
Introduction to Randomized Algorithms II

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Overview

- Discrete Probability
- Statistical Experiments and Events
- Probabilities and Random Variables
- Application Examples and Problems

DISCRETE PROBABILITY

Discrete Probability

- **Chance** enters into many attempts to understand the world we live in
- A **theory of probability** allows us to calculate the **likelihood of complex events**
- Probabilities are called “**discrete**” if we can compute the probabilities of all events by summation

Probability Space

- Probability theory starts with the idea of a probability space (Ω, \Pr)
 - A set Ω of all things that can happen
 - A rule assigning a probability $\Pr(\omega)$ to each elementary event ω in Ω

Probability Distribution

- For a **discrete** probability space
 - $\Pr(\omega) \geq 0$
 - $\sum \Pr(\omega) = 1$
- **Pr** is the probability distribution
 - It distributes the total probability among the elementary events

Example – Fair dice

- Roll one fair 6-sided die

$$D = \{ \square \cdot, \square \cdot \cdot, \square \cdot \cdot \cdot, \square \cdot \cdot \cdot \cdot, \square \cdot \cdot \cdot \cdot \cdot, \square \cdot \cdot \cdot \cdot \cdot \cdot \}$$

- Each of the 6 possibilities has probability 1/6
- Roll a pair of fair dice
 - Set of elementary events : $D^2 = ?$
 - Probability of each event ?

Example – “Loaded” dice

- Distribution of probabilities

$$\Pr_1(\boxed{\bullet}) = \Pr_1(\boxed{\begin{smallmatrix} \bullet & \bullet \\ \bullet & \bullet \end{smallmatrix}}) = \frac{1}{4};$$

$$\Pr_1(\boxed{\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}}) = \Pr_1(\boxed{\begin{smallmatrix} \bullet & \\ \bullet & \bullet \end{smallmatrix}}) = \Pr_1(\boxed{\begin{smallmatrix} & \\ \bullet & \bullet \end{smallmatrix}}) = \Pr_1(\boxed{\begin{smallmatrix} \bullet & \bullet \\ & \bullet \end{smallmatrix}}) = \frac{1}{8}$$

- Probability of **each event** in D^2 ?

$$\Pr_{11}(d d') = \Pr_1(d) \Pr_1(d')$$

Example – Fair die + “Loaded” die

- Consider the case of **one fair die** and **one loaded die**

$$\Pr_{01}(d, d') = \Pr_0(d) \Pr_1(d'), \quad \text{where } \Pr_0(d) = \frac{1}{6};$$

- Real-world dice do not turn up equally often on each side !!
 - **No perfect symmetry !!**
- **BUT, 1/6 is usually close to the truth...**

Example – Doubles are thrown

- The event that “doubles are thrown”



- Probability of an event A

$$\Pr(\omega \in A) = \sum_{\omega \in A} \Pr(\omega)$$

- $\Pr(\text{“doubles are thrown”}) = ?$
 - When is this event more probable ?

Statistical experiments and events

- Statistical experiment
 - **Repeatable** experiment where the particular **outcome** of a trial **cannot be predicted with certainty**
- Sample space, **S**
 - **Set** with the representation of all **possible outcomes** of an experiment
 - I.e., set of **elementary events**
- Examples ?

Statistical experiments and events

■ Event, E

- A set of elementary events
- Any **subset** of a **sample space**

■ Examples

- $\{2, 4, 6\}$ – Getting an even number when throwing a 6-sided die
 - Probability ?
- ...

Probabilities

- The **probability** of an event, E , describes the degree of **uncertainty** of that event
- $0 \leq P[E] \leq 1$
- $P[S] = 1$
- $P[\emptyset] = 0$
- $P[A \cup B] = P[A] + P[B]$, if A and B are **disjoint**
- If $(A \cap B) \neq \emptyset$, $P[A \cup B] = P[A] + P[B] - P[A \cap B]$

PROBABILITY DISTRIBUTIONS

Uniform probability distribution on S

- Each elementary event in S has the same probability

$$P[\{A\}] = P[\{B\}] = \frac{1}{|S|}$$

Simple problem

- Throwing a 6-sided fair die
- What is the probability of getting an even number ?
- What is the probability of getting a number larger than 2 ?
- What is the probability of getting an even number or a number larger than 2 ?

Another simple problem

- Tossing **three fair coins**
- What is the sample space S_3 ?
- How high is the probability of getting **at least one head** ?
- And **at least two heads** ?
- Idea: relate to the binary representation
- Idea: triangular representation – paths

More difficult problem

- Tossing n fair coins
- How large is the probability to get “head” exactly k times ?
- How large is the probability to get “head” at least k times ?
- You can use your code to check your answers...

Binomial probability distribution

- Characterizes the probability of obtaining **k** “successes” in **n** experiments

$$S = \{0, 1, 2, \dots, n\}$$

$$P[X = k] = \binom{n}{k} p^k (1 - p)^{n-k}, k = 0, 1, 2, \dots, n$$

- Check your previous answers !!

Tasks

- What is the probability of getting 6 heads in 15 tosses of a fair coin ?
 - Estimate the value with simulated experiments
 - Check that you got the correct value by computing the probability from the binomial distribution
- Now, consider that $P[\text{heads}] = 2 \times P[\text{tails}]$

Independent events

- Two events A and B are said to be **independent**, if the occurrence of one does not affect the occurrence of the other
- Two events A and B are independent, **if**

$$P[A \text{ and } B] = P[A] \times P[B]$$

Conditional probability

- **Conditional probability** of event A given event B

$$P[A|B] = \frac{P[A \text{ and } B]}{P[B]}, P[B] \neq 0$$

- What happens if they are **independent** ?

- **Bayes' Theorem**

$$P[A|B] = \frac{P[B|A] P[A]}{P[B]}$$

RANDOM VARIABLES

Random variable

- A random variable is a function that assigns a **real number** to each **elementary event** of the sample space
- **Discrete** r. v. – countable set of real values
 - Values obtained by throwing a dice numbered 1 to 6
 - Assigning 0 to tail and 1 to heads when tossing a coin
- **Continuous** r. v. – interval or collection of intervals
 - Examples: temperature of a room or weight of product

Example – Throwing two dice

- $S(w)$ = **sum of spots** on the dice roll w
- What is the probability that the spots total **7** ?

$$\begin{aligned} &\Pr(\boxed{\cdot} \boxed{\begin{smallmatrix} \cdot \\ \cdot \\ \cdot \end{smallmatrix}}) + \Pr(\boxed{\cdot} \boxed{\begin{smallmatrix} \cdot & \cdot \\ \cdot & \cdot \end{smallmatrix}}) + \Pr(\boxed{\cdot} \boxed{\begin{smallmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot \end{smallmatrix}}) \\ &+ \Pr(\boxed{\begin{smallmatrix} \cdot & \cdot \\ \cdot & \cdot \end{smallmatrix}} \boxed{\cdot}) + \Pr(\boxed{\begin{smallmatrix} \cdot & \cdot \\ \cdot & \cdot \end{smallmatrix}} \boxed{\cdot}) + \Pr(\boxed{\begin{smallmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot \end{smallmatrix}} \boxed{\cdot}) \end{aligned}$$

- Fair dice ?
- Loaded dice ?

Example – Throwing two dice

- A random variable is characterized by the probability distribution of its values

s	2	3	4	5	6	7	8	9	10	11	12
$P_{r00}[S=s]$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
$P_{r11}[S=s]$	$\frac{4}{64}$	$\frac{4}{64}$	$\frac{5}{64}$	$\frac{6}{64}$	$\frac{7}{64}$	$\frac{12}{64}$	$\frac{7}{64}$	$\frac{6}{64}$	$\frac{5}{64}$	$\frac{4}{64}$	$\frac{4}{64}$

Sequence of numbers – Average value

■ Mean

- Sum of all values, divided by the number of values

■ Median

- Middle value, numerically

■ Mode

- Value that occurs most often

statistics – Python module

- Computing mathematical statistics of numeric data
- Averages and measures of central location
 - `mean(...)`
 - `median(...)`
 - `mode(...)`
 - ...
- Measures of spread
 - `stdev(...)`
 - `variance(...)`
 - ...

Discrete random variable – Features

- Mean – Expected value

$$\mu = E[X] = \sum_n X_n P[X = X_n]$$

- Variance

- Measures how far a set of numbers are spread out

$$\sigma^2 = E[(X - \mu)^2] = E[X^2] - \mu^2$$

- σ is the standard deviation

Sum of independent random vars.

- Let $Z = X + Y$ be the sum of two independent random variables, defined on the same probability space

$$E[Z] = E[X + Y] = E[X] + E[Y]$$

$$\sigma_Z^2 = E[(X + Y - \mu)^2] = \sigma_X^2 + \sigma_Y^2$$

Sum of independent random vars.

- Let S_n be the sum of n independent and identically distributed random variables

$$S_n = X_1 + X_2 + \dots + X_n$$

$$E[S_n] = n \times E[X]$$

$$\sigma_{S_n}^2 = n \times \sigma_X^2$$

Estimating the mean of a rand. var.

- Set of independent empirical observations
- Sample mean is

$$\hat{\mu} = \frac{1}{n} \sum X_i$$

- Keep a record of the sum as the experiment progresses
- Update the sample mean, when needed

Estimating the mean of a rand. var.

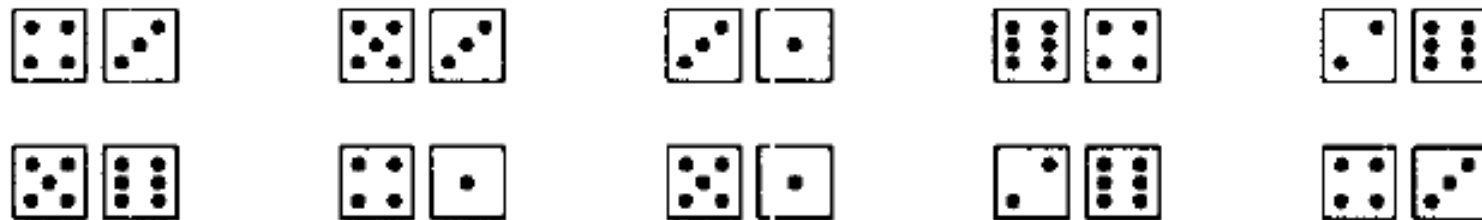
- Sample variance is

$$\hat{\sigma}^2 = \frac{1}{n-1} \sum X_i^2 - \frac{1}{n(n-1)} \left(\sum X_i \right)^2$$

- Keep a record of the sums as the experiment progresses
- Update the sample variance, when needed
- **Estimate** the mean as

$$\hat{\mu} \pm \hat{\sigma} / \sqrt{n}$$

Example – 10 rolls of two dice



- Sample mean of the spot sum

$$\hat{\mu} = 7.4$$

- Sample variance

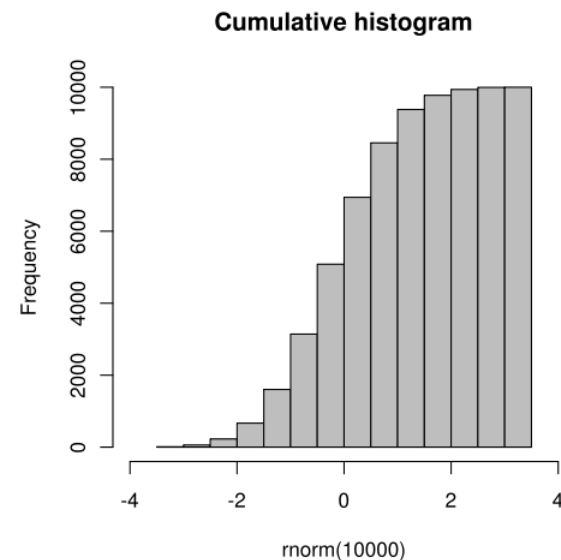
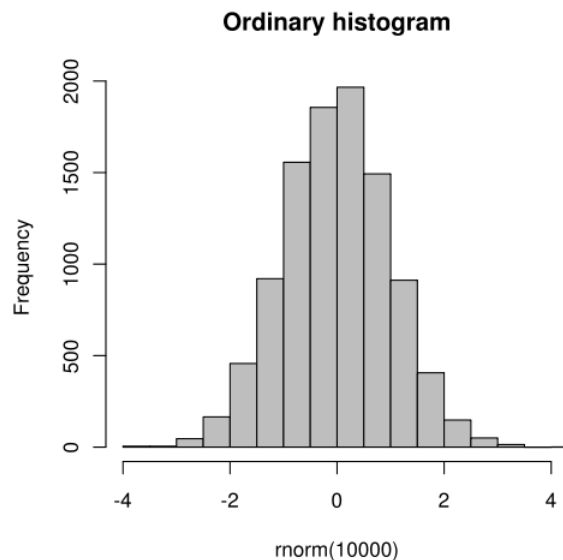
$$\hat{\sigma}^2 \approx 2.1^2$$

- Estimate

$$7.4 \pm 0.7$$

Histogram

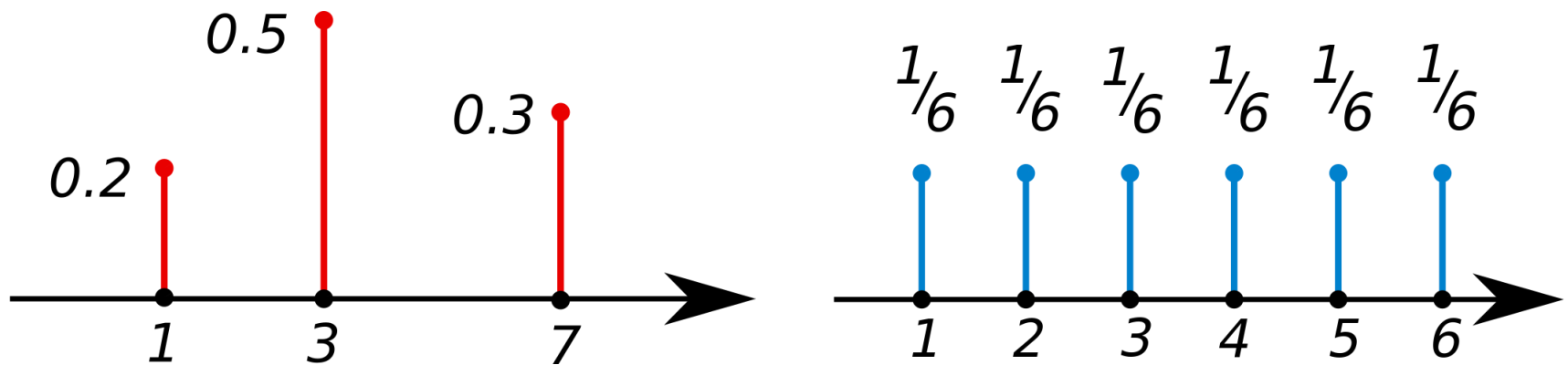
- Graphical **representation of the distribution** of numerical data
- May be **normalized** to display relative “frequencies”
- A **cumulative histogram** represents the cumulative number of observations



[Wikipedia]

Probability mass function

- Describes the **relative likelihood** of a discrete random variable to take on a given value



[Wikipedia]

APPLICATION PROBLEMS

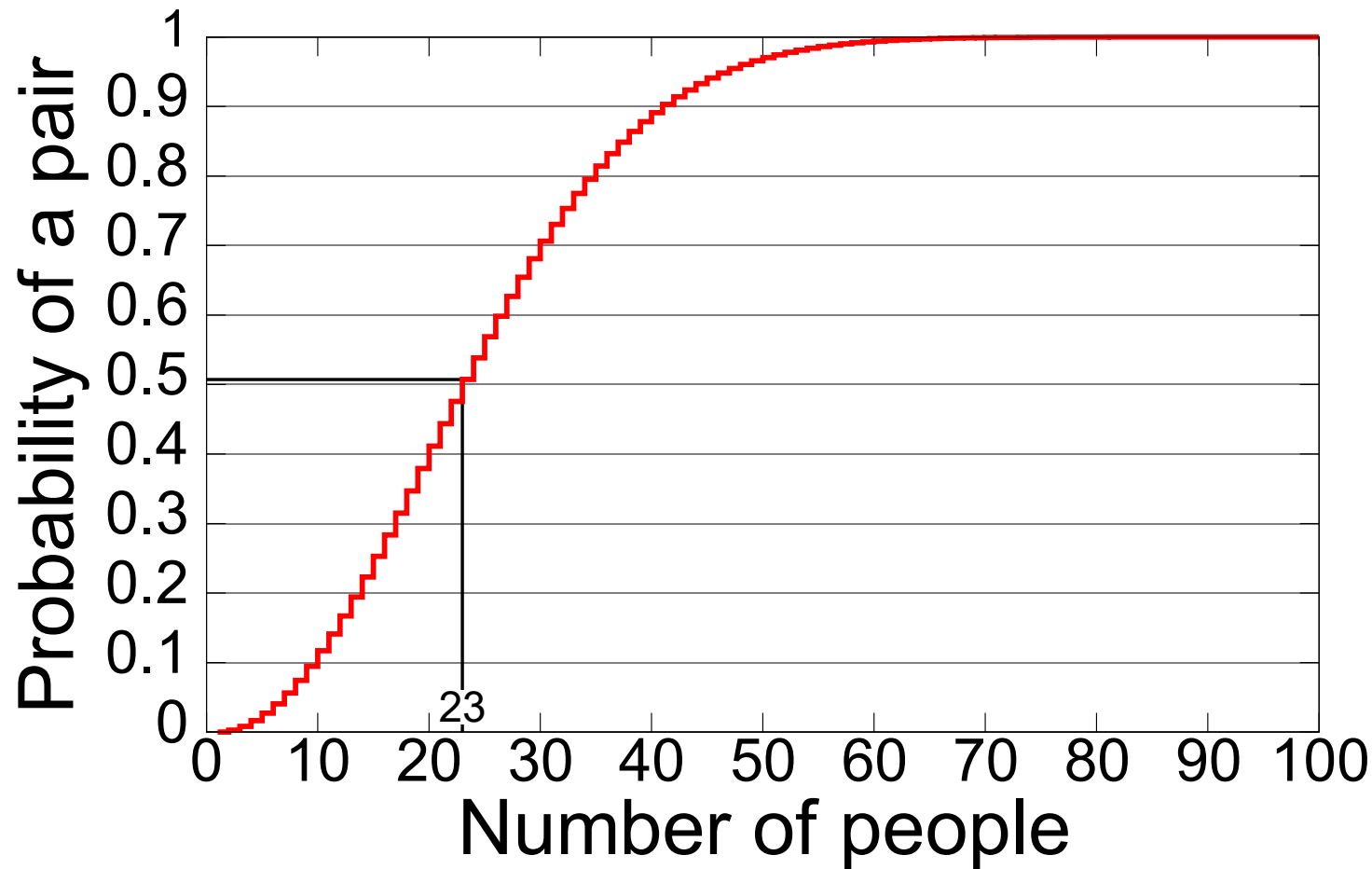
Task – Problem 1

- Consider the statistical experiment in which a fair die is thrown repeatedly until **one of the faces appears for the second time**
- An outcome of the experiment is **a list of the faces** that are thrown
 - Possible outcomes: (3, 6, 2, 6) ; (5, 5) ; (1, 4, 3, 6, 2, 4)
- Let Y denote the random variable that tells how many **throws** were **necessary** to produce a **repetition** of a face
 - For the examples above: 4, 2, 6
- Simulate an observation of Y

Task – Problem 2

- In a party with n people, what is the probability of at least two of them celebrating their birthday in the same day ?
- What is the smallest n that guaranties that the previous probability is above 50% ?
- Estimate the values with simulated experiments
- Consider that each birthday is equally likely
- Read about “The Birthday Paradox” !

The Birthday Paradox



[Wikipedia]

Task – Problem 3

- Consider $n = 4000$
- Generate random numbers in the domain $[n]$ until **two have the same value**
- How many random trials did that take ?
 - Use k to represent this value
- **Repeat** the experiment $m = 300$ times, and record for each how many random trials that took

Task – Problem 3

- Plot that data as a *cumulative density plot*
 - The x -axis records the number k of trials required, and the y -axis records the fraction of experiments that succeeded (a collision) after k trials
- Empirically estimate the **expected** number of k random trials in order to have a collision
 - That is, add up all values k , and divide by m
- **How long did it take ?**
- Carry out some tests for **much larger n and m values !!**

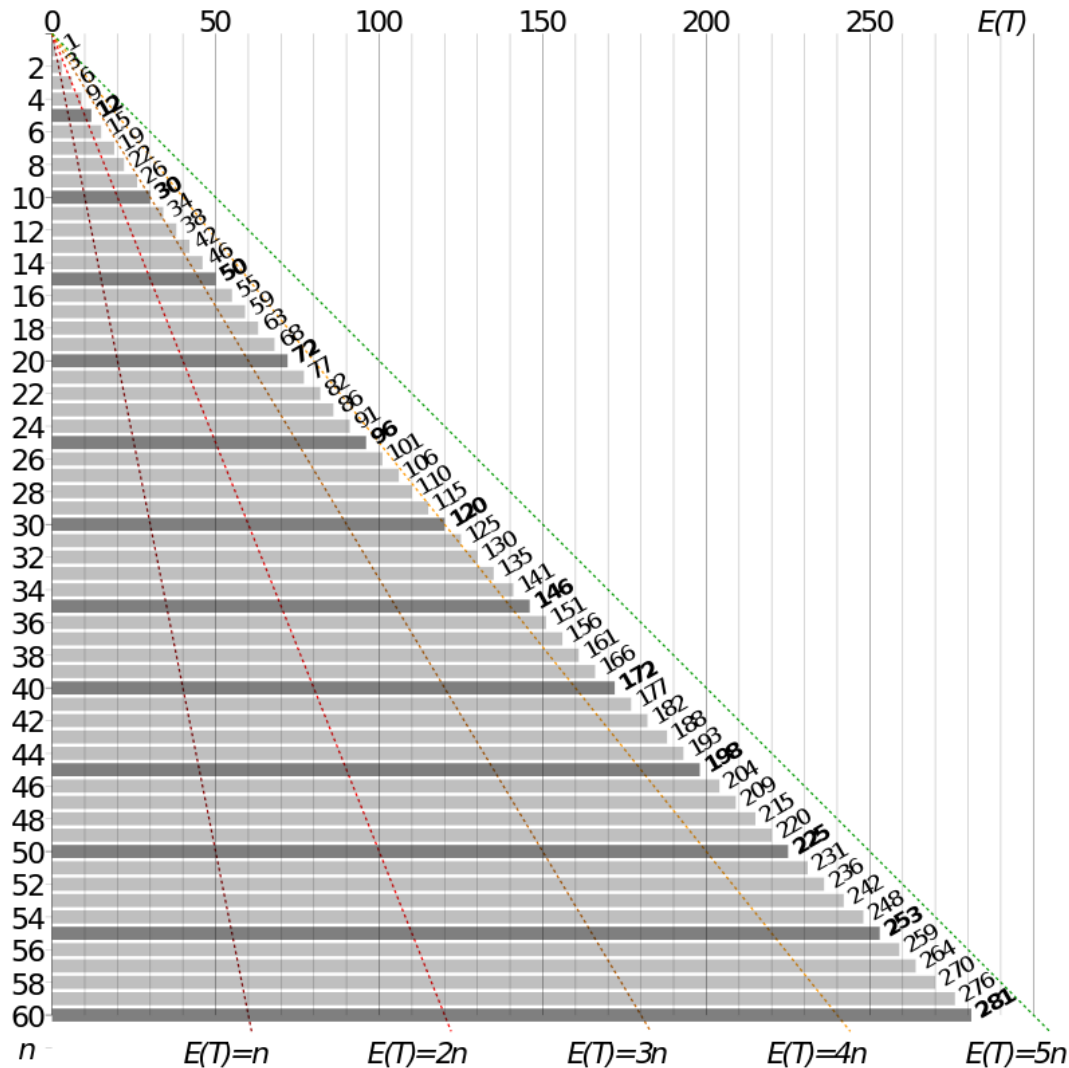
Task – Problem 4

- Consider $n = 200$
- Generate random numbers in the domain $[n]$ until every value i in $[n]$ has had one random number equal to i
- How many random trials did that take ?
 - Use k to represent this value
- Repeat the experiment $m = 300$ times, and record for each how many random trials were required to collect all values

Task – Problem 4

- Plot that data as a *cumulative density plot*
- Empirically estimate the **expected** number of k random trials in order to collect all values
- **How long did it take ?**
- Carry out some tests for **much larger n and m values !!**
- Read about **“The Coupon Collectors Problem” !!**

The Coupon Collectors Problem



$O(n \log n)$

[Wikipedia]

Extra Task – Problem 5

- Consider a blind-folded game of darts
- n darts are thrown to m targets
- Each dart reaches one and only one target !
- What is the probability of no target being hit more than once ?
- What is the probability of at least one target being hit at least twice ?

REFERENCES

References

- D. Vrajitoru and W. Knight, *Practical Analysis of Algorithms*, Springer, 2014
 - Chapter 6
- R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, 1989
 - Chapter 8
- J. Hromkovic, *Design and Analysis of Randomized Algorithms*, Springer, 2005
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