Efficient Marginalization of Discrete and Structured Latent Variables via Sparsity

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Latent Variable Models

Latent variable z can be

Latent Variable Models

Latent variable z can be continuous

Source: Bouges et al., 2013

Latent Variable Models

Latent variable z can be continuous, discrete

Source: valleyeyecareaz.com

Latent Variable Models

Latent variable z can be continuous, discrete, or structured

Source: Liu et al., 2015

Training Discrete or Structured Latent Variable Models

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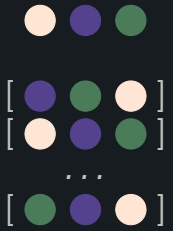
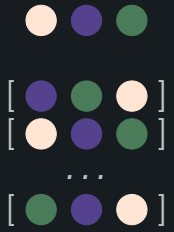
Latent variable z can be discrete or structured



Training Discrete or Structured Latent Variable Models

Latent variable z can be discrete or structured

$\pi(z|x, \theta)$: distribution over possible z



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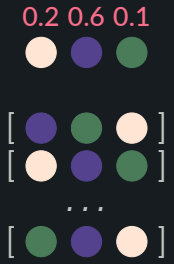

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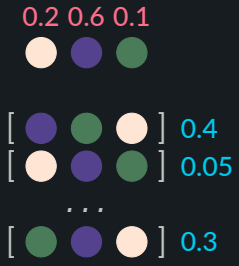

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$$\mathcal{L}_x(\theta) = \sum_{z \in \mathcal{Z}} \pi(z|x, \theta)\, \ell(x, z; \theta)$$

If $\mathcal{Z}$ is large, this sum can get very expensive due to $\ell(x, z; \theta)$! 😱

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If $\mathcal{Z}$ is combinatorial, this can be intractable to compute! 😱 😱 😱

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no need for sampling → no variance
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Does the expectation over possible z need to be expensive?

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$$\begin{aligned}\mathcal{L}_x(\theta) &= \sum_{z \in \mathcal{Z}} \pi(z|x, \theta)\, \ell(x, z; \theta) \\ &= \pi(z_1|x, \theta)\, \ell(x, z_1; \theta) + \pi(z_2|x, \theta)\, \ell(x, z_2; \theta) + \ldots \\ &+ \pi(z_i\,|x, \theta)\, \ell(x, z_i\,; \theta) + \ldots + \pi(z_N|x, \theta)\, \ell(x, z_N; \theta)\end{aligned}$$

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Usually we normalize π with softmax $\propto \exp(s) \Rightarrow \pi(z_i\,|x, \theta) > 0$

(Martins and Astudillo, 2016; Niculae, Martins, et al., 2018)

Sparse normalizers

We use sparsemax, top-k sparsemax and SparseMAP to allow efficient marginalization

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$$+ \pi(z_i|x, \theta)\, \ell(x, z_i; \theta) + \dots + \underbrace{\pi(z_N|x, \theta)}_{=0}\, \cancel{\ell(x, z_N; \theta)}$$

No need for computing $\ell(x, z; \theta)$ for all $z \in \mathcal{Z}$!

(Niculae and Blondel, 2017)

Discrete, unstructured case: sparsemax

$$\boldsymbol{\pi}_\Omega(s) = \underset{\boldsymbol{p} \in \triangle}{\arg\max}\, \boldsymbol{p}^\top s - \Omega(\boldsymbol{p})$$

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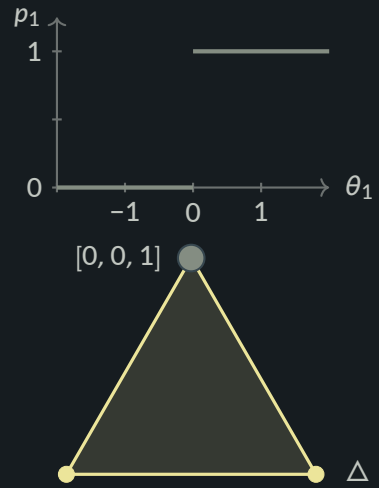

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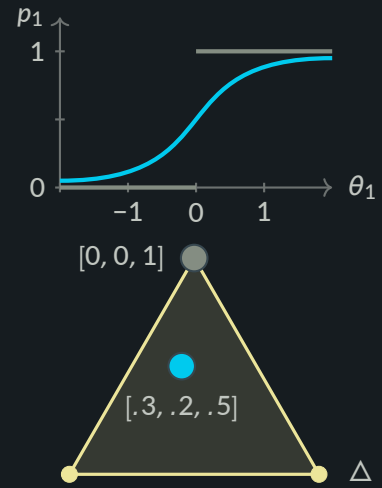

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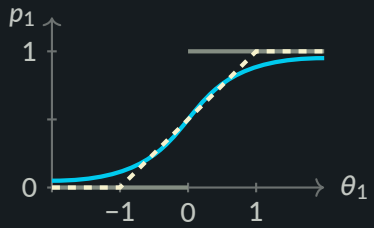
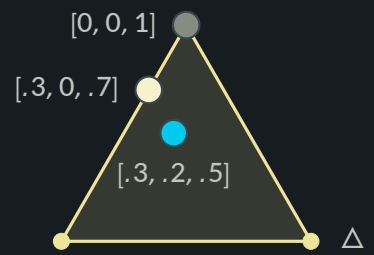
- sparsemax: $\Omega(\boldsymbol{p}) = \frac{1}{2}\|\boldsymbol{p}\|_2^2$





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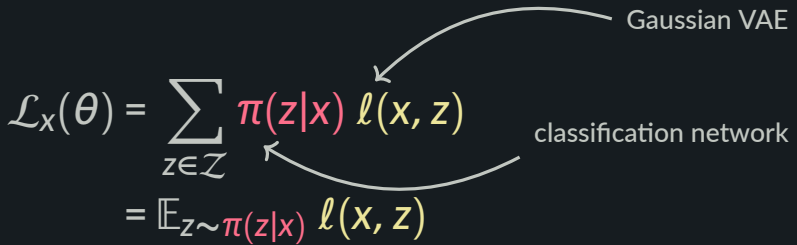
Semi-Supervised VAE

$$\mathcal{L}_x(\theta) = \sum_{z \in \mathcal{Z}} \pi(z|x)\, \ell(x, z)$$
$$= \mathbb{E}_{z \sim \pi(z|x)}\, \ell(x, z)$$

- Semi-Supervised VAE on MNIST: z is one of 10 categories

Semi-Supervised VAE



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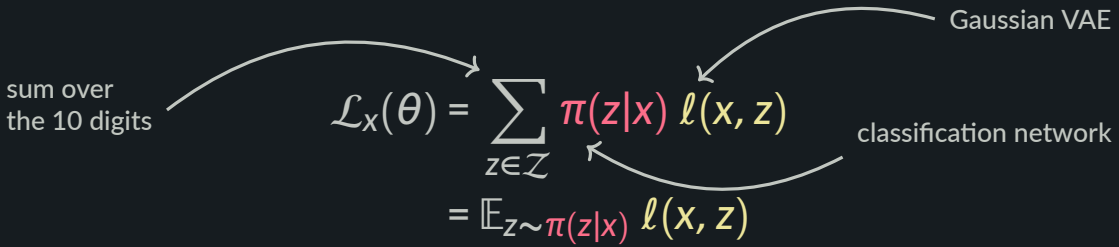

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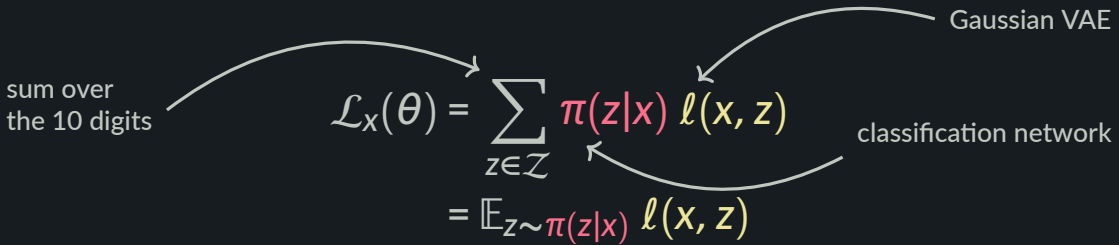

- Semi-Supervised VAE on MNIST: z is one of 10 categories
- Train this with 10% labeled data

Semi-Supervised VAE

Method	Accuracy (%)	Dec. calls
Monte Carlo		
SFE	$94.75_{\pm.002}$	1
SFE+	$96.53_{\pm.001}$	2
NVIL	$96.01_{\pm.002}$	1
Gumbel	$95.46_{\pm.001}$	1
Marginalization		
Dense	$96.93_{\pm.001}$	10

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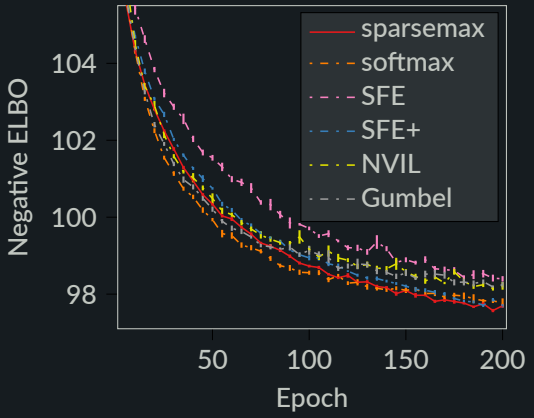

Emergent communication

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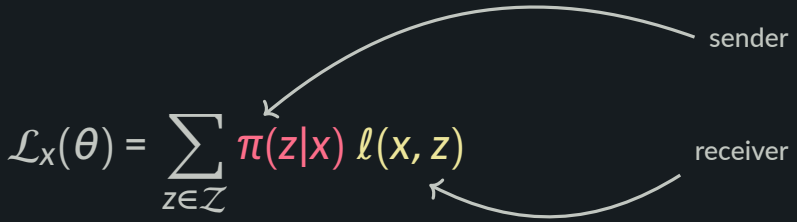

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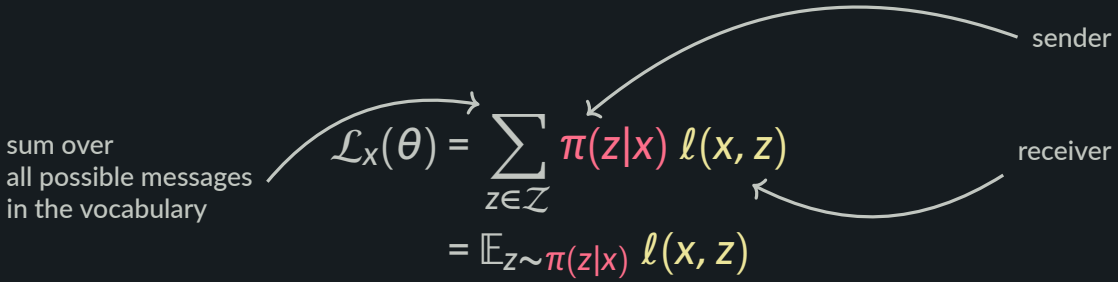

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- images come from ImageNet

(Lazaridou et al., 2017; Correia et al., 2020)

Emergent Communication

... but make it harder: $|\mathcal{Z}| = 256$, $|\mathcal{V}| = 16$

Method	success (%)	Dec. calls
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SFE	33.05 ±2.84	1
SFE+	44.32 ±2.72	2
NVIL	37.04 ±1.61	1
Gumbel	23.51 ±16.19	1
ST Gumbel	27.42 ±13.36	1
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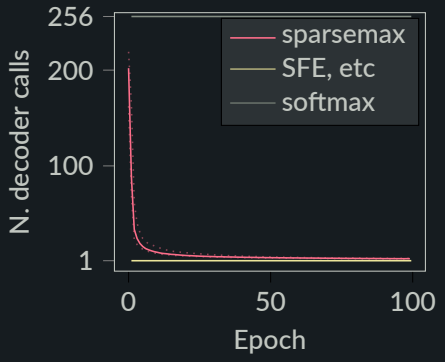

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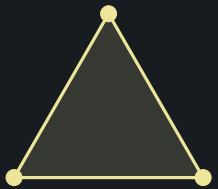
- Non-convex but easy: sparsemax over the k highest scores (Kyrillidis et al., 2013).
- Top-k oracle available for some structured problems.
- Certificate: if at least one of the top-k z gets $p(z) = 0$, **k-sparsemax = sparsemax**!
  thus, biased early on, but it goes away.



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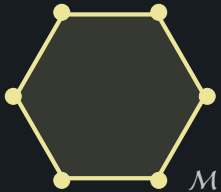
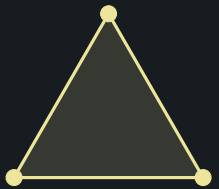


$\mathcal{M}$

$$\begin{aligned}\mathcal{M} &:= \operatorname{conv}\left\{\boldsymbol{a}_z : z \in \mathcal{Z}\right\} \\ &= \left\{\boldsymbol{A}\boldsymbol{p} : \boldsymbol{p} \in \Delta\right\} \\ &= \left\{\mathbb{E}_{Z\sim\boldsymbol{p}}\, \boldsymbol{a}_Z : \boldsymbol{p} \in \Delta\right\}\end{aligned}$$



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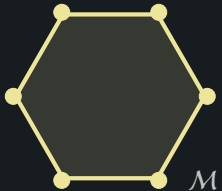
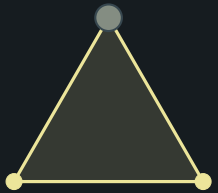
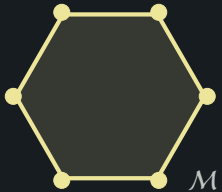


$\mathcal{M}$

- **argmax** $\arg\max_{\boldsymbol{p} \in \triangle} \boldsymbol{p}^\top \boldsymbol{s}$





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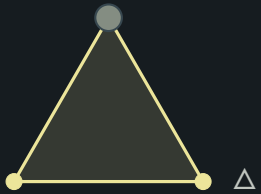
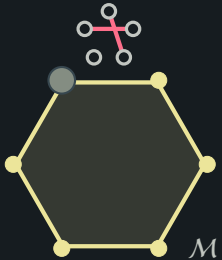
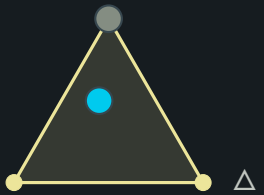


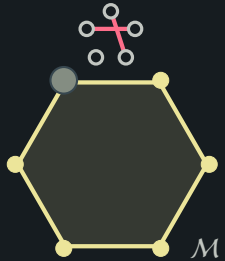
MAP $\arg\max_{\boldsymbol{\mu} \in \mathcal{M}} \boldsymbol{\mu}^\top \boldsymbol{t}$



- **argmax** $\arg\max_{\boldsymbol{p} \in \Delta} \boldsymbol{p}^\top \boldsymbol{s}$
- **softmax** $\arg\max_{\boldsymbol{p} \in \Delta} \boldsymbol{p}^\top \boldsymbol{s} + \mathrm{H}(\boldsymbol{p})$



- **MAP** $\arg\max_{\boldsymbol{\mu} \in \mathcal{M}} \boldsymbol{\mu}^\top \boldsymbol{t}$



- **argmax** $\arg\max_{p \in \triangle} p^\top s$
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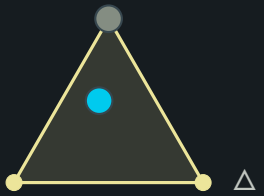
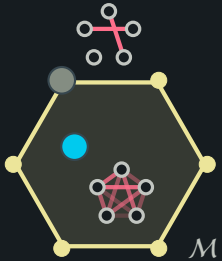
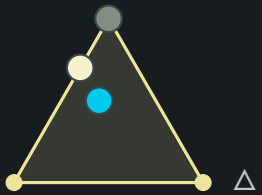


- **MAP** $\arg\max_{\mu \in \mathcal{M}} \mu^\top t$
- **marginals** $\arg\max_{\mu \in \mathcal{M}} \mu^\top t + \widetilde{\mathrm{H}}(\mu)$



- **argmax** $\arg\max_{p \in \triangle} p^\top s$
- **softmax** $\arg\max_{p \in \triangle} p^\top s + \mathrm{H}(p)$
- **sparsemax** $\arg\max_{p \in \triangle} p^\top s - 1/2 \|p\|^2$



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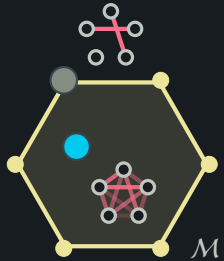

(Niculae, Martins, Blondel, and Cardie, 2018)

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- **MAP** $\arg\max_{\boldsymbol{\mu} \in \mathcal{M}} \boldsymbol{\mu}^\top \boldsymbol{t}$
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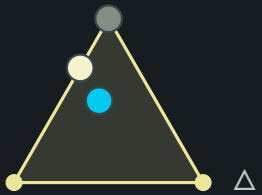
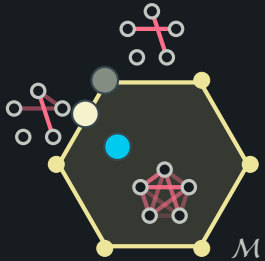
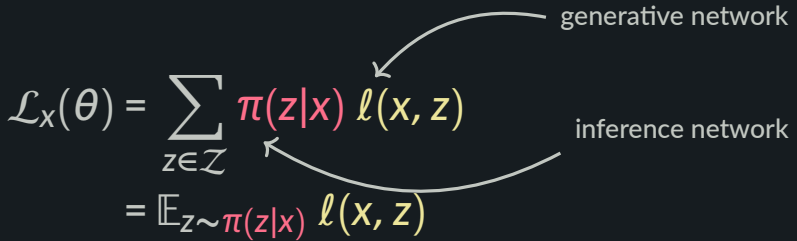




Bit-vector VAE

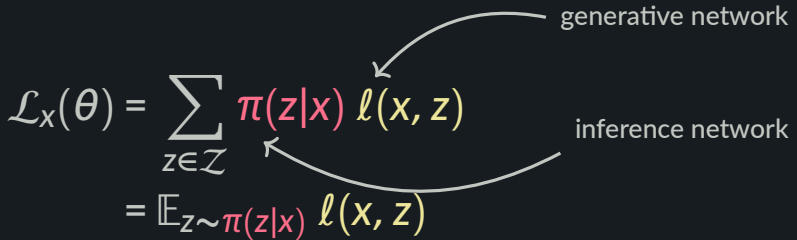
$$\mathcal{L}_x(\theta) = \sum_{z \in \mathcal{Z}} \pi(z|x)\, \ell(x, z)$$
$$= \mathbb{E}_{z \sim \pi(z|x)}\, \ell(x, z)$$

Bit-vector VAE



- VAE where z is a collection of D bits

Bit-vector VAE



- VAE where z is a collection of D bits
- Minimize the negative ELBO

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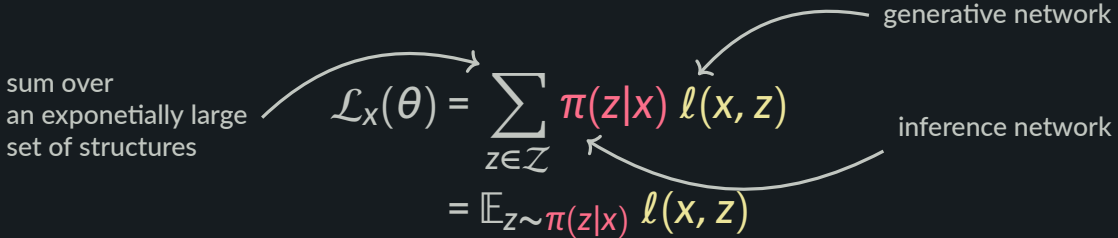

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Bit-vector VAE

Method	$D = 32$	$D = 128$
Monte Carlo		
SFE	3.74	3.77
SFE+	3.61	3.59
NVIL	3.65	3.60
Gumbel	3.57	3.49
Marginalization		
Top-k sparsemax	3.62	3.61
SparseMAP	3.72	3.67
SparseMAP (w/ budget)	3.64	3.66

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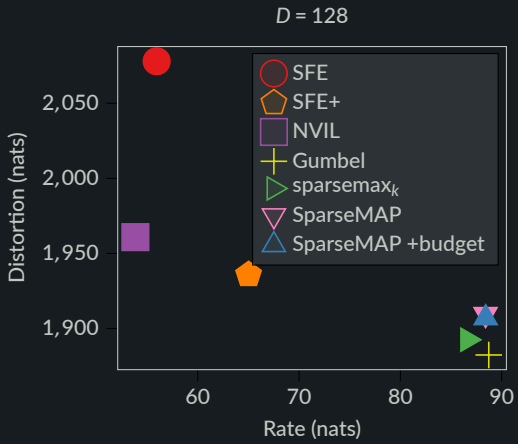

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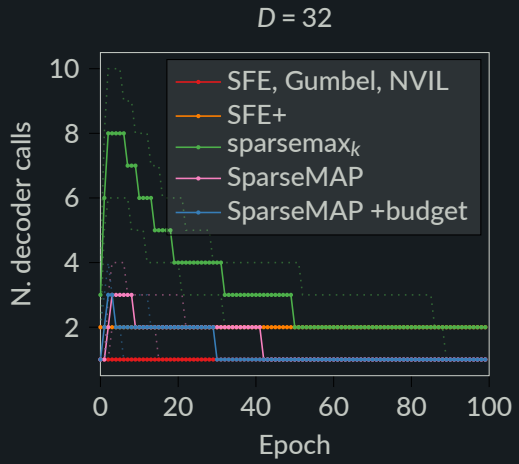

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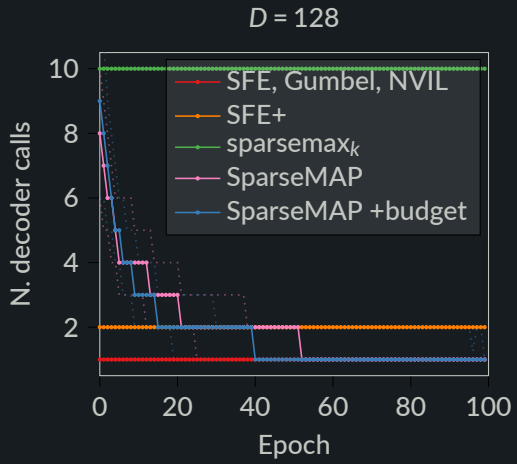

Key Takeaways

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$$\mathcal{L}_x(\theta) = \pi(z_1|x,\theta)\,\ell(x, z_1; \theta) + \underbrace{\pi(z_2|x,\theta)}_{=0}\,\ell(x, z_2; \theta) + \ldots + \pi(z_i|x,\theta)\,\ell(x, z_i; \theta) + \ldots + \underbrace{\pi(z_N|x,\theta)}_{=0}\,\ell(x, z_N; \theta)$$

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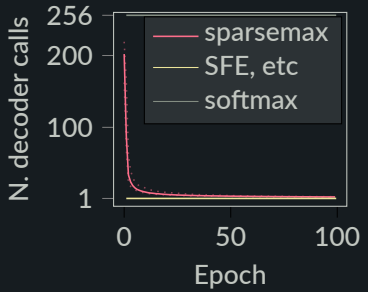
adaptive, as needed



References I

- Bouges, Pierre, Thierry Chateau, Christophe Blanc, and Gaëlle Loosli (Dec. 2013). “Handling missing weak classifiers in boosted cascade: application to multiview and occluded face detection”. In: *EURASIP Journal on Image and Video Processing* 2013, p. 55. DOI: 10.1186/1687-5281-2013-55.
- Correia, Gonçalo M., Vlad Niculae, Wilker Aziz, and André FT Martins (2020). “Efficient marginalization of discrete and structured latent variables via sparsity”. In: *Proc. NeurIPS*.
- Kyrillidis, Anastasios, Stephen Becker, Volkan Cevher, and Christoph Koch (2013). “Sparse projections onto the simplex”. In: *Proc. ICML*.
- Lazaridou, Angeliki, Alexander Peysakhovich, and Marco Baroni (2017). “Multi-agent cooperation and the emergence of (natural) language”. In: *Proc. ICLR*.
- Liu, Ziwei, Ping Luo, Xiaogang Wang, and Xiaoou Tang (Dec. 2015). “Deep Learning Face Attributes in the Wild”. In: *Proceedings of International Conference on Computer Vision (ICCV)*.
- Martins, André FT and Ramón Fernandez Astudillo (2016). “From softmax to sparsemax: A sparse model of attention and multi-label classification”. In: *Proc. of ICML*.
- Niculae, Vlad and Mathieu Blondel (2017). “A regularized framework for sparse and structured neural attention”. In: *Proc. of NeurIPS*.
- Niculae, Vlad, André FT Martins, Mathieu Blondel, and Claire Cardie (2018). “SparseMAP: Differentiable sparse structured inference”. In: *Proc. of ICML*.