Algorithm Design Strategies III

Joaquim Madeira

Version 0.3 – October 2020

Overview

- Tasks from last week – Recap + Questions ?
- The Higher-Lower Game – Analysis
- Example – Empirical Analysis
- Dynamic Programming
- Example – Fibonacci's Sequence
- Example – Linear Robot
- Example – Computing Binomial Coefficients
- Memoization

COMPUTING POWERS
BRUTE-FORCE VERSIONS

a^b – Brute-Force – Iterative algorithm

- Compute a^b, with $b \geq 0$, using

$$a^b = a \times a \times \ldots \times a$$

- Number of multiplications ?
  - Formal + Empirical analysis

a^b – Brute-Force – Iterative algorithm

```
def powerIterV1( a, b ) :
    """ Computing a**b using a loop """
    assert (type( b ) == int) and (b >= 0), "Wrong exponent!"
    assert (a != 0) or (b != 0), "Cannot compute 0**0 !"
    res = 1
    for i in range( 1, b + 1 ) :
        res *= a
    return res
```

- Number of multiplications ?

a^b – Brute-Force – Iterative algorithm

n	2**n	#Mults
0	1	0
1	2	1
2	4	2
3	8	3
4	16	4
5	32	5
6	64	6
7	128	7
8	256	8
9	512	9
10	1024	10
11	2048	11

a^b – Brute-Force – Recursive alg.

- Compute a^b, with $b \geq 0$, using

$$a^b = a \times a^{b-1}, \text{ with } a^0 = 1$$

- Number of multiplications ?
  - Formal + Empirical analysis
- Any gains ?

a^b – Brute-Force – Recursive alg.

```
def powerRecV1( a, b ) :

    """ Computing a**b recursively --- Direct algorithm """

    assert (type( b ) == int) and (b >= 0), "Wrong exponent!"

    assert (a != 0) or (b != 0), "Cannot compute 0**0 !"

    if b == 0 :

        return 1

    return a * powerRecV1( a, b - 1 )
```

- Number of multiplications ?

a^b – Brute-Force – Recursive alg.

n	2**n	#Mults
0	1	0
1	2	1
2	4	2
3	8	3
4	16	4
5	32	5
6	64	6
7	128	7
8	256	8
9	512	9
10	1024	10
11	2048	11

COMPUTING POWERS
DIVIDE-AND-CONQUER

a^b – Divide-And-Conquer

- Compute a^b, with $b \geq 0$, using

$$a^b = a^{b \text{ div } 2} \times a^{(b+1) \text{ div } 2}$$

- Base cases ?
- Always use two recursive calls !!

- Number of multiplications ?
  - Formal + Empirical analysis
- Is it better than the direct algorithm ?

a^b – Divide-And-Conquer

```
def powerRecV3( a, b ) :
    """ Computing a**b recursively --- Blind Div & Conq strategy"""
    # TWO base cases are needed !!
    # Otherwise, we would not stop when b == 1 !!
    assert (type( b ) == int) and (b >= 0), "Wrong exponent!"
    if b == 0 :
        return 1
    if b == 1 :
        return a
    return powerRecV3( a, b // 2 ) * powerRecV3( a, (b + 1) // 2 )
```

- Number of multiplications ?

Formal analysis

$$M(n) = M(n \text{ div } 2) + M((n+1) \text{ div } 2) + 1$$

- Easier to solve if n is a power of 2
  - $n = 2^k$, $k = \log_2 n$

$$M(n) = M(n / 2) + M(n / 2) + 1$$
$$= 2\,M(n / 2) + 1 = \ldots$$

- Closed formula ? Complexity order ?

a^b – Divide-And-Conquer

n	2**n	#Mults
0	1	0
1	2	0
2	4	1
3	8	2
4	16	3
5	32	4
6	64	5
7	128	6
8	256	7
9	512	8
10	1024	9
11	2048	10

COMPUTING POWERS DECREASE-AND-CONQUER

a^b – Decrease-And-Conquer

- Compute a^b, with $b \geq 0$, using

$$a^b = a^{b \text{ div } 2} \times a^{b \text{ div } 2} \text{, if } b \text{ is even}$$

$$a^b = a \times a^{(b-1) \text{ div } 2} \times a^{(b-1) \text{ div } 2} \text{, if } b \text{ is odd}$$

- Base cases ?
- Use just ONE recursive call !!
- Number of multiplications ?
  - Formal + Empirical analysis

a^b – Decrease-And-Conquer

```
def powerRecV6( a, b ) :
    """ Computing a**b recursively --- Smart Dec & Conq strategy"""
    assert (type( b ) == int) and (b >= 0), "Wrong exponent!"
    if b == 0 :
        return 1
    p = powerRecV6( a, b // 2 )
    if (b % 2) == 0 :
        return p * p
    return a * p * p
```

- Number of multiplications ?

Formal analysis

$$M(n) = M(n \text{ div } 2) + 1 \text{ , if } n \text{ is even}$$

$$M(n) = M((n-1) \text{ div } 2) + 2 \text{ , if } n \text{ is odd}$$

- Check some examples with pencil and paper
  - Do you understand what is happening?
  - Best vs. worst cases ?
- Closed formula ? Complexity order ?
- Is it better than the previous algorithms ?

a^b – Decrease-And-Conquer

n	2**n	#Mults
0	1	0
1	2	2
2	4	3
3	8	4
4	16	4
5	32	5
6	64	5
7	128	6
8	256	5
9	512	6
10	1024	6
11	2048	7
12	4096	6
13	8192	7
14	16384	7
15	32768	8
16	65536	6
17	131072	7

ARRAY: COUNTING EVEN-VALUED ELEMENTS

Task – Counting – Recap

- Given an array with non-negative integer values
- Count the number of even-valued elements
- Implement the 3 strategies:
  - Brute-Force / Dec & C / Div & C
- Formal + Empirical analysis : Comparisons

Number of array comparisons ?

- Brute-Force
  - 1 loop / n iterations / n comparisons
- Decrease & Conquer
  - C(0) = 0
  - C(n) = 1 + C(n – 1)
- Divide & Conquer
  - C(0) = 0 and C(1) = 1
  - C(n) = C(n div 2) + C((n+1) div 2)

Number of array comparisons

#Elements	#COMPS	#Even-Values
1	1	0
2	2	1
4	4	2
8	8	4
16	16	8
32	32	16
64	64	32
128	128	64
256	256	128
512	512	256

ARRAY:
SEQUENTIAL SEARCH

Task – Sequential Search – Recap

- Given an array with non-negative integer values
- Use the iterative Sequential Search algorithm to look for a given value
- Formal + Empirical analysis : Comparisons
- Best / Worst / Average Cases ?

Possible cases ?

- Best case ?
  - 1 comparison → $B(n) = O(1)$
- Worst Case ?
  - n comparisons → $W(n) = O(n)$
- Average Case ?
  - Various possible scenarios
  - BUT always → $A(n) = O(n)$

Average Case

- One possible scenario
  - No repeated array values
  - Searched value belongs to the array
  - Can be any array element – Equal probability !

- Average number of comparisons ?
  - How to compute ?

- $A(n) = (n + 1) / 2 \approx n / 2$

A different scenario

```
Sequential Search on RANDOM arrays of positive integers

Searching RANDOM values

#Elements  #Searches  #Found     #COMPS  #Average_COMPS
-------------------------------------------------------
        1          1       0          1           1.000
        2          2       0          4           2.000
        4          4       0         16           4.000
        8          8       0         64           8.000
       16         16       0        256          16.000
       32         32       2        993          31.031
       64         64       3       4007          62.609
      128        128       9      15726         122.859
      256        256      56      57621         225.082
      512        512     209     207084         404.461
     1024       1024     677     658902         643.459
     2048       2048    1751    1845775         901.257
     4096       4096    4004    3992714         974.784
```

THE HIGHER-LOWER GAME

The Higher-Lower Game

- Person A chooses a random integer in [1, 100]
- Person B guesses a number in [1, 100]
- Person A says: Yes / Low guess / High guess
- Person B keeps guessing until the answer is Yes

- How long does it take ?

- Let’s play the game !!

The Higher-Lower Game

- Naïve strategy vs. smart strategy

- How many guesses ?
  - Best case ?
  - Worst cases ?
  - Average cases ?

- Complexity order ?

Task – Game simulation

- Implement an iterative function to simulate the game
- Repeat the game many times !!

- Count the number of guesses needed in each game
  - Histogram + Simple statiscal analysis

- Complexity order ?

One simulation scenario

```
Simulation: playing the higher-lower game 100000 times
The interval of values is [1, 100]
         1 attempts:    975 - 0.97%
         2 attempts:   2062 - 2.06%
         3 attempts:   3929 - 3.93%
         4 attempts:   7935 - 7.94%
         5 attempts:  16063 - 16.06%
         6 attempts:  31973 - 31.97%
         7 attempts:  37063 - 37.06%
MIN    - The smallest number of attempts = 1
MEDIAN - The "middle" number of attempts = 6
MEAN   - The average number of attempts  = 5.8022
MAX    - The largest number of attempts = 7
```

Another simulation scenario

```
Simulation: playing the higher-lower game 100000 times
The interval of values is [1, 1000]
         1 attempts:      95 - 0.10%
         2 attempts:     185 - 0.18%
         3 attempts:     379 - 0.38%
         4 attempts:     826 - 0.83%
         5 attempts:    1612 - 1.61%
         6 attempts:    3155 - 3.16%
         7 attempts:    6403 - 6.40%
         8 attempts:   12819 - 12.82%
         9 attempts:   25820 - 25.82%
        10 attempts:   48706 - 48.71%
MIN    - The smallest number of attempts = 1
MEDIAN - The "middle" number of attempts = 9
MEAN   - The average number of attempts  = 8.98709
MAX    - The largest number of attempts = 10
```

EXPERIMENTAL ANALYSIS
- AN EXAMPLE

Example – Experimental Analysis

- Performance data for 4 sorting algorithms
- Sorting sets of n random strings

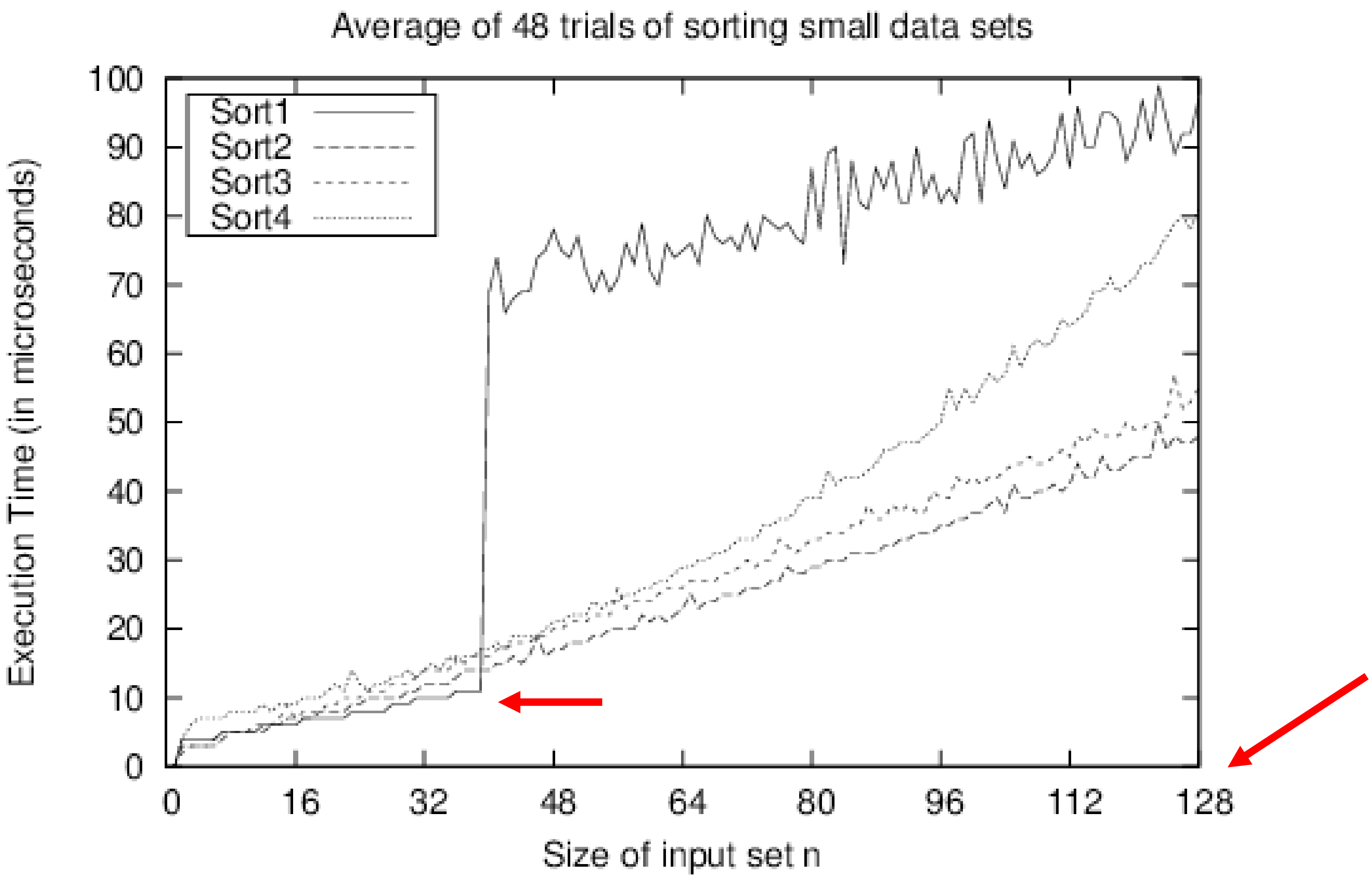
- 50 trials
- Best time and worst time were discarded

- Best algorithm ?
- Complexity order ?

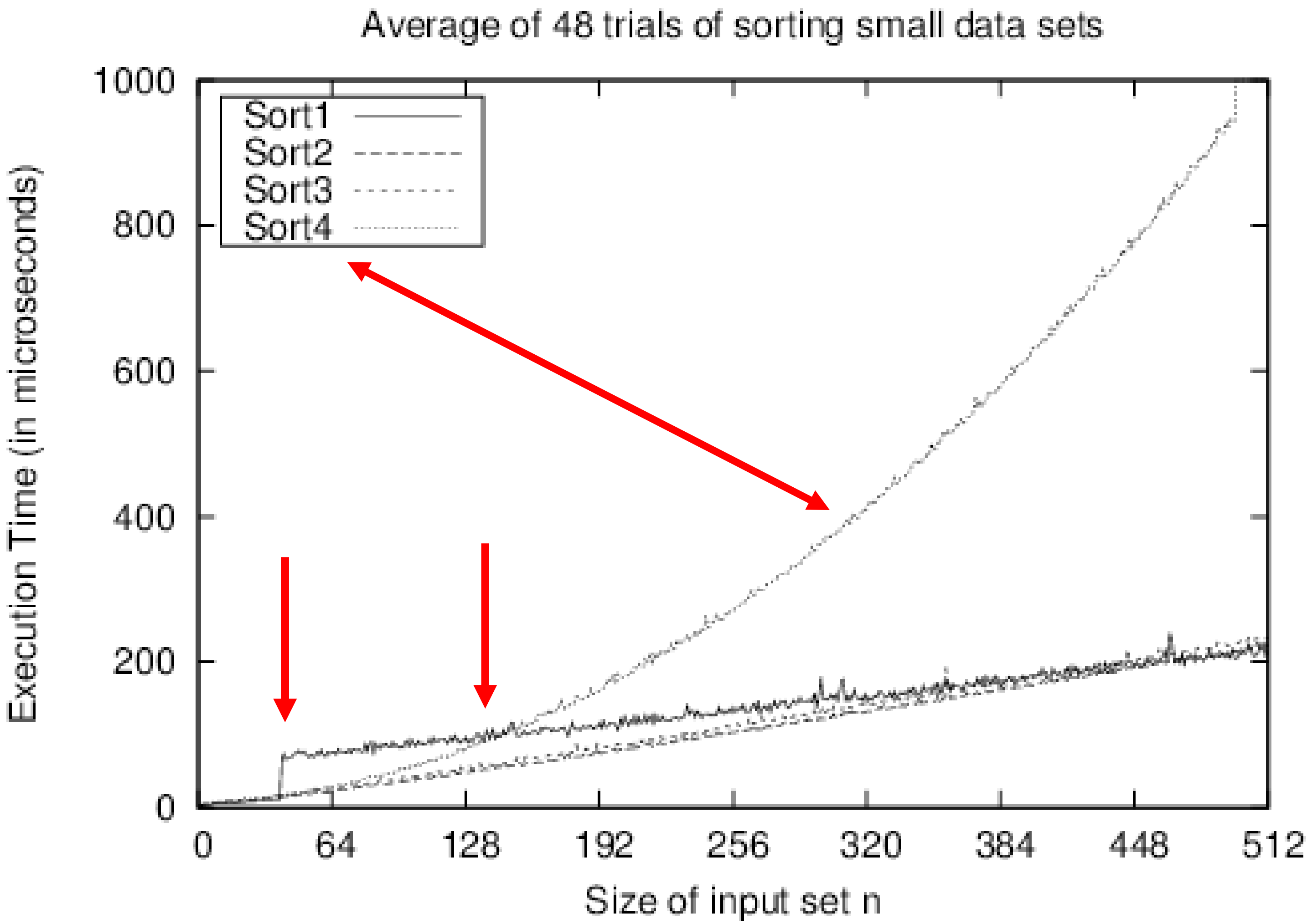
- Identify the algorithms ?

- Heineman et al. Algorithms in a Nutshell. 2nd Ed., O'Reilly, 2016

Average running time – Random data



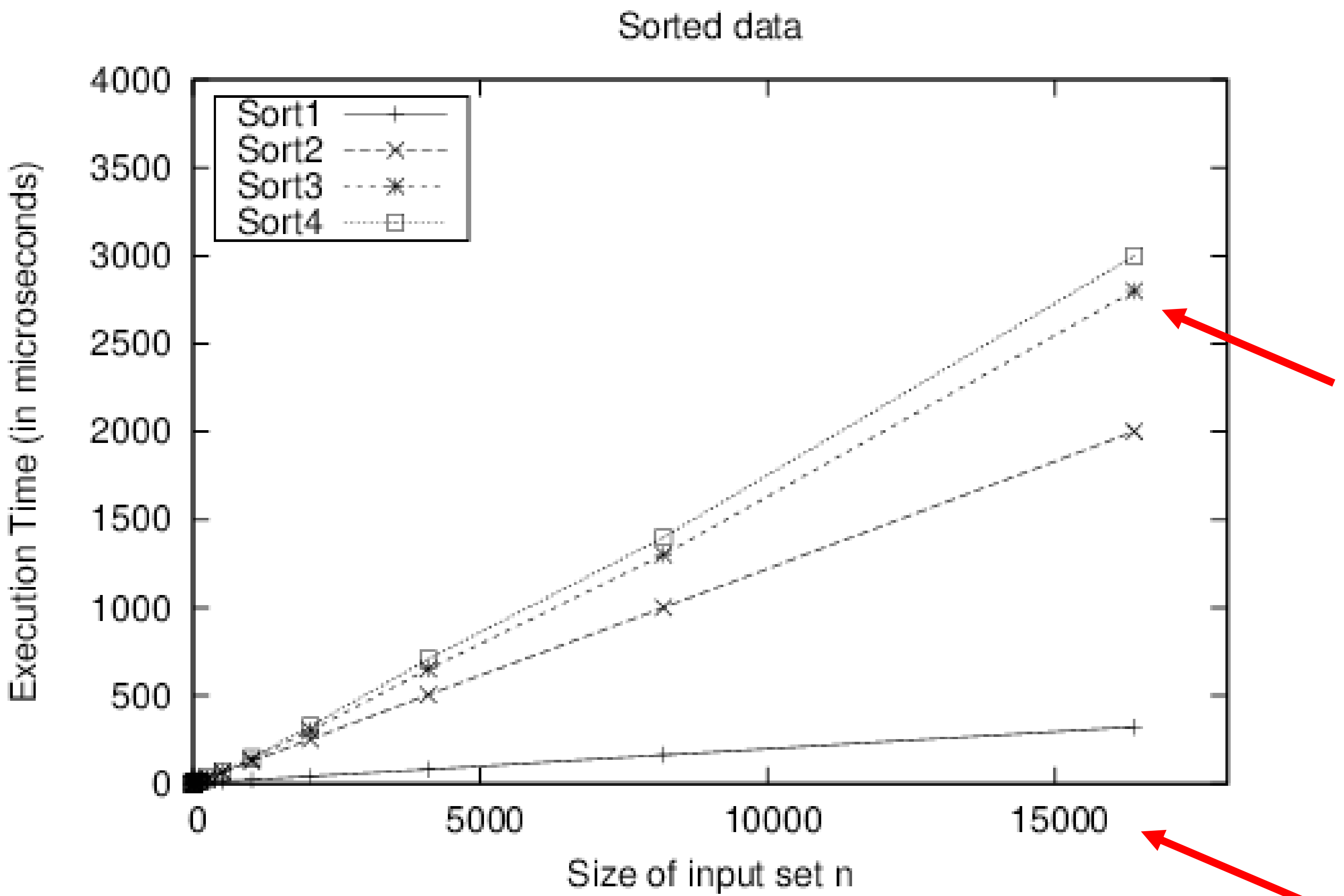
Average running time – Random data



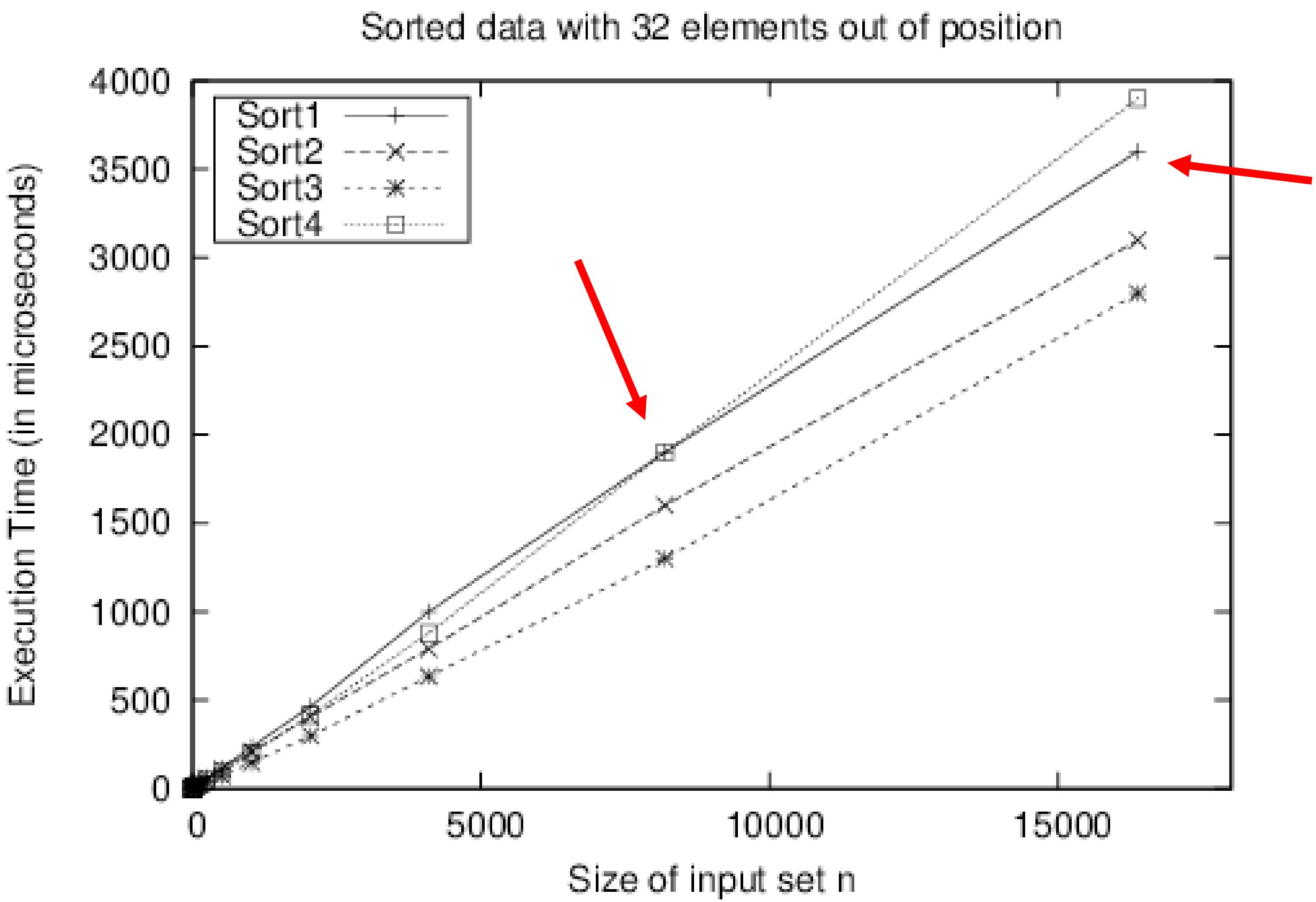
Regression analysis

- Sort-4
  - $y = 0.0053 \times n^2 - 0.3601 \times n + 39.212$
- Sort-2
  - $y = 0.05765 \times n \times log(n) + 7.9653$
- Sort-3
  - Slightly slower than Sort-2
- Sort-1
  - $n < 40$: $y = 0.0016 \times n^2 + 0.2939 \times n + 3.1838$
  - $n \geq 40$: $y = 0.0798 \times n \times log(n) + 142.7818$

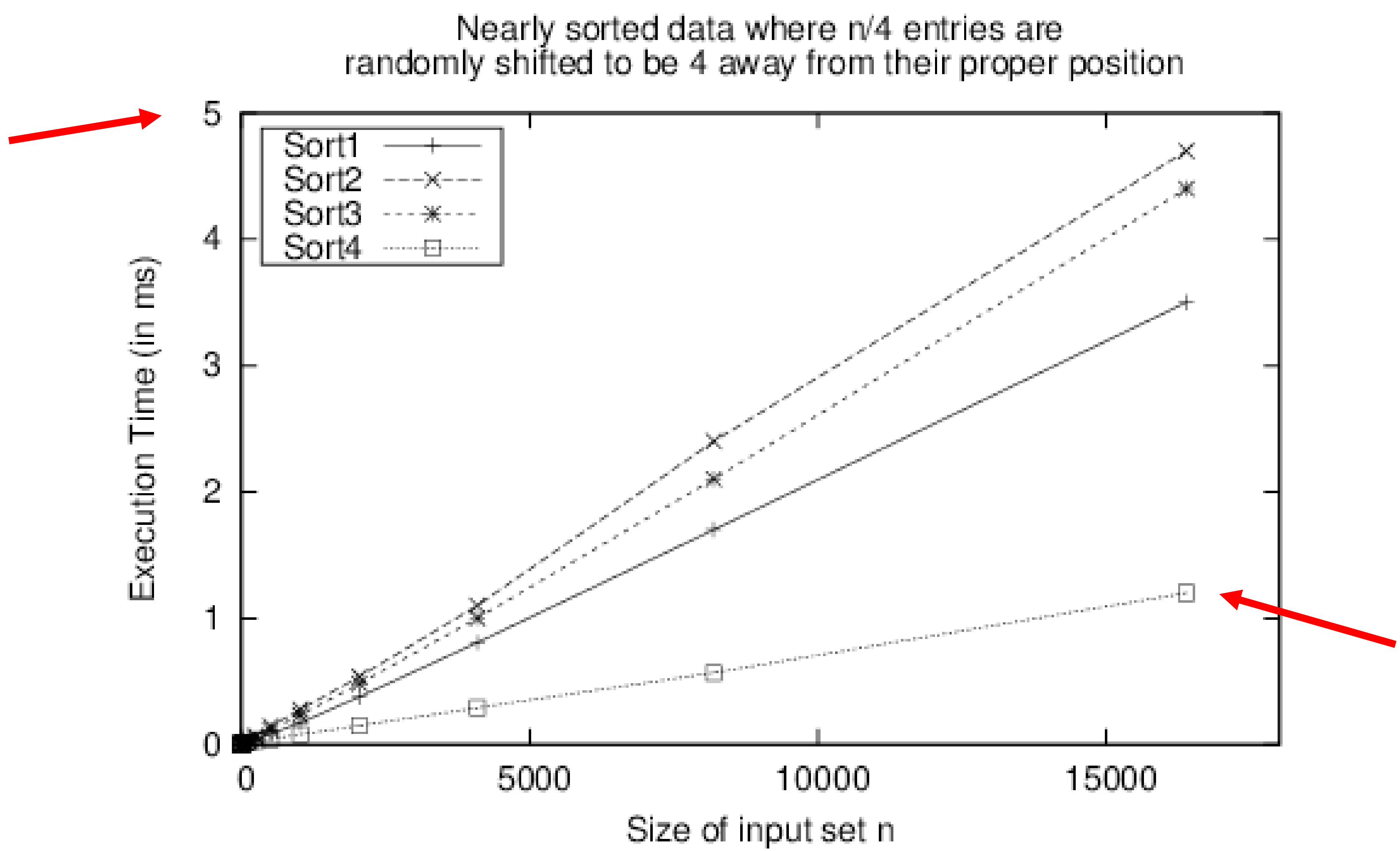
Average running time – Sorted data



Only 32 elements out of position !



Nearly-sorted



Which algorithms ?

- Sort-1 : qsort on Linux
- Sort-2 : ?
- Sort-3 : ?
- Sort-4 : ?

DYNAMIC PROGRAMMING

Dynamic Programming

- General algorithm design technique

- Apply to
  - Computing recurrences
  - Solving optimization problems

- How to store “previous” results ?
  - 2D array
  - Vector
  - A few variables

Recurrences – Top-Down

- Exploit the relationship between
  - A solution to a given problem instance
  - Solutions to smaller/simpler instances of the same problem
- Set up a recurrence !
- Decompose into smaller / simpler sub-problems
  - Parameters ?
- Identify the smallest / simplest / trivial problems
  - Base cases

Dynamic Programming – Bottom-up

- Use a recurrence: BUT go bottom-up !
- Start from the smallest / simplest / trivial problems
- Get intermediate solutions from smaller / simpler sub-problems
- Which values / results are computed in each step ?
  - How to store ?

Dynamic Programming – Advantage

- Do sub-problems overlap ?
- NOW, there is no need to repeatedly solve the same sub-problems !!
- Proceed bottom-up and store results for later use
- Compare with Divide-and-Conquer !!

COMPUTING FIBONACCI NUMBERS

Fibonacci's Sequence

- F(0) = 0 ; F(1) = 1
- F(i) = F(i – 1) + F(i – 2) ; i = 2, 3, 4,…
- F(6) = ? ➔ Number of recursive calls ?
- Do sub-problems overlap ?
- Recursion tree vs. recursion DAG !!
- Complexity order ?

Tasks – V1

- Implement the recursive function of the previous slide in Python

- Count the number of additions carried out for computing a Fibonacci number
  - Use a global variable

- Table ?
- Complexity order ?

Fibonacci's Sequence

```
def fibonacci_DC( n ) :

    """ Recursive computation of Fi """

    # Global variable, for counting the number of additions

    global num_adds

    if ( n == 0 ) or ( n == 1 ) :

        return n

    num_adds += 1

    return fibonacci_DC( n - 1 ) + fibonacci_DC( n - 2 )
```

Number of additions ?

- $A(0) = 0$; $A(1) = 0$
- $A(i) = 1 + A(i - 1) + A(i - 2)$; $i = 2, 3, 4, \ldots$
- Closed formula ?
- You can get it, if you remember Discrete Mathematics…
- BUT, we can get the complexity order from the table…

Additions – Recursive version

- How fast does F(n) grow ?
- How fast does A(n) grow ?
- From the table we get:

  A(n) = F(n+1) – 1

- Exponential growth !!
  - Why ?

$$(1+\sqrt{5})/2 = 1{,}618034$$

n	F(n)	Ratio	A(n)	Ratio
0	0		0	
1	1		0	
2	1	1	1	
3	2	2	2	2
4	3	1,5	4	2
5	5	1,666667	7	1,75
6	8	1,6	12	1,714286
7	13	1,625	20	1,666667
8	21	1,615385	33	1,65
9	34	1,619048	54	1,636364
10	55	1,617647	88	1,62963
11	89	1,618182	143	1,625
12	144	1,617978	232	1,622378
13	233	1,618056	376	1,62069
14	377	1,618026	609	1,619681
15	610	1,618037	986	1,619048
16	987	1,618033	1596	1,618661
17	1597	1,618034	2583	1,618421
18	2584	1,618034	4180	1,618273
19	4181	1,618034	6764	1,618182
20	6765	1,618034	10945	1,618125

Fibonacci's Sequence

- $F(0) = 0$; $F(1) = 1$
- $F(i) = F(i - 1) + F(i - 2)$; $i = 2, 3, 4, \ldots$

- Use Dynamic Programming !!

- Computing F(n) using an array
  - Complexity order ?

- Can we use less memory space ?

Tasks – V2 + V3

- Implement two iterative functions for computing F(i)
  - V2 : using an array
  - V3 : using just 3 variables

- Count the number of additions carried out

- Table ?
- Complexity order ?

Fibonacci's Sequence

i	f(i)	#ADDs-Rec	#ADDs_DP_1	#ADDs_DP_2
0	0	0	0	0
1	1	0	0	0
2	1	1	1	1
3	2	2	2	2
4	3	4	3	3
5	5	7	4	4
6	8	12	5	5
7	13	20	6	6
8	21	33	7	7
9	34	54	8	8
10	55	88	9	9
11	89	143	10	10
12	144	232	11	11
13	233	376	12	12
14	377	609	13	13
15	610	986	14	14

EXAMPLE
- A LINEAR ROBOT

Another example

- Linear robot
- Can move forward by 1 meter, or 2 meters, or 3 meters

- In how many ways can it move a distance of n meters ?

- Establish the recurrence !!
  - Base cases ?

Tasks – V1 + V2 + V3

- Implement three functions for computing R(i)
  - V1 : using recursion
  - V2 : using an array
  - V3 : using a few variables – how many ?
- Count the number of additions carried out
  - Formulas ?
- Tables ?
- Complexity order ?

Example – Results table

i	r(i)	#ADDs-Rec	#ADDs_DP_1	#ADDs_DP_2
1	1	0	0	0
2	2	0	0	0
3	4	0	0	0
4	7	2	2	2
5	13	4	4	4
6	24	8	6	6
7	44	16	8	8
8	81	30	10	10
9	149	56	12	12
10	274	104	14	14
11	504	192	16	16
12	927	354	18	18
13	1705	652	20	20
14	3136	1200	22	22
15	5768	2208	24	24

COMPUTING BINOMIAL COEFFICIENTS

Computing Binomial Coefficients

- $C(n,0) = 1$; $C(n,n) = 1$
- $C(n,j) = C(n-1,j) + C(n-1,j-1)$; $j = 1, 2, \ldots, n-1$

- Two arguments !!
- $C(4,3) = ?$ ➔ Number of recursive calls ?

- Do sub-problems overlap ?
- Recursion tree vs. recursion DAG !!

- Complexity order ?

Computing Binomial Coefficients

- V1 : Compute C(n,j) recursively
- V2 : Compute C(n,j) using a 2D array
  - How to proceed ?
  - Have you seen this “triangle” before ?
- Can we use less memory space ?
- And other, more efficient recurrences ?

Tasks – V1 + V2 + V3

- Implement three functions for computing C(n,j)
  - V1 : using recursion
  - V2 : using a 2D array
  - V3 : using a 1D array

- Count the number of additions carried out

- Tables ?
- Complexity order ?

Pascal's Triangle

```
Pascal's Triangle - Recursive Function

1
1       1
1       2       1
1       3       3       1
1       4       6       4       1
1       5       10      10      5       1
1       6       15      20      15      6       1
1       7       21      35      35      21      7       1
1       8       28      56      70      56      28      8       1
1       9       36      84      126     126     84      36      9       1
1       10      45      120     210     252     210     120     45      10      1
```

V1 – Number of additions

```
Number of Additions - Recursive Function

0
0       0
0       1       0
0       2       2       0
0       3       5       3       0
0       4       9       9       4       0
0       5       14      19      14      5       0
0       6       20      34      34      20      6       0
0       7       27      55      69      55      27      7       0
0       8       35      83      125     125     83      35      8       0
0       9       44      119     209     251     209     119     44      9       0
```

V2 – Number of additions

```
Number of Additions - Dynamic Programming - V. 1

0
0       0
1       1       1
3       3       3       3
6       6       6       6       6
10      10      10      10      10      10
15      15      15      15      15      15      15
21      21      21      21      21      21      21      21
28      28      28      28      28      28      28      28      28
36      36      36      36      36      36      36      36      36      36
45      45      45      45      45      45      45      45      45      45      45
```

V3 – Number of additions

```
Number of Additions - Dynamic Programming - V. 2

0
0       0
0       1       0
0       3       3       0
0       6       6       6       0
0       10      10      10      10      0
0       15      15      15      15      15      0
0       21      21      21      21      21      21      0
0       28      28      28      28      28      28      28      0
0       36      36      36      36      36      36      36      36      0
0       45      45      45      45      45      45      45      45      45      0
```

MEMOIZATION

Memoization

- Turning the results of a function into something to be remembered

- I.e., avoid repeating the calculation of results for previously processed inputs

- Use a table / array / cache to store previously computed results
  - Initialization !

- Time vs. space trade-off

Memoization

- Initialize all table entries to "null"
  - Not yet computed

- Whenever a result is to be computed for a given input
  - Check the corresponding table entry
  - If not "null", retrieve the result
  - Otherwise, compute by a recursive call(s)
  - And store the result

Fibonacci's Sequence

- Initialization

```
for(i=1, i<n, i++)   f[i] = -1;
```

- Recursive function

```
int fib( int n ) {
        int r;
        if( f[n] != -1 )   return f[n];
        if( n == 1 )   r = 1;
        else if( n == 2 )   r = 1;
        else {
                r = fib( n – 2 );
                r = r + fib( n – 1 );
        }
        f[n] = r;
        return r;
}
```

The Python way

```
# M. Hetland, Python Algorithms, Apress, 2010 - Chapter 8

from functools import wraps

def memo( func ) :

    cache = {}                              # Stored subproblem solutions

    @wraps(func)                            # Make wrap look like func

    def wrap( *args ) :                     # The memoized wrapper

        if args not in cache :              # Not already computed?

            cache[args] = func( *args )     # Compute & cache the solution

        return cache[args]                  # Return the cached solution

    return wrap                             # Return the wrapper
```

The Python way

```
# Testing the memoized version

fibonacci_DC = memo( fibonacci_DC )
```

```
 i                     f(i) #ADDs_Memo
--------------------------------------
 0                        0          0
 1                        1          0
 2                        1          1
 3                        2          1
 4                        3          1
 5                        5          1
 6                        8          1
 7                       13          1
 8                       21          1
 9                       34          1
10                       55          1

 85   259695496911122585          1
 86   420196140727489673          1
 87   679891637638612258          1
 88  1100087778366101931          1
 89  1779979416004714189          1
 90  2880067194370816120          1
```

REFERENCES

References

- A. Levitin, *Introduction to the Design and Analysis of Algorithms,* 3rd Ed., Pearson, 2012
  - Chapter 8
- R. Johnsonbaugh and M. Schaefer, *Algorithms,* Pearson Prentice Hall, 2004
  - Chapter 8
- T. H. Cormen et al., *Introduction to Algorithms*, 3rd Ed., MIT Press, 2009
  - Chapter 15