1 Preliminairies: The Closure Operation

Definition: The powerset of a set A is the set of all subsets of A, including the emptyset and A itself. It will be denoted $\mathbb{P}(A)$ here and it is denoted *Su(A)* in the book, *Su* meaning subsets.

Definition: If we are given a set A, a mapping $C : \mathbb{P}(A) \to \mathbb{P}(A)$ is called a closure operation on A, if, for $X, Y \subseteq A$, it satisfies:
(extensive)

$$X \subseteq C(X)$$

(idempotent)

$$C^2(X) = C(X)$$

(isotone)

$$X \subseteq Y \to C(X) \subseteq C(Y)$$

2 Isomorphic Algebras and Subalgebras

An isomorphism is a concept often used in particular casas like isomorphisms in group thoery or lattice theory. All these definitions derive from the general definition of isomorphisms in all algebras.

Definition: Let A and B be two algebras. Then a function $\alpha : A \to B$ is an isomorphism from A to B if α is one-to-one onto, and for every n-ary f, for $a_1, ..., a_n \in A$, we have:

$$\alpha(f^A(a_1, ..., a_n)) = f^B(\alpha(a_1), ..., \alpha(a_n))$$

In other words, an isomorphism between two algebras is a bijective morphism (a function mapping elements from A to B) that respects the proprety written above. A more formal definition, in french:

Soient M et N deux interpétations d'un langage L.

- Un L-morphisme de M dans N est une fonction $\phi : |M| \to |N|$ tele que:
  - Pour chaque symbole de constante c on a: $\phi(c_M) = c_N$
  - Pour chaque symbole de fonction n-aire f et pour $a_1, ...a_n \in |M|$ on a: $\phi(f_M(a_1, ..., a_n)) = f_N(\phi(a_1), ..., \phi(a_n))$.
  - Pour chaque symbole de relation n-aire de R (autre que $=$) et pour $a_1, ...a_n \in |M|$ on a: $(a_1, ..., a_n) \in R_M$ ssi $(\phi(a_1), ..., \phi(a_n)) \in R_N$.
- Un L-isomorphisme est un L-morphisme bijectif.
- M et N sont L-isomorphes s'il existe un L-isomorphisme de M dans N.

Remarque et exemple

- La notion de morphisme dépend du langage. Soit $L = \{0, +, -, \times\}$ et $L' = \{1\} \cup L$. Soient $\mathbb{Z}/3\mathbb{Z}$ et $\mathbb{Z}/12\mathbb{Z}$ les interprétations usuelles. La fonction $\phi : n \to 4n$ de $\mathbb{Z}/3\mathbb{Z}$ dans $\mathbb{Z}/12\mathbb{Z}$ est un L-morphisme puisque $\phi(0_L) = 0'_L = 0_L$, $\phi(0_M +_M 0_M) = \phi(0_M) +_N \phi(0_M) = 0_M +_M +0_M = 0_M = 4 * 0_M$ ect. Par contre, ϕ n'est pas L'-morphisme puisque $\phi(1_M) = 4 \neq 1_N$.

En guise d'exemple, on peut vérifier que si $L = \{c, f, S\}$ et M et N sont définies par:

- $|M| = \mathbb{R}, c_M = 0, f_M(a, b) = a + b$ et $S_M = \{(a, b)/a \leq b\}$
- $|N| =]0, \infty[, c_N = 1, f_N(a, b) = ab$ et $S_N = \{(a, b)/a \leq b\}$

Alors la fonction $\phi : x \to e^x$ est un isomorphisme de M dans N.
En effet:

- $\phi(c_M) = c_N$ est vérifié puisque $e^0 = 1$.
- $\phi(f_M(a, b)) = f_N(\phi(a), \phi(b))$ est vérifié puisque $e^{a+b} = e^a e^b$
- $a, b \in S_M$ ssi $\phi(a), \phi(b) \in S_N$ est aussi vérifié puisque $a \leq b$ ssi $e^a \leq e^b$ par croissance de la fonction exponentielle.

Definition: Let A and B be two algebras. Then B is a *subalgebra* of A if $B \subseteq A$ and every fundamental operation of B is the restriction of the corresponding operation of A, ie. for each function symbol f, f^B is f^A restricted to B; we write simply $B \leq A$. A *subuniverse* of A is a subset B of A which is closed under the fundemental opperations of A, ie. if f is a fundamental n-ary operation of A and $a_1, ..., a_n \in B$ we would require $f(a_1, ..., a_n) \in B$.

This definition of subalgebras comes with some limitations: for example, we would like a subalgebra of a group to be a group but a subalgebra would only mean subsemigroup (the positive integers are a subsemigroup of the group of all integers). We should consider a suitable modification (enlrargement) so the concept of subalgebras so it coincides with the ususal notion for several examples of algebras (section 1 of the book).

Definition: A function $\alpha : A \to B$ is an *embedding* of A into B if α is one-to-one (bijective) and satisfies $\alpha(f^A(a_1, ..., a_n)) = f^B(\alpha(a_1), ..., \alpha(a_n))$ (such an α is also called a *monomorphism*) We say A can be *embedded* in B if there is an embedding of A into B.

Theorem

if $\alpha : A \to B$ is an *embedding* then $\alpha(A)$ is a subuniverse of B.

Proof: Let $\alpha : A \to B$ be an embedding. Then for any n-ary function f and $a_1, ..., a_n \in A$:

$$f^B(\alpha(a_1), ..., \alpha(a_n)) = \alpha(f^A(a_1, ..., a_n))$$

Given that α is a bijection it is clear that $\alpha(A)$ is a subset of A. In addition, $\alpha(A)$ is closed on B given that any element of $\alpha(A)$ is the result of a function f^B in B like the previous equation states.

3 Algebraic Lattices and Subuniverses

Definition: Given an algebra A, for every $X \subseteq A$,

$$Sg(X) = \bigcap\{B : X \subseteq B \text{ and } B \text{ is a subuniverse of } A\}$$

$Sg(X)$ is the "subuniverse generated by X".