Finite Atomized Semilattices - Notes

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1 Preliminaries

1.1 Semilllatice

Definition: A *semillatice* is an algebra $(M, \odot)$ where $\odot$ is a commutative, associative and idempotent binary operation.

From this very simple definition we can already define a partial order we denote $\leq$. Indeed, a partial order is a binary relation R that is reflexive, antisymetric and transitive. All these propreties derive from the commutativity, associativity and idempotence of $\odot$:

Let us define $\leq$ such that $a \leq b$ iff $a \odot b = b$.

The idempotence proprety gives us the reflexivity $a \leq a = a \odot a = a$.

From the commutativity of $\odot$ we get: $a \odot b = b \odot a$. So if $a \leq b$ and $b \leq a$ and therefore $a \odot b = b$ and $b \odot a = a$, then $a = b$.

From the associativity of $\odot$: $a \odot (b \odot c) = (a \odot b) \odot c$; so if $a \leq b$ and $b \leq c$ we have $a \odot b = b$ and $b \odot c = c$; so $a \odot (b \odot c) = a \odot c = (a \odot b) \odot c = b \odot c = c$. Therefore we've got $a \leq c$ given that we prooved $a \odot c = c$.

Definition: We say $a \leq b$ in a semilattice M if and only if $M \Vdash (b = b \odot a)$.

1.2 Terms, constants and atoms

1.2.1 In the first paper

The first paper describing AML uses semilattices to embed problems into an algebra. The binary operation is called merge (symbol $\odot$) and the set S has three types of of elements: constants, terms and atoms.

Terms should describe the objects of the problem. For exemple, terms can represent images as matrices. Here's a simple exemple of a 4-pixel image described by a term T:

$$T = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$$

Here c_1, c_2, c_3 and c_4 are constants, the primitive description elements of any embedding problem, and so the previous representation of T in the algebra S is:

$$T = c_1 \odot c_2 \odot c_3 \odot c_4$$

Atoms are elements created by the learning algorithm, represented by greek letters, and each constant is a merge of atoms. Here, T is therefore a merge of atoms too.

1.2.2 In the second paper

In the second paper, the elements of the free algebra $F_C(\emptyset)$ are called terms. A duple $r = (r_L, r_R)$ is defined as an ordered pair of elements of $F_C(\emptyset)$. Positive and negative duples, written r^+ and r^-, are used as follows: we say $M \Vdash r^+$ iff $M \Vdash r_L \leq r_R$ and we say $M \Vdash r^-$ iff $M \nVdash r_L \leq r_R$.

A semilattice model M is said to be *atomized* when it is extended with a set of additional elements called atoms. Each semilattice is a partial order and an atomization of the semillatice is an extension of the partial order. However, an atomized semilattice is not a semilattice extension; meaning in atomized semilattices the idempotent operator is defined exclusively for regular elements while the order relation is defined for all, atoms and regular elements. Formaly, this means that $\leq$ is defined for all elements of the atomized semilattice but $\odot$ is restricted to the regular elements of the atomized semilattice..

Given that $\leq$ is well defined for all elements of the atomized semilattice; it is therefore a **partially ordered set**. Let us formalize this into a definition.

1.3 Formal definition of an atomized semilattice

Definition: An atomized semilattice over a set of constants C is a structure M with elements of two sorts, the regular elements $\{a, b, c, ...\}$ and the atoms $\{\phi, \psi, ...\}$, with an itempotent, commutative and associative binary operator $\odot$ defined for regular elements and a partial order relation $<$ that is defined for all elements, such that the regular elements are either constants or idempotent summations of constants, and M satisfies the axioms of the operations and the additional:

AS1:

$$\forall \phi \exists c, (c \in C) \wedge (\phi < c)$$

AS2:

$$\forall \phi \forall a (a \not\leq \phi)$$

AS3:

$$\forall a \forall b (a \leq b \Leftrightarrow \neg \exists \phi, ((\phi < a) \wedge (\phi \not< b))$$

AS4:

$$\forall \phi \forall a \forall b (\phi < a \odot b \Leftrightarrow (\phi < a) \vee (\phi < b))$$

AS5:

$$\forall c \in C ((\phi < c) \Leftrightarrow (\psi < c)) \Rightarrow (\phi = \psi)$$

AS6:

$$\forall a \exists \phi, (\phi < a)$$

Notes

- AS1 states that for each atom, there exists a constant in C edeged to it. So every atom is edeged to at least one constant. ie; there are no "hanging" atoms.
- AS2 states that atoms are at the bottom of the relation order (if we visualize the order relation in a graph)
- AS3 states that for every couple of regular elements, if b is higher than a, then there is no atom that is edged to a but not to b; ie. if $a \leq b$, all atoms ,edeged to a are also edged to b.
- AS4 states the following: every atom edeged to the merge of a and b must be edged to either a or b.
- AS5 states that every couple of atoms that are edged to the same constants must be equal.
- AS6 states that each regular element is edged to at least one atom.

Here, $<$ represents the partial order relation between any element of the atomized semilattice. In this formal definiton, $\leq$ is the partial order relation defined from the operator $\odot$ like it was shown in section 1.1

However, $a \leq b \Leftrightarrow a \odot b = b$ is not an axiom and it should be verified that $\leq$ still holds. Let's define a new formula called AS3b: $\forall a \forall b (a \leq b \Leftrightarrow a \odot b = b)$.

Theorem 1

Assume AS4 and the antisymmetry of the order relation.

- i) AS3 implies AS3b
- ii) Assume $\forall a \forall b (\forall \phi ((\phi < a) \Leftrightarrow (\phi < b)) \Rightarrow (a = b))$. Then AS3b $\Rightarrow$ AS3.
- iii) AS3 implies $\forall a \forall b (\forall \phi ((\phi < a) \Leftrightarrow (\phi < b)) \Rightarrow (a = b))$.

Proof for i): Let a and b be regular elements of an atomized semilattice. Let's suppose that $a \leq b$ holds. From AS3 we get $\neg \exists \phi, ((\phi < a) \wedge (\phi \not< b))$. Developping the negation and using the logical implication gives us the expression $\forall \phi ((\phi < a) \Rightarrow (\phi < b))$ which states very clearly that each atom edged to a must also be edged to b.

Therefore we have $(\phi < a \vee \phi < b \Leftrightarrow \phi < b)$. Indeed, if we imagine a tree proof, the only proposition that wouldn't be imediatly true is $\phi < a \Rightarrow \phi < b$ but we just showed this is also true.

Given the last proposition, AS4 becomes $(\phi < a \odot b \Leftrightarrow \phi < b)$. The antisymmetry of $<$ gives us a way of getting $a \odot b = b$ if we proove $(a \odot b < b) \wedge (b < a \odot b)$. This can be done by splitting the equivalence we got from AS4 into $(\phi < a \odot b) \Rightarrow (\phi < b)$ and $(\phi < b) \Rightarrow (\phi < a \odot b)$. By using AS3 from right to left we get both $a \odot b < b$ and $b < a \odot b$, giving us the desired result.

Now let's assume that $a \odot b = b$. From AS4 we get $(\phi < b \Leftrightarrow (\phi < a) \vee (\phi < b))$ so $\phi < a \Rightarrow \phi < b$. Applying AS3 directly gives us $a \leq b$.

Proof of ii): Let a and b be rergular elements of an atomized semilattice such that for each atom ϕ: $((\phi < a) \Leftrightarrow (\phi < b)) \Rightarrow a = b$. Let's assume that AS3b is true: $a \leq b \Leftrightarrow a \odot b = b$.

Assuming $a \leq b$, we want to proove $(\phi < a) \Rightarrow (\phi < b)$. From $a \leq b$, AS4 becomes: $(\phi < b) \Leftrightarrow (\phi < a) \vee (\phi < b)$; which gives us our proof.

Now we assume $(\phi < a) \Rightarrow (\phi < b)$ and we would like to prove $a \leq b$, wich is equivalent to $a \odot b = b$ given AS3b. From this first assumption we get $(\phi < a \vee \phi < b) \Leftrightarrow \phi < b$ (cf. previous proof) and so AS4 gives us the formula $\phi < a \odot b \Leftrightarrow \phi < b$. Using the formula we assumed $((\phi < p) \Leftrightarrow (\phi < q)) \Rightarrow p = q$ with $p = a \odot b$ and $q = b$ we get $a \odot b = b$.