetilde{D}$ at our disposal, we state the following

Proposition 1.9. $\mathfrak{D}^{\mu}{}_{\nu}$ and $\mathscr{D}_{\nu}{}^{\mu}$ satisfy

$$\mathfrak{D}^{\mu}{}_{\sigma}\mathscr{D}_{\nu}{}^{\sigma} = \delta^{\mu}_{\nu}, \tag{1.38-I}$$

and hence act as mutual inverses.

Proof. This is straightforward from equations (1.36-I) and (1.37-I). ∎

Using equations (1.25-I) and (1.33-I) and the coordinate expressions for $\mathfrak{D}^{\mu}{}_{\nu}$ and $\mathscr{D}_{\nu}{}^{\mu}$, it is easy to show the following

Proposition 1.10. The diagrams

$$\begin{tikzcd} \Gamma(T\mathscr{M}) \arrow[r, "\vec{D}"] \arrow[dr, "\hat{g}"'] & \Gamma(T\mathscr{M}) \arrow[d, "g"] & \Gamma(T^*\mathscr{M}) \arrow[r, "\widetilde{D}"] \arrow[d, "\hat{h}"'] & \Gamma(T^*\mathscr{M}) \arrow[dl, "h"] \\ & \Gamma(T^*\mathscr{M}) & \Gamma(T\mathscr{M}) & \end{tikzcd} \tag{1.39-I}$$

are not commutative, i.e., $g(\vec{D}(V),\cdot) \neq \hat{g}(V,\cdot)$ and $h(\widetilde{D}(\omega),*) \neq \hat{h}(\omega,*)$.

From the theory of differentiable manifolds it is known that for each point $p \in \mathscr{M}$, the tangent $T_p\mathscr{M}$ and cotangent $T^*_p\mathscr{M}$ spaces of a differentiable manifold $\mathscr{M}$ are naturally isomorphic linear spaces, although this isomorphism is basis dependent. In the presence of a metric tensor on $\mathscr{M}$, there is a canonical (basis independent) isomorphism between $T_p\mathscr{M}$ and $T^*_p\mathscr{M}$, namely, for a given vector $X \in T_p\mathscr{M}$, there is a unique $\omega \in T^*_p\mathscr{M}$ satisfying $g(X,\cdot) = \omega$. Since we are now dealing with a manifold endowed with two metric tensors, the proposition 1.10 indicates an ambiguity when taking duals since the $g-$dual of a disformal vector is not the $\hat{g}-$dual of the vector. This ambiguity is always present when the manifold under consideration has more than one metric tensor. Therefore, it is important to distinguish which metric tensor is being used when raising and lowering indices. One shall deal with this problem by explicitly writing the metric in all formulas in which indices are raised or lowered.

There exists, however, a commutative manner to deform vectors and covectors and take their duals. It is not difficult to show that the diagram

$$\begin{tikzcd} \Gamma(T\mathscr{M}) \arrow[r, "\vec{D}"] \arrow[d, "\hat{g}"'] & \Gamma(T\mathscr{M}) \arrow[d, "g"] \\ \Gamma(T^*\mathscr{M}) \arrow[r, "\widetilde{D}"'] & \Gamma(T^*\mathscr{M}) \end{tikzcd} \tag{1.40-I}$$

is *commutative*. In fact, we could have started with equation (1.25-I) and the definition of $\vec{D}$ given by equation (1.26-I) and then defined $\widetilde{D}$ to be the only operator able to make 1.40-I a commutative diagram. In doing so, one can recover the coordinate expression for $\widetilde{D}$ and the other properties associated with it. For completeness, it should be stressed that if the direction of the arrows labeled by g and $\hat{g}$ is reversed, and g and $\hat{g}$ are replaced by their inverses, the diagram is also commutative. Thus, one can start with the cometric and the definition of $\widetilde{D}$ to define $\vec{D}$ and everything else in terms of it.

1.4.2 Disformal group structure revisited

In this section we revisit the discussion incepted in section 1.2, but now in the context of disformal operators. Let $\overrightarrow{D}_i$ be disformal operators with disformal parameters α_i and β_i, for $i = 1,2$, given by

$$\overrightarrow{D}_1(\cdot) = \sqrt{\alpha_1+\beta_1}(\cdot)_\parallel + \sqrt{\alpha_1}(\cdot)_\perp, \tag{1.41-I}$$

$$\overrightarrow{D}_2(\cdot) = \sqrt{\alpha_2+\beta_2}(\cdot)_\parallel + \sqrt{\alpha_2}(\cdot)_\perp. \tag{1.42-I}$$

Since the action of $\overrightarrow{D}_i$ on a vector field is also a vector field, it is easy to verify that for any vector field X the following holds:

$$\overrightarrow{D}_1\left(\overrightarrow{D}_2(X)\right) = \overrightarrow{D}_2\left(\overrightarrow{D}_1(X)\right) = \sqrt{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}X_\parallel + \sqrt{\alpha_1\alpha_2}X_\perp. \tag{1.43-I}$$

One could use this equation to define the composition of two disformal operators as

$$\left(\overrightarrow{D}_1 \bullet \overrightarrow{D}_2\right)(X) = \left(\overrightarrow{D}_2 \bullet \overrightarrow{D}_1\right)(X) = \sqrt{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}X_\parallel + \sqrt{\alpha_1\alpha_2}X_\perp. \tag{1.44-I}$$

One can also verify that the set of all disformal operators with the composition law given above is closed. It means that the composition of two disformal operators is itself another disformal operator. The associativity and commutativity are easily verified, characterizing an Abelian group structure for the set of disformal operators, where the identity operator has parameters $\alpha = 1$ and $\beta = 0$ and the inverse of an operator with parameters α and β has parameters $\alpha' = \alpha^{-1}$ and $\beta' = -\beta[\alpha(\alpha+\beta)]^{-1}$. A similar result is obtained for $\widetilde{D}$, since they share the same structure.

As it was shown in section 1.2, there is a group structure associated with disformal metrics. Comparing the composition law in equation (1.44-I) with the approach presented in section 1.2, it is neater and more elegant if one deals with operators instead of the abstract group action. Finally, it should be pointed that particularly interesting examples of disformal subgroups take place when all conformal coefficients are equal to 1 – which renders disformal metrics similar to those from the spin−2 field theory formulation, but with finite inverse metric [**?**] – and the cases in which the disformal coefficients are zero ($\beta's = 0$), coinciding with the usual conformal group.

1.4.3 Uniqueness of the disformal operator

It has been shown that there is an Abelian group acting on the space of metrics on $\mathscr{M}$ giving rise to the disformal ones. Besides, we have seen that $\overrightarrow{D}$ and $\widetilde{D}$ also satisfy an Abelian group structure, and in equation (1.25-I) we proposed that a disformal metric arises when we deform vectors and use the background metric. So, if we want to characterize the disformal metric in terms of a disformal operator, once $\overrightarrow{D}$ is defined, $\hat{g}$ must inherit its properties. Recalling that there are four possible disformal operators satisfying equation (1.25-I) and that we have claimed $\overrightarrow{D}_{\{+,+\}}$ is unique in a certain sense, we then state and prove the following

Theorem 1.11. $\overrightarrow{D}_{\{+,+\}}$ is the only class of disformal operator satisfying equation (1.25-I) and the disformal group structure. The same holds for the disformal cometric and the class $\widetilde{D}_{\{+,+\}}$.

Proof. Consider the set $S = \left\{\overrightarrow{D}_{\{\pm,\pm\}}\right\}$ of all admissible classes of disformal operators given by equation (1.30-I) with a composition law ($\bullet$) derived from the left hand side of (1.43-I). By admissible we mean those operators with disformal parameters (α's and β's) that satisfy $\alpha > 0$ and $\alpha + \beta > 0$. For example,

$$\begin{aligned}\left(\overrightarrow{D}^1_{\{-,+\}} \bullet \overrightarrow{D}^2_{\{+,-\}}\right)(X) &= \left(\overrightarrow{D}^2_{\{+,-\}} \bullet \overrightarrow{D}^1_{\{-,+\}}\right)(X) = \overrightarrow{D}_{\{-,-\}}(X) \\ &= -\sqrt{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}X_{\parallel} - \sqrt{\alpha_1\alpha_2}X_{\perp},\end{aligned}$$

for any $X \in \Gamma(T\mathscr{M})$, where the disformal parameters are $\alpha' = \alpha_1\alpha_2$ and $\beta' = \alpha_1\beta_2 + \beta_1\alpha_2 + \beta_1\beta_2$. It is easy to ascertain that the sign algebra of operators in $(S,\bullet)$ is isomorphic to $(\mathbb{Z}_2 \oplus \mathbb{Z}_2, +)$ in example 1.7 with identity element $\{+,+\}$. Therefore, the only closed composition of disformal operators (i.e., the only composition between disformal operators resulting in a disformal operator *of the same kind*) holds when they are both from the class represented by $\overrightarrow{D}_{\{+,+\}}$. The proof that the elements of the class $\overrightarrow{D}_{\{+,+\}}$ satisfies a group structure was given in section 1.4.2. The proof for the class $\widetilde{D}_{\{+,+\}}$ is entirely analogous. ∎

A geometrical argument to rule out the other three candidates for the disformal operator (the ones with at least one negative sign) is that, because of the negative sign, they must include a reflection in the direction perpendicular to V and/or a change in direction along with V. Therefore, whenever they are applied an even number of times, a positive sign must appear, implying that the operation is not closed. Thus, this ensures that there is a unique class of disformal operators characterizing the disformal metric.

1.4.4 The disformal operator as the square root of the disformal metric

It was shown that given a local coordinate system $\{x^\mu\}$ the coordinate expression for the disformal operator $\overrightarrow{D}$ takes the form (1.36-I). Lowering the μ index with the background metric g, one gets

$$\mathfrak{D}_{\mu\nu} \equiv g_{\mu\sigma}\mathfrak{D}^{\sigma}{}_{\nu} = \alpha' g_{\mu\nu} + \frac{\beta'}{V^2}V_\mu V_\nu,$$

with $\alpha' = \sqrt{\alpha} > 0$ and $\alpha' + \beta' = \sqrt{\alpha+\beta} > 0$ and, therefore, $\mathfrak{D}_{\mu\nu}$ can be seen as a disformal metric tensor on $\mathscr{M}$.

Using the composition law between disformal metrics given in (1.15-I)

$$\Big(\lfloor\alpha_1,\beta_1\rfloor \star \lfloor\alpha_2,\beta_2\rfloor\Big)g = (\alpha_1\alpha_2)g_{\mu\nu} + \frac{\beta_1\alpha_2+\beta_1\beta_2+\alpha_1\beta_2}{V^2}V_\mu V_\nu,$$

we obtain that the square of $\mathfrak{D}$ is precisely $\hat{g}$:

$$\Big(\lfloor\alpha',\beta'\rfloor \star \lfloor\alpha',\beta'\rfloor\Big)g = \hat{g}_{\mu\nu}. \tag{1.45-I}$$

Another way to see this is by inspecting the eigenvalues of $\hat{g}^{\mu}{}_{\nu} = \hat{g}_{\sigma\nu}g^{\sigma\mu}$ and $\mathfrak{D}^{\mu}{}_{\nu}$: from the definition 1.26-I, the eigenvalue problem associated with the operator $\overrightarrow{D}$ is trivially solvable. One could check that V is an eigenvector related to the eigenvalue $\lambda_V = \sqrt{\alpha+\beta}$, while the other eigenvalues are degenerate and equal to $\sqrt{\alpha}$, with linearly independent eigenvectors lying on the orthogonal complement of V. Using the result in the end of section 1.1, it means that the eigenvalues of $\hat{g}^{\mu}{}_{\nu}$ are exactly the eigenvalues of $\mathfrak{D}^{\mu}{}_{\nu}$ squared, and hence they are equal as operators.

1.5 Causal analysis

For this section we have two tasks: 1) study the relationship between light cones of the background geometry and the disformal one, and 2) explore how equation (1.25-I) gives rise to two interesting, yet equivalent, interpretations of causality in the context of disformally related metrics and disformal operators acting on the background metric.

For the first task, let $(\mathcal{M}, g, \upharpoonright_V, \alpha, \beta) \mapsto (\mathcal{M}, \hat{g})$ be a disformal transformation. Let us fix a point $p \in \mathcal{M}$ and at that point consider an orthonormal basis $\{e_A\}$, with respect to the background metric g, such that $V = \sqrt{V^2} e_0$. Thus, for any $X \in T_p\mathcal{M}$, one can write $X = X^A e_A$ and obtain

$$\hat{g}(X,X) = (\alpha+\beta)\left(X^0\right)^2 - \alpha\,\delta_{AB} X^A X^B, \tag{1.46-I}$$

where δ_{AB} is the Kronecker delta, for $A, B = 1, 2, 3$.

Since we want to compare light cones of both metrics, let us assume that X is a null-like vector with respect to $\hat{g}$. This assumption leads to the equation

$$\left(X^0\right)^2 - \delta_{AB} X^A X^B = -\frac{\beta\left(X^0\right)^2}{\alpha}. \tag{1.47-I}$$

Therefore, at $p \in \mathcal{M}$, we have the following conditions:

1. If $\beta = 0$, then X is also a null-like vector with respect to g and the disformal light cone is the same as the background one;
2. If $\beta < 0$, then X is a time-like vector with respect to g and the disformal light cone lies inside the background one;
3. If $\beta > 0$, then X is a space-like vector with respect to g and the background light cone lies inside the disformal one.

With this, it is proven the following

Proposition 1.12. Let $(\mathcal{M}, g, \upharpoonright_V, \alpha, \beta) \mapsto (\mathcal{M}, \hat{g})$ be a disformal transformation. Thus:

1. If $\beta = 0$ throughout $\mathcal{M}$, then the disformal and background light cones coincide.
2. If $\beta < 0$ throughout $\mathcal{M}$, then the disformal light cones lie inside the background ones.
3. If $\beta > 0$ throughout $\mathcal{M}$, then the background light cones lie inside the disformal ones.

For the second task, note that the existence of the tensor field $\overrightarrow{D}$, such that a disformal transformation $(\mathcal{M}, g, \upharpoonright_V, \alpha, \beta) \mapsto (\mathcal{M}, \hat{g})$ can be written as equation (1.25-I) for any fields Y and Z, allows us to interpret the left-hand side of equation (1.25-I) as a new metric tensor for $\mathcal{M}$. Thus, the light cones of the metric $\hat{g}$ are, in general, different from those of g and hence have a different causal structure on $\mathcal{M}$. Conversely, the right-hand side of equation (1.25-I) indicates that one can consider just the background metric g, but applied to deformed vectors. Therefore, although light cones are preserved, causal relations change because the vectors have done so. Since (1.25-I) is a tensor identity, the important feature here is that the causal relation must agree whether you apply the disformal metric to vectors or the background metric to the deformed vectors. For practical reasons, it is provided below a simple and merely illustrative example, without any physical meaning *a priori*.

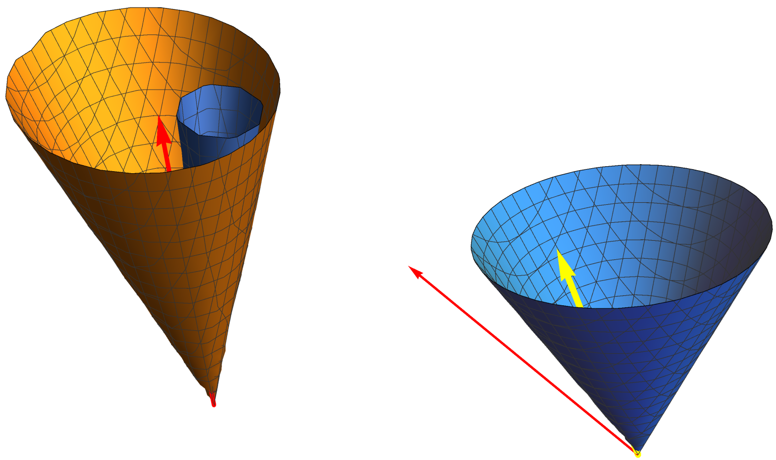

Figure 1.1. On the left, the vector is kept and the light cone is changed. In this case, X is space-like in g and time-like in $\hat{g}$. On the right, the light cone is kept and the vector is deformed. The original vector in red is space-like and its deformed counterpart in yellow (not to scale) is time-like.

Example 1.13. Let us fix the background metric to be the Minkowski one in standard Cartesian coordinates $g = \mathrm{diag}(1, -1, -1, -1)$ and that, at a fixed point p, we have:

$$\begin{aligned} V(p) &= (2, 1, 0, 0), \\ X(p) &= 5(-1, 1, 1, 0), \\ \alpha(p) &= 2 \text{ and } \beta(p) = 3. \end{aligned} \tag{1.48-I}$$

Because $\beta(p) > 0$, by proposition 1.12 it is already known that, at p, the disformal light cone contains the background one. The resulting disformal metric has components given by

$$\hat{g}_{\mu\nu} = 2g_{\mu\nu} + V_\mu V_\nu, \tag{1.49-I}$$

where $V_\mu = g_{\mu\nu} V^\nu$ is the dual of V. In matrix notation we have

$$\hat{g} = \begin{pmatrix} 6 & -2 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \tag{1.50-I}$$

and, using equation (1.9-I), we have $\det \hat{g} = \alpha^3 (\alpha + \beta) \det g = 40 \det g$. Additionaly, it is easily verified that

$$\begin{aligned} g(X,X) &= -25 < 0 \\ \hat{X} &= \left(5\sqrt{2} - 10\sqrt{5}, 10\sqrt{2} - 5\sqrt{5}, 5\sqrt{2}, 0\right) \\ \hat{g}(X,X) &= g(\hat{X},\hat{X}) = 175 > 0, \end{aligned} \tag{1.51-I}$$

which shows that a vector field could be space-like in the background metric and time-like in the disformal one. This situation is depicted on the left panel of Fig. 1.1. Conversely, as depicted on the right panel of Fig. 1.1, the vector is originally space-like and its disformal counterpart (not to scale) is time-like. □

We thus complete the mathematical aspects of disformal transformations we intended to develop and present here. Now, we shall see some of the roles that disformal metrics can play in physics.

2
Applications to physics

A creative man is motivated by the desire to achieve, not by the desire to beat others.

Ayn Rand

In this chapter, the disformal metrics are going to be used in two different scenarios. The first one relates to the search of new kinds of symmetry in physics, where the disformal invariance of the Dirac equation is investigated. The second scenario associates a purely mathematical structure to a physical theory, in which the connection between disformal and rainbow metrics is achieved via the CLB formalism.

2.1 Disformal invariance of the Dirac equation

First, let us recall the Dirac equation in flat space-time. All the generalizations needed for curved space-times are introduced as demanded. Subsequently, a sufficient condition that keeps the Dirac equation invariant under disformal transformations of the metric tensor is found.

2.1.1 The Dirac equation in flat space-time

The Schrödinger equation describes all atomic phenomena except those involving magnetism and relativity. The Schrödinger-Pauli equation takes care of magnetism by including the spin of the electron. The relativistic phenomena can be taken into consideration if one starts with the dispersion relation

$$\frac{E^2}{c^2} - \vec{p}^{\,2} = m^2c^2. \tag{2.1-I}$$

Inserting the energy and momentum operators into this equation, i.e., replacing

$$E \longrightarrow i\hbar\frac{\partial}{\partial t},\ p_{x_i} \longrightarrow -i\hbar\frac{\partial}{\partial x_i}, \tag{2.2-I}$$

results in the Klein-Gordon equation

$$\hbar^2\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} + \frac{\partial^2}{\partial {x_3}^2}\right)\Psi = m^2c^2\Psi. \tag{2.3-I}$$

It is easy to grasp that the Klein-Gordon equation has a quadratic dependence in $p_0 = i\hbar\frac{\partial}{\partial t} = E$, since it was derived from the dispersion relation (2.1-I). However, being quadratic in p_0 is not a desirable thing in quantum mechanics, where, since the derivation of Schrödinger's equation

$$i\hbar\frac{\partial}{\partial t}\Psi = \hat{H}\Psi, \tag{2.4-I}$$

one is familiar with the need to obtain wave equations that are linear in p_0. In order to avoid all of the problems associated with the Klein-Gordon equation, one might wonder about the possibility of taking the square root of equation (2.3-I) (naively shortened to $\partial^2 - m^2$) and then apply it to the wave function. A first guess might be treating the object ∂^2 as an ordinary algebraic symbol in order to write

$$\partial^2 - m^2 = \left(\sqrt{\partial^2} + m\right)\left(\sqrt{\partial^2} - m\right). \tag{2.5-I}$$

Unfortunately, $\sqrt{\partial^2}$ is not defined. Then, how could one take the square root of the ∂^2 operator? The great P. A. M. Dirac, in 1928, found the answer obtaining the equation

$$i\hbar\left(\gamma_0\frac{1}{c}\frac{\partial}{\partial t} - \gamma_1\frac{\partial}{\partial x_1} - \gamma_2\frac{\partial}{\partial x_2} - \gamma_3\frac{\partial}{\partial x_3}\right)\Psi = mc\Psi. \tag{2.6-I}$$

The Dirac equation above implies the Klein-Gordon equation provided the symbols γ_μ satisfy the relations

$$\begin{gathered}(\gamma_0)^2 = \mathbf{1}, \quad (\gamma_1)^2 = (\gamma_2)^2 = (\gamma_3)^2 = -\mathbf{1} \\ \gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu, \quad \text{for } \mu \neq \nu.\end{gathered} \tag{2.7-I}$$

Dirac found a set of $4\times 4-$ matrices satisfying these relations, namely, the following (constant) Dirac matrices:

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{2.8-I}$$

In terms of the 2×2 Pauli spin matrices σ_k the Dirac matrices γ_μ can be expressed as:

$$\gamma_0 = \gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \gamma_k = -\gamma^k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}. \tag{2.9-I}$$

It is easily verified that the matrices $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$ also satisfy the anti-commutation relations (2.7-I), summarized as

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}. \tag{2.10-I}$$

The matrices $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$ together the relations (2.10-I) generate a matrix representation for the Clifford algebra $C\ell_{1,3}(\mathbb{R})$, called the *Dirac representation*. In fact, a set of *hypercomplex numbers* satisfying the relations described in equation (2.10-I) is called a Clifford algebra. It can be shown that there exists a Clifford algebra in any set with a smooth metric $g^{\mu\nu}$ [**?**], flat or not.

If one writes $x_0 = ct$, the Dirac equation can be summarized into the form

$$i\hbar\gamma^{\mu}\partial_{\mu}\Psi = mc\Psi, \tag{2.11-I}$$

in which $\partial_{\mu} = \frac{\partial}{\partial x_{\mu}}$ denotes the partial derivative. The wave function is then a *column spinor*, depending on the coordinates $\mathbf{x} = (x_0, x_1, x_2, x_3) = (ct, x, y, z)$, on which the gamma matrices act as operators. That is,

$$\Psi(\mathbf{x}) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \in \mathbb{C}^4, \tag{2.12-I}$$

where each $\psi_i \in \mathbb{C}, i \in \{1,2,3,4\}$, is coordinate dependent. The *Dirac adjoint* of a column spinor $\Psi \in \mathbb{C}^4$ is row matrix

$$\overline{\Psi}(\mathbf{x}) \equiv \Psi^{\dagger}(\mathbf{x})\gamma_0 = \begin{pmatrix} \psi_1^* & \psi_2^* & -\psi_3^* & -\psi_4^* \end{pmatrix}, \tag{2.13-I}$$

where $\dagger$ is the operation of matrix transposition together with complex conjugation $(*)$. A column spinor Ψ and its Dirac adjoint $\Psi^{\dagger}\gamma_0$ can be used to define four real-valued functions

$$J^{\mu}(\mathbf{x}) = \Psi^{\dagger}(\mathbf{x})\gamma_0\gamma^{\mu}\Psi(\mathbf{x}) \tag{2.14-I}$$

which are components of an authentic space-time vector field, called the Dirac current,

$$\mathbf{J}(\mathbf{x}) = \gamma_{\mu}J^{\mu}(\mathbf{x}) \in \Gamma(T\mathscr{M}), \tag{2.15-I}$$

which is invariant under any Lorentz transformation. The reader interested in Clifford algebras and spinors is encouraged to read the beautiful content of [**?**].

Example 2.1 (Plane wave solution for the massless Dirac equation)**.** Let us consider the Dirac equation given by equation (2.11-I) in units where $c = \hbar = 1$. Let $\Psi \in \mathbb{C}^4$ be a massless spinor given by

$$\Psi = \Psi_0 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} e^{-iE(t+z)}. \tag{2.16-I}$$

As the reader can check, the associated covariant Dirac current associated with the spinor Ψ is given by $J_{\mu} = 2|\Psi_0|^2(1,0,0,1)$. It is easy to verify that Ψ is a solution for the massless Dirac equation: Since Ψ depends only on t and z and the massless Dirac equation reduces to

$$i\gamma^0\partial_t\Psi + i\gamma^3\partial_z\Psi = 0. \tag{2.17-I}$$

The computation of the derivatives yields

$$\partial_\mu \Psi = -iE\frac{J_\mu}{J_0}\Psi. \tag{2.18-I}$$

Combining equations (2.17-I) and (2.18-I) we obtain that $(\gamma^0+\gamma^3)\Psi = 0$. Using the relations given in 2.9-I to raise the index of the gamma matrices, a simple calculation shows that Ψ indeed satisfies the massless Dirac equation. □

2.1.2 Construction of the disformal transformation for spinors

In this section, the mathematical tools one needs to deal with spinor fields in a curved space-time are developed. As ensured by proposition 1.4, we shall use space-time components to favour the calculations and to present the results in a standard language. Also, we mainly concentrate on the Clifford algebra that the Dirac matrices must satisfy in the disformal and background metrics and on the construction of the disformal map between these metrics through the Weyl-Cartan formalism.

With the Dirac matrices γ^μ and an arbitrary Dirac spinor field Ψ defined in a space-time with metric $g_{\mu\nu}$, one can construct two Hermitian scalars $A \equiv \overline{\Psi}\Psi$ and $B \equiv i\overline{\Psi}\gamma_5\Psi$, where again $\overline{\Psi} \equiv \Psi^\dagger\gamma^0$ and $\gamma_5 \equiv \frac{i}{4!}\eta_{\alpha\beta\mu\nu}\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu$, with $\eta_{\alpha\beta\mu\nu} = \sqrt{|g|}\epsilon_{\alpha\beta\mu\nu}$ corresponding to the Levi-Civita tensor in terms of the Levi-Civita symbol $\epsilon_{\alpha\beta\mu\nu}$ **[?]**, and assuming the convention in which $\epsilon_{0123} = 1$. The reader can check that γ_5 anti-commutes with the other gamma matrices and that $(\gamma_5)^2 = \mathbf{1}$. One can also define two space-time vectors depending algebraically on Ψ, which are the already known Dirac current $J^\mu \equiv \overline{\Psi}\gamma^\mu\Psi$ and the axial current $I^\mu \equiv \overline{\Psi}\gamma^\mu\gamma_5\Psi$. Using Pauli-Kofink's identity

$$\left(\overline{\Psi}Q\gamma_\lambda\Psi\right)\gamma^\lambda\Psi = \left(\overline{\Psi}Q\Psi\right)\Psi - \left(\overline{\Psi}Q\gamma_5\Psi\right)\gamma_5\Psi, \tag{2.19-I}$$

where Q is an arbitrary element of the Clifford algebra, it is straightforward to prove the following

Proposition 2.2 (Fierz identities)**.** The Dirac current J^μ, the axial current I^μ, and the Hermitian scalars A and B are linked through:

$$J^\mu J_\mu = -I^\mu I_\mu = A^2 + B^2 > 0, \tag{2.20-I}$$

$$J^\mu I_\mu = 0. \tag{2.21-I}$$

Proof. Using Pauli-Kofink's identity for $Q = \mathbf{1}$ yields

$$\begin{aligned}\left(\overline{\Psi}\gamma_\lambda\Psi\right)\gamma^\lambda\Psi = J_\lambda\gamma^\lambda\Psi &= \left(\overline{\Psi}\Psi\right)\Psi - \left(\overline{\Psi}\gamma_5\Psi\right)\gamma_5\Psi \\ &= A\Psi + iB\gamma_5\Psi.\end{aligned} \tag{2.22-I}$$

Multiplying the expression above by $\overline{\Psi}$ results in $J^\mu J_\mu = A^2 + B^2$, proving the first identity. Setting $Q = \gamma_5$, however, produces

$$\begin{aligned}\left(\overline{\Psi}\gamma_5\gamma_\lambda\Psi\right)\gamma^\lambda\Psi = -I_\lambda\gamma^\lambda\Psi &= \left(\overline{\Psi}\gamma_5\Psi\right)\Psi - \left(\overline{\Psi}\gamma_5\gamma_5\Psi\right)\gamma_5\Psi \\ &= -iB\Psi - A\gamma_5\Psi.\end{aligned} \tag{2.23-I}$$

Multiplying the expression above by $\overline{\Psi}\gamma_5$ results in $I^\mu I_\mu = -A^2 - B^2$, proving the second identity. For the third identity one could set $Q = \mathbf{1}$, obtaining equation (2.22-I). Multiplying by $\overline{\Psi}\gamma_5$, because J_λ is a function, one gets

$$\begin{aligned} J_\lambda \overline{\Psi}\gamma_5\gamma^\lambda\Psi = -J_\lambda I^\lambda &= A\overline{\Psi}\gamma_5\Psi + iB\overline{\Psi}\gamma_5\gamma_5\Psi \\ &= -iAB + iAB \\ &= 0, \end{aligned}$$

concluding the proof. ■

Corollary 2.3. From equation (2.22-I), if the Dirac current is light-like, i.e., $A = B = 0$, then $J_\mu\gamma^\mu\Psi = 0$.

According to the spin-2 field theory formulation [?], any space-time metric $\hat{g}_{\mu\nu}$ can be always split into two parts,

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + \Sigma_{\mu\nu}, \tag{2.24-I}$$

with a background geometry $g_{\mu\nu}$ and a rank two tensor field $\Sigma_{\mu\nu}$ responsible for the spin-2 particle description. Under this form, the inverse metric $\hat{g}^{\mu\nu}$ is, in general, given by an infinite series. However, in the case of disformal metrics expressed by equation (2.24-I), their inverse admits the same binomial form if we choose $\Sigma_{\mu\nu}$ such that the condition $\Sigma^{\mu\nu}\Sigma_{\nu\lambda} = p\,\delta^\mu_\lambda + q\,\Sigma^\mu{}_\lambda$ holds, where p and q are arbitrary functions of the coordinates.

In this way, one shall proceed with the disformal transformation using directly the components of the space-time geometry, as we said before. Thereby, since the Dirac equation involves only first order derivatives of Ψ, the most general expression one can attribute to the disformal metric making use only of the spinor currents is

$$\hat{g}^{\mu\nu} = \alpha g^{\mu\nu} + \beta\frac{J^\mu J^\nu}{J^2} - \gamma\frac{I^\mu I^\nu}{J^2} + \delta\frac{J^{(\mu}I^{\nu)}}{J^2}, \tag{2.25-I}$$

where parentheses indicate symmetrization, $J^2 = g^{\mu\nu}J_\mu J_\nu$ and α, β, γ and δ are arbitrary functions of the scalars A and B. Using the relation $\hat{g}^{\mu\nu}\hat{g}_{\nu\lambda} = \delta^\mu_\lambda$ to guarantee that $\hat{g}^{\mu\nu}$ corresponds indeed to a metric tensor, the components of the inverse metric tensor can be written as

$$\hat{g}_{\mu\nu} = \frac{1}{\alpha}g_{\mu\nu} - \frac{(\alpha+\gamma)\beta+\delta^2}{\alpha\Upsilon}\frac{J_\mu J_\nu}{J^2} + \frac{(\alpha+\beta)\gamma+\delta^2}{\alpha\Upsilon}\frac{I_\mu I_\nu}{J^2} - \frac{\delta}{\Upsilon}\frac{J_{(\mu}I_{\nu)}}{J^2}, \tag{2.26-I}$$

with $\Upsilon \equiv (\alpha+\beta)(\alpha+\gamma)+\delta^2$. It should be emphasized that $\hat{g}^{\mu\nu}$ and $g^{\mu\nu}$ have no gravitational character, that is, the fact that the geometry (2.25-I) and $g^{\mu\nu}$ are not necessarily flat has nothing to do with any geometric theory of gravity.

Let us now introduce the tetrad frames [?] (or vierbeins [?]), in order to perform properly the disformal transformation on the spinorial dynamics (see details on the Weyl-Cartan formalism also in [?]). This simplifies considerably the calculations since we are dealing with an object (spinor) that has a particular internal space and is defined on a curved background, at the same time. We then start by rewriting the quantities defined above in terms of the tetrads. We denote by $\hat{e}^\mu{}_{(A)}$ the tetrad basis acting on the disformal metric $\hat{g}^{\mu\nu}$ and $e^\mu{}_{(A)}$ the one acting on the background metric $g^{\mu\nu}$.

Remark 2.4. Formally, given a local coordinate system $\{x^\mu\}$ for $\mathscr{M}$, we are choosing a set of four vector fields $\vec{e}_{(A)} = e^\mu{}_{(A)}\partial_\mu$ (resp. $\hat{e}_{(A)} = \hat{e}^\mu{}_{(A)}\partial_\mu$) that comprise an orthonormal basis with respect to the metric g ($\hat{g}$) and expressing them in terms of the coordinate basis $\partial_\mu = \partial/\partial x^\mu$. The elements of the basis $\{e_{(A)}\}_{A=1,2,3,4}$ ($\{\hat{e}_{(A)}\}_{A=1,2,3,4}$) are then represented in terms of their components $e^\mu{}_{(A)}$ ($\hat{e}^\mu{}_{(A)}$) in the coordinate basis as usual, just like any vector field $V = V^\mu\partial_\mu$ would simply be represented by V^μ. From the mathematical viewpoint, a global tetrad field defines a *parallelization* of $\mathscr{M}$, which is equivalent to an isomorphism between the tangent bundle $\pi : T\mathscr{M} \longrightarrow \mathscr{M}$ and the trivial bundle $\iota : \mathscr{M} \times \mathbb{R}^4 \longrightarrow \mathscr{M}$, i.e., $T\mathscr{M} \cong \mathscr{M} \times \mathbb{R}^4$ [**?**]. Since not every manifold is parallelizable[1], in general, tetrads can only be chosen locally. On the other hand, as shown by R. Geroch in [**?**], a four-dimensional and noncompact Lorentz manifold admits a spinor structure if, and only if, it is parallelizable. Therefore, if convenient, the reader should assume these hypotheses in what follows, although it is possible to assume a local Dirac spinor structure on $\mathscr{M}$ so that the local results are valid as well [**?**].

Each orthonormal basis is composed by one time-like and three space-like vectors, with both satisfying the conditions

$$\eta_{AB} = \hat{g}_{\mu\nu}\,\hat{e}^\mu{}_{(A)}\,\hat{e}^\nu{}_{(B)} = g_{\mu\nu}\,e^\mu{}_{(A)}\,e^\nu{}_{(B)}. \tag{2.27-I}$$

Note that the Greek indices (running from 0 to 3) are lowered and raised by their corresponding space-time metric ($g_{\mu\nu}$ or $\hat{g}_{\mu\nu}$) and the Latin labels (running from 1 to 4) are lowered and raised with $\eta_{AB} = \mathrm{diag}(1, -1, -1, -1)$.

Example 2.5 (Tetrads of the standard Reissner-Nordström metric)**.** Consider the Reissner-Nordström metric given in standard Schwarzschild-like coordinates

$$g = g_{\mu\nu}dx^\mu \otimes dx^\nu = f(r)dt \otimes dt - \frac{1}{f(r)}dr \otimes dr - r^2 d\theta \otimes d\theta - r^2 \sin^2\theta d\phi \otimes d\phi$$
$$f(r) = 1 - \frac{M}{r} + \frac{Q^2}{r^2}.$$

The set of four vector fields $\vec{e}_{(A)} = e^\mu{}_{(A)}\partial_\mu$, $A \in \{1,2,3,4\}$ and $\mu \in \{0,1,2,3\}$, given by

$$\vec{e}_{(1)} = \frac{1}{\sqrt{f(r)}}\partial_t, \quad \vec{e}_{(2)} = \sqrt{f(r)}\partial_r,$$
$$\vec{e}_{(3)} = \frac{1}{r}\partial_\theta, \quad \vec{e}_{(4)} = \frac{1}{r\sin\theta}\partial_\phi$$

are shown to satisfy the condition given in equation (2.27-I). As particular cases of this one we have the Schwarzschild tetrads (setting $Q = 0$) and the Minkowski tetrads (setting both Q and M equal to zero), both in the coordinate system (t, r, θ, ϕ). □

For the sake of simplicity, the tetrad indices are denoted without parentheses. Using equation (2.27-I), we can define the inverse tetrad bases $e_\mu{}^A$ and $\hat{e}_\mu{}^A$ from the

[1] A classical result in algebraic topology, with the funny name of *hairy ball theorem* (cf. reference [**?**], for instance), asserts that every vector field defined on any even dimensional sphere $\mathbb{S}^{2n}$, $n \geq 1$, must be zero at least in one point. Therefore, $\mathbb{S}^{2n}$, $n \geq 1$, is not parallelizable.

conditions ${e_\mu}^A {e^\nu}_A = {\hat{e}_\mu}^A {\hat{e}^\nu}_A = \delta^\nu_\mu$ and ${e_\mu}^A {e^\mu}_B = {\hat{e}_\mu}^A {\hat{e}^\mu}_B = \delta^A_B$. The Dirac matrices and the tetrad bases associated with each space-time are such that

$$\gamma^A = {\hat{e}_\mu}^A \hat{\gamma}^\mu = {e_\mu}^A \gamma^\mu, \tag{2.28-I}$$

where γ^A's are the constant Dirac matrices. Then, we can state and prove the following

Lemma 2.6. Each γ_5 matrix constructed with either $\hat{\gamma}^\mu$, γ^μ or γ^A corresponds indeed to the same matrix.

Proof. We are going to use equations (2.27-I) and (2.28-I) and the content available in section 2.8 of [**?**] in order to prove this lemma. For simplicity, it is going to be shown that γ_5 constructed with the constant Dirac matrices γ^A and the ones constructed with the metric g (γ^μ) are the same. The case for γ^A and $\hat{\gamma}^\mu$ is entirely analogous and the result holds by transitivity.

By definition,

$$\begin{aligned} \gamma_5 &= \frac{i}{4!}\eta_{\alpha\beta\mu\nu}\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu \\ &= \frac{i}{4!}\sqrt{|g|}\epsilon_{\alpha\beta\mu\nu}\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu \\ &= \frac{i}{4!}\sqrt{|g|}\epsilon_{\alpha\beta\mu\nu}{e^\alpha}_A{e^\beta}_B{e^\mu}_C{e^\nu}_D\gamma^A\gamma^B\gamma^C\gamma^D, \end{aligned} \tag{2.29-I}$$

where the last equality comes from the inverse relations implied by equation (2.28-I). Using equation (2.27-I), equation (2.68) from [**?**], and the fact that $\det(\eta_{AB}) = -1$, one can show that

$$\det(g) \equiv g = -\det\left({e_\mu}^A\right)^2, \tag{2.30-I}$$

implying that $\sqrt{|g|} = \left|\det\left({e_\mu}^A\right)\right|$. Replacing that relation in the last equality of equation (2.29-I) yields

$$\begin{aligned} \gamma_5 &= \frac{i}{4!}\underbrace{\left|\det\left({e_\mu}^A\right)\right|\epsilon_{\alpha\beta\mu\nu}{e^\alpha}_A{e^\beta}_B{e^\mu}_C{e^\nu}_D}_{=\epsilon_{ABCD}}\gamma^A\gamma^B\gamma^C\gamma^D \\ &= \frac{i}{4!}\epsilon_{ABCD}\gamma^A\gamma^B\gamma^C\gamma^D, \end{aligned} \tag{2.31-I}$$

which is simply the γ_5 matrix written in terms of the constant Dirac matrices, and where the identity highlighted by the underbrace comes from equation (2.67) in [**?**] and also using $|\det(\eta_{AB})| = 1$. ■

Furthermore, the $\hat{\gamma}^\mu$, γ^μ and γ^A must satisfy their respective closure relation, namely

$$\{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = 2\hat{g}^{\mu\nu}\mathbf{1}, \qquad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}, \quad \text{and} \quad \{\gamma^A, \gamma^B\} = 2\eta^{AB}\mathbf{1}, \tag{2.32-I}$$

where $\mathbf{1}$ is the identity element of the algebra and the curly brackets represent the anti-commutation operator.

For consistency with the definitions above, both tetrad bases must be related somehow. Hence, motivated by the algebraic form of the metric (2.25-I), we set

$$\hat{e}^{\mu}{}_{A} = a\, e^{\mu}{}_{A} + \Theta^{\mu}{}_{A}, \tag{2.33-I}$$

where the most general expression for $\Theta^{\mu}{}_{A}$ is given by

$$\Theta^{\mu}{}_{A} = \frac{1}{J^2}[J^{\mu}(bJ_A + cI_A) + I^{\mu}(dJ_A + fI_A)], \tag{2.34-I}$$

with functions a, b, c, d and f linked to the coefficients of the metric through

$$\alpha = a^2, \quad \beta = b(b+2a) - c^2, \tag{2.35-I}$$

$$\gamma = f(f-2a) - d^2, \quad \text{and} \quad \delta = (a-f)c + (a+b)d. \tag{2.36-I}$$

It is straightforward to verify that the inverse tetrad basis will have a similar form to (2.33-I), as follows

$$\begin{aligned} \hat{e}_{\mu}{}^{A} &= \frac{1}{a} e_{\mu}{}^{A} \\ &- \frac{1}{J^2 \Upsilon}\left[J_{\mu}\left(\frac{b(a-f)+cd}{a}J^A + dI^A\right) + I_{\mu}\left(cJ^A + \frac{f(a+b)-cd}{a}I^A\right)\right], \end{aligned} \tag{2.37-I}$$

where, in terms of the tetrad coefficients, $\Upsilon \equiv (a+b)(a-f)+cd$. In conclusion, these are the mathematical ingredients of the Weyl-Cartan formalism we need to apply the disformal transformation to the Dirac equation. Nevertheless, we have to analyze first the conservation laws for the spinor currents in both space-times before entering into the details of the dynamical equations for Ψ on the space-times in concern.

2.1.3 Conservation of the currents

Once we are interested in the disformal invariance of a given dynamics for Ψ, it is expected that the action of the disformal transformation on the conservation laws of the spinor currents in a given metric implies the conservation of the corresponding currents in the other metric if these geometries are linked. Though it is not a direct consequence of the map, it can be true when the arbitrary functions of $\hat{g}^{\mu\nu}$ satisfy some extra constraint equations.

Let us define the Dirac currents $J^{\mu} = \overline{\Psi}\gamma^{\mu}\Psi$ and $\hat{J}^{\mu} = \overline{\Psi}\hat{\gamma}^{\mu}\Psi$ associated with the metrics $g_{\mu\nu}$ and $\hat{g}_{\mu\nu}$, respectively and, using lemma 2.6, the axial currents as $I^{\mu} = \overline{\Psi}\gamma^{\mu}\gamma_5\Psi$ and $\hat{I}^{\mu} = \overline{\Psi}\hat{\gamma}^{\mu}\gamma_5\Psi$. This is necessary in order to guarantee the conservation of the currents just as a consequence of the dynamical equation for Ψ in the disformal metric. Thus, imposing the following laws for the currents

$$\hat{\nabla}_{\mu}\hat{J}^{\mu} = 0, \quad \text{and} \quad \hat{\nabla}_{\mu}\hat{I}^{\mu} = S_d(A,B), \tag{2.38-I}$$

in which $\hat{\nabla}_{\mu}$ corresponds to the covariant derivative compatible with $\hat{g}_{\mu\nu}$ and $S_d(A,B)$ is a source term obtained from the combination of the dynamics for Ψ and its complex conjugate both defined in $\hat{g}^{\mu\nu}$. If Ψ satisfy the Dirac equation, then S_d is proportional to the mass.

Using equation (1.9-I), we relate the determinant $\hat{g}$ of $\hat{g}_{\mu\nu}$ given by equation (2.26-I) with the determinant g of $g_{\mu\nu}$, that is

$$\hat{g} = \frac{g}{a^4 \Upsilon}. \tag{2.39-I}$$

The definitions of $\hat{J}^\mu$ and $\hat{I}^\mu$ together with equation (2.28-I) allow us to write $\hat{J}^\mu = (a+b)J^\mu + dI^\mu$ and $\hat{I}^\mu = (a-f)I^\mu - cJ^\mu$ and, consequently, equations (2.38-I) have their analogs in $g_{\mu\nu}$, as follows

$$\hat{\nabla}_\mu \hat{J}^\mu = \frac{a^2\Upsilon}{\sqrt{-g}}\partial_\mu \left\{ \frac{\sqrt{-g}}{a^2\Upsilon}[(a+b)J^\mu + dI^\mu] \right\} = 0, \tag{2.40-I}$$

$$\hat{\nabla}_\mu \hat{I}^\mu = \frac{a^2\Upsilon}{\sqrt{-g}}\partial_\mu \left\{ \frac{\sqrt{-g}}{a^2\Upsilon}[(a-f)I^\mu - cJ^\mu] \right\} = S_d, \tag{2.41-I}$$

where ∂_μ denotes partial derivative. Introducing new variables $X \equiv \ln[(a+b)/a^2\Upsilon]$, $Y \equiv \ln[(a-f)/a^2\Upsilon]$, $Z \equiv \ln(d/a^2\Upsilon)$ and $W \equiv \ln(c/a^2\Upsilon)$ and rearranging the terms, one gets

$$\nabla_\mu J^\mu + J^\mu \partial_\mu X + e^{Z-X}(\nabla_\mu I^\mu + I^\mu \partial_\mu Z) = 0, \tag{2.42-I}$$

$$e^Y(\nabla_\mu I^\mu + I^\mu \partial_\mu Y) - e^W(\nabla_\mu J^\mu + J^\mu \partial_\mu W) = \frac{S_d}{a^2\Upsilon}, \tag{2.43-I}$$

with ∇_μ corresponding to the covariant derivative compatible with $g_{\mu\nu}$. In order that the conservation of the currents also holds in the background metric, that is $\nabla_\mu J^\mu = 0$ and $\nabla_\mu I^\mu = S_b(A,B)$[2], it is sufficient that the coefficients of the metric $\hat{g}_{\mu\nu}$ satisfy the conditions

$$J^\mu \partial_\mu \left(e^X\right) + I^\mu \partial_\mu \left(e^Z\right) = 0, \tag{2.44-Ia}$$

$$I^\mu \partial_\mu \left(e^Y\right) - J^\mu \partial_\mu \left(e^W\right) = \frac{S_d + (f-a)S_b}{a^2\Upsilon}. \tag{2.44-Ib}$$

It should be noted that the existence of conserved currents describing the probability flux of the spinor lead us necessarily to requirements other than the pure disformal map between the dynamics of Ψ, contrary to the scalar and electromagnetic cases [?, ?]; otherwise, the map would be mathematically well-defined, but meaningless from the physical point of view. The system 2.44-I involving a, b, c, d and f suggest that the spinor case cannot be naively compared with the previous ones [?, ?], which impose only algebraic constraints on the metric coefficients. Therefore, disformal transformations applied to spinor fields induce first-order differential equations for the metric coefficients, apart from the Clifford algebra which must be satisfied.

2.1.4 The disformal transformation of the Dirac equation

In this section, we shall describe how the disformal transformations act on dynamical equations for spinor fields. In particular, we consider a spinor field Ψ satisfying a given dynamics in the disformal metric $\hat{g}_{\mu\nu}$ and then applying the disformal

[2] The indices “d" and “b" are used to indicate different sources for the equation of the axial current resulting from the dynamics of Ψ in the disformal and background metrics, respectively.

map we induce a dynamical equation for Ψ in the background metric $g_{\mu\nu}$. After, we also look for conditions on the metric coefficients such that the induced dynamics corresponds to the exact Dirac equation. The way we will apply the disformal transformation here corresponds to a mathematical strategy, of course. The dynamics for Ψ is reasonably solvable only if we start from the background metric (which is Ψ-independent and given *a priori*) and then construct the disformal metric, where the equation for Ψ defined in terms of $\hat{g}_{\mu\nu}$ is automatically verified; otherwise, we need to integrate a nonlinear equation for Ψ in the disformal metric.

From the beginning, we know that the disformal invariance of the massive Dirac equation will be valid only in specific cases, once the disformal transformations contain the conformal ones as a particular case and the latter do not let this equation invariant (see reference [**?**] and references therein). As we shall see, the break of the disformal invariance is due solely to the presence of the conformal factor. An alternative to bypass this difficulty would be, for instance, to resort to scenarios where conformal transformations play an important role in the definition of the mass (cf. reference [**?**]).

The choice of $\hat{g}_{\mu\nu}$ and the symmetries associated with J^{μ} and I^{μ}, evidenced by the equations above, indicate that there is no need of all disformal terms in (2.25-I). So, the disformal transformation can be equally implemented by using only one of the spinor currents. Thus, as a matter of simplicity, we set $\gamma = \delta = 0$ in equations (2.25-I) and (2.26-I) and define a normalized four-vector $V^{\mu} \equiv J^{\mu}/\sqrt{J^2}$ so that the disformal metric (2.25-I) becomes[3]

$$\hat{g}^{\mu\nu} = \alpha g^{\mu\nu} + \beta V^{\mu} V^{\nu}. \tag{2.45-I}$$

In this way, we start with a modified Dirac equation for Ψ, where the mass term takes into account the problem introduced by the conformal transformations, written in terms of the metric (2.45-I) which reads[4]

$$i\hat{\gamma}^{\mu}\hat{\nabla}_{\mu}\Psi - \sqrt{\alpha}\, m\Psi = 0, \tag{2.46-I}$$

with $\hat{\nabla}_{\mu} \equiv \partial_{\mu} - \hat{\Gamma}_{\mu}$ and the Fock-Ivanenko connection $\hat{\Gamma}_{\mu}$ given by

$$\hat{\Gamma}_{\mu} = -\frac{1}{8}\left([\hat{\gamma}^{\alpha}, \partial_{\mu}\hat{\gamma}_{\alpha}] - \hat{\Gamma}^{\rho}_{\alpha\mu}[\hat{\gamma}^{\alpha}, \hat{\gamma}_{\rho}]\right). \tag{2.47-I}$$

The squared brackets denote the usual commutator and the Christoffel symbols $\hat{\Gamma}^{\rho}_{\alpha\mu}$ are constructed with $\hat{g}_{\mu\nu}$. We are using units in which $c = \hbar = 1$. The main result of this section can be summarized in the following

Theorem 2.7. Let $\mathscr{M}$ be a space-time with metric tensors g and $\hat{g}$, with inverses respectively given by h and $\hat{h}$, and smooth scalar functions α and β, such that $\alpha > 0$ and $\alpha + \beta > 0$. Consider a smooth (at least C^2), normalized, time-like vector field V and a disformal relation between the cometrics as

$$\hat{h}(*,\cdot) = \alpha\, h(*,\cdot) + \beta\, h(\widetilde{V}, *) \otimes h(\widetilde{V}, \cdot), \tag{2.48-I}$$

[3]Analogously, one could set $\beta = \delta = 0$ and define $V_{\mu} = I_{\mu}/\sqrt{J^2}$, but the special form of (2.25-I) would lead to similar conclusions (altough one should redifine the disformal transformation for the case of space-like vector fields). Assuming $\beta = \gamma = 0$ and $\delta \neq 0$ do not provide a genuine disformal transformation once the inverse metric is not given by a binomial expression (see equation (2.26-I)).

[4]Note that when the conformal factor α of equation (2.45-I) is constant, we recover the genuine mass term of the Dirac equation.

where $\widetilde{V}$ is the metric dual of V constructed with g. Let Ψ be a Dirac spinor field satisfying the modified Dirac equation (2.46-I) in the disformal metric $\hat{g}$ and let $V^\mu \equiv \overline{\Psi}\gamma^\mu\Psi/\sqrt{g^{\alpha\beta}J_\alpha J_\beta}$ in space-time coordinates, then there exists a class of Ψ's which verifies the Dirac equation (massive or not) written in terms of the background metric g.

Corollary 2.8. If the conformal coefficient a of (2.45-I) is constant or the spinor is massless, then Ψ is a disformally invariant solution of the Dirac operator.

The complete proof of the theorem and corollary corresponds to the rest of this section. Let us introduce a tetrad basis and its inverse with the same disformal symmetry as its associated metric (2.45-I), that is

$$\hat{e}^\mu{}_A = a\, e^\mu{}_A + bV^\mu V_A, \quad \text{and} \quad \hat{e}_\mu{}^A = \frac{1}{a} e_\mu{}^A - \frac{b}{a(a+b)} V_\mu V^A. \tag{2.49-I}$$

The Fock-Ivanenko connection (2.47-I) can be rewritten as

$$\hat{\Gamma}_\mu = \hat{e}_\mu{}^A \hat{\Gamma}_A, \qquad \text{with} \qquad \hat{\Gamma}_A = -\frac{1}{8}\hat{\gamma}_{BCA}[\gamma^B, \gamma^C], \tag{2.50-I}$$

where $\hat{\gamma}_{ABC}$ is the spin connection defined as

$$\hat{\gamma}_{ABC} = \frac{1}{2}(\hat{C}_{ABC} - \hat{C}_{BAC} - \hat{C}_{CAB}) \quad \text{and} \tag{2.51-I}$$

$$\hat{C}_{ABC} = \hat{e}_{\nu A}(\hat{e}^\mu{}_C\, \partial_\mu \hat{e}^\nu{}_B - \hat{e}^\mu{}_B\, \partial_\mu \hat{e}^\nu{}_C), \tag{2.52-I}$$

with symmetry $\hat{C}_{ABC} = -\hat{C}_{ACB}$, implying $\hat{\gamma}_{ABC} = -\hat{\gamma}_{BAC}$.

The next step is to use equation (2.49-I) and rewrite equation (2.46-I) completely in terms of the objects defined at the background metric. Thus, a first manipulation of equation (2.46-I) produces

$$i\gamma^A(\hat{\partial}_A - \hat{\Gamma}_A)\Psi \; - am\Psi = \; i\gamma^A\left(a\partial_A + bV_A V^B\partial_B + \frac{1}{8}\hat{\gamma}_{BCA}[\gamma^B, \gamma^C]\right)\Psi - am\Psi = 0, \tag{2.53-I}$$

where $\hat{\partial}_A \equiv \hat{e}^\mu{}_A \partial_\mu = a\partial_A + bV_A V^B \partial_B$. Then, using the algebraic identity between the γ^A matrices

$$\gamma^A\gamma^B\gamma^C = \eta^{AB}\gamma^C + \eta^{BC}\gamma^A - \eta^{AC}\gamma^B + i\epsilon^{ABC}{}_D\gamma^D\gamma_5 \tag{2.54-I}$$

to compute the expression $\hat{\gamma}_{BCA}\gamma^A\gamma^B\gamma^C$, one gets

$$i\left[\gamma^A\partial_A - m + \frac{b}{a}\gamma^A V_A V^B \partial_B + \frac{1}{4a}\left(2\hat{C}^A{}_{BA}\gamma^B + \frac{i}{2}\hat{C}_{ABC}\epsilon^{ABC}{}_D\gamma^D\gamma_5\right)\right]\Psi = 0. \tag{2.55-I}$$

From equation (2.51-I), one should find that

$$\hat{C}^A{}_{BA} = aC^A{}_{BA} - \partial_\mu a\left[\frac{3a+2b}{a+b}\, e^\mu{}_B + \frac{b\,(4a+3b)}{a\,(a+b)} V^\mu V_B\right] + \nabla_\mu(bV^\mu)V_B + b\dot{V}_B \tag{2.56-I}$$

and

$$\hat{C}_{ABC}\epsilon^{ABC}{}_D = \left[aC_{ABC} + 2b\, e_{\nu A}V_C\left(\dot{e}^\nu{}_B + \frac{b}{a+b}\nabla_\mu V^\nu e^\mu{}_B\right)\right]\epsilon^{ABC}{}_D, \tag{2.57-I}$$

where dot ($\dot{}$) stands for covariant derivative in the background metric projected along V^μ.

Finally, using the condition resulting from the conservation of the vector current (2.44-Ia) restricted to the case we are dealing with, which is merely

$$\dot{a} = 0, \tag{2.58-I}$$

the equation (2.46-I) written in the disformal geometry becomes the following when written in the background space-time

$$\begin{aligned} i\gamma^A \nabla_A \Psi - m\Psi + \frac{ib}{a}\gamma^A V_A V^B \nabla_B \Psi \\ -\frac{ib}{2a}\left[\frac{3a+2b}{b(a+b)}\partial_B a - \frac{\dot{b}}{b}V_B - (\nabla_\mu V^\mu)V_B - \dot{V}_B\right]\gamma^B\Psi + \\ +\frac{b}{4a}\left\{e_\nu{}^B V_D\left[\dot{e}^\nu{}_C + \frac{b}{a+b}\nabla_\mu V^\nu e^\mu{}_C\right]\epsilon_{AB}{}^{CD}\gamma^A\gamma_5\right\}\Psi = 0. \end{aligned} \tag{2.59-I}$$

Note that this is a nonlinear dynamical equation for Ψ on $\mathscr{M}$ endowed with $g_{\mu\nu}$: its first two terms correspond to the Dirac operator $i\gamma^A\nabla_A - m\mathbf{1}$ fully defined on the background metric and all the other self-interacting terms are originated by the relation between the tetrad bases. As far as it is known, the equation (2.59-I) does not fit any well-known nonlinear dynamics for a Dirac spinor [**?**]. Notwithstanding, we have shown that all solutions of equation (2.46-I) defined in the disformal metric given by (2.45-I) are the same for this highly nonlinear equation in the background metric. In the next subsection, we will see that there are special classes of solutions for this equation which satisfies the exact Dirac equation in the background metric.

2.1.5 Classes of Inomata-type solutions

With no loss of generality, we consider an arbitrary point $p \in \mathscr{M}$ and Fermi normal coordinates [**?**] around p and considering the geodesic at p to have velocity vector V. Under these conditions, the background metric is the Minkowski one at p (in fact, along the geodesic), the matrix representation of the tetrad basis $e^\mu{}_A$ at p is the Kronecker delta, the covariant derivatives reduce to partial derivatives ($\nabla_\mu \to \partial_\mu$), and the tetrad is parallel-transported along the geodesic ($\dot{e}^\nu{}_C = 0$ along the geodesic, in particular at p). In this case, equation (2.59-I) is considerably simplified, yielding

$$\begin{aligned} i\gamma^A\partial_A\Psi \quad &- \quad m\Psi + \frac{ib}{a}\gamma^A V_A V^B \partial_B \Psi \\ &- \quad \frac{ib}{2a}\left[\frac{3a+2b}{b(a+b)}\partial_B a - \frac{\dot{b}}{b}V_B - (\partial_\mu V^\mu)V_B - \dot{V}_B\right]\gamma^B\Psi \\ &- \quad \frac{b^2}{2a(a+b)}\omega_A\gamma^A\gamma_5\Psi = 0, \end{aligned} \tag{2.60-I}$$

where $\omega^A \equiv -\frac{1}{2}\epsilon^{ABCD}\omega_{BC}V_D$ is the vorticity vector and $\omega_{BC} \equiv \frac{1}{2}h_B{}^\mu h_C{}^\nu(\partial_\nu V_\mu - \partial_\mu V_\nu)$ is the vorticity tensor both associated with V_A, and h is the projector operator onto the subspace of the tangent space perpendicular to V, given by $h_C{}^\nu = \delta^\nu_C - V_C V^\nu$.

In reference [**?**], Inomata found classes of Ψs satisfying the Heisenberg equation by assuming that the derivative of the spinor field could be written as a linear combination of the elements of the Clifford algebra using semilinear coefficients depending on Ψ. However, in our case, some self-interacting terms depend also on derivatives of Ψ, in particular, the vorticity associated with the Dirac current and, thus, Inomata's condition cannot be applied in general. Therefore, we propose a generalization of it assuming as an *ansatz* that the coefficients of the linear combination of the elements of the algebra could involve also quasi-linear terms, that is

$$\partial_B\Psi = \left(s_0 - \frac{mB\gamma_5}{3\sqrt{J^2}}\right)V_B\Psi - \left(s_0 + \frac{imA}{\sqrt{J^2}} - \frac{4mB\gamma_5}{3\sqrt{J^2}}\right)\frac{(A-iB\gamma_5)\gamma_B}{4\sqrt{J^2}}\Psi$$
$$+\gamma_B\gamma_C\omega^C(s_1+s_2\gamma_5)\Psi + \omega_B(s_3+s_4\gamma_5)\Psi, \qquad (2.61\text{-I})$$

where s_j for $j = 0,\dots,4$ are arbitrary functions of Ψ to be determined. If we apply γ^B to equation (2.61-I), we get the Dirac operator on the left-hand side. In order to have the Dirac equation satisfied, that is $i\gamma^A\partial_A\Psi - m\Psi = 0$, we impose the following constraints on the free functions

$$s_3 = -4s_1, \quad \text{and} \quad s_4 = -4s_2.$$

Therefore, Ψ given by equation (2.61-I) is a solution of the massive Dirac equation in the vicinity of $p \in \mathcal{M}$.

However, we need to guarantee that this class also verify the remaining equation constituted by the other terms of equation (2.60-I). Then, applying $\gamma^A V_A V^B$ to equation (2.61-I) and substituting the outcome into equation (2.60-I), we get

$$\left[\left(\frac{3s_0}{4} - \frac{imA}{4\sqrt{J^2}} + \frac{\dot b}{2b} + \frac{1}{2}\partial_\mu V^\mu\right)V_A + s_1\omega_A - \frac{3a+2b}{b(a+b)}h_A{}^C\partial_C a + \frac{1}{2}\dot V_A\right]\gamma^A\Psi$$
$$+ \quad \left(is_2 - \frac{b}{2(a+b)}\right)\omega_A\gamma^A\gamma_5\Psi = 0, \qquad (2.62\text{-I})$$

where we gather the terms according to the linear independence of the algebra elements and the orthogonality with respect to V_A. At the end, we obtain three equations determining completely the remaining free functions of (2.61-I)

$$s_0 = \frac{imA}{3\sqrt{J^2}} + \frac{2}{3}\left(\ln\frac{\sqrt{J^2}}{b}\right)^{\bullet}, \qquad (2.63\text{-Ia})$$

$$s_2 = -\frac{ib}{2(a+b)}, \qquad (2.63\text{-Ib})$$

$$\dot V_A = \frac{3a+2b}{b(a+b)}h_A{}^C\partial_C a - 2s_1\omega_A \qquad (2.63\text{-Ic})$$

plus the conservation law associated with the axial current (2.44-Ib), with $S_d = 2amB$ and $S_b = 2mB$, restricted to the case (2.49-I), that is

$$2\frac{a'}{a} + \frac{a'+b'}{a+b} = 0, \qquad (2.64\text{-I})$$

where prime means covariant derivative projected along to I^μ. Note that equations (2.63-Ia) and (2.63-Ib) provides s_0 and s_2 algebraically in terms of Ψ and the tetrad coefficients, but instead we have to solve the differential equation (2.63-Ic) to find s_1. Due to Helmholtz's decomposition[5] (also known as the fundamental theorem of the vector calculus) and the relation between a and b given by equation (2.64-I), this equation always admits a solution for s_1, once on the right hand side the first term is the gradient of a function of a and the second one is a rotational. We have also used the conservation law for J^μ to write $\partial_\mu J^\mu = 0$ as $\partial_\mu V^\mu = -(\ln\sqrt{J^2})^\bullet$. It should be remarked that the disformal invariance of the Dirac equation can be restored without choosing a Fermi normal coordinates if we modify the special class of solutions (2.61-I) properly through

$$\partial_B \longrightarrow \nabla_B, \quad \text{and} \quad \omega_A \longrightarrow \omega_A + \epsilon_{AB}{}^{CD} e_\nu{}^B \dot{e}^\nu{}_C V_D.$$

Since the set of vectors $\{e^\mu{}_B\}$ is an orthonormal basis, $\dot{e}^\mu{}_C$ is precisely the acceleration associated with the tetrad frame and, therefore, the modifications caused by this in the equations above are not purely mathematical, but instead they have a clear physical interpretation in terms of the kinematical quantities of a congruence of curves.

Summarizing the results obtained above, we have shown that the disformal transformation of a modified Dirac equation written in $\hat{g}_{\mu\nu}$ leads to a nonlinear equation for Ψ in $g_{\mu\nu}$. It means that the solutions of the former also satisfy the latter. Then, we select a sub-class of Ψs satisfying a generalized Inomata's condition in order that this nonlinear equation in $g_{\mu\nu}$ reduces to the Dirac equation

$$i\gamma^\mu \nabla_\mu \Psi - m\Psi = 0. \tag{2.65-I}$$

Therefore, it was provided sufficient conditions for the disformal map of the Dirac equation, as claimed by theorem 2.7, achieving the disformal invariance case the hypothesis of the corollary 2.8 holds true. It is important to note that the search for explicit solutions of the Dirac equation satisfying the generalized Inomata condition (2.61-I) is a hard task. In particular, this quasi-linear system of PDEs is not verified for simple solutions of the Dirac equation.

2.1.6 The case of light-like Dirac current

The results presented above can be easily extended for the case when the Dirac current is light-like ($A = B = 0$). However, the expression (2.45-I) for the disformal metric is not appropriate because V^μ has J^2 in the denominator. Therefore, the procedure corresponds to replace $\beta \longrightarrow \beta J^2$ and $b \longrightarrow bJ^2$ in the disformal metric and its corresponding tetrad basis and substitute V^μ explicitly in terms of the Dirac

[5] This is subtle point: the Helmholtz decomposition is only valid in dimension three. Fortunately it can be applied in our case, since $V^A \dot{V}_A = 0$. In fact, when contracted, the first term vanishes when the projector is applied to V and the second vanishes because the vorticity has a term with V_D and the Levi-Civita symbol. Hence, $\dot{V}^A$ lies in a three dimensional subspace (namely, the one orthogonal to V^A).

current J^μ. We then have

$$\hat{g}^{\mu\nu} = \alpha g^{\mu\nu} + \beta J^\mu J^\nu, \quad \text{and} \quad \hat{g}_{\mu\nu} = \frac{1}{\alpha} g_{\mu\nu} - \frac{\beta}{\alpha^2} J_\mu J_\nu, \tag{2.66-I}$$

with the tetrad basis and its inverse given respectively by

$$\hat{e}^\mu{}_A = a\, e^\mu{}_A + b J^\mu J_A, \quad \text{and} \quad \hat{e}_\mu{}^A = \frac{1}{a} e_\mu{}^A - \frac{b}{a^2} J_\mu J^A. \tag{2.67-I}$$

Once light-like currents are linked only to massless particles, the disformal map consists in rewriting the massless Dirac equation defined in the disformal metric (equation (2.46-I) with $m = 0$) in terms of the objects associated with the background metric. A straightforward calculation shows that the expression corresponding to equation (2.59-I) is

$$i\gamma^A \nabla_A \Psi - \frac{i}{2a}(b\dot{J}_B + 3\partial_B a)\gamma^B \Psi - \frac{b}{4a}\sigma_A \gamma^A \gamma_5 \Psi = 0, \tag{2.68-I}$$

where $\sigma_A \equiv \epsilon_{AB}{}^{CD} e_\nu{}^B J_D \dot{e}^\nu{}_C$, and we used the identity for light-like currents derived in corollary 2.3. This identity was also used to show that

$$\gamma^A J_A J^B \partial_B \Psi = -\gamma^A \dot{J}_A \Psi.$$

Thus, the sufficient conditions to have the disformal invariance of the Dirac equation are

$$\partial_B a = -\frac{b}{3}\dot{J}_B, \quad \text{and} \quad \sigma_A = 0. \tag{2.69-I}$$

Note that these equations take into account only space-time objets, i.e., no mention to the elements of the Clifford algebra, constraining only the two coefficients of the disformal metric and the tetrad basis of the background metric. In this case, the generalized Inomata condition is superfluous and it is particularly easier to provide examples of the disformal invariance of the massless Dirac equation.

Example 2.9. There is a simple example in this case, if we assume the background metric as the Minkowski space-time in standard coordinates (i.e., $e^\nu{}_B$ is the Kronecker delta and $\dot{e}^\nu{}_C = 0$, for every point) and Ψ as a plane wave solution of the massless Dirac equation in this background (exactly the one in example 2.1):

$$\Psi = \Psi_0 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} e^{-iE(t+z)}, \tag{2.70-I}$$

where Ψ_0 is a constant amplitude and E is the energy of the particle described by Ψ. The vector and axial currents are, in lower indices form, $J_B = 2|\Psi_0|^2(1,0,0,1) = -I_B$. Therefore, equations (2.69-I) are identically satisfied if we set a equal to a constant. The conservation law of the axial current (2.64-I) impose that $b = b(t-z,x,y)$.

Finally, introducing auxiliary coordinates $\eta = (t-z)/a$, $\xi = (t+z)/a$, $\tilde{x} = x/a$ and $\tilde{y} = y/a$, the disformal metric associated with (2.70-I) gives the infinitesimal line element

$$\begin{aligned} \widehat{ds}^2 &= \left(\frac{1}{\alpha}\eta_{\mu\nu} - \frac{\beta}{\alpha^2} J_\mu J_\nu\right) dx^\mu dx^\nu \\ &= d\eta d\xi - d\tilde{x}^2 - d\tilde{y}^2 - \tilde{\beta}(\eta,x,y) d\xi^2, \end{aligned} \tag{2.71-I}$$

with $\tilde{\beta} = 4\beta|\Psi_0|^4/a^2$. The only nonzero component of the Riemann curvature tensor [?] is $R_{\eta\xi\eta\xi} = \tilde{\beta}_{,\eta\eta}/2$, showing that the disformal metric is not flat in general. If we write

$$\partial_B \Psi = -iE\,\frac{J_B}{J_0}\Psi, \quad \text{and} \quad \gamma^A \hat{\Gamma}_A \Psi = -\frac{1}{2}\dot{b} J_A \gamma^A \Psi, \tag{2.72-I}$$

and use Pauli-Kofink (2.19-I) which gives $\gamma_A J^A \Psi = 0$ (see corollary 2.3). Using these equations, it is easy to see that the spinor in equation (2.70-I) satisfies the massless Dirac equation in the disformal metric (2.53-I). □

2.2 The CLB formalism in the scenario of rainbow gravity

Disformal transformations can naturally be seen as pure mathematics that are attractive on their own and have various physical applications. In particular, we have seen here different representations for performing disformal transformations in the context of differential geometry, essentially by means of metric tensors.

Nevertheless, we are now interested in the applications of the new formalism developed in chapter 1 in what concerns phenomenological approaches to quantum gravity, specifically rainbow gravity [?]. It is believed that disformal transformations as presented above provide a unified language for deforming a background space-time metric in this scenario and can shed light on some fundamental problems there, like covariance and causality. It should be stressed that the validity of rainbow gravity as a theory is not being investigated or even discussed here.

2.2.1 Rainbow gravity and disformal metrics

The formulation of rainbow gravity is a phenomenological modification of general relativity that incorporates some properties of the doubly special relativity (DSR) program [?]. DSR models deform the kinematics of special relativity, modifying also the energy-momentum conservation laws and the Lorentz symmetry group, by admitting an invariant energy scale associated with quantum-gravitational effects: the Planck scale. The motivation behind this stems from the path used to go from Galilean relativity to the special relativity, modifying the kinematic equations of the former in order to arrive at an invariant velocity scale. Following the same lines, it is possible to deform the latter by taking into account an invariant energy scale, which is generally believed to correspond to quantum-space-time effects, and thus derive the DSR without violation of the relativity principle. For pioneering works see references [?, ?, ?], and for a broad review see reference [?].

Following the prescription presented in [?] to perform such modifications, one can deform the momentum space of a particle with momentum $\pi = (p_0, p_i)$ using a function U that depends on the ratio between the particle energy p_0 and the Planck energy $\kappa \equiv 1/\sqrt{G}$, where G is Newton's constant, as follows[6]:

$$U(p_0, p_i) = \Big(f_1(p_0/\kappa)p_0, f_2(p_0/\kappa)p_i\Big), \tag{2.73-I}$$

[6] Considering units in which $c = \hbar = 1$.

leading to the modified dispersion relation (MDR)

$$\|\pi\|^2 = \eta^{AB}\,[U(\pi)]_A\,[U(\pi)]_B = (f_1)^2 p_0^2 - (f_2)^2 |\vec{p}|^2. \tag{2.74-I}$$

In order to guarantee the invariance of this MDR, Lorentz symmetry transformations also need to be deformed. Although this deformation was initially intended to take place in the Minkowski space, the idea of rainbow gravity is that such a MDR can be described by energy-dependent tetrad fields, which in turn produce an energy-dependent (rainbow) metric. In fact, from equation (2.74-I), one can write

$$\begin{aligned} \|\pi\|^2 &= \tilde{p}_0^2 - \delta^{IJ}\tilde{p}_I\tilde{p}_J \\ &= \left(\tilde{e}_0^{\ \mu}\pi_\mu\right)^2 - \delta^{IJ}\left(\tilde{e}_I^{\ \mu}\pi_\mu\right)\left(\tilde{e}_J^{\ \nu}\pi_\nu\right) \\ &= (f_1)^2 p_0^2 - (f_2)^2\delta^{IJ} p_I p_J \\ &= (f_1)^2\left(e_0^{\ \mu}\pi_\mu\right)^2 - (f_2)^2\delta^{IJ}\left(e_I^{\ \mu}\pi_\mu\right)\left(e_J^{\ \nu}\pi_\nu\right). \end{aligned} \tag{2.75-I}$$

Denoting the inverse of $e_A^{\ \mu}$ by $\theta^A_{\ \mu}$ [7], the squared line element is then given by

$$\begin{aligned} ds^2 &= \tilde{g}_{\mu\nu}dx^\mu dx^\nu = \tilde{\theta}^A_{\ \mu}\tilde{\theta}^B_{\ \nu}\eta_{AB}dx^\mu dx^\nu \\ &= \left(\theta^0_{\ 0}\right)^2\frac{\left(dx^0\right)^2}{f_1^2} - \delta_{IJ}\theta^I_{\ i}\theta^J_{\ j}\frac{1}{f_2^2}dx^i dx^j, \end{aligned} \tag{2.76-I}$$

where δ_{IJ} is the Kronecker delta, for $I,J = 1,2,3$. This means that space-time is deformed in a way that is the inverse way of the deformation in the momentum space (for details, see reference [**?**]) [8]. The U transformation defined in equation (2.73-I) resembles the ones we have considered throughout chapter 1, for a suitable choice of the disformal operator $\widetilde{D}$.

In fact, considering a time-like 1-form field $\widetilde{V}$ as defining a preferred direction in space-time leads to the definition of energy as the projection of the four-momentum π onto the direction of the corresponding normalized 1-form vector $v \equiv \widetilde{V}/\sqrt{h(\widetilde{V},\widetilde{V})}$, that is, $p_0 \equiv h(\pi, v)$. Therefore, the covector responsible for the disformal transformation introduces a natural time-like direction to the reference frame. Thus, using an orthonormal basis $\{v, \theta^I\}$, an immediate conclusion one can get from this analysis is that the disformal momentum assumes the form

$$\hat{\pi} = \widetilde{D}(\pi) = \frac{1}{\sqrt{\alpha+\beta}}\,p_0\,v + \frac{1}{\sqrt{\alpha}}\,p_I\,\theta^I, \tag{2.77-I}$$

where α and β are now scalar functions depending on π and v, and are linked to the rainbow functions f_1 and f_2 through equation (2.73-I):

$$\alpha = (f_2)^{-2} \quad \text{and} \quad \beta = (f_1)^{-2} - (f_2)^{-2}. \tag{2.78-I}$$

Furthermore, from the definition of the particle mass as the norm of π, a MDR naturally appears from this map in complete analogy with equation (2.74-I):

$$\hat{m}_\pi^2 \equiv \hat{h}(\pi,\pi) = \left(\frac{1}{\alpha+\beta}\right)p_0^2 - \frac{1}{\alpha}\,\delta^{IJ}\,p_I\,p_J. \tag{2.79-I}$$

[7] This relation implies that $\tilde{\theta}^A_{\ \mu} = (f_{1,2})^{-1}\theta^A_{\ \mu}$.

[8] This way, since the spacetime probed by each particle depends on its energy/frequency, we refer to this as a rainbow spacetime in the sense that the trajectories and the geometry of the probed spacetime depends on the "color" of the particle.

Finally, the equivalence between the formalism we developed here and rainbow gravity is fulfilled by deriving the induced space-time metric (see equation (2.76-I))

$$\widehat{ds}^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu = (\alpha+\beta)\left(\theta^0{}_0\right)^2\left(dx^0\right)^2 - \alpha\,\delta_{IJ}\theta^I{}_i\theta^J{}_j dx^i dx^j. \tag{2.80-I}$$

Thus, one could unequivocally identify the energy that appears in equation (2.73-I) as $p_0 = h(\pi,\nu)$ as well as the respective time-like direction that defines the deformation. It should be stressed that although this formalism seems to impose a preferred inertial frame in space-time, which would break the local canonical Lorentz symmetry, in the light of a DSR formulation this is not at all the case, since a *deformed version of the Lorentz transformation is the one that preserves the local relativity principle*: this is the main conceptual achievement of DSR. Not taking the existence of deformations into account, then this would lead to a formalism with Lorentz invariance violation and, consequently, a preferred reference frame. It should also be noted that the formalism we developed here is intrinsically geometric and that it is fully covariant under coordinate transformations.

We now make use of the literature on rainbow gravity (cf. references [**?**, **?**, **?**]) to illustrate with some examples how this relation works in practice.

Example 2.10 (The case with $f_1 = f_2$)**.** This will not alter the light cone. If f_1 is equal to f_2, then $\beta = 0$ and the disformal transformation reduces to a conformal one. A well-known example of this was first proposed in reference [**?**]:

$$f_1(E/\kappa) = f_2(E/\kappa) = \frac{1}{1-E/\kappa} \tag{2.81-I}$$

implying that

$$\alpha = \left[1 - \frac{h(\pi,\nu)}{\kappa}\right]^2. \tag{2.82-I}$$

This choice of deformation yields a maximum energy for a one-particle system, given by κ, and the causal structure is, of course, kept invariant. □

Example 2.11 (The case with $f_2 = 1$)**.** This second example (cf. details in reference [**?**]) has an invariant spatial contribution for the dispersion relation. Let

$$f_1(E/\kappa) = \frac{e^{E/\kappa}-1}{E/\kappa} \quad \text{and} \quad f_2(E/\kappa) \equiv 1. \tag{2.83-I}$$

In terms of the disformal functions, we get

$$\alpha = 1, \quad \text{and} \quad \beta = \left(\frac{h(\pi,\nu)/\kappa}{e^{h(\pi,\nu)/\kappa}-1}\right)^2 - 1. \tag{2.84-I}$$

For this dispersion relation, one can calculate the speed of light as

$$\left.\left(\frac{dE}{dp}\right)\right|_{m=0} \approx 1 - E/\kappa. \tag{2.85-I}$$

Therefore, ultraviolet (high-energy) photons propagate with a speed smaller than infrared (low-energy) ones within this model. Note that this is completely consistent with the causal structure analysed in proposition 1.12. Since $-1 < \beta < 0$, the disformal light cone lies inside the undeformed one. □

Example 2.12 (The case with $f_1 = 1$)**.** This third example is the counterpart of the previous one (see reference [**?**] and references therein), in the sense that the time contribution is now kept invariant. Consider

$$f_1 = 1 \quad ,\text{and} \quad f_2 = \left[1 + \left(\frac{|\vec{p}|}{\kappa}\right)^4\right]^{\frac{1}{2}}. \tag{2.86-I}$$

In terms of the metric coefficients, we obtain

$$\alpha = \frac{\kappa^4}{k^4 + [h^2(\pi,\nu) - h(\pi,\pi)]^2} \tag{2.87-I}$$

$$\beta = \frac{[h^2(\pi,\nu) - h(\pi,\pi)]^2}{\kappa^4 + [h^2(\pi,\nu) - h(\pi,\pi)]^2}. \tag{2.88-I}$$

For this dispersion relation the deformed speed of light is

$$\left.\left(\frac{dE}{dp}\right)\right|_{m=0} \approx 1 + \frac{5}{2}(|\vec{p}\,|/\kappa)^4, \tag{2.89-I}$$

which means that high-energy photons propagate with a speed larger than low-energy ones. Again, this is compatible with the causal structure. Since $0 < \beta < 1$, the disformal light cone lies outside the undeformed one. □

Conclusions

It was shown that disformal (co)metrics can be written in terms of a linear isomorphism acting on the (co)tangent space and that they actually inherit the properties of what we called the disformal operators. From the reasons presented in the text, these operators can be seen as a more fundamental quantity than the disformal metric, providing a new mathematical framework for disformal transformations in general. We then analyzed this new facet of disformal transformations in the light of the causal structure, where it gives rise to an alternative interpretation of the modified causal cones in purely algebraic terms.

Then, we have analyzed the action of the disformal transformations on the case of propagating spinor fields making use of the Weyl-Cartan formalism. In particular, we have shown that generalizing the Inomata condition it is possible to find a class of solutions which let the Dirac equation almost invariant under these maps, up to a conformal factor in the mass term. That is, they verify this equation in the disformal and background metrics once a set of conditions are fulfilled. However, explicit expressions for Ψ satisfying the hypotheses of theorem 2.7 are not simple. Preliminary attempts have pointed out that if Ψ is a superposition of plane waves this could be solved, the complete analysis is postponed for future projects.

The situation in which the norm of J^{μ} is zero could be easily obtained from previous results by modifying the disformal term in the metric and the tetrad basis, and then taking the limit $J^2 \to 0$. In comparison with the case $J^2 \neq 0$, this leads to a simpler set of equations for Ψ in the background metric and requiring only two conditions upon space-time objects for the disformal invariance of the massless Dirac equation. In this context, a simple example in terms of plane waves allows for the construction of a curved disformal metric. It should be noted that the equation (2.66-I) could correspond to the Kerr-Schild metrics of general relativity, which means that the disformal invariance of the Dirac equation could be useful in the study of propagating spinor fields on these backgrounds.

Finally, as a direct application of the CLB formalism developed previously, we verified that the most relevant models in rainbow gravity are perfectly described in terms of disformal transformations. In this vein, it was possible to obtain the missing covariant approach for such a phenomenological theory, with a well-behaved causal structure and a clear mathematical interpretation of the physical quantities involved.

Part II

Scalar field self-force effects on a particle orbiting a Reissner-Nordström black hole

INTRODUCTION

Scalar self-force (SSF) effects arise when a *scalar charge*, moving along a given orbit in a curved space-time, interacts with its own gravitational field, i.e., its self-field. The associated scalar field satisfies a d'Alembert-like equation with source term singular at the position of the particle, mimicking the more interesting situation of gravitational perturbations induced by a small mass moving in a gravitational background modified by its own presence. The interaction of the particle with its own gravitational field in this case gives rise to a gravitational self-force (GSF) (see, e.g., reference [**?**] and references therein). It is a matter of fact that the latter problem is physically more interesting than the former. However, the study of the first problem is easier than the second, even though the approaches, as well as the computational techniques, used in both cases are similar. This explains why in the literature the SSF problem has been considered as a preliminary study to the GSF one, scouting/solving all technical difficulties also affecting the more general gravitational perturbation problem. The existing literature on this topic is very rich. Indeed, besides the various pioneering works developing the fundamental formalism for self-force calculations in a curved space-time [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**], a number of interesting papers has been produced over the years, aiming at understanding self-force effects in black hole space-times, mostly Schwarzschild and Kerr [**?**, **?**].

The present part concerns to scalar field self-force effects on a scalar charge moving along a spatially circular equatorial orbit around a Reissner-Nordström (RN) black hole. The interaction between the particle and the background field is therefore of the gravitational type only – the particle carrying no electromagnetic charge. We are interested in studying the coupling between the scalar charge of the associated field with the mass and the electromagnetic charge of the nonvacuum background, which was never explored before. This coupling can be seen as a gravitationally induced scalar interaction, which is complementary to existing studies on "gravitationally induced electromagnetic radiation" as well as "electromagnetically induced gravitational radiation" processes, initiated long ago by Zerilli and coworkers [**?**, **?**, **?**]. The only available analytical study of such a kind of perturbation problem in a RN space-time involves an electromagnetic charge at rest in a perturbed RN space-time [**?**, **?**, **?**], where the effect of charge induction on the horizon is also investigated. Allowing the electromagnetic charge to move around the hole complicates matters considerably. On the other hand, having a scalar charge in circular motion is an intermediate step toward such a more general situation, the advantage of which is the possibility to perform the computations fully analytically. Switching off the black hole charge one ends up with the corresponding SSF problem in the vacuum Schwarzschild space-time, which has been already addressed in the literature from both analytical and numerical perspectives [**?**, **?**, **?**, **?**].

The main technical difficulties associated with self-force calculations are related to the regularization procedure of the scalar field and its derivatives, allowing one to extract the correct, physically meaningful, self-force components.

Summary of part II

For convenience, a detailed structure of the subsequent chapters of the second part of this thesis is given below.

Chapter 1: This chapter starts reviewing the RN solution for the coupled Einstein-Maxwell equations. A brief review of differential geometry, concerning geodesics and the box operator $\Box$, is given for completeness. Then, in section 1.2, a comprehensive introduction to the scalar self-force problem is given and discussed. In section 1.3 we derive the fundamental equation we are interested in. Finally, in section 1.4, the definition of the scalar self-force is provided.

Chapter 2: In section, 2.1 standard techniques, given in reference [**?**], are used to compute the self-field decomposed into spherical harmonics and frequency modes, and regularize it at the position of the particle, mode by mode, by subtracting the diverging large-l limit, obtaining the self-field up to 7.5 post-Newtonian order. In section 2.2, the components of the scalar self-force are computed analytically (also up to 7.5PN order) and the results are compared with previous numerical studies in the Schwarzschild case [**?**, **?**], obtaining a good agreement. Finally, in section 2.3, the analysis is completed by providing explicit expressions for the scalar radiation both at infinity and on the outer horizon.

1
Preliminaries

Lasciate ogne speranza, voi ch'intrate.

Dante Alighieri

This chapter starts revising the Reissner-Nordström space-time as a solution of the coupled Einstein-Maxwell equations and some of its geometric aspects. Subsequently, the hypothesis under which the scalar self-force problem relies on, as well as the problem itself, are discussed.

1.1 The Reissner-Nordström black hole

In what follows, consider that the units were chosen in order to have $G = c = 1$. Consider a spherical massive body that is also electrically charged and consider that the electromagnetic potential is denoted by A_μ. This electromagnetic potential is used to construct a rank-2 skew-symmetric tensor field defined by $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, in terms of which Maxwell's (source free) equations are given, in tensor notation, by

$$g^{\mu\nu}\nabla_\mu F_{\nu\alpha} = 0 \tag{1.1-II}$$
$$\nabla_{[\mu}F_{\nu\alpha]} = 0. \tag{1.2-II}$$

For completeness, using differential forms, Maxwell's equations are elegantly written as

$$df = 0 \tag{1.3-II}$$
$$\delta f = 0, \tag{1.4-II}$$

where d stands for exterior derivative operator, $\delta = *d*$ is the co-derivative operator, $*$ is the Hodge dual operator [?], $f = \frac{1}{2}F_{\mu\nu}\omega^\mu \wedge \omega^\nu$, and ω^α, $\alpha = 0,1,2,3$, are the basis forms. In our case,

$$f = -\frac{Q}{r^2}dt \wedge dr. \tag{1.5-II}$$

Intending to solve Einstein's equations [?, ?],

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.6-II}$$

together with Maxwell's (1.3-II) and (1.4-II), where the energy-momentum is given by

$$T_{\mu\nu} = \frac{1}{4\pi}\left[F_{\mu\rho}F_\nu^{\ \rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right], \tag{1.7-II}$$

the unique spherically symmetric solution is the so-called Reissner-Nordström solution. The RN solution has its squared line element given, in standard Schwarzschild-like coordinates (t, r, θ, ϕ) and metric signature $(-+++)$[1], by

$$ds^2 = -\frac{\Delta}{r^2}dt^2 + \frac{r^2}{\Delta} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.8-II}$$

with $\Delta = r^2 - 2Mr + Q^2$, and the constants M and Q should be interpreted as the gravitational mass and the electric charge of the central body, respectively. Additionally, the condition $\Delta = 0$ defines the two horizons at radii $r_\pm = M \pm \sqrt{M^2 - Q^2} \equiv M(1 \pm \kappa)$, with $\kappa = \sqrt{1 - Q^2/M^2}$. The so-called extreme RN case corresponds to $|Q| = M$ (or $\kappa = 0$), and the two horizons coalesce into one. It will be convenient to introduce also the new variable $w = 1 - \kappa^2 = (Q/M)^2$. Therefore, under the limit $w = 0$, the RN metric is equivalent to the Schwarzschild one.

In contrast to the Schwarzschild and Kerr solutions for Einstein's field equations, which are vacuum solutions, the Reissner-Nordström solution is a nonvacuum solution. In fact, it represents the space-time outside a spherically symmetric charged body carrying an electric charge (but with no spin or magnetic dipole [2]). An electrically charged body will be surrounded by an electric field yielding a nonzero energy-momentum tensor throughout space – namely, the one from the electromagnetic field in the space-time given by equation (1.7-II).

In an astrophysical situation, the total amount of charge is expected to be very small, especially when compared with the gravitational mass (in terms of the relative gravitational effects). Nevertheless, charged black holes provide a useful testing ground for various thought experiments and toy models in theoretical astrophysics, so they are worth considering.

1.1.1 Some differential geometry

In this section it is provided the definitions of a geodesic and the box operator $\Box$. First, consider a curve $\gamma : I \subset \mathbb{R} \longrightarrow \mathscr{M}$ parametrized by arc-length (proper time) τ. The curve γ is said to be a geodesic if it is *parallel transported*[3] along itself. In symbols,

$$\Gamma(T\mathscr{M}) \ni \nabla_{\dot{\gamma}}\dot{\gamma} = 0, \tag{1.9-II}$$

where $\dot{\gamma}$ represents the velocity vector of the curve γ, i.e. $\dot{\gamma} = d\gamma/d\tau$. Given a local coordinate system $\{x^\mu\}$ for $\mathscr{M}$, recall that the covariant derivative of a vector field is another vector field with components given by

$$\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu_{\mu\lambda} X^\lambda, \tag{1.10-II}$$

[1] It is assumed this signature convention throughout this part.

[2] So, for instance, this is not a good representation of the field outside an electron.

[3] Some authors define geodesics as curves solving the variational problem of locally extremize the distance between two points instead of autoparallel curves. Since we are using the Levi-Civita connection, the two definitions are equivalent.

from which one can recover the usual geodesic equations

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\tau}\frac{dx^\rho}{d\tau} = 0, \tag{1.11-II}$$

where $x^\mu(\tau)$ is the coordinate expression for γ and $\Gamma^\mu_{\sigma\rho}$ are the Christoffel symbols. The system of ordinary differential equations summarized in equation (1.11-II) has a unique solution given a starting point and an initial (normalized) velocity vector.

The second geometric object that we need to define is the box operator $\Box$ acting on real valued functions defined on $\mathscr{M}$. Given a function $\Phi : \mathscr{M} \longrightarrow \mathbb{R}$, $\Box\Phi$ is defined as the trace of the second (covariant) derivative of Φ. The final expression is shown to be [**?**]

$$\Box\Phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi). \tag{1.12-II}$$

In our case, since we are dealing with a metric with Lorentzian signature, the box operator defined above is called the d'Alembertian of Φ and has a hyperbolic character, whereas for Riemannian metrics the box operator has an elliptic character and coincides with the usual Laplace-Beltrami operator[4]. The nonzero connection coefficients (Christoffel symbols) of the RN space-time, computed using the standard formula [**?**]

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right) \tag{1.13-II}$$

are given below, for completeness:

$$\begin{aligned}
\Gamma^0_{01} &= \frac{Mr-Q^2}{r\Delta}, \;\; \Gamma^1_{00} = \frac{\Delta\left(Mr-Q^2\right)}{r^5} \\
\Gamma^1_{11} &= -\frac{Mr-Q^2}{r\Delta}, \;\; \Gamma^1_{22} = -\frac{\Delta}{r} \\
\Gamma^1_{33} &= -\frac{\Delta(\sin\theta)^2}{r}, \;\; \Gamma^2_{12} = \frac{1}{r} \\
\Gamma^2_{33} &= -\sin\theta\cos\theta, \;\; \Gamma^3_{13} = \frac{1}{r}, \;\; \Gamma^3_{23} = \cot\theta.
\end{aligned} \tag{1.14-II}$$

Example 1.1 (Geodesics and d'Alembertian in Minkowski space-time)**.** Consider the Minkowski space-time with the standard coordinate system (t,x,y,z), in units where $c = 1$. Using equation (1.13-II) it is easy to show that all Christoffel symbols are zero in that case, reducing the geodesic equation to

$$\frac{d^2x^\mu}{d\tau^2} = 0. \tag{1.15-II}$$

Given an initial point p and a initial unitary velocity vector $\vec{v}$, the previous equation implies that the geodesic passing through p with velocity $\vec{v}$ is a straight line in the direction of $\vec{v}$.

[4]This point was missed, as we know, for a long time, since relativity was developed much later than potential theory and the theory of surfaces.

Using equation (1.12-II), and considering a function $f : \mathscr{M} \longrightarrow \mathbb{R}$, one can easily show that the d'Alembertian of f is given by

$$\Box f = -\frac{\partial^2 f}{\partial t^2} + \Delta f, \tag{1.16-II}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is minus the usual Laplace operator in $\mathbb{R}^3$ (here assuming the convention in which the spectrum of the Laplacian is bounded from below). Therefore, the condition $\Box f = 0$ is simply the usual wave equation. □

Example 1.2 (The d'Alembertian in the Reissner-Nordström space-time)**.** Consider a real-valued function $f : \mathscr{M} \longrightarrow \mathbb{R}$. Given the coordinate system $\{t, r, \theta, \phi\}$ for the RN space-time, one can express the function as $f(x^\mu) = f(t, r, \theta, \phi)$ and use equation (1.12-II) to find

$$\begin{aligned} \Box_{\mathrm{RN}} f &= -\frac{1}{g(r)}\frac{\partial^2 f}{\partial t^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 g(r)\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \\ g(r) &\equiv \frac{\Delta(r)}{r^2}. \end{aligned}$$

□

1.1.2 The Reissner-Nordström metric as a disformal metric

For the sake of consistency with the first part of this thesis, this subsection is dedicated to show how could one express the Reissner-Nordström metric as a disformal transformation of the Minkowski metric. The Reissner-Nordström metric is a special case of the Kerr-Newman metric when one sets the angular momentum parameter to be equal to zero. Furthermore, the Kerr-Newman metric can also can be written in another formulation, called the Kerr-Schild formulation. For the Reissner-Nordström case we have the Kerr-Schild metric with components in Cartesian coordinates (t, x, y, z) given by

$$g_{\mu\nu} = \eta_{\mu\nu} + \beta V_\mu V_\nu \tag{1.17-II}$$

where η is the Minkowski metric β is a function and $V \in \Gamma(T^*\mathscr{M})$, given by

$$\beta = \frac{2Mr - Q^2}{r^2}, \tag{1.18-II}$$

$$V = \left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right), \tag{1.19-II}$$

$$r^2 = x^2 + y^2 + z^2. \tag{1.20-II}$$

It easy to check that V is light-like ($V_\mu V^\mu = 0$) in the background metric. Additionally, the electromagnetic potential is given by

$$A_\mu = \frac{Q}{r} V_\mu \tag{1.21-II}$$

from which one can recover the electromagnetic tensor $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. The metric in equation (1.17-II) is shown to be a solution for the coupled Einstein-Maxwell equations (1.3-II), (1.4-II), and (1.6-II).

The conditions $\alpha > 0$ and $\alpha + \beta > 0$ we found in part I are necessary when the disformal transformation of the metric is made with a time-like vector field. For the case of Kerr-Schild metrics, in which the transformation is made with a light-like vector, the determinant of the disformal metric is known to be always negative and the signature is the same as the background metric, being automatically a metric tensor.

1.2 Statement and hypotheses of the scalar self-force problem

Suppose that we have an infinitesimal mass (compared to the mass M) orbiting around a circular and equatorial geodesic of a RN space-time. In general relativity, a particle of infinitesimal mass will orbit around a black hole of much larger mass along a worldline which is an exact geodesic in the background geometry. If the orbiting particle is not infinitesimal, having a small finite mass compared to the larger one, its orbit will no longer be a geodesic in the background of the larger mass, and gravitational waves will be emitted by the system – at infinity. The diference of the worldline of the particle from a geodesic in the background geometry is said to arise from the interaction from the interaction of the orbiting particle with its own gravitational field. It is said to result from a *self-force*, even though, in the perturbed geometry determined by both the small and large masses, the orbit would be observed to be a geodesic. The main task in this part is to calculate the *regularized* self-force acting on a pointlike particle with mass $\mu \ll M$ and scalar charge in orbit around a RN black hole. By *scalar charge* it is meant a pointlike, spin-0, particle carrying a charge q that acts as a source for a scalar field ψ. In this case, there will be just a gravitational interaction between the central massive object and the infinitesimal particle, and the background geometry suffers just a small perturbation as shown below. As a consequence, all the geometrical entities considered (covariant derivatives, geodesics, differential operators, etc.) are related to the background metric alone.

In analogy with the case in which μ is not infinitesimal compared to M, where in a first-order (in μ/M) perturbative calculation the particle moves on a geodesic of the background space-time, we take the orbit of the particle to be a geodesic and calculate the self-force as a vector field on this geodesic. It is important to emphasize that, in this work, the back reaction from the scalar self-force on the motion of the particle is not going to be considered. The goal is merely to calculate the scalar self-force that would be felt by a particle fixed on a geodesic orbit.

1.3 Derivation of the fundamental equation

We now turn to derive the equation we need to solve for the scalar field associated with this charged moving particle. Let ψ be a real-valued, minimally coupled scalar field associated with a pointlike particle with mass μ carrying a scalar charge q moving along a circular equatorial time-like curve parametrized by proper time according to

$$z^{\mu}(\tau):\ t = \Gamma\tau\,,\ r = r_0\,,\ \theta = \frac{\pi}{2}\,,\ \phi = \Gamma\Omega\tau = \Omega t. \qquad (1.22\text{-II})$$

Taking the derivative with respect to the proper time τ, the 4-velocity is given by

$$U = \frac{dz(\tau)}{d\tau} = \Gamma(\partial_t + \Omega\partial_\phi), \tag{1.23-II}$$

where $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$ are the basis vectors associated with the coordinate system $\{t,r,\theta,\phi\}$. Let us find the normalization factor Γ and the angular velocity Ω in terms of the inverse dimensionless radial distance $u = M/r$ in the case that the equatorial circular curve we defined is a geodesic. First, requiring that the time-like 4−velocity U is normalized ($U^\mu U_\mu = -1$), one finds the condition

$$\Gamma^{-2} = \frac{\Delta}{r_0^2} - \Omega^2 r_0^2. \tag{1.24-II}$$

To find Ω^2, one imposes the orbit to be a geodesic. Using equation, (1.9-II) one finds:

$$\begin{aligned} \nabla_U U &= \left(0, \Delta\Gamma^2\left(-\Omega^2 r + \frac{M}{r^2} - \frac{Q^2}{r^3}\right), 0, 0\right) \\ &\equiv (0,\ 0,\ 0,\ 0), \end{aligned} \tag{1.25-II}$$

which produces

$$\Omega^2 = \frac{Mr_0 - Q^2}{r_0{}^4}, \tag{1.26-II}$$

along the worldline. Replacing equation (1.26-II) in the right hand side of equation (1.24-II), and writing in terms of u and w, yields

$$\begin{aligned} M\Omega &= u^{3/2}\sqrt{1-wu}, \\ \Gamma &= \frac{1}{\sqrt{1-3u+2wu^2}}. \end{aligned} \tag{1.27-II}$$

A particle of mass μ and scalar charge q moving on a worldline $z(\tau)$ in a curved space-time with metric tensor $g_{\mu\nu}$ will create a scalar field ψ, and the dynamics of the scalar field ψ is described by the action principle

$$S(\psi) = S_{\text{free}}(\psi) + S_{\text{int}}(\psi), \tag{1.28-II}$$

where the free field action is the standard one

$$S_{\text{free}}(\psi) = \int \left(\frac{1}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi + \frac{1}{2}\mu^2\psi^2\right)\sqrt{-g}\,d^4x \tag{1.29-II}$$

while the interaction action is written in the form

$$S_{\text{int}}(\psi) = -4\pi\int \psi(x^\mu)\varrho(x^\mu)\sqrt{-g}\,d^4x. \tag{1.30-II}$$

Here, the minus sign and the factor 4π are purely conventional and $\varrho(x^\mu)$ is assumed to have support[5] along the worldline of the particle

$$\varrho(x^\mu) = q\int (-g)^{-1/2}\delta^4(x^\mu - z^\mu(\tau))\,d\tau. \tag{1.31-II}$$

[5] The support of a function is defined to be the smallest closed set of its domain containing all points not mapped to zero.

Requiring the extremality of the action with respect to the field ψ, i.e., $\delta S/\delta\psi(x) = 0$, leads to

$$0 = \int \left(g^{\mu\nu}\delta(\partial_\mu\psi)\partial_\nu\psi + \mu^2\psi(\delta\psi)\right)\sqrt{-g}d^4x - 4\pi\int(\delta\psi)\varrho(x^\mu)\sqrt{-g}d^4x\,. \qquad (1.32\text{-II})$$

Commuting δ with ∂_μ, integrating by parts the first term and assuming that there are no boundary contributions, one finds

$$0 = \int \delta\psi\left(-\frac{\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu\psi)}{\sqrt{-g}} + \mu^2\psi\right)\sqrt{-g}d^4x - 4\pi\int(\delta\psi)\varrho(x^\mu)\sqrt{-g}d^4x\,, \qquad (1.33\text{-II})$$

or equivalently, recalling the definition of d'Alembertian in equation (1.12-II),

$$\Box\psi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi), \qquad (1.34\text{-II})$$

we write

$$0 = \int \delta\psi\left(-\Box\psi + \mu^2\psi\right)\sqrt{-g}d^4x - 4\pi\int\delta\psi\varrho(x^\mu)\sqrt{-g}d^4x\,. \qquad (1.35\text{-II})$$

Finally,

$$-\Box\psi + \mu^2\psi - 4\pi\varrho(x^\mu) = 0\,. \qquad (1.36\text{-II})$$

At first order in μ (i.e., at the approximation order of interest here) the above equation reduces to

$$\Box\psi = -4\pi\varrho\,, \qquad (1.37\text{-II})$$

and we aim at solving perturbatively this equation in order to evaluate the solution field ψ along the worldline of the particle.

1.4 The scalar self-force

Once the solution $\psi(x^\mu)$ for equation (2.1-II) is found, the scalar self-force can be naively calculated as (for the definition of the scalar self-force see for instance [**?**])

$$F_\alpha(\tau) = q\nabla_\alpha\psi\left(z^\mu(\tau)\right), \qquad (1.38\text{-II})$$

and a physical analysis regarding ψ and F_α can be carried out. Following the lines discussed in [**?**] and references therein, it is not going to be assumed that the mass μ is changing with time. Since the relation between the scalar self-force and the derivative $d\mu/d\tau$ is given by

$$\frac{d\mu}{d\tau} = -U^\alpha F_\alpha, \qquad (1.39\text{-II})$$

in this stationary and circular orbit $U^\alpha F_\alpha = 0$, implying the relation

$$F_t + \Omega F_\phi = 0. \qquad (1.40\text{-II})$$

This trivial relation between the components of the self-force means that in the subsequent analysis one needs only to compute one of these components.

The appropriate way to regularize the scalar field ψ, described in [?], is to perform a decomposition

$$\psi_0 = \psi_0^{\mathrm{reg}} + B \qquad (1.41\text{-II})$$

in terms of the singular part B and the regular reminder ψ_0^{reg}. The function ψ_0^{reg} is differentiable and regular at the worldline of the particle, satisfies the homogeneous equation associated with equation (2.1-II), and is solely responsible for the self-force acting on the particle. On the other hand, B satisfies equation (2.1-II), is singular only at the position of the particle as the retarded solution, and produces no force on the particle. Rearranging equation (1.41-II) and differentiating once, one can write the regularized self-force as

$$F_\alpha \equiv q \nabla_\alpha \psi_0^{\mathrm{reg}} = q \left(\nabla_\alpha \psi_0 - \nabla_\alpha B \right). \qquad (1.42\text{-II})$$

These are the ingredients we need to proceed and solve the problem in the next chapter.

2
Computations

As coisas findas, muito mais que
lindas, essas ficarão.

Carlos Drummond de Andrade

This chapter aims at computing the scalar field ψ, the scalar self-force F_α and energy fluxes both at infinity and at the horizon. For the sake of simplicity, the techniques used in this chapter are better explained in the appendix A.

2.1 Computation of the scalar field

It was shown in the previous chapter that the field produced by the particle can be treated as a small perturbation on the fixed RN background. This hypothesis was shown to imply that the scalar field obeys the massless Klein-Gordon equation

$$\Box\psi = -4\pi\varrho\,, \qquad \text{(2.1-II)}$$

with

$$\Box\psi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi) \qquad \text{(2.2-II)}$$

denoting the d'Alembertian (g denoting the determinant of the metric) and

$$\begin{aligned} \varrho(x^\mu) &\equiv q\int(-g)^{-1/2}\delta^4(x^\mu - z^\mu(\tau))d\tau \\ &= \frac{q}{r_0^2\Gamma}\delta(r-r_0)\delta\left(\theta-\frac{\pi}{2}\right)\delta(\phi-\Omega t), \end{aligned} \qquad \text{(2.3-II)}$$

is the charge density of the scalar particle with support only along the worldline of the particle (1.22-II). Decomposing ϱ into spherical harmonics [1] then gives

$$\varrho = \frac{1}{4\pi r_0}\delta(r-r_0)\sum_{l=0}^{\infty}\sum_{m=-l}^{l} q_{lm}e^{-im\Omega t}Y_{lm}(\theta,\phi), \qquad \text{(2.4-II)}$$

for

$$q_{lm} = \frac{4\pi q}{r_0\Gamma}Y^*_{lm}(\pi/2,0). \qquad \text{(2.5-II)}$$

Similarly, the scalar field ψ dependence on temporal, radial and angular variables can be separated as

$$\psi(t,r,\theta,\phi) = \frac{1}{2\pi}\int\sum_{l,m}\psi_{lm\omega}(r)e^{-i\omega t}Y_{lm}(\theta,\phi)d\omega\,, \qquad \text{(2.6-II)}$$

[1] For an auxiliary reference in multipole decomposition, spherical harmonics, Dirac's δ function and the Green function method that used later, the reader should consult, for instance, [?].

for $\omega = m\Omega$. The decomposition (2.6-II) allows the wave equation (2.1-II) to be fully decoupled, thus reducing to the following equation for the radial part,

$$\mathscr{L}_{(r)}(\psi_{lm\omega}(r)) = S_{lm\omega}\delta(r-r_0), \tag{2.7-II}$$

with

$$\mathscr{L}_{(r)}(\psi_{lm\omega}(r)) \equiv \frac{d^2}{dr^2}\psi_{lm\omega}(r) + \frac{2(r-M)}{\Delta}\frac{d}{dr}\psi_{lm\omega}(r) + \left[\frac{\omega^2 r^4}{\Delta^2} - \frac{l(l+1)}{\Delta}\right]\psi_{lm\omega}(r), \tag{2.8-II}$$

whereas

$$S_{lm\omega} = -2\pi\frac{r_0}{\Delta_0}q_{lm}\delta(\omega - m\Omega), \tag{2.9-II}$$

with $\Delta_0 = \Delta(r_0)$, comes from taking the Fourier transform of the charge density (2.4-II).

The radial part of the scalar field is computed by using the Green function method as

$$\begin{aligned} \psi_{lm\omega}(r) &= \int G_{lm\omega}(r,r')\Delta(r')S_{lm\omega}\delta(r'-r_0)dr' \\ &= G_{lm\omega}(r,r_0)\Delta_0 S_{lm\omega}, \end{aligned} \tag{2.10-II}$$

for the Green function $G_{lm\omega}(r,r')$ satisfying the equation

$$\mathscr{L}_{(r)}(G_{lm\omega}(r,r')) = \Delta^{-1}(r')\delta(r-r').$$

It reads as

$$G_{lm\omega}(r,r') = \frac{1}{W_{lm\omega}}\left[R_{\rm in}^{lm\omega}(r)R_{\rm up}^{lm\omega}(r')H(r'-r) + R_{\rm in}^{lm\omega}(r')R_{\rm up}^{lm\omega}(r)H(r-r')\right], \tag{2.11-II}$$

$H(x)$ denoting the Heaviside step function, $R_{\rm in}^{lm\omega}(r)$ and $R_{\rm up}^{lm\omega}(r)$ are the two independent homogeneous solutions of the radial wave equation having the correct behavior at the outer horizon and at infinity, respectively, and

$$W_{lm\omega} = \Delta(r)\left[R_{\rm in}^{lm\omega}(r)R'^{lm\omega}_{\rm up}(r) - R'^{lm\omega}_{\rm in}(r)R_{\rm up}^{lm\omega}(r)\right] \tag{2.12-II}$$

is the associated constant Wronskian. Substituting then into equation (2.6-II) gives

$$\psi(x^\mu) = -\sum_{l,m} G_{lm\omega}(r,r_0)\Big|_{\omega=m\Omega} r_0 q_{lm} e^{-im\Omega t} Y_{lm}(\theta,\phi), \tag{2.13-II}$$

which, once evaluated along the worldline of the particle (1.22-II), becomes

$$\psi_0 = -\frac{4\pi q}{\Gamma}\sum_{l,m} G_{lm\omega}(r_0,r_0)\Big|_{\omega=m\Omega} |Y_{lm}(\pi/2,0)|^2, \tag{2.14-II}$$

only depending on r_0. The above expression for ψ_0 actually requires taking the limit $r \to r_0^\pm$ properly, and must be suitably regularized in order to remove its singular part, because the field has a divergent behavior there.

In order to compute the Green function one has first to solve the homogeneous radial wave equation (2.7-II) up to a certain post-Newtonian order to obtain the "in" and "up" solutions, which are of the form

$$\begin{aligned} R^{lm\omega}_{\mathrm{in(PN)}}(r) &= r^l[1 + A^{lm\omega}_2(r)\eta^2 + A^{lm\omega}_4(r)\eta^4 \\ &\quad + A^{lm\omega}_6(r)\eta^6 + A^{lm\omega}_8(r)\eta^8 + \ldots], \\ R^{lm\omega}_{\mathrm{up(PN)}}(r) &= R^{(-l-1)m\omega}_{\mathrm{in(PN)}}(r). \end{aligned} \tag{2.15-II}$$

However, these solutions do not automatically fulfill the correct boundary conditions. A consequence of this fact is the presence of divergent terms in the coefficients A_i for certain values of l. Therefore, high-order post-Newtonian solutions usually require a technique first introduced by Mano, Suzuki, and Takasugi (MST) [**?**, **?**]. Some details will be shown in Appendix A.

Turning then to equation (2.14-II), the sum over m is straightforwardly computed by using standard formulas. Before summing over l, instead, one has to remove the divergent term for large l, i.e.,

$$\psi_0^{\mathrm{reg}} = \sum_{l=0}^{\infty}(\psi_0^l - B). \tag{2.16-II}$$

The subtraction term turns out to be, in units of q,

$$\begin{aligned} B &= u - \frac{1}{4}u^2 + \left(\frac{9}{64} - \frac{3}{4}\kappa^2\right)u^3 + \left(-\frac{73}{32}\kappa^2 + \frac{199}{256}\right)u^4 \\ &\quad + \left(\frac{39625}{16384} - \frac{39}{64}\kappa^4 - \frac{1425}{256}\kappa^2\right)u^5 \\ &\quad + \left(-\frac{907}{256}\kappa^4 - \frac{52585}{4096}\kappa^2 + \frac{451007}{65536}\right)u^6 \\ &\quad + \left(-\frac{1926415}{65536}\kappa^2 - \frac{109317}{8192}\kappa^4 \right. \\ &\quad \left. + \frac{20043121}{1048576} - \frac{171}{256}\kappa^6\right)u^7 + O(u^8). \end{aligned} \tag{2.17-II}$$

Following the lines given in [**?**], this expression can be shown to be the Taylor expansion of

$$B_{\mathrm{analytic}} = \frac{u}{\sqrt{1-3u}}\frac{\sqrt{1-\sigma}}{\Gamma}\frac{2}{\pi}\mathrm{EllipticK}(\sigma), \tag{2.18-II}$$

where

$$\sigma = \frac{u[1 + u(1-\kappa^2)]}{1 - 2u + u^2(1-\kappa^2)}. \tag{2.19-II}$$

It is useful to introduce the dimensionless angular velocity variable

$$y = (M\Omega)^{2/3} = u(1 + wu)^{1/3}, \tag{2.20-II}$$

as from equation (1.27-II), with inverse relation

$$\begin{aligned} u &= y - \frac{1}{3}wy^2 + \frac{1}{3}w^2y^3 - \frac{35}{81}w^3y^4 + \frac{154}{243}w^4y^5 \\ &\quad - w^5y^6 + \frac{10868}{6561}w^6y^7 + O(y^8), \end{aligned} \tag{2.21-II}$$

for $w = 1-\kappa^2$.

By applying the MST technique [?, ?] to the multipoles up to $l = 4$ (included), one gets the following final result for the regularized field valid up to the 7.5 PN order

$$
\begin{aligned}
\psi_0^{\rm reg} \quad = \quad & -y^3 + \left[\frac{35}{18} + \left(-\frac{7}{32} + \frac{w}{32}\right)\pi^2 - \frac{4}{3}\gamma - \frac{4}{3}\ln(2) - \frac{2}{3}\ln(y)\right] y^4 \\
& + \left[\frac{1141}{360} - \frac{35}{54}w + \left(\frac{29}{512} + \frac{97}{1536}w - \frac{w^2}{96}\right)\pi^2 + \left(\frac{2}{3} - \frac{8}{9}w\right)\gamma + \left(-\frac{18}{5} - \frac{8}{9}w\right)\ln(2) + \left(\frac{1}{3} - \frac{4}{9}w\right)\ln(y)\right] y^5 \\
& + \left(-\frac{38}{45} + \frac{8}{45}w\right)\pi y^{11/2} \\
& + \left[-\frac{23741}{1680} + \frac{4607}{540}w + \frac{23}{54}w^2 + \left(-\frac{279}{1024} - \frac{397}{1536}w - \frac{11}{512}w^2 + \frac{1}{96}w^3\right)\pi^2 + \left(\frac{77}{6} - \frac{46}{9}w + \frac{4}{9}w^2\right)\gamma \right. \\
& \left. + \left(\frac{1627}{42} - \frac{54}{5}w + \frac{4}{9}w^2\right)\ln(2) - \frac{729}{70}\ln(3) + \left(\frac{77}{12} - \frac{23}{9}w + \frac{2}{9}w^2\right)\ln(y)\right] y^6 \\
& + \left(-\frac{3}{35} - \frac{2696}{4725}w + \frac{16}{135}w^2\right)\pi y^{13/2} \qquad (2.22\text{-II}) \\
& + \left\{-\frac{1515589307}{27216000} + \frac{3098381}{378000}w - \frac{3497}{3240}w^2 - \frac{793}{1458}w^3 \right. \\
& + \left(-\frac{58}{45} + \frac{8}{45}w\right)(1-w)^{3/2} - \frac{2}{3}(2-w)(1-w)\ln(1-w) \\
& + \left(-\frac{6059603}{983040} + \frac{1892003}{983040}w + \frac{2287}{9216}w^2 + \frac{871}{20736}w^3 - \frac{35}{2592}w^4\right)\pi^2 + \left(\frac{76585}{262144} - \frac{14281}{131072}w + \frac{2665}{262144}w^2\right)\pi^4 \\
& + \left[-\frac{5321}{900} + \frac{4312}{675}w - \frac{4}{27}w^2 - \frac{112}{243}w^3 + \left(\frac{152}{45} - \frac{32}{45}w\right)\gamma + \left(\frac{304}{45} - \frac{64}{45}w\right)\ln(2) + \left(\frac{152}{45} - \frac{32}{45}w\right)\ln(y)\right]\gamma \\
& + \left[-\frac{1786621}{18900} + \frac{149404}{4725}w - \frac{8}{15}w^2 - \frac{112}{243}w^3 + \left(\frac{152}{45} - \frac{32}{45}w\right)\ln(2) + \left(\frac{152}{45} - \frac{32}{45}w\right)\ln(y)\right]\ln(2) \\
& \left. + \left(\frac{12393}{140} - \frac{729}{35}w\right)\ln(3) - \frac{16}{3}\zeta(3) + \left[-\frac{10121}{1800} + \frac{4856}{675}w - \frac{38}{27}w^2 - \frac{56}{243}w^3 + \left(\frac{38}{45} - \frac{8}{45}w\right)\ln(y)\right]\ln(y)\right\} y^7 \\
& + \left(\frac{35633}{3780} - \frac{192541}{33075}w + \frac{5062}{4725}w^2 - \frac{8}{135}w^3\right)\pi y^{15/2} + O(y^8).
\end{aligned}
$$

In the Schwarzschild case (i.e., in the limit $w \to 0$) it reduces to

$$
\begin{aligned}
\psi_0^{\rm reg,\, schw} \quad = \quad & -y^3 + \left(\frac{35}{18} - \frac{7}{32}\pi^2 - \frac{4}{3}\gamma - \frac{4}{3}\ln(2) - \frac{2}{3}\ln(y)\right) y^4 + \left(\frac{1141}{360} + \frac{29}{512}\pi^2 + \frac{2}{3}\gamma - \frac{18}{5}\ln(2) + \frac{1}{3}\ln(y)\right) y^5 \\
& - \frac{38}{45}\pi y^{11/2} \\
& + \left(-\frac{23741}{1680} - \frac{279}{1024}\pi^2 + \frac{77}{6}\gamma + \frac{1627}{42}\ln(2) - \frac{729}{70}\ln(3) + \frac{77}{12}\ln(y)\right) y^6 \qquad (2.23\text{-II}) \\
& - \frac{3}{35}\pi y^{13/2} \\
& + \left[-\frac{1515589307}{27216000} - \frac{6059603}{983040}\pi^2 + \frac{76585}{262144}\pi^4 + \left(-\frac{5321}{900} + \frac{152}{45}\gamma + \frac{304}{45}\ln(2) + \frac{152}{45}\ln(y)\right)\gamma \right. \\
& \left. + \left(-\frac{1786621}{18900} + \frac{152}{45}\ln(2) + \frac{152}{45}\ln(y)\right)\ln(2) + \frac{12393}{140}\ln(3) - \frac{16}{3}\zeta(3) + \left(-\frac{10121}{1800} + \frac{38}{45}\ln(y)\right)\ln(y)\right] y^7 \\
& + \frac{35633}{3780}\pi y^{15/2} + O(y^8),
\end{aligned}
$$

which was never shown before in the literature [2]. Replacing ordinary logarithms by "eulerlogs," i.e.,

$$\mathrm{eulerlog}_m(x) = \gamma + \ln(2) + \frac{1}{2}\ln(y) + \ln(m), \qquad m = 1, 2, 3, \dots, \qquad (2.24\text{-II})$$

first introduced in reference [?], in order to absorb also the Euler γ constant, highlights the transcendental structure of the various PN orders. For example, the lowest order ($O(y^4)$) contains only $\mathrm{eulerlog}_1$, at $O(y^5)$ a combination of $\mathrm{eulerlog}_1$ and

[2] The first terms of this expansion (up to $O(y^5)$ included) agree with unpublished results by Bini and Damour [?].

eulerlog$_2$ appears, etc. However, starting from $O(y^7)$ this replacement is not enough to completely remove the Euler γ terms, meaning that the transcendental structure is more involved.

Scalar self-force effects on a Schwarzschild background were numerically studied in reference [?]. The comparison between our analytical results and these numerical values shows a reasonable agreement (see Table 2.1 and Fig. 2.1). It is also interesting to study the behavior of this scalar field at the light ring $y = 1/3$ (see reference [?] for the case of a massive particle orbiting a Schwarzschild black hole). Below it is provided a simple numerical fit of the data of Table 2.1

$$\psi_0^{\rm reg,\,schw\,fit} = -\frac{y^3}{(1-3y)^2}(1-7.84y+47.36y^2-8.65y^3+81.77y^3\ln(y)) \qquad (2.25\text{-II})$$

(with a maximal residual of about 2.4×10^{-4}), suggesting a blow-up of the form $(1-3y)^{-2}$. However, this is an indication only, and a more conclusive statement requires strong field numerical data still currently unavailable.

Finally, in the extreme RN case (i.e., in the limit $w\to1$),

$$\begin{aligned}\psi_0^{\rm reg,extr} \;=\;& -y^3+\left(\frac{35}{18}-\frac{3}{16}\pi^2-\frac{4}{3}\gamma-\frac{4}{3}\ln(2)-\frac{2}{3}\ln(y)\right)y^4+\left(\frac{2723}{1080}+\frac{7}{64}\pi^2-\frac{2}{9}\gamma-\frac{202}{45}\ln(2)-\frac{1}{9}\ln(y)\right)y^5\\
&-\frac{2}{3}\pi y^{11/2}\\
&+\left(-\frac{78233}{15120}-\frac{555}{1024}\pi^2+\frac{49}{6}\gamma+\frac{17881}{630}\ln(2)-\frac{729}{70}\ln(3)+\frac{49}{12}\ln(y)\right)y^6\\
&-\frac{121}{225}\pi y^{13/2}\\
&+\Bigg[-\frac{160402001}{3265920}-\frac{438259}{110592}\pi^2+\frac{99}{512}\pi^4+\left(-\frac{647}{4860}+\frac{8}{3}\gamma+\frac{16}{3}\ln(2)+\frac{8}{3}\ln(y)\right)\gamma\\
&+\left(-\frac{2174033}{34020}+\frac{8}{3}\ln(2)+\frac{8}{3}\ln(y)\right)\ln(2)+\frac{9477}{140}\ln(3)-\frac{16}{3}\zeta(3)+\left(-\frac{647}{9720}+\frac{2}{3}\ln(y)\right)\ln(y)\Bigg]y^7\\
&+\frac{203629}{44100}\pi y^{15/2}+O(y^8). \end{aligned} \qquad (2.26\text{-II})$$

Note that the term with $\ln(1-w)$ in equation (2.22-II) is proportional to $(1-w)\ln(1-w)$, which vanishes in the limit $w\to1$, so that the final expression is finite.

Figure 2.1. The behavior of the regularized scalar field (2.23-II) in the Schwarzschild case ($w = 0$) as a function of y is shown in comparison with existing numerical values. The data points are taken from Table I of reference [?].

Table 2.1. Comparison between the analytical prediction (2.23-II) for the regularized scalar field in the Schwarzschild case ($w=0$) and the numerical values taken from Table I of reference [?]. The difference $\Delta\psi_0^{\rm schw}=\psi_0^{\rm schw,num}-\psi_0^{\rm schw}$ and the relative error $\Delta\psi_0^{\rm schw}/\psi_0^{\rm schw}$ are shown in the third and fourth columns, respectively. The superscript "reg" has been suppressed for simplicity.

y	$\psi_0^{\rm schw}$	$\Delta\psi_0^{\rm schw}$	$\Delta\psi_0^{\rm schw}/\psi_0^{\rm schw}$
1/4	-0.02304519610	-9.43×10^{-4}	0.0409
1/5	-0.01022371010	-1.05×10^{-5}	0.00102
1/6	-0.005468782560	1.40×10^{-5}	-0.00255
1/7	-0.003282635718	7.29×10^{-6}	-0.00222
1/8	-0.002130877461	3.37×10^{-6}	-0.00158
1/10	-0.001050586634	7.94×10^{-7}	-7.55×10^{-4}
1/14	$-3.701411742\times10^{-4}$	7.66×10^{-8}	-2.07×10^{-4}
1/20	$-1.246786056\times10^{-4}$	5.81×10^{-9}	-4.66×10^{-5}
1/30	$-3.661740186\times10^{-5}$	3.02×10^{-10}	-8.24×10^{-6}
1/50	$-7.889525256\times10^{-6}$	7.26×10^{-12}	-9.20×10^{-7}
1/70	$-2.877222881\times10^{-6}$	8.81×10^{-13}	-3.06×10^{-7}
1/100	$-9.884245218\times10^{-7}$	2.18×10^{-14}	-2.21×10^{-8}
1/200	$-1.239865750\times10^{-7}$	-2.50×10^{-14}	2.02×10^{-7}

2.2 Computation of the scalar self-force

As discussed before, the scalar self-force, still not regularized, is given by

$$\begin{aligned} F_\alpha(x^\mu) &= q\nabla_\alpha\psi(x^\mu) \\ &= -q\nabla_\alpha\left(\sum_{l,m} G_{lm\omega}(r,r_0)\Big|_{\omega=m\Omega} r_0 q_{lm} e^{-im\Omega t} Y_{lm}(\theta,\phi)\right), \end{aligned} \tag{2.27-II}$$

with nonvanishing components F_t, F_ϕ and F_r related by

$$\begin{aligned} F^0_{t(\pm)} &= -\Omega F^0_{\phi(\pm)} = i\Omega\frac{4\pi q^2}{\Gamma}\sum_{l,m} m\, G_{lm\omega}(r,r_0)\Big|_{r=r_0^\pm,\omega=m\Omega}|Y_{lm}(\pi/2,0)|^2, \\ F^0_{r(\pm)} &= -\frac{4\pi q^2}{\Gamma}\sum_{l,m}\partial_r G_{lm\omega}(r,r_0)\Big|_{r=r_0^\pm,\omega=m\Omega}|Y_{lm}(\pi/2,0)|^2, \end{aligned} \tag{2.28-II}$$

once evaluated at the position of the scalar charge, i.e., in the limit $r\to r_0^\pm$. After summing over m, the divergent behavior for large l is removed by

$$F_\alpha^{0\,\rm reg} = \sum_{l=0}^{\infty}\left[\frac{1}{2}\left(F^{0l}_{\alpha(+)}+F^{0l}_{\alpha(-)}\right)-B_\alpha\right], \tag{2.29-II}$$

where $F^{0l}_{\alpha(\pm)}$ denote the limits $r\to r_0^\pm$ of each mode and the l-independent quantities $B_\alpha=\nabla_\alpha B$ are suitable regularization parameters. The subtraction term for the radial component turns out to be, in units of q,

$$B_r = -\frac{1}{2}y^2+\left(-\frac{1}{8}+\frac{1}{3}w\right)y^3+\left(-\frac{21}{128}+\frac{1}{2}w-\frac{7}{18}w^2\right)y^4+\left(-\frac{53}{512}+\frac{57}{64}w-\frac{2}{3}w^2+\frac{44}{81}w^3\right)y^5$$
$$+\left(\frac{12607}{32768}+\frac{97}{96}w-\frac{331}{384}w^2+w^3-\frac{5}{6}w^4\right)y^6$$
$$+\left(\frac{306759}{131072}-\frac{18433}{16384}w+\frac{4517}{2304}w^2+\frac{1055}{864}w^3-\frac{130}{81}w^4+\frac{988}{729}w^5\right)y^7+O(y^8), \qquad (2.30\text{-II})$$

whereas $B_t = B_\phi = B_\theta = 0$.

The final result for the regularized temporal and radial components of the self-force valid through 7.5PN order is, in units of q,

$$F_t^{0\,\mathrm{reg}} = \frac{1}{3}y^4+\left(-\frac{1}{6}+\frac{2}{9}w\right)y^5+\frac{2}{3}\pi y^{11/2}+\left(-\frac{77}{24}+\frac{23}{18}w-\frac{1}{9}w^2\right)y^6+\left(\frac{9}{5}+\frac{4}{9}w\right)\pi y^{13/2}$$
$$+\left[\frac{7721}{3600}-\frac{1753}{675}w+\frac{10}{27}w^2+\frac{28}{243}w^3+\frac{2}{3}(1-w)^{3/2}+\frac{4}{9}\pi^2+\left(-\frac{76}{45}+\frac{16}{45}w\right)\gamma+\left(-\frac{76}{45}+\frac{16}{45}w\right)\ln(2)\right.$$
$$\left.+\left(-\frac{38}{45}+\frac{8}{45}w\right)\ln(y)\right]y^7$$
$$+\left(-\frac{3761}{420}+\frac{27}{5}w-\frac{2}{9}w^2\right)\pi y^{15/2}+O(y^8),$$

$$F_r^{0\,\mathrm{reg}} = \left[-\frac{2}{9}+\left(\frac{7}{64}-\frac{1}{64}w\right)\pi^2-\frac{4}{3}\gamma-\frac{4}{3}\ln(2)-\frac{2}{3}\ln(y)\right]y^5$$
$$+\left[\frac{604}{45}-\frac{41}{27}w+\left(\frac{29}{1024}-\frac{239}{3072}w+\frac{1}{96}w^2\right)\pi^2-\left(\frac{14}{3}+\frac{4}{9}w\right)\gamma-\left(\frac{66}{5}+\frac{4}{9}w\right)\ln(2)-\left(\frac{7}{3}+\frac{2}{9}w\right)\ln(y)\right]y^6$$
$$+\left(-\frac{38}{45}+\frac{8}{45}w\right)\pi y^{13/2}$$
$$+\left[\frac{1511}{140}+\frac{473}{90}w+\frac{103}{81}w^2+\left(\frac{1335}{2048}-\frac{1}{16}w+\frac{151}{2304}w^2-\frac{7}{576}w^3\right)\pi^2+\left(\frac{31}{2}-\frac{28}{3}w+\frac{8}{27}w^2\right)\gamma\right.$$
$$\left.+\left(\frac{857}{14}-\frac{268}{15}w+\frac{8}{27}w^2\right)\ln(2)-\frac{2187}{70}\ln(3)+\left(\frac{31}{4}-\frac{14}{3}w+\frac{4}{27}w^2\right)\ln(y)\right]y^7$$
$$+\left(-\frac{139}{35}+\frac{2378}{4725}w+\frac{8}{135}w^2\right)\pi y^{15/2}+O(y^8), \qquad (2.31\text{-II})$$

respectively.

In the Schwarzschild limit ($w \to 0$),

$$F_t^{0\,\mathrm{reg,schw}} = \frac{1}{3}y^4-\frac{1}{6}y^5+\frac{2}{3}\pi y^{11/2}-\frac{77}{24}y^6+\frac{9}{5}\pi y^{13/2}$$
$$+\left[\frac{10121}{3600}+\frac{4}{9}\pi^2-\frac{76}{45}\gamma-\frac{76}{45}\ln(2)-\frac{38}{45}\ln(y)\right]y^7$$
$$-\frac{3761}{420}\pi y^{15/2}+O(y^8),$$

$$F_r^{0\,\mathrm{reg,schw}} = \left[-\frac{2}{9}+\frac{7}{64}\pi^2-\frac{4}{3}\gamma-\frac{4}{3}\ln(2)-\frac{2}{3}\ln(y)\right]y^5$$
$$+\left[\frac{604}{45}+\frac{29}{1024}\pi^2-\frac{14}{3}\gamma-\frac{66}{5}\ln(2)-\frac{7}{3}\ln(y)\right]y^6$$
$$-\frac{38}{45}\pi y^{13/2}$$
$$+\left[\frac{1511}{140}+\frac{1335}{2048}\pi^2+\frac{31}{2}\gamma+\frac{857}{14}\ln(2)-\frac{2187}{70}\ln(3)+\frac{31}{4}\ln(y)\right]y^7$$
$$-\frac{139}{35}\pi y^{15/2}+O(y^8). \qquad (2.32\text{-II})$$

The leading 3PN and 4PN terms of the previous expressions agree with those of reference [**?**]. Furthermore, the comparison with available numerical results of references [**?**, **?**] shows again a very good agreement (see Table 2.2 and Fig. 2.2, where we refer to the most recent work [**?**]).

Table 2.2. Comparison between the analytical expressions (2.32-II) for the regularized temporal and radial components of the self-force (in units of q, the superscript "reg" being suppressed for simplicity) in the Schwarzschild case ($w = 0$) and the numerical values taken from Tables II and III of reference [**?**]. The second and third columns display the values obtained by our analytical expressions, whereas the fourth and fifth columns display the difference with the corresponding numerical values (i.e., $\Delta F_t^{0\,\mathrm{schw}} = F_t^{0\,\mathrm{schw,num}} - F_t^{0\,\mathrm{schw}}$ and $\Delta F_r^{0\,\mathrm{schw}} = F_r^{0\,\mathrm{schw,num}} - F_t^{0\,\mathrm{schw}}$). Finally, the last two columns show the associated relative errors $\Delta F_t^{0\,\mathrm{schw}}/F_t^{0\,\mathrm{schw}}$ and $\Delta F_r^{0\,\mathrm{schw}}/F_r^{0\,\mathrm{schw}}$, respectively.

y	$F_t^{0\,\mathrm{schw}}$	$F_r^{0\,\mathrm{schw}}$	$\Delta F_t^{0\,\mathrm{schw}}$	$\Delta F_r^{0\,\mathrm{schw}}$	$\Delta F_t^{0\,\mathrm{schw}}/F_t^{0\,\mathrm{schw}}$	$\Delta F_r^{0\,\mathrm{schw}}/F_r^{0\,\mathrm{schw}}$
1/6	$3.088678309 \times 10^{-4}$	$2.069192430 \times 10^{-4}$	5.20×10^{-5}	-3.92×10^{-5}	0.168	−0.189
1/7	$1.621300383 \times 10^{-4}$	$9.086682872 \times 10^{-5}$	1.46×10^{-5}	-1.24×10^{-5}	0.0901	−0.136
1/8	$9.278324817 \times 10^{-5}$	$4.535558187 \times 10^{-5}$	4.94×10^{-6}	-4.53×10^{-6}	0.0532	−0.0999
1/10	$3.667967766 \times 10^{-5}$	$1.462629728 \times 10^{-5}$	8.23×10^{-7}	-8.42×10^{-7}	0.0224	−0.0576
1/14	$9.180183375 \times 10^{-6}$	$2.785864322 \times 10^{-6}$	5.66×10^{-8}	-6.58×10^{-8}	0.00616	−0.0236
1/20	$2.148236416 \times 10^{-6}$	$4.981418996 \times 10^{-7}$	3.36×10^{-9}	-4.35×10^{-9}	0.00156	−0.00874
1/30	$4.175425467 \times 10^{-7}$	$7.191466709 \times 10^{-8}$	1.36×10^{-10}	-1.96×10^{-10}	3.26×10^{-4}	−0.00272
1/50	$5.359926673 \times 10^{-8}$	$6.350626477 \times 10^{-9}$	2.40×10^{-12}	-3.93×10^{-12}	4.47×10^{-5}	-6.18×10^{-4}
1/70	$1.391199738 \times 10^{-8}$	$1.284814550 \times 10^{-9}$	1.67×10^{-13}	-3.15×10^{-13}	1.20×10^{-5}	-2.45×10^{-4}
1/100	$3.335029050 \times 10^{-9}$	$2.356682550 \times 10^{-10}$	9.90×10^{-15}	-6.83×10^{-14}	2.97×10^{-6}	-2.90×10^{-4}

Figure 2.2. Comparison of numerical data from reference [**?**] for the regularized temporal and radial components of the self force (in units of q) in the Schwarzschild case ($w = 0$) with the behavior of the corresponding analytical expressions (2.32-II).

2.3 Computation of the scalar radiation

Let us compute the amount of scalar radiation either flowing into the hole or transmitted at spatial infinity. In order to do so, one needs to construct the solution to the nonhomogeneous wave equation (2.7-II) which satisfies purely ingoing-wave boundary conditions at the black hole horizon and purely outgoing-wave boundary conditions at infinity. This is accomplished by using the two kinds of solutions $R^H_{lm\omega}$ and $R^\infty_{lm\omega}$ to the corresponding homogeneous equation with asymptotic behavior [**?**,

?, ?]

$$
R^H_{lm\omega} \quad \rightarrow \quad \begin{cases} B^{\rm trans} e^{-i\omega r_*}, & r \rightarrow r_+ \\ B^{\rm ref}\dfrac{e^{i\omega r_*}}{r} + B^{\rm inc}\dfrac{e^{-i\omega r_*}}{r}, & r \rightarrow \infty \end{cases},
$$

$$
R^\infty_{lm\omega} \quad \rightarrow \quad \begin{cases} C^{\rm up} e^{i\omega r_*} + C^{\rm ref} e^{-i\omega r_*}, & r \rightarrow r_+ \\ C^{\rm trans}\dfrac{e^{i\omega r_*}}{r}, & r \rightarrow \infty \end{cases}, \tag{2.33-II}
$$

where r_* is the tortoise-like coordinate defined by $dr_*/dr = r^2/\Delta$, i.e.,

$$
r_* = r + \frac{2Mr_+}{r_+ - r_-} \ln \frac{r - r_+}{2M} - \frac{2Mr_-}{r_+ - r_-} \ln \frac{r - r_-}{2M}. \tag{2.34-II}
$$

The final solution is given by [?]

$$
R_{lm\omega}(r) = Z^H_{lm\omega}(r) R^\infty_{lm\omega}(r) + Z^\infty_{lm\omega}(r) R^H_{lm\omega}(r), \tag{2.35-II}
$$

where

$$
\begin{aligned}
Z^H_{lm\omega}(r) &= \frac{-1}{W_{lm\omega}} \int_{r_+}^{r} R^H_{lm\omega}(r')\Delta(r') S_{lm\omega} \delta(r' - r_0) dr', \\
Z^\infty_{lm\omega}(r) &= \frac{-1}{W_{lm\omega}} \int_{r}^{\infty} R^\infty_{lm\omega}(r')\Delta(r') S_{lm\omega} \delta(r' - r_0) dr',
\end{aligned} \tag{2.36-II}
$$

and $W_{lm\omega} = 2i\omega C^{\rm trans} B^{\rm inc}$ is the constant Wronskian. The asymptotic behaviors of $R_{lm\omega}$ at the horizon and at infinity are then

$$
\begin{aligned}
R_{lm\omega}(r \rightarrow r_+) &\rightarrow B^{\rm trans} Z^\infty_{lm\omega}(r_+) e^{-i\omega r_*}, \\
R_{lm\omega}(r \rightarrow \infty) &\rightarrow C^{\rm trans} Z^H_{lm\omega}(\infty) \frac{e^{i\omega r_*}}{r},
\end{aligned} \tag{2.37-II}
$$

respectively, so that one can define the amplitudes

$$
\mathcal{Z}^H_{lm\omega} = B^{\rm trans} Z^\infty_{lm\omega}(r_+), \qquad \mathcal{Z}^\infty_{lm\omega} = C^{\rm trans} Z^H_{lm\omega}(\infty). \tag{2.38-II}
$$

Explicitly one finds

$$
\begin{aligned}
\mathcal{Z}^H_{lm\omega} &= 2\pi \frac{B^{\rm trans}}{W_{lm\omega}} r_0 q_{lm\omega} R^\infty_{lm\omega}(r_0) \delta(\omega - m\Omega) \\
&\equiv 2\pi \tilde{\mathcal{Z}}^H_{lm} \delta(\omega - m\Omega), \\
\mathcal{Z}^\infty_{lm\omega} &= 2\pi \frac{C^{\rm trans}}{W_{lm\omega}} r_0 q_{lm\omega} R^H_{lm\omega}(r_0) \delta(\omega - m\Omega) \\
&\equiv 2\pi \tilde{\mathcal{Z}}^\infty_{lm} \delta(\omega - m\Omega).
\end{aligned} \tag{2.39-II}
$$

The energy flux at infinity is thus given by [?, ?, ?]

$$
\frac{dE^\infty}{dt} = \sum_{l,m} \frac{\omega^2}{4\pi} |\tilde{\mathcal{Z}}^\infty_{lm}|^2, \tag{2.40-II}
$$

while the energy flux at the event horizon is

$$\frac{dE^H}{dt} = \sum_{l,m} \frac{M\omega^2 r_+}{2\pi} |\tilde{\mathscr{Z}}^H_{lm}|^2 , \qquad (2.41\text{-II})$$

with $\tilde{\mathscr{Z}}^{H,\infty}_{lm}$ defined in equation (2.39-II) and $\omega = m\Omega$.

For the computation of the amplitudes and the transmission coefficients we have used the MST ingoing and upgoing solutions, which satisfy the proper boundary conditions at the horizon and at infinity for any given value of l, i.e., $R^H_{lm\omega}(r) = R^{\text{in(MST)}}_{lm\omega}(r)$ and $R^\infty_{lm\omega}(r) = R^{\text{up(MST)}}_{lm\omega}(r)$. The corresponding transmission coefficients are given by [?]

$$\begin{aligned} B^{\text{trans}} &= e^{i\frac{\kappa}{2}(\epsilon+\tau)\left(1+\frac{2\ln\kappa}{1+\kappa}\right)} \sum_{n=-\infty}^{\infty} a_n , \\ C^{\text{trans}} &= \omega^{-1} e^{i\epsilon\left(\ln\epsilon - \frac{1-\kappa}{2}\right)} A^\nu_- , \end{aligned} \qquad (2.42\text{-II})$$

for

$$A^\nu_- = 2^{-1+i\epsilon} e^{-\frac{\pi}{2} i(\nu+1)} e^{-\frac{\pi}{2}\epsilon} \sum_{n=-\infty}^{\infty} (-1)^n \frac{(\nu+1-i\epsilon)_n}{(\nu+1+i\epsilon)_n} a_n . \qquad (2.43\text{-II})$$

The definitions of the various quantities ϵ, τ, ν, a_n are given in Appendix A.2 for convenience.

One finds, in units of q,

$$\begin{aligned} \frac{dE^\infty}{dt} &= \left(\frac{dE^\infty}{dt}\right)_N \Bigg\{ 1 + \left(-2 + \frac{2}{3}w\right) y + 2\pi y^{3/2} + \left(-10 + \frac{13}{3}w - \frac{1}{3}w^2\right) y^2 + \left(\frac{12}{5} + \frac{4}{3}w\right)\pi y^{5/2} \\ &+ \Bigg[\frac{1331}{75} - \frac{2203}{225}w + \frac{4}{9}w^2 + \frac{28}{81}w^3 + \frac{4}{3}\pi^2 \\ &+ \left(-\frac{76}{15} + \frac{16}{15}w\right)\gamma + \left(-\frac{76}{15} + \frac{16}{15}w\right)\ln(2) + \left(-\frac{38}{15} + \frac{8}{15}w\right)\ln(y) \Bigg] y^3 \\ &+ \left(-\frac{521}{14} + \frac{86}{5}w - \frac{2}{3}w^2\right)\pi y^{7/2} + O(y^4) \Bigg\} , \\ \frac{dE^H}{dt} &= \left(\frac{dE^\infty}{dt}\right)_N 2(1-w)(1+\sqrt{1-w}) \Bigg\{ y^3 + \left(2 - \frac{4}{3}w\right) y^4 + \left(3 - \frac{7}{3}w + 2w^2\right) y^5 \\ &+ \Bigg[\frac{1231}{75} - \frac{793}{225}w + \frac{32}{9}w^2 - \frac{260}{81}w^3 + \frac{1}{1-w} + \left(-\frac{38}{15} + \frac{8}{15}w\right)\ln(1-w) + \frac{1}{(1-w)^{1/2}}\left(\frac{4}{3} - \frac{2}{3}w\right)\pi^2 \\ &- \frac{8}{3}\gamma + \left(-\frac{116}{15} + \frac{16}{15}w\right)\ln(2) + \left(-\frac{32}{5} + \frac{16}{15}w\right)\ln(y) \Bigg] y^6 \\ &+ \Bigg[\frac{1769}{225} - \frac{46}{15}w + \frac{907}{675}w^2 - \frac{473}{81}w^3 + \frac{1309}{243}w^4 + \frac{1}{1-w}\left(4 - \frac{10}{3}w\right) + \left(-\frac{76}{15} + \frac{40}{9}w - \frac{32}{45}w^2\right)\ln(1-w) \\ &+ \frac{1}{(1-w)^{1/2}}\left(\frac{8}{3} - \frac{28}{9}w + \frac{8}{9}w^2\right)\pi^2 + \left(\frac{64}{15} + \frac{8}{9}w\right)\gamma + \left(-\frac{88}{15} + \frac{88}{9}w - \frac{64}{45}w^2\right)\ln(2) \\ &+ \left(-8 + \frac{28}{3}w - \frac{64}{45}w^2\right)\ln(y) \Bigg] y^7 \\ &+ \left(-\frac{56}{45} + \frac{16}{45}w\right)\pi y^{15/2} \\ &+ \Bigg[-\frac{208762}{7875} + \frac{10034656}{165375}w - \frac{4518949}{165375}w^2 + \frac{1162}{675}w^3 + 10w^4 - \frac{28}{3}w^5 + \frac{1}{1-w}\left(9 - \frac{44}{3}w + \frac{25}{3}w^2\right) \\ &+ \left(-\frac{1818}{175} + \frac{17086}{1575}w - \frac{10804}{1575}w^2 + \frac{16}{15}w^3\right)\ln(1-w) + \frac{1}{(1-w)^{1/2}}\left(\frac{36}{5} - \frac{446}{45}w + \frac{262}{45}w^2 - \frac{4}{3}w^3\right)\pi^2 \\ &+ \left(\frac{608}{105} - \frac{344}{45}w - \frac{8}{9}w^2\right)\gamma + \left(-\frac{1124}{75} + \frac{22132}{1575}w - \frac{23008}{1575}w^2 + \frac{32}{15}w^3\right)\ln(2) \\ &+ \left(-\frac{9388}{525} + \frac{3128}{175}w - \frac{7436}{525}w^2 + \frac{32}{15}w^3\right)\ln(y) \Bigg] y^8 \\ &+ \left(\frac{88}{225} + \frac{196}{675}w - \frac{16}{135}w^2\right)\pi y^{17/2} + O(y^9) \Bigg\} , \end{aligned} \qquad (2.44\text{-II})$$

where

$$\left(\frac{dE^\infty}{dt}\right)_N = \frac{1}{3}y^4, \tag{2.45-II}$$

in terms of the gauge-invariant variable y (see equation (2.21-II)). Note that the flux at infinity is computed up to the 3.5PN order, i.e., at $O(y^{7/2})$ included (see Appendix A.3). The leading contribution to the flux on the horizon, instead, enters at three PN orders beyond the lowest order, and is computed through $O(y^{17/2})$.

In the Schwarzschild case ($w \to 0$) the previous expressions reduce to

$$\begin{aligned}
\frac{dE^\infty}{dt} &= \left(\frac{dE^\infty}{dt}\right)_N \Bigg[1-2y+2\pi y^{3/2}-10y^2+\frac{12}{5}\pi y^{5/2}+\left(\frac{1331}{75}+\frac{4}{3}\pi^2-\frac{76}{15}\gamma-\frac{76}{15}\ln(2)-\frac{38}{15}\ln(y)\right)y^3 \\
&\quad -\frac{521}{14}\pi y^{7/2}+O(y^4)\Bigg], \\
\frac{dE^H}{dt} &= \left(\frac{dE^\infty}{dt}\right)_N 4y^3\Bigg[1+2y+3y^2+\left(\frac{1306}{75}+\frac{4}{3}\pi^2-\frac{8}{3}\gamma-\frac{116}{15}\ln(2)-\frac{32}{5}\ln(y)\right)y^3 \\
&\quad +\left(\frac{2669}{225}+\frac{8}{3}\pi^2+\frac{64}{15}\gamma-\frac{88}{15}\ln(2)-8\ln(y)\right)y^4-\frac{56}{45}\pi y^{9/2} \\
&\quad +\left(-\frac{137887}{7875}+\frac{36}{5}\pi^2+\frac{608}{105}\gamma-\frac{1124}{75}\ln(2)-\frac{9388}{525}\ln(y)\right)y^5+\frac{88}{225}\pi y^{11/2}+O(y^6)\Bigg],
\end{aligned} \tag{2.46-II}$$

whereas in the extreme RN case ($w \to 1$),

$$\begin{aligned}
\frac{dE^\infty}{dt} &= \left(\frac{dE^\infty}{dt}\right)_N \Bigg[1-\frac{4}{3}y+2\pi y^{3/2}-6y^2+\frac{56}{15}\pi y^{5/2}+\left(\frac{3542}{405}+\frac{4}{3}\pi^2-4\gamma-4\ln(2)-2\ln(y)\right)y^3 \\
&\quad -\frac{4343}{210}\pi y^{7/2}+O(y^4)\Bigg], \\
\frac{dE^H}{dt} &= \left(\frac{dE^\infty}{dt}\right)_N 2y^6\left[1+\frac{2}{3}y+\frac{8}{3}y^2+O(y^3)\right].
\end{aligned} \tag{2.47-II}$$

Therefore, when the black hole is extremely charged the horizon-absorbed flux starts three more PN orders beyond, with respect to the nonextreme case.

Table 2.3. Comparison between the total energy flux $\dot{E}^{\rm tot} = \dot{E}^\infty + \dot{E}^H$ (in units of q, the overdot denoting d/dt) computed by using the analytical expressions (2.46-II) in the Schwarzschild case ($w = 0$) and the numerical values taken from Table I of reference [**?**]. The last two columns display the difference $\Delta\dot{E}^{\rm tot} = \dot{E}^{\rm tot,num} - \dot{E}^{\rm tot}$ and the corresponding relative error $\Delta\dot{E}^{\rm tot}/\dot{E}^{\rm tot}$.

y	$\dot{E}^{\rm tot}$	$\Delta\dot{E}^{\rm tot}$	$\Delta\dot{E}^{\rm tot}/\dot{E}^{\rm tot}$
1/6	2.174332404×10^{-4}	3.78×10^{-5}	0.174
1/8	7.334363785×10^{-5}	3.91×10^{-6}	0.0533
1/10	3.06988477×10^{-5}	6.78×10^{-7}	0.0221
1/20	1.980748387×10^{-6}	2.92×10^{-9}	0.00147
1/40	1.262548016×10^{-7}	-2.81×10^{-11}	-2.22×10^{-4}

It is shown in Table 2.3 the comparison between the total energy flux

$$\frac{dE^{\rm tot}}{dt} = \frac{dE^\infty}{dt} + \frac{dE^H}{dt} \tag{2.48-II}$$

computed by using equation (2.46-II) in the Schwarzschild case and the numerical values taken from Table I of reference [**?**]. We find that the agreement is reasonably good in the weak-field regime, with fractional errors ranging from 10^{-1} (at

$y = 1/6$) to 10^{-4} (at $y = 1/40$). Furthermore, we have checked that our analytic expression (2.48-II) for the total energy flux agrees with the analogous quantity obtained from the self-force energy balance relation $dE^{\rm tot}/dt = \Gamma^{-1} F_t^{0\,\rm reg}$ up to the order $O(y^8)$, which is our accuracy in the computation of the self-force components (see equation (2.32-II)).

Finally, we note that the (dimensionless) angular momentum fluxes can be easily calculated through [**?**]

$$\frac{dJ^{H,\infty}}{dt} = y^{-3/2}\frac{dE^{H,\infty}}{dt}\,. \tag{2.49-II}$$

Conclusions

After introducing the Reissner-Nordström metric, we have analyzed self-force effects on a scalar charge moving along a circular orbit around a Reissner-Nordström black hole. The scalar wave equation is separated by using standard spherical harmonics, available here because of the underlying spherical symmetry of the background, and the field is decomposed into frequency modes. The associated radial equation is solved perturbatively in a PN framework by using the Green function method. The scalar field as well as the components of the self-force are then regularized at the position of the particle by subtracting the divergent term mode by mode, summing then the infinite series up to a certain PN order. The MST approach has also been adopted for computing a number of radiative multipoles (up to $l = 4$), so that our final result for the field is accurate up to the 7.5PN order, i.e., up to the order $O(y^{15/2})$ included in terms of the dimensionless gauge-invariant frequency variable $y = (M\Omega)^{2/3}$. Since the scalar charge interacts only gravitationally with the background field, the coupling with the black hole electromagnetic charge is quadratic. The two limiting cases of a Schwarzschild black hole, which was missing in the literature and represents by itself an interesting byproduct of our work, and of an extreme Reissner-Nordström black hole are discussed explicitly. The comparison of the analytically computed regularized field and self-force components with existing numerical results in the Schwarzschild case [**?**, **?**] has shown a good agreement (see Tables 2.1 and 2.2). We have also evaluated the radiation fluxes both at infinity and on the outer horizon up to $O(y^{7/2})$ and $O(y^{17/2})$ included, respectively. It was found that, when the black hole is extremely charged, the horizon-absorbed flux starts three more PN orders beyond the nonextreme case. The problem of radiation due to an electromagnetic charge orbiting a RN black hole generalizing the present analysis will be considered in future works.

A
Homogeneous solutions to the scalar wave equation and energy fluxes

A.1 Perturbative solutions

PN solutions have the form (2.15-II), i.e.,

$$
\begin{aligned}
R^{lm\omega}_{\mathrm{in(PN)}}(r) &= r^l[1+A^{lm\omega}_2(r)\eta^2+A^{lm\omega}_4(r)\eta^4+A^{lm\omega}_6(r)\eta^6+A^{lm\omega}_8(r)\eta^8+\ldots],\\
R^{lm\omega}_{\mathrm{up(PN)}}(r) &= R^{(-l-1)m\omega}_{\mathrm{in(PN)}}(r).
\end{aligned}
\tag{A.1.1}
$$

The first four coefficients are given by

$$
\begin{aligned}
A^{lm\omega}_2(r) &= -\frac{Ml}{r}-\frac{\omega^2r^2}{2(2l+3)},\\
A^{lm\omega}_4(r) &= \frac{M^2}{r^2}\frac{l(l-1)(2l-1-\kappa^2)}{2(2l-1)}+M\omega^2r\frac{l^2-5l-10}{2(2l+3)(l+1)}+\frac{\omega^4r^4}{8(2l+3)(2l+5)},\\
A^{lm\omega}_6(r) &= -\frac{M^3}{r^3}\frac{l(l-1)(l-2)(2l-1-3\kappa^2)}{6(2l-1)}-2M^2\omega^2\frac{[3(2l-1)(2l+3)+(3l^2+3l-2)\kappa^2][(2l+1)\ln(r/R)-1]}{(2l-1)(2l+3)(2l+1)^2}\\
&\quad -M\omega^4r^3\frac{3l^3-27l^2-142l-136}{24(l+1)(l+2)(2l+3)(2l+5)}-\frac{\omega^6r^6}{48(2l+3)(2l+5)(2l+7)},\\
A^{lm\omega}_8(r) &= \frac{M^4}{r^4}\frac{l(l-1)(l-2)(l-3)}{24(2l-1)(2l-3)}[(2l-1)(2l-3)-3\kappa^2(4l-6-\kappa^2)]\\
&\quad +\frac{M^3\omega^2}{r}\left\{-\frac{4l^6-32l^5-99l^4-241l^3-436l^2-276l-36}{6l(2l+3)(2l+1)^2}+\frac{48l^5+348l^4+540l^3+126l^2-120l-36}{6l(2l+3)(2l-1)(2l+1)^2}\kappa^2\right.\\
&\quad \left.+\left[\frac{6l}{2l+1}+\frac{2l(3l^2+3l-2)}{(2l+3)(2l-1)(2l+1)}\kappa^2\right]\ln(r/R)\right\}\\
&\quad +M^2\omega^2r^2\left\{-\frac{24l^7+156l^6-1766l^5-13267l^4-29512l^3-23465l^2-2058l+2784}{48(l+1)(2l+3)^2(2l+5)(2l+1)^2(l+2)}\right.\\
&\quad +\frac{16l^6+80l^5-440l^4-2432l^3-2803l^2+11l+784}{16(2l-1)(2l+3)^3(2l+5)(2l+1)^2}\kappa^2\\
&\quad \left.+\left[\frac{3}{(2l+3)(2l+1)}+\frac{3l^2+3l-2}{(2l-1)(2l+3)^2(2l+1)}\kappa^2\right]\ln(r/R)\right\}\\
&\quad +M\omega^6r^5\frac{5l^4-60l^3-625l^2-1548l-1108}{240(l+3)(l+2)(2l+7)(2l+5)(2l+3)(l+1)}+\frac{\omega^8r^8}{384(2l+9)(2l+7)(2l+5)(2l+3)},
\end{aligned}
\tag{A.1.2}
$$

where R is a length scale. This solution, which we need, however, to compute the sum over all multipoles, becomes immediately inadequate, and one should use the MST technique. In fact, the coefficient A_4 of the "up" solution (obtained from $A^{lm\omega}_4(r)$ with $l\to -l-1$) diverges for $l=0$; similarly, higher order coefficients diverge for $l=1,2,\ldots$ etc. The technique described in the next section aims at circumventing this problem.

A.2 MST solutions

The MST technique [?, ?] allows one to find homogeneous solution to the radial equation which satisfies retarded boundary conditions at the horizon ($R^{lm\omega}_{\mathrm{in(MST)}}(r)$) and radiative boundary conditions at infinity ($R^{lm\omega}_{\mathrm{up(MST)}}(r)$).

The ingoing solution (formally given by the confluent Heun function) can be formally written as a convergent (at any finite value of r) series of hypergeometric functions

$$R^{lm\omega}_{\rm in(MST)}(x) = C_{\rm (in)}(x) \sum_{n=-\infty}^{\infty} a_n F(n+\nu+1-i\tau, -n-\nu-i\tau, 1-i\epsilon-i\tau; x), \qquad \text{(A.2.3)}$$

with

$$C_{\rm (in)}(x) = e^{i\epsilon\kappa x}(-x)^{-i(\epsilon+\tau)/2}(1-x)^{i(\epsilon-\tau)/2}, \qquad \text{(A.2.4)}$$

where the new variable $x = (r_+ - r)/2M\kappa$ has been introduced and

$$\epsilon = 2M\omega, \quad \tau = \frac{1}{2}\frac{\epsilon(\kappa^2+1)}{\kappa}. \qquad \text{(A.2.5)}$$

The hypergeometric functions above are better evaluated by using the standard identity found in [?]

$$F(a,b;c;x) = y^a \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} F(a, c-b, a-b+1; y) + y^b \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} F(b, c-a, b-a+1; y), \qquad \text{(A.2.6)}$$

involving the "small" variable $y = 1/(1-x)$. Note that the overall factor $\Gamma(c)$ does not depend on n, so that it can be factored out.

Following the lines described in section 2.3 of [?], the expansion coefficients a_n satisfy the following three-term recurrence relation

$$\alpha^\nu_n a_{n+1} + \beta^\nu_n a_n + \gamma^\nu_n a_{n-1} = 0, \qquad \text{(A.2.7)}$$

for

$$\begin{aligned} \alpha^\nu_n &= \frac{i\epsilon\kappa(n+\nu+1+i\epsilon)(n+\nu+1-i\epsilon)(n+\nu+1+i\tau)}{(n+\nu+1)(2n+2\nu+3)}, \\ \beta^\nu_n &= -l(l+1) + (n+\nu)(n+\nu+1) + \epsilon\kappa\tau + \epsilon^2 + \frac{\epsilon^3\kappa\tau}{(n+\nu)(n+\nu+1)}, \\ \gamma^\nu_n &= -\frac{i\epsilon\kappa(n+\nu+i\epsilon)(n+\nu-i\tau)(n+\nu-i\epsilon)}{(n+\nu)(2n+2\nu-1)}. \end{aligned} \qquad \text{(A.2.8)}$$

Once the recurrence system has been solved for $n \in \{1, \ldots, N\}$ and $n \in \{-N, \ldots, -1\}$ for a given N such that $a_N = 0 = a_{-N}$, the case $n = 0$ with $a_0 = 1$ becomes a compatibility condition which yields the parameter

$$\nu = l + \sum_{k=1}^{\infty} \nu_k \epsilon^{2k}. \qquad \text{(A.2.9)}$$

The solution of the recurrence system is rather involved (even in this relatively simple case). The structure of the expansion coefficients

$$a_n = \sum_{k=i}^{j} c_{nk}\epsilon^k \qquad \text{(A.2.10)}$$

is summarized in Table A.1 for $l = 1$ and $N = 15$, as an example.

Table A.1. The structure of the expansion coefficients (A.2.10) of the recurrence relation is shown for $l = 1$ and $N = 15$, so that $a_{-15} = 0 = a_{15}$ and $a_0 = 1$.

n	−14	−13	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
i	13	12	11	10	9	8	7	6	5	4	3	2	3	1	–	1	2	3	4	5	6	7	8	9	10	11	12	13	14
j	14	14	14	14	14	14	14	14	14	14	14	14	14	14	–	14	14	14	14	14	14	14	14	14	14	14	14	14	14

The upgoing solution can be formally written as a convergent (at spatial infinity) series of irregular confluent hypergeometric functions with the same series coefficients

$$R^{lm\omega}_{\rm up(MST)}(z) = C_{\rm (up)}(z) \sum_{n=-\infty}^{\infty} a_n \frac{(\nu+1-i\epsilon)_n}{(\nu+1+i\epsilon)_n} (2iz)^n \Psi[n+\nu+1-i\epsilon, 2n+2\nu+2; -2iz]\,, \tag{A.2.11}$$

with

$$C_{\rm (up)}(z) = (2z)^\nu e^{-\pi\epsilon} e^{-i\pi(\nu+1)} e^{iz} \left(1-\frac{\epsilon\kappa}{z}\right)^{-i(\epsilon+\tau)/2}, \tag{A.2.12}$$

where the new variable $z = \omega(r-r_-) = \epsilon\kappa(1-x)$ has been introduced and $(A)_n = \Gamma(A+n)/\Gamma(A)$ is the Pochammer symbol. The irregular confluent hypergeometric functions above can be conveniently split into two pieces by using the identity

$$\Psi(a,b;\zeta) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} F(a,b;\zeta) + \zeta^{1-b}\frac{\Gamma(b-1)}{\Gamma(a)} F(a-b+1, 2-b;\zeta)\,. \tag{A.2.13}$$

For instance, for $l = 1$ we get

$$\begin{aligned}
R^{l=1}_{\rm in(MST)}(r) &= \frac{r}{M\kappa} - \frac{1}{\kappa}\left(1+\frac{\omega^2 r^3}{10M}\right)\eta^2 + \frac{i\omega r}{\kappa^2}\left[1+2\kappa+2\kappa^2-(1+\kappa)^2\gamma\right]\eta^3 - \frac{\omega^2 r^2}{\kappa}\left(\frac{7}{10}-\frac{\omega^2 r^3}{280M}\right)\eta^4 \\
&\quad - \frac{iM\omega}{\kappa^2}\left(1+\frac{\omega^2 r^3}{10M}\right)\left[1+2\kappa+2\kappa^2-(1+\kappa)^2\gamma\right]\eta^5 \\
&\quad - \left\{\left[\frac{16875+120375\kappa^2+87075\kappa^4+23790\kappa^6+1616\kappa^8}{75(15+4\kappa^2)^2} + \left[1+2\kappa+2\kappa^2-(1+\kappa)^2\gamma\right]^2\right.\right. \\
&\quad \left.\left. -\frac{1}{6}(1+6\kappa^2+\kappa^4)\pi^2 - \frac{4}{15}\kappa^2(15+4\kappa^2)\ln\left(\frac{2M\kappa\eta^2}{r}\right)\right]\frac{M\omega^2 r}{2\kappa^3} - \frac{\omega^4 r^4}{\kappa}\left(\frac{151}{2520}-\frac{\omega^2 r^3}{15120M}\right)\right\}\eta^6 \\
&\quad + O(\eta^7)\,, \\
R^{l=1}_{\rm up(MST)}(r) &= -\frac{i}{2\omega^2 r^2} - \frac{i}{4}\left(1+\frac{4M}{\omega^2 r^3}\right)\eta^2 + \frac{M}{\omega r^2}\left(1+\frac{\omega^2 r^3}{6M}-\gamma+i\pi\right)\eta^3 - i\left(\frac{M}{r}-\frac{\omega^2 r^2}{16}+\frac{3(5+\kappa^2)M^2}{10\omega^2 r^4}\right)\eta^4 \\
&\quad + \left[\left(\frac{1}{3}-\frac{\gamma}{2}+\frac{i\pi}{2}\right)M\omega + (1-\gamma+i\pi)\frac{2M^2}{\omega r^3} - \frac{\omega^3 r^3}{60}\right]\eta^5 \\
&\quad + \left\{-\frac{2iM^3}{5\omega^2 r^5}(5+3\kappa^2) + \frac{iM^2}{15r^2}\left[-(15+4\kappa^2)\ln(2\omega r\eta) + 15(\gamma-i\pi)^2 - (45+4\kappa^2)\gamma + 30i\pi - \frac{5}{2}\pi^2\right.\right. \\
&\quad \left.\left. + \frac{101250+67800\kappa^2+15435\kappa^4+848\kappa^6}{10(15+4\kappa^2)^2}\right] + \frac{iM\omega^2 r}{3}\left(2\ln(2\omega r\eta)+\gamma-\frac{61}{24}\right) - \frac{i\omega^4 r^4}{288}\right\}\eta^6 \\
&\quad + O(\eta^7)\,,
\end{aligned} \tag{A.2.14}$$

having rescaled the "in" solution by the constant factor $\Gamma(c)$, with

$$\nu = 1 - \left(\frac{1}{2}+\frac{2}{15}\kappa^2\right)\epsilon^2 - \frac{496125+680400\kappa^2+135990\kappa^4+12688\kappa^6}{189000(15+4\kappa^2)}\epsilon^4 + O(\epsilon^6)\,. \tag{A.2.15}$$

A.3 Energy fluxes

Using the notation of Ref. [?], the energy fluxes (2.40-II) and (2.41-II) can be written as

$$\begin{aligned}\frac{dE^{\infty}}{dt} &= \left(\frac{dE^{\infty}}{dt}\right)_N \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \eta^{\infty}_{lm}, \\ \frac{dE^{H}}{dt} &= \left(\frac{dE^{\infty}}{dt}\right)_N 2(1-w)(1+\sqrt{1-w})y^3 \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \eta^{H}_{lm},\end{aligned} \tag{A.3.16}$$

with $\eta^{H,\infty}_{l-m} = \eta^{H,\infty}_{lm}$. For small values of the dimensionless angular velocity variable y the expansion coefficients behave as $\eta^{\infty}_{lm} \sim y^{l-1}$ and $\eta^{H}_{lm} \sim y^{2(l-1)}$, for fixed values of m. Therefore, since we have used the MST solutions up to $l=4$, our calculations of the flux at infinity and on the horizon are accurate up to the order $O(y^{7/2})$ and $O(y^{15/2})$ beyond the lowest order, respectively. For example, for $l=1$ the first few terms of the expansion are given by

$$\begin{aligned}\eta^{\infty}_{11} &= \frac{1}{2} + \left(-\frac{13}{5}+\frac{1}{3}w\right)y + \pi y^{3/2} + \left(\frac{1123}{350}+\frac{1}{30}w-\frac{1}{6}w^2\right)y^2 + \left(-\frac{26}{5}+\frac{2}{3}w\right)\pi y^{5/2} \\ &+ \left[\frac{10958}{945}-\frac{12427}{3150}w+\frac{26}{45}w^2+\frac{14}{81}w^3+\frac{2}{3}\pi^2+\left(-\frac{38}{15}+\frac{8}{15}w\right)\gamma+\left(-\frac{38}{15}+\frac{8}{15}w\right)\ln(2)\right. \\ &\left.+\left(-\frac{19}{15}+\frac{4}{15}w\right)\ln(y)\right]y^3 + \left(\frac{1123}{175}+\frac{1}{15}w-\frac{1}{3}w^2\right)\pi y^{7/2} + O(y^4), \\ \eta^{H}_{11} &= \frac{1}{2} + \left(1-\frac{2}{3}w\right)y + \left(\frac{11}{10}-\frac{23}{30}w+w^2\right)y^3 \\ &+ \left[\frac{971}{150}+\frac{347}{450}w+\frac{44}{45}w^2-\frac{130}{81}w^3+\frac{1}{2(1-w)}+\left(-\frac{19}{15}+\frac{4}{15}w\right)\ln(1-w)+\frac{1}{(1-w)^{1/2}}\left(\frac{2}{3}-\frac{1}{3}w\right)\pi^2\right. \\ &\left.-\frac{4}{3}\gamma+\left(-\frac{58}{15}+\frac{8}{15}w\right)\ln(2)+\left(-\frac{16}{5}+\frac{8}{15}w\right)\ln(y)\right]y^4 + O(y^5).\end{aligned} \tag{A.3.17}$$