Algorithm Design Strategies II

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Version 0.3 – October 2020

Overview

- Counting basic operations – Recap
- Deterministic vs Non-Deterministic Algorithms
- Problem Types and Design Strategies
- Algorithm Efficiency and Complexity Analysis
- Brute-Force
- Divide-and-Conquer
- Decrease-and-Conquer
- Example: computing powers

RECAP
– COUNTING OPERATIONS

Running Time vs. Operations Count

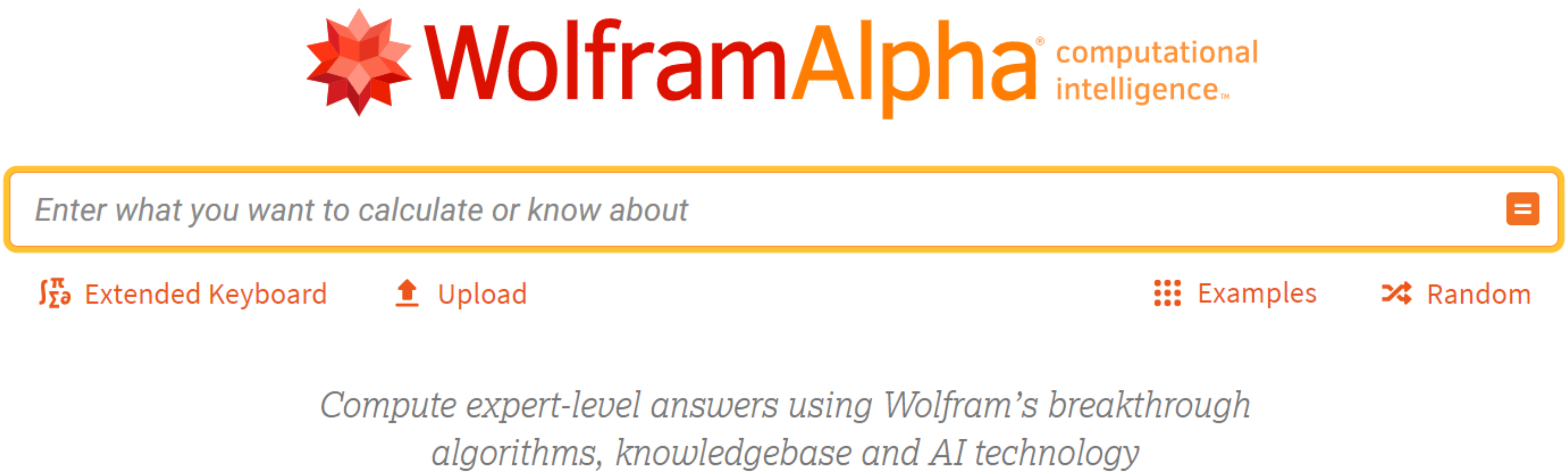
- Running time is not (very) useful for comparing algorithms
  - Speed of particular computers
  - Chosen computer language
  - Quality of programming implementation
  - Compiler optimizations

- Evaluate efficiency in an independent way
  - Count the "basic operations" !!
    - Contribute the most to overall running time

Formal Analysis – Pencil and paper

- Understand algorithm behavior
  - Count arithmetic operations / comparisons
  - Find a closed formula !!
  - Identify best, worst and average case situations, if that is the case
- Iterative algorithms
  - Loops : how many iterations ?
  - Set a sum for the basic operation counts
- Recursive algorithms
  - How many recursive calls ?
  - Establish and solve appropriate recurrences

WolframAlpha – Did you use it?



https://www.wolframalpha.com/

Return value? – Number of iterations?

```
int f1(int n) {
   int i,r=0;
   for(i = 1; i <= n; i++)
      r += i;
   return r;
}
```

```
int f2(int n) {
   int i,j,r=0;
   for(i = 1; i <= n; i++)
      for(j = 1; j <= n; j++)
         r += 1;
   return r;
}
```

```
int f3(int n) {
   int i,j,r=0;
   for(i = 1; i <= n; i++)
      for(j = i; j <= n; j++)
         r += 1;
   return r;
}
```

```
int f4(int n) {
   int i,j,r=0;
   for(i = 1; i <= n; i++)
      for(j = 1; j <= i; j++)
         r += j;
   return r;
}
```

Closed formulas? – Comput. tests?

- f1(n) = n (n + 1) / 2 n_iters1(n) = n
- f2(n) = n^2 n_iters2(n) = f2(n)
- f3(n) = n (n + 1) / 2 n_iters3(n) = f3(n)
- f4(n) = n (n + 1) (n + 2) / 6
- n_iters4(n) = n (n + 1) / 2
- Use WolframAlpha to get / check results !

Return value? – Number of calls?

```
unsigned int
r1(unsigned int n) {
   if(n == 0) return 0;
   return 1 + r1(n – 1);
}
```

```
unsigned int
r2(unsigned int n) {
   if(n == 0) return 0;
   if(n == 1) return 1;
   return n + r2(n – 2);
}
```

```
unsigned int
r3(unsigned int n) {
  if(n == 0) return 0;
   return 1 + 2 * r3(n – 1);
}
```

```
unsigned int
r4(unsigned int n) {
  if(n == 0) return 0;
   return 1 + r4(n – 1) + r4(n – 1);
}
```

Closed formulas? – Comput. tests?

- r1(n) = n   n_calls1(n) = r1(n)

- r2(n) = n (n + 2) / 4, if n is even
- r2(n) = 1 + (n − 1) (n + 3) / 4, if n is odd

- n_calls2(n) = *floor*(n / 2)

- Use WolframAlpha to get / check results !

Closed formulas? – Comput. tests?

- r3(n) $= 2^n - 1$ n_calls3(n) = n_calls1(n)
- r4(n) = r3(n) $= 2^n - 1$
- n_calls4(n) $= 2 \times (2^n - 1) = 2 \times$ r4(n)

- r3 and r4 compute the same result
- BUT, r4 will take much more time…
  - How far can you go with your computer?

DETERMINISTIC VS NON-DETERMINISTIC

Algorithms

- Algorithm
  - Sequence of non-ambiguous instructions
  - Finite amount of time

- Input to an algorithm
  - An <u>instance</u> of the problem the algorithm solves

- How to classify / group algorithms?
  - Type of problems solved
  - Design techniques
  - Deterministic vs non-deterministic

Deterministic Algorithms

- A deterministic algorithm
  - Returns the same answer no matter how many times it is called on the same data.
  - Always takes the same steps to complete the task when applied to the same data.
- The most familiar kind of algorithm !
- There is a more formal definition in terms of state machines…

Non-Deterministic Algorithms

- A non-deterministic algorithm
  - Can exhibit different behavior, for the same input data, on different runs.
  - As opposed to a deterministic algorithm !
- Often used to obtain approximate solutions to given problem instances
  - When it is too costly to find exact solutions using a deterministic algorithm

Non-Deterministic Algorithms

- How to behave differently from run to run ?

- Factors of non-deterministic behavior
  - External state other than the input data
    - User input / timer values / random values
  - Timing-sensitive operations on multiple processor machines
  - Hardware errors might force state to change in unexpected ways

PROBLEM TYPES

Problem Types

- Searching
- Sorting
- String Processing
- Graph / Network problems
- Combinatorial problems
- Bioinformatics
- …

- Examples of algorithms ?

Searching

- Which items?
  - Numbers, strings, records (key?), etc.

- Possible representations?
  - Arrays, lists, trees, etc.

- Ordered vs. non-ordered items
- Dynamically changing set?

- Sequential vs. binary search
- Others?

Sorting

- Which items?
  - Numbers, strings, records (key?), etc.

- Possible representations?
  - Arrays, lists, trees, etc.

- Use an indexing array?

- Which ordering? Repeated items?
- Stable? In-place?

- How many algorithms do you know?
- Which ones are the "most efficient"? When?

String Processing

- Text strings, bit strings, gene sequences, etc.

- String matching?
- Longest-common substring?
- String-edit distance?
- Other problems / algorithms?

Graph / Network Problems

- Modeling the real-world!

- Dense vs. sparse graphs / networks

- Representations
  - Adjacency matrices vs. lists
  - Forward-star and reverse-star forms

- Depth vs. breadth traversals
- Shortest path? K-shortest paths?
- Minimum spanning tree?
- Traveling salesman !
- Other problems?

Combinatorial Problems

- Find a permutation, combination or subset !!

- What are the constraints?
- Are we optimizing some property?
  - Max value, min cost, etc.

- The most difficult problems in computing !!
- No (known?) polynomial algorithms for some problems !!

- Instance size vs. execution time
  - Exhaustive search?
- Optimal solutions vs. approximations

- Examples
  - N-Queens / Knapsack / Traveling salesman

Bioinformatics

- Applications in molecular biology

- Dealing with sequences (DNA or proteins)
  - Storing
  - Mapping and analyzing
  - Aligning

ALGORITHM DESIGN TECHNIQUES

Algorithm Design Techniques

- Design techniques / strategies / paradigms
- General approaches to problem solving
- Apply to
  - Various problem types
  - Different application areas

Algorithm Design Techniques

- Brute-Force
- Divide-and-Conquer
- Decrease-and-Conquer
- Transform-and-Conquer
- Dynamic Programming
- Greedy Algorithms

- Examples of algorithms ?

- What about problems / instances that cannot be solved within a reasonable amount of time ?

Brute-Force

- Direct approaches
  - Selection sort
  - Sequential search
  - …

- Exhaustive search
  - Problem instances of small (?!) size
  - Traveling salesman
  - Knapsack
  - …

Divide-and-Conquer

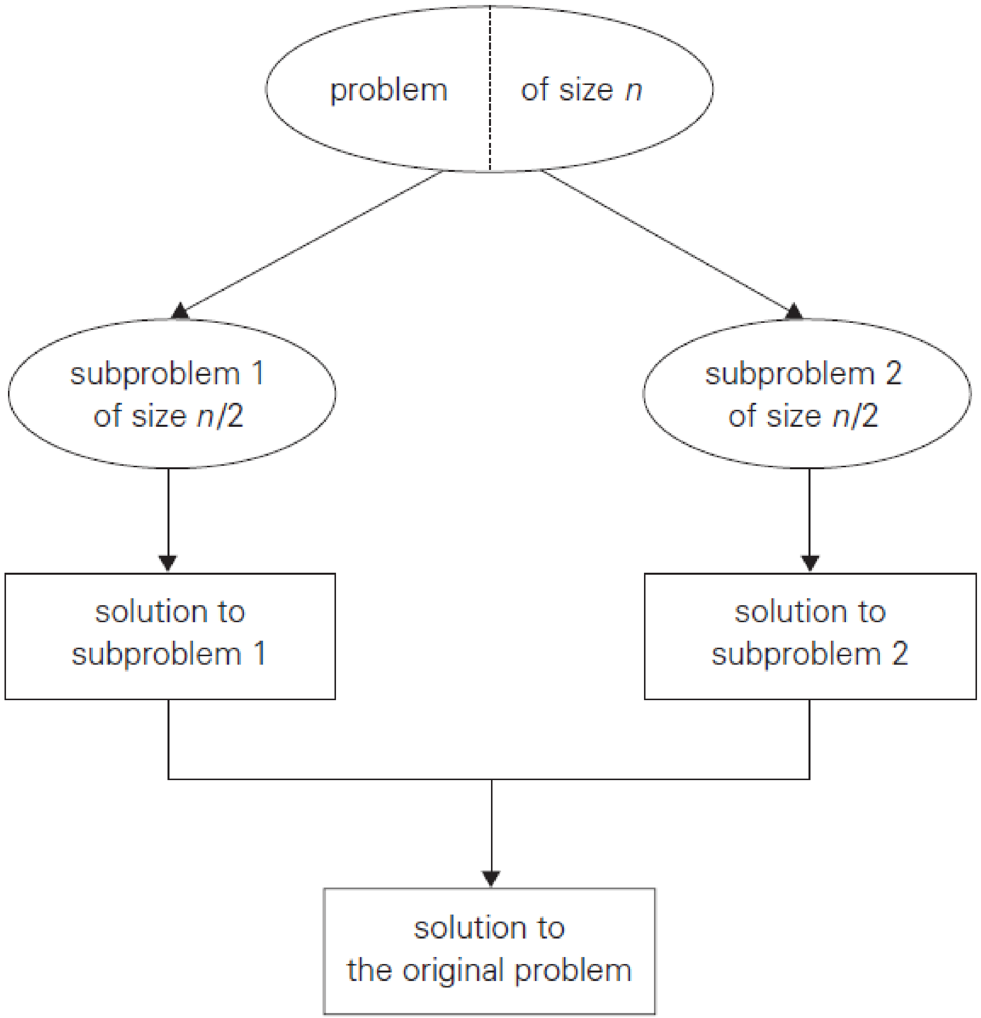
- Recursive decomposition into "smaller" prob. instances
- Solve them all !

- Sorting
  - Mergesort
  - Quicksort

- Multiplication
  - Multiplying large integers
  - Strassen matrix multiplication

- …

Divide-and-Conquer



[Levitin]

FIGURE 5.1 Divide-and-conquer technique (typical case).

Decrease-and-Conquer

- Successive decomposition into a “smaller” problem instance

- How small is it?
  - Decrease-by-one
  - Decrease by a constant factor
  - Variable-size decrease

- Examples
  - Binary search
  - Interpolation search
  - Fake-coin problem

Decrease-and-Conquer

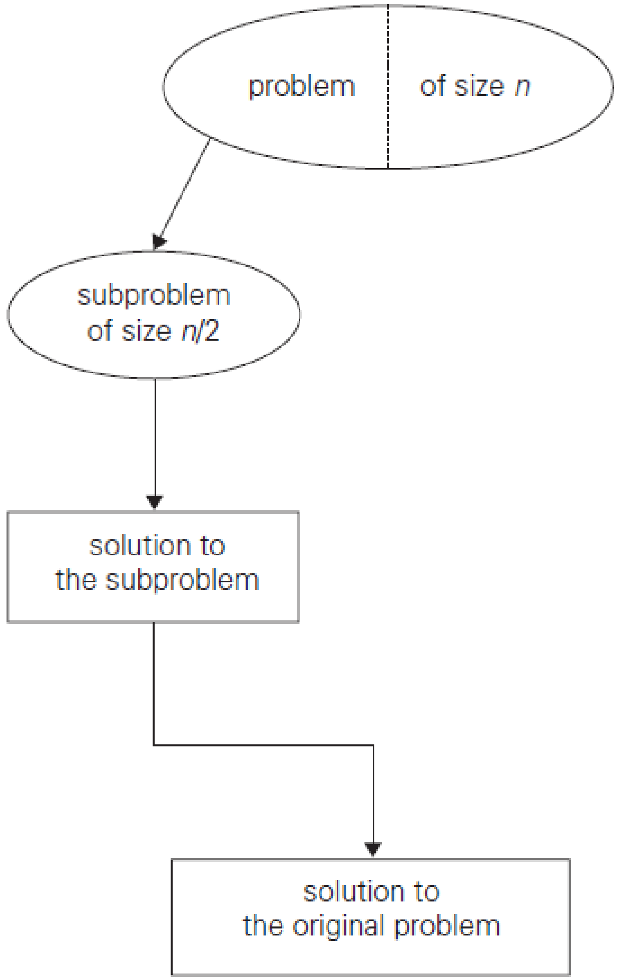


FIGURE 4.2 Decrease-(by half)-and-conquer technique.

[Levitin]

Transform-and-Conquer

- Solve a different problem and get the desired result
  - Problem reduction
- Sometimes, perform some kind of pre-processing on the data
- Examples
  - Searching on ordered and balanced trees
    - AVL and 2-3 trees
  - Heapsort

Dynamic Programming

- Decomposition into overlapping (smaller !) sub-problems
  - Avoid solving them all !!
  - Proceed bottom-up
  - Store results and use them !!
- Simple examples
  - Computing Fibonacci numbers
  - Computing binomial coefficients
  - …
- Other
  - Graphs: Warshall alg.; Floyd alg; etc.
  - Knapsack

Greedy Algorithms

- Construct a solution through a sequence of steps
  - Expand a partially constructed solution

- The choice made at each step is
  - Feasible : satisfies constraints
  - Locally optimal : best choice at each step
  - Irrevocable

- Examples
  - Coin-changing problem
  - Graphs
    - Dijkstra’s shortest-path algorithm
    - Prim’s minimum-spanning tree algorithm
    - Kruskal’s minimum-spanning tree algorithm

Limitations of Algorithmic Power

- How to cope?

- Backtracking
  - N-Queens problem
  - …

- Branch-and-Bound
  - Assignment problem
  - Knapsack problem
  - TSP
  - …

- Approximation algorithms for NP-hard problems
  - Knapsack problem
  - TSP
  - ...

DATA STRUCTURES & ABSTRACT DATA TYPES

Fundamental Data Structures

- Algorithms operate on data !
- How to organize and store related data items?
  - Data structures (DS)
- Which operations should be provided?
  - Abstract data types (ADT) or classes (in OO languages)
- How to choose?
  - Identify the most common operations on the data
  - Identify the needs of particular algorithms
- Different algorithms for the same problem often require different data structures
  - Efficiency !!

Fundamental Data Structures

- Arrays
  - 1D, 2D, …

- Linked Lists
  - Single pointer vs. two pointers per node
  - List of lists
  - …

- Trees
  - Binary tree
  - Quaternary tree
  - …

Common Abstract Data Types

- Stack
- Queue
- Priority Queue
- Ordered List
- Binary Search Tree
- …
- Graph / Network
- …

ALGORITHM EFFICIENCY ANALYSIS

Algorithm Efficiency

- Analyze algorithm efficiency
  - Running time ?
  - Memory space ?
- Time
  - How fast does an algorithm run?
- Space
  - Does an algorithm require additional memory?

Efficiency Analysis

- How fast does an algorithm run ?
  - Most algorithms run longer on larger inputs !
- How to relate running time to input size ?
- How to rank / compare algorithms ?
  - If there is more than one available…
- How to estimate running time for larger problem instances ?

Running Time vs. Operations Count

- Running time is not (very) useful for comparing algorithms
  - Speed of particular computers
  - Chosen computer language
  - Quality of programming implementation
  - Compiler optimizations

- Evaluate efficiency in an independent way
  - Count the "basic operations" !!
    - Contribute the most to overall running time

Input Size

- Relate operations count / running time to input size !!
  - Number of array / matrix / list elements
  - …

- Relate size metric to the main operations of an algorithm
  - Working with individual chars vs. with words
  - Number of bits in binary rep., when checking if n is prime
  - …

Formal Analysis – Pencil and paper

- Understand algorithm behavior
  - Count arithmetic operations / comparisons
  - Find a closed formula !!
  - Identify best, worst and average case situations, if that is the case
- Iterative algorithms
  - Loops : how many iterations ?
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Worst, Best and Average Cases

- Running time depends on input size
- BUT, for some algorithms, it might also depend on particular data configurations !!

- Sequential search on a n-element array
  - Non-ordered array ?
  - Ordered array ?
  - Increasing vs. decreasing order
  - Probability of a successful search ?

Worst, Best and Average Cases

- Worst case : W(n)
  - Input(s) of size n for which an algorithm runs longest
  - Upper bound for operations count
- Best case : B(n)
  - Input(s) of size n for which an algorithm runs fastest
  - Lower bound for operations count
  - Not very useful…
- Average case : A(n)
  - Behavior for “typical” or “random” inputs
  - Establish assumptions about possible inputs of size n
  - For some algorithms, much better than worst case !!

Growth Rate

- Identify algorithm efficiency for large input sizes
- How fast does the running time (i.e., number of operations) of an algorithm grow, when input size becomes (much) larger ?
- What happens when the input size
  - doubles ?
  - increases ten-fold ?
  - …
- How to represent such growth rate?

Orders of Growth

- Approximate values for some common functions

n	$\log_2 n$	n	$n \log_2 n$	n^2	n^3	2^n	$n!$
10	3.3	10	3.3×10^1	10^2	10^3	10^3	3.6×10^6
10^2	6.6	10^2	6.6×10^2	10^4	10^6	1.3×10^{30}	9.3×10^{157}
10^3	10	10^3	10^4	10^6	10^9	?	?
10^4	13	10^4	1.3×10^5	10^8	10^{12}	?	?
10^5	17	10^5	1.7×10^6	10^{10}	10^{15}	?	?
10^6	20	10^6	2.0×10^7	10^{12}	10^{18}	?	?

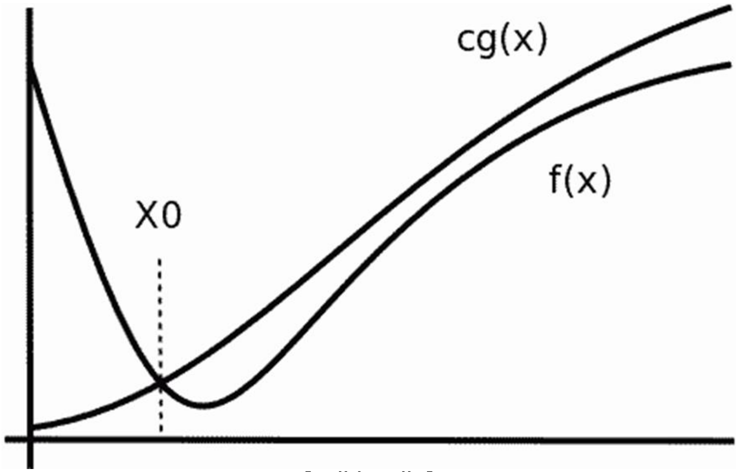
Asymptotic Notations

- Order of growth of operations count indicates efficiency

- How to compare / rank algorithms for the same problem?
  - Compare their orders of growth !!

- Useful notations: $O(n)$, $\Omega(n)$, $\Theta(n)$

Big-Oh Notation

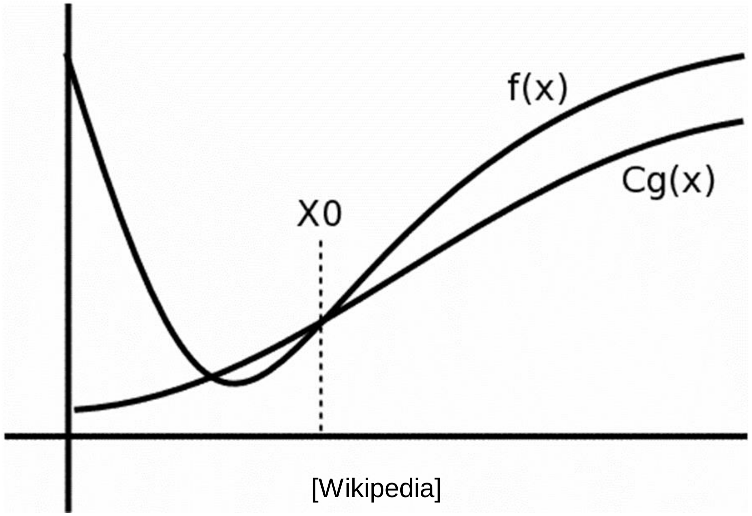


[Wikipedia]

- Asymptotic upper bound

- O(g(n)) : set of all functions with smaller or same order of growth as g(n)
  - $t(n) \leq c\ g(n)$, for all $n \geq n_0$, positive constant c
  - t(n), g(n) : non-negative functions on the set of natural numbers

Big-Omega Notation

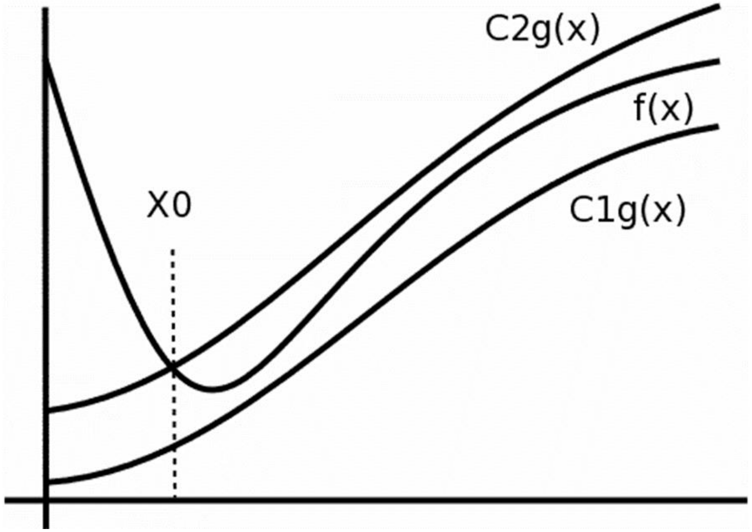


- Asymptotic lower bound

- $\Omega(g(n))$: set of all functions with larger or same order of growth as g(n)

  - $t(n) \geq c\, g(n)$, for all $n \geq n_0$, positive constant c

Big-Theta Notation



[Wikipedia]

- Asymptotic tight bound

- $\Theta(g(n))$: set of all functions with the same order of growth as g(n)

  - $c_1\ g(n) \leq t(n) \leq c_2\ g(n)$, for all $n \geq n_0$, positive constants c_1 , c_2

  - t(n) in O(g(n)) and t(n) in Ω(g(n))

Asymptotic Notation

- Hide unimportant details about how fast a function grows
  - Forget constants and lower-order terms

- $T_1(n) = 2\ n^2 + 3000\ n + 5$
- $T_2(n) = 10\ n^2 + 100\ n - 23$
- For large values of n, $T_2(n)$ grows faster than $T_1(n)$
- BUT both grow quadratically : $\Theta(n^2)$

Asymptotic Notation – Example

- $T(n) = 10\,n^2 + 100\,n - 23$

$T(n) = O(n^2)$ $T(n) = O(n^3)$ $T(n) \neq O(n)$

$T(n) = \Omega(n^2)$ $T(n) \neq \Omega(n^3)$ $T(n) = \Omega(n)$

$T(n) = \Theta(n^2)$ $T(n) \neq \Theta(n^3)$ $T(n) \neq \Theta(n)$

Efficiency Classes

- O(1) : constant
  - Which algorithms?
- O(log n) : logarithmic
  - E.g., decrease-and-conquer
- O(n) : linear
  - Processing all elements of an array, list, etc.
- O(n log n) : n-log-n
  - E.g., divide-and-conquer

Efficiency Classes

- $O(n^k)$: polynomial (quadratic, cubic, etc.)
  - k nested loops
- $O(2^n)$: exponential
  - Generating all subsets of an n-element set
- $O(n!)$: factorial
  - Generating all permutations of an n-element set

Empirical Analysis

- Run the algorithm on a sample of test inputs
  - Input data should represent all possible cases
  - Input data should encompass large (set) sizes
  - Pseudo-random data
- Record and analyze – Tables
  - operation counts
  - running times (?)
- Identify best, worst and average case behavior
  - If that is the case…
- Identify complexity classes

Example – Table of operations count

n	1	2	4	8	16	32	64	128	256
$M(n)$	1	3	10	36	136	528	2080	8256	32896

- M(n) : the number of operations carried out
- Complexity order ?
- Closed formula for the number of operations ?

Another table of operations count

n	1	2	3	4	5	6	7	8	9	10
$M(n)$	1	3	7	15	31	63	127	255	511	1023

- M(n) : the number of operations carried out
- Complexity order ?
- Closed formula for the number of operations ?

Empirical Analysis

- Problems
  - Inadequate sample input data
    - Size? Configurations?
  - Dependence of running times

- Advantages
  - Avoid difficult formal analysis
  - Allow predicting the running time for different input data sets
    - Interpolation and extrapolation (?)

- BUT, some problems / instances cannot be solved quickly enough…

BRUTE-FORCE

Brute-Force

- The (most) straightforward approach to solving a problem

- Directly based on
  - The problem statement
  - The definitions involved

- Strengths
  - Simplicity
  - Applicable to different kinds of problems

- Weaknesses
  - (Very!) Low efficiency in some cases
  - Useful only for instances of (relatively) small size !!

Brute-Force

- Where to apply?

- Numerical problems, searching, sorting, etc.
  - Acceptable efficiency
  - Can be used for large problem instances

- Combinatorial problems
  - Exhaustive search
  - Set of candidate solutions grows very fast
  - Used only for reduced size instances

Brute-Force

- How many examples do you know?

- Add n numbers
- Direct matrix multiplication
- Sequential search
- Selection sort
- Bubble sort
- …

TASK 1 – DIRECT ALGORITHM ITERATIVE VS RECURSIVE

Brute-Force – Tasks

- Compute a^b, with $b \geq 0$, using

$$a^b = a \times a \times \ldots \times a \qquad\qquad a^b = a \times a^{b-1}$$

- Base cases for the recursion ?
- Number of multiplications ?
  - Formal + Empirical analysis
- Any gains from the recursive approach ?

DIVIDE-AND-CONQUER

Divide-And-Conquer

- The best-known algorithm design technique

- General framework
  - Divide a problem instance into (two or more) similar, smaller instances
  - The smaller instances are solved recursively
  - Solutions for smaller instances are combined to get the solution of the original problem, if needed

Divide-And-Conquer

- In each subdivision step, the smaller instances should have approx. the same size !
  - This might not happen, for some particular instances

- All smaller problem instances have to be solved !!
  - Usually two new smaller instances, at each step

- When do we stop the subdivision process ?
  - Base cases ? Just one or more ?
  - Smaller instances might be solved by another algorithm

Divide-And-Conquer

- This recursive strategy can be implemented
  - Using recursive functions / procedures (obvious solution !)
  - Iteratively, using a stack, queue, etc.
    - Choose which sub-problem to solve next !!

- Problems ?
  - Recursion is slow !
    - Identify all possible base cases
    - Solve small instances using other algorithms
  - Not the best approach for simple problems !
    - E.g., adding N numbers
  - Sub-problems might overlap !
    - Reuse previous results / solutions !

TASK 2 – DIVIDE & CONQUER RECURSIVE FUNCTION

Divide-And-Conquer – Tasks

- Compute a^b, with $b \geq 0$, using

$$a^b = a^{b \text{ div } 2} \times a^{(b+1) \text{ div } 2}$$

- Base cases ?
- Always use two recursive calls !!

- Number of multiplications ?
  - Formal + Empirical analysis

DECREASE-AND-CONQUER

Decrease-And-Conquer

- Exploit the relationship between
  - A solution to a given problem instance
  - A solution to a smaller instance of the same problem
- General framework (Top-Down)
  - Identify ONE similar and smaller problem instance
  - The smaller instance is solved recursively
  - Solutions for smaller instances are processed to get the solution of the original problem, if needed
- Compare with Divide-and-Conquer !!

TASK 3 – DIVIDE & CONQUER RECURSIVE FUNCTION

Decrease-And-Conquer – Tasks

- Compute a^b, with $b \geq 0$, using

$$a^b = a^{b \text{ div } 2} \times a^{b \text{ div } 2} \text{, if b is even}$$

$$a^b = a \times a^{(b-1) \text{ div } 2} \times a^{(b-1) \text{ div } 2} \text{, if b is odd}$$

- Base cases ?
- Use just ONE recursive call !!

- Number of multiplications ?
  - Formal + Empirical analysis

EXTRA TASK – D & C ITERATIVE FUNCTION

Decrease-And-Conquer – Extra-Task

- Compute a^b, with $b \geq 0$, using

$$a^b = a^{b \text{ div } 2} \times a^{b \text{ div } 2} \text{ , if b is even}$$

$$a^b = a \times a^{(b-1) \text{ div } 2} \times a^{(b-1) \text{ div } 2} \text{ , if b is odd}$$

- Develop an iterative version !!
- It should have the same behavior as the recursive version
  - Same algorithm, but a different implementation

ADDITIONAL TASKS
- TRY DOING IT AT HOME

New Task – Counting

- Given an array with non-negative integer values

- Count the number of even-valued elements

- Implement the 3 strategies:
  - Brute-Force / Div & C / Dec & C

- Formal + Empirical analysis : Comparisons

New Task – Sequential Search

- Given an array with non-negative integer values
- Use the iterative Sequential Search algorithm to look for a given value
- Formal + Empirical analysis : Comparisons
- Best / Worst / Average Cases ?

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