

Finite Atomized Semilattices - Notes

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1 Preliminaries

1.1 Semillattice

Definition: A *semillattice* is an algebra (M, \odot) where \odot is a commutative, associative and idempotent binary operation.

From this very simple definition we can already define a partial order we denote \leq . Indeed, a partial order is a binary relation R that is reflexive, antisymmetric and transitive. All these properties derive from the commutativity, associativity and idempotence of \odot :

Let us define \leq such that $a \leq b$ iff $a \odot b = b$.

The idempotence property gives us the reflexivity $a \leq a = a \odot a = a$.

From the commutativity of \odot we get: $a \odot b = b \odot a$. So if $a \leq b$ and $b \leq a$ and therefore $a \odot b = b$ and $b \odot a = a$, then $a = b$.

From the associativity of \odot : $a \odot (b \odot c) = (a \odot b) \odot c$; so if $a \leq b$ and $b \leq c$ we have $a \odot b = b$ and $b \odot c = c$; so $a \odot (b \odot c) = a \odot c = (a \odot b) \odot c = b \odot c = c$. Therefore we've got $a \leq c$ given that we proved $a \odot c = c$.

Definition: We say $a \leq b$ in a semilattice M if and only if $M \models (b = b \odot a)$.

1.2 Terms, constants and atoms

1.2.1 In the first paper

The first paper describing AML uses semilattices to embed problems into an algebra. The binary operation is called merge (symbol \odot) and the set S has three types of elements: constants, terms and atoms.

Terms should describe the objects of the problem. For example, terms can represent images as matrices. Here's a simple example of a 4-pixel image described by a term T :

$$T = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$$

Here c_1, c_2, c_3 and c_4 are constants, the primitive description elements of any embedding problem, and so the previous representation of T in the algebra S is:

$$T = c_1 \odot c_2 \odot c_3 \odot c_4$$

Atoms are elements created by the learning algorithm, represented by greek letters, and each constant is a merge of atoms. Here, T is therefore a merge of atoms too.

1.2.2 In the second paper

In the second paper, the elements of the free algebra $F_C(\emptyset)$ are called terms. A duple $r = (r_L, r_R)$ is defined as an ordered pair of elements of $F_C(\emptyset)$. Positive and negative duples, written r^+ and r^- , are used as follows: we say $M \Vdash r^+$ iff $M \Vdash r_L \leq r_R$ and we say $M \Vdash r^-$ iff $M \nVdash r_L \leq r_R$.

A semilattice model M is said to be *atomized* when it is extended with a set of additional elements called atoms. Each semilattice is a partial order and an atomization of the semilattice is an extension of the partial order. However, an atomized semilattice is not a semilattice extension; meaning in atomized semilattices the idempotent operator is defined exclusively for regular elements while the order relation is defined for all, atoms and regular elements. Formally, this means that \leq is defined for all elements of the atomized semilattice but \odot is restricted to the regular elements of the atomized semilattice..

Given that \leq is well defined for all elements of the atomized semilattice; it is therefore a **partially ordered set**. Let us formalize this into a definition.

1.3 Formal definition of an atomized semilattice

Definition: An atomized semilattice over a set of constants C is a structure M with elements of two sorts, the regular elements $\{a, b, c, \dots\}$ and the atoms $\{\phi, \psi, \dots\}$, with an idempotent, commutative and associative binary operator \odot defined for regular elements and a partial order relation $<$ that is defined for all elements, such that the regular elements are either constants or idempotent summations of constants, and M satisfies the axioms of the operations and the additional:

AS1:

$$\forall \phi \exists c, (c \in C) \wedge (\phi < c)$$

AS2:

$$\forall \phi \forall a (a \not\leq \phi)$$

AS3:

$$\forall a \forall b (a \leq b \Leftrightarrow \neg \exists \phi, ((\phi < a) \wedge (\phi \not\leq b)))$$

AS4:

$$\forall \phi \forall a \forall b (\phi < a \odot b \Leftrightarrow (\phi < a) \vee (\phi < b))$$

AS5:

$$\forall c \in C ((\phi < c) \Leftrightarrow (\psi < c)) \Rightarrow (\phi = \psi)$$

AS6:

$$\forall a \exists \phi, (\phi < a)$$

Notes

- AS1 states that for each atom, there exists a constant in C edged to it. So every atom is edged to at least one constant. ie; there are no "hanging" atoms.
- AS2 states that atoms are at the bottom of the relation order (if we visualize the order relation in a graph)
- AS3 states that for every couple of regular elements, if b is higher than a , then there is no atom that is edged to a but not to b ; ie. if $a \leq b$, all atoms edged to a are also edged to b .
- AS4 states the following: every atom edged to the merge of a and b must be edged to either a or b .
- AS5 states that every couple of atoms that are edged to the same constants must be equal.
- AS6 states that each regular element is edged to at least one atom.

Here, $<$ represents the partial order relation between any element of the atomized semilattice. In this formal definition, \leq is the partial order relation defined from the operator \odot like it was shown in section 1.1

However, $a \leq b \Leftrightarrow a \odot b = b$ is not an axiom and it should be verified that \leq still holds. Let's define a new formula called AS3b: $\forall a \forall b (a \leq b \Leftrightarrow a \odot b = b)$.

Theorem 1

Assume AS4 and the antisymmetry of the order relation.

- i) AS3 implies AS3b
- ii) Assume $\forall a \forall b (\forall \phi ((\phi < a) \Leftrightarrow (\phi < b)) \Rightarrow (a = b))$. Then $\text{AS3b} \Rightarrow \text{AS3}$.
- iii) AS3 implies $\forall a \forall b (\forall \phi ((\phi < a) \Leftrightarrow (\phi < b)) \Rightarrow (a = b))$.

Proof for i): Let a and b be regular elements of an atomized semilattice. Let's suppose that $a \leq b$ holds. From AS3 we get $\neg \exists \phi, ((\phi < a) \wedge (\phi \not< b))$. Developing the negation and using the logical implication gives us the expression $\forall \phi ((\phi < a) \Rightarrow (\phi < b))$ which states very clearly that each atom edged to a must also be edged to b .

Therefore we have $(\phi < a \vee \phi < b \Leftrightarrow \phi < b)$. Indeed, if we imagine a tree proof, the only proposition that wouldn't be immediately true is $\phi < a \Rightarrow \phi < b$ but we just showed this is also true.

Given the last proposition, AS4 becomes $(\phi < a \odot b \Leftrightarrow \phi < b)$. The antisymmetry of $<$ gives us a way of getting $a \odot b = b$ if we prove $(a \odot b < b) \wedge (b < a \odot b)$. This can be done by splitting the equivalence we got from AS4 into $(\phi < a \odot b) \Rightarrow (\phi < b)$ and $(\phi < b) \Rightarrow (\phi < a \odot b)$. By using AS3 from right to left we get both $a \odot b < b$ and $b < a \odot b$, giving us the desired result.

Now let's assume that $a \odot b = b$. From AS4 we get $(\phi < b \Leftrightarrow (\phi < a) \vee (\phi < b))$ so $\phi < a \Rightarrow \phi < b$. Applying AS3 directly gives us $a \leq b$.

Proof of ii): Let a and b be regular elements of an atomized semilattice such that for each atom ϕ : $((\phi < a) \Leftrightarrow (\phi < b)) \Rightarrow a = b$. Let's assume that AS3b is true: $a \leq b \Leftrightarrow a \odot b = b$.

Assuming $a \leq b$, we want to prove $(\phi < a) \Rightarrow (\phi < b)$. From $a \leq b$, AS4 becomes: $(\phi < b) \Leftrightarrow (\phi < a) \vee (\phi < b)$; which gives us our proof.

Now we assume $(\phi < a) \Rightarrow (\phi < b)$ and we would like to prove $a \leq b$, which is equivalent to $a \odot b = b$ given AS3b. From this first assumption we get $(\phi < a \vee \phi < b) \Leftrightarrow \phi < b$ (cf. previous proof) and so AS4 gives us the formula $\phi < a \odot b \Leftrightarrow \phi < b$. Using the formula we assumed $((\phi < p) \Leftrightarrow (\phi < q)) \Rightarrow p = q$ with $p = a \odot b$ and $q = b$ we get $a \odot b = b$.

Proof of iii): Let's suppose AS3 for a and b in the set of regular elements:
 $(a \leq b \Leftrightarrow \forall \phi (\phi < a) \Rightarrow (\phi < b))$ (we're using the formulation discussed in proof i.).

Assuming $(\phi < a) \Leftrightarrow (\phi < b)$ we have, from AS3 applied two times where only the order of a and b changes, $a \leq b$ and $b \leq a$. This gives us $a = b$ because of the antisymmetry property.