Data Structures And Algorithms Made Easy

Second Edition

Success Key for:

- Programming Questions for Interviews
- Campus Preparation
- Degree/Masters Course Preparation
- Instructor's
- Big Job hunters: Microsoft, Google, Amazon & many more
- Reference Manual for Working People

Multiple smart solutions with different complexities

Narasimha Karumanchi

M-Tech, IIT Bombay

Founder of CareerMonk.com

CareerMonk Publications

Table of Contents

Chapter-1

INTRODUCTION

The objective of this chapter is to explain the importance of analysis of algorithms, their notations, relationships and solving as many problems as possible. We first concentrate on understanding the basic elements of algorithms, importance of analysis and then slowly move towards analyzing the algorithms with different notations and finally the problems. After completion of this chapter you should be able to find the complexity of any given algorithm (especially recursive functions).

1.1 Variables

Before going to the definition of variables, let us relate them to old mathematical equations. All of us have solved many mathematical equations since childhood. As an example, consider the below equation:

$$x^2 + 2y - 2 = 1$$

We don't have to worry about the use of above equation. The important thing that we need to understand is, the equation has some names (x and y) which hold values (data). That means, the *names* (x and y) are the place holders for representing data. Similarly, in computer science we need something for holding data and *variables* are the facility for doing that.

1.2 Data types

In the above equation, the variables x and y can take any values like integral numbers (10, 20 etc...), real numbers ($0.23, 5.5$ etc...) or just 0 and 1. To solve the equation, we need to relate them to kind of values they can take and *data type* is the name being used in computer science for this purpose.

A *data type* in a programming language is a set of data with values having predefined characteristics. Examples of data types are: integer, floating point unit number, character, string etc...

Computer memory is all filled with zeros and ones. If we have a problem and wanted to code it, it's very difficult to provide the solution in terms of zeros and ones. To help users, programming languages and compilers are providing the facility of data types.

For example, *integer* takes 2 bytes (actual value depends on compiler), *float* takes 4 bytes etc... This says that, in memory we are combining 2 bytes (16 bits) and calling it as *integer*. Similarly, combining 4 bytes (32 bits) and calling it as *float*. A data type reduces the coding effort. Basically, at the top level, there are two types of data types:

- System defined data types (also called *Primitive* data types)
- User defined data types

System defined data types (Primitive data types)

Data types which are defined by system are called *primitive* data types. The primitive data types which are provided by many programming languages are: int, float, char, double, bool, etc...

The number of bits allocated for each primitive data type depends on the programming languages, compiler and operating system. For the same primitive data type, different languages may use different sizes. Depending on the size of the data types the total available values (domain) will also changes. For example, "*int*" may take 2 bytes or 4 bytes. If it takes 2 bytes (16 bits) then the total possible values are -32,768 to +32,767 (-2^{15} *to* $2^{15}-1$). If it takes, 4 bytes (32 bits), then the possible values are between −2,147,483,648 to +2,147,483,648 (-2^{31} *to* $2^{31}-1$). Same is the case with remaining data types too.

User defined data types

If the system defined data types are not enough then most programming languages allows the users to define their own data types called as user defined data types. Good example of user defined data types are: structures in $C/C++$ and classes in *Java*.

For example, in the below case, we are combining many system defined data types and called it as user defined data type with name "*newType*". This gives more flexibility and comfort in dealing with computer memory.

```
struct newType {
        int data1;
        float data 2;
        …
        char data;
};
```

1.3 Data Structure

Based on the above discussion, once we have data in variables, we need some mechanism for manipulating that data to solve problems. *Data structure* is a particular way of storing and organizing data in a computer so that it can be used efficiently. That means, a *data structure* is a specialized format for organizing and storing data. General data structure types include arrays, files, linked lists, stacks, queues, trees, graphs and so on.

Depending on the organization of the elements, data structures are classified into two types:

1) *Linear data structures*: Elements are accessed in a sequential order but it is not compulsory to store all elements sequentially (say, Linked Lists). *Examples*: Linked Lists, Stacks and Queues.
2) *Non − linear data structures*: Elements of this data structure are stored/accessed in a non-linear order. *Examples*: Trees and graphs.

1.4 Abstract Data Types (ADTs)

Before defining abstract data types, let us consider the different view of system defined data types. We all know that, by default, all primitive data types (int, float, et..) supports basic operations like addition, subtraction etc… The system is providing the implementations for the primitive data types. For user defined data types also we need to define operations. The implementation for these operations can be done when we want to actually use them. That means, in general user defined data types are defined along with their operations.

To simplify the process of solving the problems, we generally combine the data structures along with their operations and are called *Abstract Data Types* (ADTs). An ADT consists of *two* parts:

1. Declaration of data
2. Declaration of operations

Commonly used ADTs *include*: Linked Lists, Stacks, Queues, Priority Queues, Binary Trees, Dictionaries, Disjoint Sets (Union and Find), Hash Tables, Graphs, and many other. For example, stack uses LIFO (Last-In-First-Out) mechanism while storing the data in data structures. The last element inserted into the stack is the first element that gets deleted. Common operations of it are: creating the stack, pushing an element onto the stack, popping an element from stack, finding the current top of the stack, finding number of elements in the stack etc...

While defining the ADTs do not care about implementation details. They come in to picture only when we want to use them. Different kinds of ADTs are suited to different kinds of applications, and some are highly specialized to specific tasks. By the end of this book, we will go through many of them and you will be in a position to relate the data structures to the kind of problems they solve.

1.5 What is an Algorithm?

Let us consider the problem of preparing an omelet. For preparing omelet, general steps we follow are:

1) Get the frying pan.
2) Get the oil.
    a. Do we have oil?
        i. If yes, put it in the pan.
        ii. If no, do we want to buy oil?
            1. If yes, then go out and buy.
            2. If no, we can terminate.
3) Turn on the stove, etc...

What we are doing is, for a given problem (preparing an omelet), giving step by step procedure for solving it. Formal definition of an algorithm can be given as:

An algorithm is the step-by-step instructions to solve a given problem.

Note: we do not have to prove each step of the algorithm.

1.6 Why Analysis of Algorithms?

To go from city "*A*" to city "*B*", there can be many ways of accomplishing this: by flight, by bus, by train and also by cycle. Depending on the availability and convenience we choose the one which suits us. Similarly, in computer science there can be multiple algorithms exist for solving the same problem (for example, sorting problem has many algorithms like insertion sort, selection sort, quick sort and many more). Algorithm analysis helps us determining which of them is efficient in terms of time and space consumed.

1.7 Goal of Analysis of Algorithms

The goal of *analysis of algorithms* is to compare algorithms (or solutions) mainly in terms of running time but also in terms of other factors (e.g., memory, developers effort etc.)

1.8 What is Running Time Analysis?

It is the process of determining how processing time increases as the size of the problem (input size) increases. Input size is number of elements in the input and depending on the problem type the input may be of different types. In general, we encounter the following types of inputs.

- Size of an array
- Polynomial degree
- Number of elements in a matrix
- Number of bits in binary representation of the input
- Vertices and edges in a graph

1.9 How to Compare Algorithms?

To compare algorithms, let us define few *objective measures*:

Execution times? *Not a good measure* as execution times are specific to a particular computer.

Number of statements executed? *Not a good measure*, since the number of statements varies with the programming language as well as the style of the individual programmer.

Ideal Solution? Let us assume that we expressed running time of given algorithm as a function of the input size n (i.e., $f(n)$) and compare these different functions corresponding to running times. This kind of comparison is independent of machine time, programming style, etc...

1.10 What is Rate of Growth?

The rate at which the running time increases as a function of input is called *rate of growth*. Let us assume that you went to a shop for buying a car and a cycle. If your friend sees you there and asks what you are buying then in general we say *buying a car*. This is because, cost of car is too big compared to cost of cycle (approximating the cost of cycle to cost of car).

$$Total\ Cost = cost\_of\_car + cost\_of\_cycle$$
$$Total\ Cost \approx cost\_of\_car\ (approximation)$$

For the above example, we can represent the cost of car and cost of cycle in terms of function and for a given function ignore the low order terms that are relatively insignificant (for large value of input size, n). As an example in the below case, n^4, $2n^2$, $100n$ and 500 are the individual costs of some function and approximate it to n^4. Since, n^4 is the highest rate of growth.

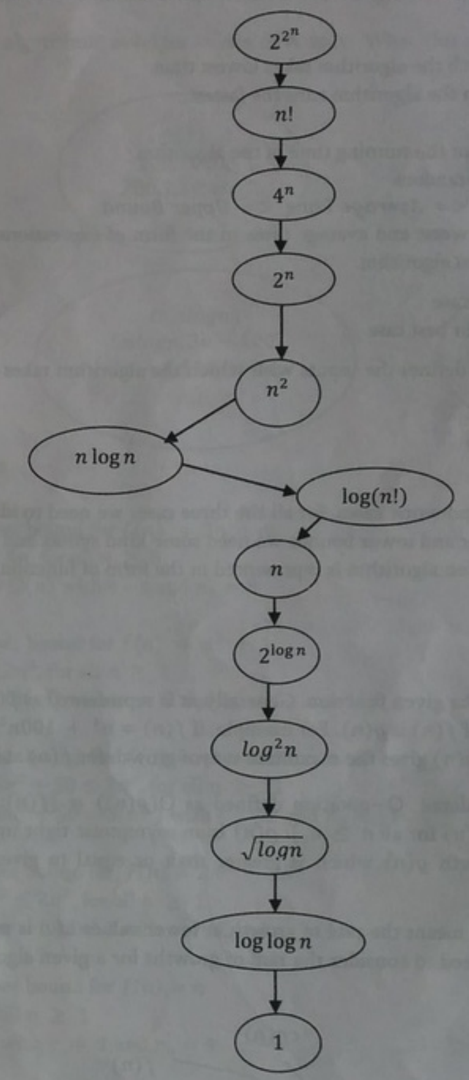
$$n^4 + 2n^2 + 100n + 500 \approx n^4$$

1.11 Commonly used Rate of Growths

Below is the list of rate of growths which come across in remaining chapters.

Time complexity	Name	Example
1	Constant	Adding an element to the front of a linked list
$logn$	Logarithmic	Finding an element in a sorted array
n	Linear	Finding an element in an unsorted array
$nlogn$	Linear Logarithmic	Sorting n items by 'divide-and-conquer'-Mergesort
n^2	Quadratic	Shortest path between two nodes in a graph
n^3	Cubic	Matrix Multiplication
2^n	Exponential	The Towers of Hanoi problem

Below diagram shows the relationship between different rates of growth.



1.12 Types of Analysis

To analyze the given algorithm we need to know on what inputs the algorithm is taking less time (performing well) and on what inputs the algorithm is taking huge time. We have already seen that an algorithm can be represented in the form of an expression. That means we represent the algorithm with multiple expressions: one for case where it is taking the less time and other for case where it is taking the more time. In general the first case is called the *best case* and second case is called the *worst case* of the algorithm. To analyze an algorithm we need some kind of syntax and that forms the base for asymptotic analysis/notation. There are three types of analysis:

- **Worst case**
  - Defines the input for which the algorithm takes huge time.
  - Input is the one for which the algorithm runs the slower.

- **Best case**
  - Defines the input for which the algorithm takes lowest time.
  - Input is the one for which the algorithm runs the fastest.
- **Average case**
  - Provides a prediction about the running time of the algorithm
  - Assumes that the input is random

Lower Bound <= Average Time <= Upper Bound

For a given algorithm, we can represent best, worst and average cases in the form of expressions. As an example, let $f(n)$ be the function which represents the given algorithm.

$f(n) = n^2 + 500$, for worst case
$f(n) = n + 100n + 500$, for best case

Similarly, for average case too. The expression defines the inputs with which the algorithm takes the average running time (or memory).

1.13 Asymptotic Notation

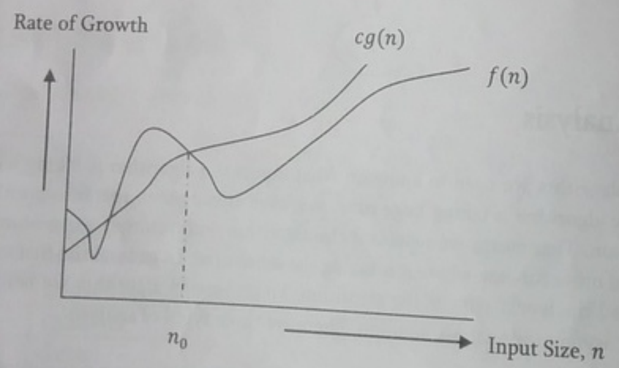
Having the expressions for best, average case and worst cases, for all the three cases we need to identify the upper and lower bounds. In order to represent these upper and lower bounds we need some kind syntax and that is the subject of following discussion. Let us assume that the given algorithm is represented in the form of function $f(n)$.

1.14 Big-O Notation

This notation gives the *tight* upper bound of the given function. Generally, it is represented as $f(n) = O(g(n))$. That means, at larger values of n, the upper bound of $f(n)$ is $g(n)$. For example, if $f(n) = n^4 + 100n^2 + 10n + 50$ is the given algorithm, then n^4 is $g(n)$. That means, $g(n)$ gives the maximum rate of growth for $f(n)$ at larger values of n.

Let us see the O−notation with little more detail. O−notation defined as $O(g(n)) = \{f(n)$: there exist positive constants c and n_0 such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0\}$. $g(n)$ is an asymptotic tight upper bound for $f(n)$. Our objective is to give smallest rate of growth $g(n)$ which is greater than or equal to given algorithms rate of growth $f(n)$.

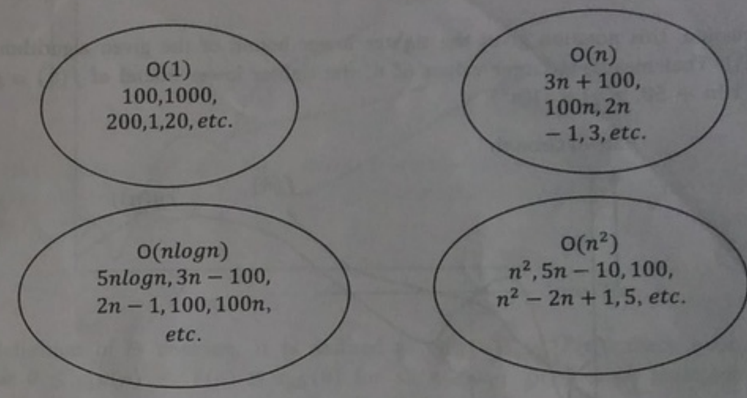
In general, we discard lower values of n. That means the rate of growth at lower values of n is not important. In the below figure, n_0 is the point from which we need to consider the rate of growths for a given algorithm. Below n_0 the rate of growths could be different.



Big-O Visualization

$O(g(n))$ is the set of functions with smaller or same order of growth as $g(n)$. For example, $O(n^2)$ includes $O(1), O(n), O(nlogn)$ etc..

Note: Analyze the algorithms at larger values of n only. What this means is, below n_0 we do not care for rate of growth.



Big-O Examples

Example-1 Find upper bound for $f(n) = 3n + 8$
Solution: $3n + 8 \leq 4n$, for all $n \geq 8$
$\therefore 3n + 8 = O(n)$ with c = 4 and $n_0 = 8$

Example-2 Find upper bound for $f(n) = n^2 + 1$
Solution: $n^2 + 1 \leq 2n^2$, for all $n \geq 1$
$\therefore n^2 + 1 = O(n^2)$ with $c = 2$ and $n_0 = 1$

Example-3 Find upper bound for $f(n) = n^4 + 100n^2 + 50$
Solution: $n^4 + 100n^2 + 50 \leq 2n^4$, for all $n \geq 11$
$\therefore n^4 + 100n^2 + 50 = O(n^4)$ with $c = 2$ and $n_0 = 11$

Example-4 Find upper bound for $f(n) = 2n^3 - 2n^2$
Solution: $2n^3 - 2n^2 \leq 2n^3$, for all $n \geq 1$
$\therefore 2n^3 - 2n^2 = O(2n^3)$ with $c = 2$ and $n_0 = 1$

Example-5 Find upper bound for $f(n) = n$
Solution: $n \leq n$, for all $n \geq 1$
$\therefore n = O(n)$ with $c = 1$ and $n_0 = 1$

Example-6 Find upper bound for $f(n) = 410$
Solution: $410 \leq 410$, for all $n \geq 1$
$\therefore 410 = O(1)$ with $c = 1$ and $n_0 = 1$

No Uniqueness?

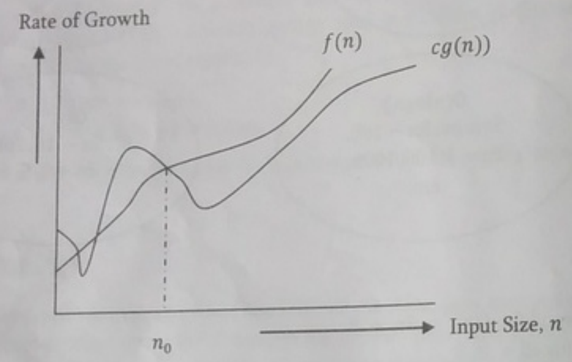
There are no unique set of values for n_0 and c in proving the asymptotic bounds. Let us consider, $100n + 5 = O(n)$. For this function there are multiple n_0 and c values possible.

Solution1: $100n + 5 \leq 100n + n = 101n \leq 101n$, for all $n \geq 5$, $n_0 = 5$ and $c = 101$ is a solution.

Solution2: $100n + 5 \leq 100n + 5n = 105n \leq 105n$, for all $n \geq 1$, $n_0 = 1$ and $c = 105$ is also a solution.

1.15 Omega-Ω Notation

Similar to O discussion, this notation gives the tighter lower bound of the given algorithm and we represent it as $f(n) = \Omega(g(n))$. That means, at larger values of n, the tighter lower bound of $f(n)$ is $g(n)$. For example, if $f(n) = 100n^2 + 10n + 50$, $g(n)$ is $\Omega(n^2)$.



The Ω notation can be defined as Ω(g(n)) = {f(n): there exist positive constants c and n_0 such that $0 \le cg(n) \le f(n)$ for all $n \ge n_0$}. $g(n)$ is an asymptotic tight lower bound for $f(n)$. Our objective is to give largest rate of growth $g(n)$ which is less than or equal to given algorithms rate of growth $f(n)$.

Ω Examples

Example-1 Find lower bound for $f(n) = 5n^2$

Solution: $\exists\, c,\ n_0$ Such that: $0 \le cn \le 5n^2 \Rightarrow cn \le 5n^2 \Rightarrow c = 1$ and $n_0 = 1$
$\therefore 5n^2 = \Omega(n^2)$ with $c = 1$ and $n_0 = 1$

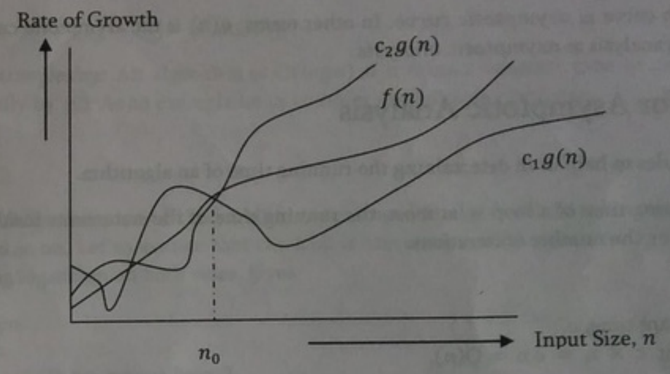
Example-2 Prove $f(n) = 100n + 5 \ne \Omega(n^2)$.

Solution: $\exists\, c,\ n_0$ Such that: $0 \le cn^2 \le 100n + 5$
$100n + 5 \le 100n + 5n\ (\forall n \ge 1) = 105n$
$cn^2 \le 105n \Rightarrow n(cn - 105) \le 0$
Since n is positive $\Rightarrow cn - 105 \le 0 \Rightarrow n \le 105/c$
$\Rightarrow$ Contradiction: n cannot be smaller than a constant

Example-3 $2n = \Omega(n)$, $n^3 = \Omega(n^3)$, $logn = \Omega(logn)$.

1.16 Theta-Θ Notation

This notation decides whether the upper and lower bounds of a given function (algorithm) are same or not. The average running time of algorithm is always between lower bound and upper bound. If the upper bound (O) and lower bound (Ω) gives the same result then Θ notation will also have the same rate of growth. As an example, let us assume that $f(n) = 10n + n$ is the expression. Then, its tight upper bound $g(n)$ is $O(n)$. The rate of growth in best case is $g(n) = O(n)$. In this case, rate of growths in best case and worst are same. As a result, the average case will also be same. For a given function (algorithm), if the rate of growths (bounds) for O and Ω are not same then the rate of growth Θ case may not be same. In this case, we need to consider all possible time complexities and take average of those (for example, quick sort average case, refer *Sorting* chapter).



Now consider the definition of Θ notation. It is defined as $\Theta(g(n)) = \{f(n)$: there exist positive constants c_1, c_2 and n_0 such that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0\}$. $g(n)$ is an asymptotic tight bound for $f(n)$. $\Theta(g(n))$ is the set of functions with the same order of growth as $g(n)$.

Θ Examples

Example-1 Find Θ bound for $f(n) = \frac{n^2}{2} - \frac{n}{2}$

Solution: $\frac{n^2}{5} \le \frac{n^2}{2} - \frac{n}{2} \le n^2$, for all, $n \ge 1$
$\therefore \frac{n^2}{2} - \frac{n}{2} = \Theta(n^2)$ with $c_1 = 1/5, c_2 = 1$ and $n_0 = 1$

Example-2 Prove $n \ne \Theta(n^2)$
Solution: $c_1 n^2 \le n \le c_2 n^2 \Rightarrow$ only holds for: $n \le 1/c_1$
$\therefore n \ne \Theta(n^2)$

Example-3 Prove $6n^3 \ne \Theta(n^2)$
Solution: $c_1 n^2 \le 6n^3 \le c_2 n^2 \Rightarrow$ only holds for: $n \le c_2/6$
$\therefore 6n^3 \ne \Theta(n^2)$

Example-4 Prove $n \ne \Theta(logn)$
Solution: $c_1 logn \le n \le c_2 log\, n \Rightarrow c_2 \ge \frac{n}{\log n}, \forall n \ge n_0$ – Impossible

Important Notes

For analysis (best case, worst case and average) we try to give upper bound (O) and lower bound (Ω) and average running time (Θ). From the above examples, it should also be clear that, for a given function (algorithm) getting upper bound (O) and lower bound (Ω) and average running time (Θ) may not be possible always. For example, if we are discussing the best case of an algorithm, then we try to give upper bound (O) and lower bound (Ω) and average running time (Θ). In the remaining chapters we generally concentrate on upper bound (O) because knowing lower bound (Ω) of an algorithm is of no practical importance and we use Θ notation if upper bound (O) and lower bound (Ω) are same.

1.17 Why is it called Asymptotic Analysis?

From the above discussion (for all the three notations: worst case, best case and average case), we can easily understand that, in every case for a given function $f(n)$ we are trying to find other function $g(n)$ which approximates $f(n)$ at higher values of n. That means, $g(n)$ is also a curve which approximates $f(n)$ at higher values of n. In

mathematics we call such curve as *asymptotic curve*. In other terms, $g(n)$ is the asymptotic curve for $f(n)$. For this reason, we call algorithm analysis as *asymptotic analysis*.

1.18 Guidelines for Asymptotic Analysis

There are some general rules to help us in determining the running time of an algorithm.

1) **Loops**: The running time of a loop is, at most, the running time of the statements inside the loop (including tests) multiplied by the number of iterations.

```
// executes n times
for (i=1; i<=n; i++)
   m = m + 2; // constant time, c
```

Total time = a constant $c \times n = cn = O(n)$.

2) **Nested loops**: Analyze from inside out. Total running time is the product of the sizes of all the loops.

```
//outer loop executed n times
for (i=1; i<=n; i++) {
    // inner loop executed n times
    for (j=1; j<=n; j++)
            k = k+1; //constant time
}
```

Total time = $c \times n \times n = cn^2 = O(n^2)$.

3) **Consecutive statements**: Add the time complexities of each statement.

```
x = x +1; //constant time
// executed n times
for (i=1; i<=n; i++)
    m = m + 2; //constant time
//outer loop executed n times
for (i=1; i<=n; i++) {
    //inner loop executed n times
    for (j=1; j<=n; j++)
            k = k+1; //constant time
}
```

Total time = $c_0 + c_1 n + c_2 n^2 = O(n^2)$.

4) **If-then-else statements**: Worst-case running time: the test, plus *either* the *then* part *or* the *else* part (whichever is the larger).

```
//test: constant
if(length( ) == 0 ) {
    return false; //then part: constant
}
else {// else part: (constant + constant) * n
    for (int n = 0; n < length( ); n++) {
            // another if : constant + constant (no else part)
            if(!list[n].equals(otherList.list[n]))
                    //constant
                    return false;
    }
}
```


Total time = $c_0 + c_1 + (c_2 + c_3) * n = O(n)$.

5) **Logarithmic complexity**: An algorithm is $O(logn)$ if it takes a constant time to cut the problem size by a fraction (usually by ½). As an example let us consider the following program:

```
for (i=1; i<=n;)
        i = i*2;
```

If we observe carefully, the value of i is doubling every time. Initially $i = 1$, in next step $i = 2$, and in subsequent steps $i = 4, 8$ and so on. Let us assume that the loop is executing some k times. At k^{th} step $2^i = n$ and we come out of loop. Taking logarithm on both sides, gives

$log(2^i) = logn$
$ilog2 = logn$
$i = logn$ //if we assume base-2

Total time = $O(logn)$.

Note: Similarly, for the below case also, worst case rate of growth is $O(logn)$. The same discussion holds good for decreasing sequence as well.

```
for (i=n; i<=1;)
    i = i/2;
```

Another example: binary search (finding a word in a dictionary of n pages)

- Look at the center point in the dictionary
- Is word towards left or right of center?
- Repeat process with left or right part of dictionary until the word is found

1.19 Properties of Notations

- Transitivity: $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$. Valid for O and Ω as well.
- Reflexivity: $f(n) = \Theta(f(n))$. Valid for O and Ω also.
- Symmetry: $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$.
- Transpose symmetry: $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$.

1.20 Commonly used Logarithms and Summations

Logarithms

$log\ x^y = y\ log\ x$ $\quad$ $logn = log_{10}^n$
$log\ xy = logx + logy$ $\quad$ $log^k n = (logn)^k$
$log\ logn = log(logn)$ $\quad$ $log\frac{x}{y} = logx - logy$
$a^{log_b^x} = x^{log_b^a}$ $\quad$ $log_b^x = \frac{log_a^x}{log_a^b}$

Arithmetic series

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Geometric series

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 \ldots + x^n = \frac{x^{n+1} - 1}{x - 1} (x \neq 1)$$

Harmonic series

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \approx log\ n$$

Other important formulae

$$\sum_{k=1}^{n} log\, k \approx nlogn$$

$$\sum_{k=1}^{n} k^p = 1^p + 2^p + \cdots + n^p \approx \frac{1}{p+1} n^{p+1}$$

1.21 Master Theorem for Divide and Conquer

All divide and conquer algorithms (In detail, we will discuss them in *Divide and Conquer* chapter) divides the problem into subproblems, each of which is part of the original problem, and then perform some additional work to compute the final answer. As an example, merge sort algorithm [for details, refer *Sorting* chapter] operates on two subproblems, each of which is half the size of the original and then performs O(n) additional work for merging. This gives the running time equation:

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

The following theorem can be used to determine the running time of divide and conquer algorithms. For a given program (algorithm), first we try to find the recurrence relation for the problem. If the recurrence is of the below form then we can directly give the answer without fully solving it.

If the recurrence is of the form $T(n) = aT(\frac{n}{b}) + \Theta(n^k log^p n)$, where $a \geq 1$, $b > 1$, $k \geq 0$ and p is a real number, then:

1) If $a > b^k$, then $T(n) = \Theta(n^{log_b^a})$
2) If $a = b^k$
   a. If $p > -1$, then $T(n) = \Theta(n^{log_b^a} log^{p+1} n)$
   b. If $p = -1$, then $T(n) = \Theta(n^{log_b^a} loglogn)$
   c. If $p < -1$, then $T(n) = \Theta(n^{log_b^a})$
3) If $a < b^k$
   a. If $p \geq 0$, then $T(n) = \Theta(n^k log^p n)$
   b. If $p < 0$, then $T(n) = O(n^k)$

1.22 Problems on Divide and Conquer Master Theorem

For each of the following recurrences, give an expression for the runtime $T(n)$ if the recurrence can be solved with the Master Theorem. Otherwise, indicate that the Master Theorem does not apply.

Problem-1 $T(n) = 3T(n/2) + n^2$

Solution: $T(n) = 3T(n/2) + n^2 \Rightarrow T(n) = \Theta(n^2)$ (Master Theorem Case 3.a)

Problem-2 $T(n) = 4T(n/2) + n^2$

Solution: $T(n) = 4T(n/2) + n^2 \Rightarrow T(n) = \Theta(n^2 logn)$ (Master Theorem Case 2.a)

Problem-3 $T(n) = T(n/2) + n^2$

Solution: $T(n) = T(n/2) + n^2 \Rightarrow \Theta(n^2)$ (Master Theorem Case 3.a)

Problem-4 $T(n) = 2^n T(n/2) + n^n$

Solution: $T(n) = 2^n T(n/2) + n^n \Rightarrow$ Does not apply (a is not constant)

Problem-5 $T(n) = 16T(n/4) + n$

Solution: $T(n) = 16T(n/4) + n \Rightarrow T(n) = \Theta(n^2)$ (Master Theorem Case 1)

Problem-6 $T(n) = 2T(n/2) + nlogn$

Solution: $T(n) = 2T(n/2) + nlogn \Rightarrow T(n) = \Theta(nlog^2 n)$ (Master Theorem Case 2.a)

Problem-7 $T(n) = 2T(n/2) + n/logn$

Solution: $T(n) = 2T(n/2) + n/logn \Rightarrow T(n) = \Theta(nloglogn)$ (Master Theorem Case 2.b)

Problem-8 $T(n) = 2T(n/4) + n^{0.51}$

Solution: $T(n) = 2T(n/4) + n^{0.51} \Rightarrow T(n) = \Theta(n^{0.51})$ (Master Theorem Case 3.b)

Problem-9 $T(n) = 0.5T(n/2) + 1/n$

Solution: $T(n) = 0.5T(n/2) + 1/n \Rightarrow$ Does not apply ($a < 1$)

Problem-10 $T(n) = 6T(n/3) + n^2 logn$

Solution: $T(n) = 6T(n/3) + n^2 logn \Rightarrow T(n) = \Theta(n^2 logn)$ (Master Theorem Case 3.a)

Problem-11 $T(n) = 64T(n/8) - n^2 logn$

Solution: $T(n) = 64T(n/8) - n^2 logn \Rightarrow$ Does not apply (function is not positive)

Problem-12 $T(n) = 7T(n/3) + n^2$

Solution: $T(n) = 7T(n/3) + n^2 \Rightarrow T(n) = \Theta(n^2)$ (Master Theorem Case 3.as)

Problem-13 $T(n) = 4T(n/2) + logn$

Solution: $T(n) = 4T(n/2) + logn \Rightarrow T(n) = \Theta(n^2)$ (Master Theorem Case 1)

Problem-14 $T(n) = 16T(n/4) + n!$

Solution: $T(n) = 16T(n/4) + n! \Rightarrow T(n) = \Theta(n!)$ (Master Theorem Case 3.a)

Problem-15 $T(n) = \sqrt{2}T(n/2) + logn$

Solution: $T(n) = \sqrt{2}T(n/2) + logn \Rightarrow T(n) = \Theta(\sqrt{n})$ (Master Theorem Case 1)

Problem-16 $T(n) = 3T(n/2) + n$

Solution: $T(n) = 3T(n/2) + n \Rightarrow T(n) = \Theta(n^{log3})$ (Master Theorem Case 1)

Problem-17 $T(n) = 3T(n/3) + \sqrt{n}$

Solution: $T(n) = 3T(n/3) + \sqrt{n} \Rightarrow T(n) = \Theta(n)$ (Master Theorem Case 1)

Problem-18 $T(n) = 4T(n/2) + cn$

Solution: $T(n) = 4T(n/2) + cn \Rightarrow T(n) = \Theta(n^2)$ (Master Theorem Case 1)

Problem-19 $T(n) = 3T(n/4) + nlogn$

Solution: $T(n) = 3T(n/4) + nlogn \Rightarrow T(n) = \Theta(nlogn)$ (Master Theorem Case 3.a)

Problem-20 $T(n) = 3T(n/3) + n/2$

Solution: $T(n) = 3T(n/3) + n/2 \Rightarrow T(n) = \Theta(nlogn)$ (Master Theorem Case 2.a)

1.23 Master Theorem for Subtract and Conquer Recurrences

Let $T(n)$ be a function defined on positive n, and having the property

$$T(n) = \begin{cases} c, & \text{if } n \le 1 \\ aT(n-b) + f(n), & \text{if } n > 1 \end{cases}$$

for some constants $c, a > 0, b > 0, k \ge 0$, and function $f(n)$. If $f(n)$ is in $O(n^k)$, then

$$T(n) = \begin{cases} O(n^k), & \text{if } a < 1 \\ O(n^{k+1}), & \text{if } a = 1 \\ O\left(n^k a^{\frac{n}{b}}\right), & \text{if } a > 1 \end{cases}$$

1.24 Variant of subtraction and conquer master theorem

The solution to the equation $T(n) = T(\alpha n) + T((1-\alpha)n) + \beta n$, where $0 < \alpha < 1$ and $\beta > 0$ are constants, is $O(nlogn)$.

1.25 Amortized Analysis

Amortized analysis refers to determining the time-averaged running time for a sequence of operations. It is different from average case analysis, because amortized analysis does not make any assumption about the distribution of the data values, whereas average case analysis assumes the data are not "bad" (e.g., some sorting algorithms do well on "average" over all input orderings but very badly on certain input orderings). That is, amortized analysis is a worst case analysis, but for a sequence of operations, rather than for individual operations.

The motivation for amortized analysis is to better understand the running time of certain techniques, where standard worst case analysis provides an overly pessimistic bound. Amortized analysis generally applies to a method that consists of a sequence of operations, where the vast majority of the operations are cheap, but some of the operations are expensive. If we can show that the expensive operations are particularly rare we can "charge them" to the cheap operations, and only bound the cheap operations.

The general approach is to assign an artificial cost to each operation in the sequence, such that the total of the artificial costs for the sequence of operations bounds total of the real costs for the sequence. This artificial cost is called the amortized cost of an operation. In order to analyze the running time, the amortized cost thus is a correct way of understanding the overall running time — but note that particular operations can still take longer so it is not a way of bounding the running time of any individual operation in the sequence.

When one event in a sequence affects the cost of later events:

- One particular task may be expensive.
- But it may leave data structure in a state that next few operations becomes easier.

Example: Let us consider an array of elements from which we want to find k^{th} smallest element. We can solve this problem using sorting. After sorting the given array, we just need to return the k^{th} element from it. Cost of performing sort (assuming comparison based sorting algorithm) is $O(nlogn)$. If we perform n such selections then the average cost of each selection is $O(nlogn/n) = O(logn)$. This clearly indicates that sorting once is reducing the complexity of subsequent operations.

1.26 Problems on Algorithms Analysis

Note: From the following problems, try to understand the cases which give different complexities ($O(n), O(logn), O(loglogn)$ etc...).

Problem-21 Find the complexity of the below recurrence:

$$T(n) = \begin{cases} 3T(n-1), if\ n > 0, \\ 1, \quad otherwise \end{cases}$$

Solution: Let us try solving this function with substitution.

$$T(n) = 3T(n-1)$$
$$T(n) = 3(3T(n-2)) = 3^2T(n-2)$$
$$T(n) = 3^2(3T(n-3))$$
$$\vdots$$
$$T(n) = 3^nT(n-n) = 3^nT(0) = 3^n$$

This clearly shows that the complexity of this function is $O(3^n)$.
Note: We can use the *Subtraction and Conquer* master theorem for this problem.

Problem-22 Find the complexity of the below recurrence:

$$T(n) = \begin{cases} 2T(n-1) - 1, if\ n > 0, \\ 1, \quad otherwise \end{cases}$$

Solution: Let us try solving this function with substitution.

$$T(n) = 2T(n-1) - 1$$
$$T(n) = 2(2T(n-2) - 1) - 1 = 2^2T(n-2) - 2 - 1$$
$$T(n) = 2^2(2T(n-3) - 2 - 1) - 1 = 2^3T(n-4) - 2^2 - 2^1 - 2^0$$
$$T(n) = 2^nT(n-n) - 2^{n-1} - 2^{n-2} - 2^{n-3} \ldots . 2^2 - 2^1 - 2^0$$
$$T(n) = 2^n - 2^{n-1} - 2^{n-2} - 2^{n-3} \ldots . 2^2 - 2^1 - 2^0$$
$$T(n) = 2^n - (2^n - 1)\ [note: 2^{n-1} + 2^{n-2} + \cdots + 2^0 = 2^n]$$
$$T(n) = 1$$

∴Complexity is O(1). Note that while the recurrence relation looks exponential the solution to the recurrence relation here gives a different result.

Problem-23 What is the running time of the following function?

```
void Function(int n) {
        int i=1, s=1;
        while( s <= n) {
                i++;
                s= s+i;
                printf("*");
        }
}
```

Solution: Consider the comments in below function:

```
void Function (int n) {
        int i=1, s=1;
        // s is increasing not at rate 1 but i
        while( s <= n) {
                i++;
                s= s+i;
                printf("*");
        }
}
```

We can define the terms 's' according to the relation $s_i = s_{i-1} + i$. The value of 'i' increases by one for each iteration. The value contained in 's' at the i^{th} iteration is the sum of the first 'i' positive integers. If k is the total number of iterations taken by the program, then *while* loop terminates if:

$$1 + 2 + \ldots + k = \frac{k(k+1)}{2} > n \Rightarrow k = O(\sqrt{n}).$$

Problem-24 Find the complexity of the function given below.

```
void Function(int n) {
    int i, count =0;
    for(i=1; i*i<=n; i++)
        count++;
}
```

Solution:

```
void Function(int n) {
    int i, count =0;
    for(i=1; i*i<=n; i++)
        count++;
}
```

In the above function the loop will end, if $i^2 \leq n \Rightarrow T(n) = O(\sqrt{n})$. The reasoning is same as that of Problem-23.

Problem-25 What is the complexity of the below program:

```
void function(int n) {
    int i, j, k , count =0;
    for(i=n/2; i<=n; i++)
        for(j=1; j + n/2<=n; j= j++)
            for(k=1; k<=n; k= k * 2)
                count++;
}
```

Solution: Consider the comments in the following function.

```
void function(int n) {
    int i, j, k , count =0;
    //outer loop execute n/2 times
    for(i=n/2; i<=n; i++)
        //Middle loop executes n/2 times
        for(j=1; j + n/2<=n; j= j++)
            //outer loop execute logn times
            for(k=1; k<=n; k= k * 2)
                count++;
}
```

The complexity of the above function is $O(n^2 logn)$.

Problem-26 What is the complexity of the below program:

```
void function(int n) {
    int i, j, k , count =0;
    for(i=n/2; i<=n; i++)
        for(j=1; j<=n; j= 2 * j)
            for(k=1; k<=n; k= k * 2)
                count++;
}
```

Solution: Consider the comments in the following function.

```
void function(int n) {
    int i, j, k , count =0;
    //outer loop execute n/2 times
    for(i=n/2; i<=n; i++)
```



```
        //Middle loop executes logn times
        for(j=1; j<=n; j= 2 * j)
            //outer loop execute logn times
            for(k=1; k<=n; k= k*2)
                count++;
}
```

The complexity of the above function is $O(nlog^2 n)$.

Problem-27 Find the complexity of the below program.

```
function( int n ) {
    if(n == 1) return;
    for(int i = 1 ; i <= n ; i + + ) {
        for(int j= 1 ; j <= n ; j + + ) {
            printf("*" );
            break;
        }
    }
}
```

Solution: Consider the comments in the following function.

```
function( int n ) {
    //constant time
    if( n == 1 ) return;
    //outer loop execute n times
    for(int i = 1 ; i <= n ; i + + ) {
        // inner loop executes only time due to break statement.
        for(int j= 1 ; j <= n ; j + + ) {
            printf("*" );
            break;
        }
    }
}
```

The complexity of the above function is $O(n)$. Even though the inner loop is bounded by n, but due to the break statement it is executing only once.

Problem-28 Write a recursive function for the running time $T(n)$ of the function given below. Prove using the iterative method that $T(n) = \Theta(n^3)$.

```
function( int n ) {
    if( n == 1 ) return;
    for(int i = 1 ; i <= n ; i + + )
        for(int j = 1 ; j <= n ; j + + )
            printf("*" ) ;
    function( n-3 );
}
```

Solution: Consider the comments in below function:

```
function (int n) {
    //constant time
    if( n == 1 ) return;
    //outer loop execute n times
    for(int i = 1 ; i <= n ; i + + )
        //inner loop executes n times
```

```
        for(int j = 1 ; j <= n ; j + + )
                //constant time
                printf("*" ) ;
    function( n-3 );
}
```

The recurrence for this code is clearly $T(n) = T(n-3) + cn^2$ for some constant $c > 0$ since each call prints out n^2 asterisks and calls itself recursively on n - 3. Using the iterative method we get: $T(n) = T(n-3) + cn^2$. Using the *Subtraction and Conquer* master theorem, we get $T(n) = \Theta(n^3)$.

Problem-29 Determine Θ bounds for the recurrence relation: $T(n) = 2T\left(\frac{n}{2}\right) + nlogn$.

Solution: Using Divide and Conquer master theorem, we get $O(nlog^2n)$.

Problem-30 Determine Θ bounds for the recurrence: $T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + T\left(\frac{n}{8}\right) + n$.

Solution: Substituting in the recurrence equation, we get: $T(n) \leq c1 * \frac{n}{2} + c2 * \frac{n}{4} + c3 * \frac{n}{8} + cn \leq k * n$, where k is a constant. This clearly says $\Theta(n)$.

Problem-31 Determine Θ bounds for the recurrence relation: $T(n) = T(\lceil n/2 \rceil) + 7$.

Solution: Using Master Theorem we get $\Theta(logn)$.

Problem-32 Prove that the running time of the code below is $\Omega(logn)$.

```
void Read(int n) {
        int k = 1;
        while( k < n )
                k = 3*k;
}
```

Solution: The *while* loop will terminate once the value of 'k' is greater than or equal to the value of 'n'. In each iteration the value of 'k' is multiplied by 3. If i is the number of iterations, then 'k' has the value of $3i$ after i iterations. The loop is terminated upon reaching i iterations when $3^i \geq n \leftrightarrow i \geq \log_3 n$, which shows that $i = \Omega(logn)$.

Problem-33 Solve the following recurrence.

$$T(n) = \begin{cases} 1, & if\ n = 1 \\ T(n-1) + n(n-1), & if\ n \geq 2 \end{cases}$$

Solution: By iteration:

$$T(n) = T(n-2) + (n-1)(n-2) + n(n-1)$$

...

$$T(n) = T(1) + \sum_{i=1}^{n} i(i-1)$$

$$T(n) = T(1) + \sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i$$

$$T(n) = 1 + \frac{n((n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$

$$T(n) = \Theta(n^3)$$

Note: We can use the *Subtraction and Conquer* master theorem for this problem.

Problem-34 Consider the following program:

```
Fib[n]
if(n==0) then return 0
else if(n==1) then return 1
else return Fib[n-1]+Fib[n-2]
```

Solution: The recurrence relation for running time of this program is: $T(n) = T(n-1) + T(n-2) + c$. Notice $T(n)$ has two recurrence calls indicating a binary tree. Each step recursively calls the program for n reduced by 1 and 2, so the depth of the recurrence tree is $O(n)$. The number of leaves at depth n is 2^n since this is a full binary tree, and each leaf takes at least $O(1)$ computation for the constant factor. Running time is clearly exponential in n and it is $O(2^n)$.

Problem-35 Running time of following program?

```
function(n) {
        for(int i = 1; i <= n ; i + + )
                for(int j = 1 ; j <= n ; j+ = i )
                        printf(" * ") ;
}
```

Solution: Consider the comments in below function:

```
function (n) {
        //this loop executes n times
        for(int i = 1;  i <= n ; i + + )
                //this loop executes j times with j increase by the rate of i
                for(int j = 1 ; j <= n ; j+ = i )
                        printf( " * " ) ;
}
```

In the above code, inner loop executes n/i times for each value of i. Its running time is $n \times (\sum_{i=1}^{n} n/i) = O(nlogn)$.

Problem-36 What is the complexity of $\sum_{i=1}^{n} log\ i$?

Solution: Using the logarithmic property, $logxy = logx + logy$, we can see that this problem is equivalent to

$$\sum_{i=1}^{n} logi = log\ 1 + log\ 2 + \cdots + log\ n = log(1 \times 2 \times \ldots \times n) = log(n!) \leq log(n^n) \leq nlogn$$

This shows that that the time complexity = $O(nlogn)$.

Problem-37 What is the running time of the following recursive function (specified as a function of the input value n)? First write the recurrence formula and then find its complexity.

```
function(int n) {
        if(n <= 1) return;
        for (int i=1 ; i <= 3; i++ )
                f(⌈n/3⌉);
}
```

Solution: Consider the comments in below function:

```
function (int n) {
        //constant time
        if(n <= 1) return;
        //this loop executes with recursive loop of n/3 value
        for (int i=1 ; i <= 3; i++ )
                f(⌈n/3⌉);
}
```

We can assume that for asymptotical analysis $k = \lceil k \rceil$ for every integer $k \geq 1$. The recurrence for this code is $T(n) = 3T(\frac{n}{3}) + \Theta(1)$. Using master theorem, we get $T(n) = \Theta(n)$.

Problem-38 What is the running time of the following recursive function (specified as a function of the input value n)? First write a recurrence formula, and show its solution using induction.

```
function(int n) {
```

```
        if(n <= 1) return;
        for (int i=1 ; i <= 3 ; i++ )
                function (n – 1).
}
```

Solution: Consider the comments in below function:

```
function (int n) {
        //constant time
        if(n <= 1) return;
        //this loop executes 3 times with recursive call of n-1 value
        for (int i=1 ; i <= 3 ; i++ )
                function (n – 1).
}
```

The *if* statement requires constant time [O(1)]. With the *for* loop, we neglect the loop overhead and only count three times that the function is called recursively. This implies a time complexity recurrence:

$$T(n) = c, if\ n \leq 1;$$
$$= c + 3T(n-1), if\ n > 1.$$

Using the *Subtraction and Conquer* master theorem, we get $T(n) = \Theta(3^n)$.

Problem-39 Write a recursion formula for the running time $T(n)$ of the function whose code is below.

```
function (int n) {
        if(n <= 1)  return;
        for(int i = 1; i < n; i + +)
                printf(" * ");
        function ( 0.8n ) ;
}
```

Solution: Consider the comments in below function:

```
function (int n) {
        if(n <= 1)  return;          //constant time
        // this loop executes n times with constant time loop
        for(int i = 1; i < n; i + +)
                printf(" * ");
        //recursive call with 0.8n
        function ( 0.8n ) ;
}
```

The recurrence for this piece of code is $T(n) = T(.8n) + O(n) = T(4/5n) + O(n) = 4/5\ T(n) + O(n)$. Applying master theorem, we get $T(n) = O(n)$.

Problem-40 Find the complexity of the recurrence: $T(n) = 2T(\sqrt{n}) + logn$

Solution: The given recurrence is not in the master theorem form. Let us try to convert this to master theorem format by assuming $n = 2^m$. Applying logarithm on both sides gives, $logn = mlog2 \Rightarrow m = logn$. Now, the given function becomes,

$$T(n) = T(2^m) = 2T(\sqrt{2^m}) + m = 2T\left(2^{\frac{m}{2}}\right) + m.$$

To make it simple we assume $S(m) = T(2^m) \Rightarrow S(\frac{m}{2}) = T(2^{\frac{m}{2}}) \Rightarrow S(m) = 2S\left(\frac{m}{2}\right) + m$. Applying the master theorem would result $S(m) = O(mlogm)$. If we substitute $m = logn$ back, $T(n) = S(logn) = O((logn)\ loglogn)$.

Problem-41 Find the complexity of the recurrence: $T(n) = T(\sqrt{n}) + 1$

Solution: Applying the logic of Problem-40, gives $S(m) = S\left(\frac{m}{2}\right) + 1$. Applying the master theorem would result $S(m) = O(logm)$. Substituting $m = logn$, gives $T(n) = S(logn) = O(loglogn)$.

Problem-42 Find the complexity of the recurrence: $T(n) = 2T(\sqrt{n}) + 1$

Solution: Applying the logic of Problem-40, gives: $S(m) = 2S\left(\frac{m}{2}\right) + 1$. Using the master theorem results $S(m) = O(m^{log_2^2}) = O(m)$. Substituting $m = logn$ gives $T(n) = O(logn)$.

Problem-43 Find the complexity of the below function.

```
int Function (int n) {
        if(n <= 2) return 1;
        else return (Function (floor(sqrt(n))) + 1);
}
```

Solution: Consider the comments in below function:

```
int Function (int n) {
        //constant time
        if(n <= 2) return 1;
        else    // executes √n + 1 times
                return (Function (floor(sqrt(n))) + 1);
}
```

For the above code, the recurrence function can be given as: $T(n) = T(\sqrt{n}) + 1$. This is same as that of Problem-41.

Problem-44 Analyze the running time of the following recursive psuedocode as a function of n.

```
void function(int n) {
        if( n < 2 ) return;
        else counter = 0;
        for i = 1 to 8 do
                function (n/2);
        for i =1 to n^3 do
                counter = counter + 1;
}
```

Solution: Consider the comments in below psuedocode and call running time of function(n) as $T(n)$.

```
void function(int n) {
        if( n < 2 ) return;   //constant time
        else    counter = 0;
        // this loop executes 8 times with n value half in every call
        for i = 1 to 8 do
                function (n/2);
        // this loop executes n^3 times with constant time loop
        for i =1 to n^3 do
                counter = counter + 1;
}
```

$T(n)$ can be defined as follows:

$$T(n) = 1\ if\ n < 2,$$
$$= 8T(\frac{n}{2}) + n^3 + 1\ otherwise.$$

Using the master theorem gives, $T(n) = \Theta(n^{log_2^8} logn) = \Theta(n^3 logn)$.

Problem-45 Find the complexity of the below psuedocode.

```
temp = 1
repeat
        for i = 1 to n
                temp = temp + 1;
        n = n/2;
```

until n <= 1

Solution: Consider the comments in below psuedocode:

```
temp = 1  //const time
repeat  // this loops executes n times
        for i = 1 to n
                temp = temp + 1;
        //recursive call with n/2 value
        n = n/2;
until n <= 1
```

The recurrence for this function is $T(n) = T(n/2) + n$. Using master theorem we get, $T(n) = O(nlogn)$.

Problem-46 Running time of following program?

```
function(int n) {
        for(int i = 1 ; i <= n ; i + + )
                for(int j = 1 ; j <= n ; j * = 2 )
                        printf( " * " );
}
```

Solution: Consider the comments in below function:

```
function(int n) {
        for(int i = 1 ; i <= n ; i + + ) // this loops executes n times
                // this loops executes logn times from our logarithms guideline
                for(int j = 1 ; j <= n ; j * = 2 )
                        printf( " * " );
}
```

Complexity of above program is : $O(nlogn)$.

Problem-47 Running time of following program?

```
function(int n) {
        for(int i = 1 ; i <= n/3 ; i + + )
                for(int j = 1 ; j <= n ; j += 4 )
                        printf( " * " );
}
```

Solution: Consider the comments in below function:

```
function(int n) { // this loops executes n/3 times
        for(int i = 1 ; i <= n/3 ; i + + )
                // this loops executes n/4 times
                for(int j = 1 ; j <= n ; j += 4)
                        printf( " * " );
}
```

The time complexity of this program is : $O(n^2)$.

Problem-48 Find the complexity of the below function.

```
void function(int n) {
  if(n <= 1) return;
  if(n > 1) {
        printf (" * ");
        function(n/2);
        function(n/2);
  }
}
```

Solution: Consider the comments in below function:


```
void function(int n) {
        //constant time
        if(n <= 1) return;
        if(n > 1) {
                //constant time
                printf (" * ");
                //recursion with n/2 value
                function( n/2 );
                //recursion with n/2 value
                function( n/2 );
        }
}
```

The recurrence for this function is: $T(n) = 2T\left(\frac{n}{2}\right) + 1$. Using master theorem, we get $T(n) = O(n)$.

Problem-49 Find the complexity of the below function.

```
function(int n) {
        int i=1;
        while (i < n) {
                int j=n;
                while(j > 0)
                        j = j/2;
                i=2*i;
        } // i
}
```

Solution:

```
function(int n) {
        int i=1;
        while (i < n) {
                int j=n;
                while(j > 0)
                        j = j/2;  //logn code
                i=2*i; //logn times
        } // i
}
```

Time Complexity: $O(logn * logn) = O(log^2 n)$.

Problem-50 $\sum_{1 \le k \le n} O(n)$, where $O(n)$ stands for order n is:

(a) $O(n)$ (b) $O(n^2)$ (c) $O(n^3)$ (d) $O(3n^2)$ (e) $O(1.5n^2)$

Solution: (b). $\sum_{1 \le k \le n} O(n) = O(n) \sum_{1 \le k \le n} 1 = O(n^2)$.

Problem-51 Which of the following three claims are correct

I $(n + k)^m = \Theta(n^m)$, where k and m are constants II $2^{n+1} = O(2^n)$ III $2^{2n+1} = O(2^n)$

(a) I and II (b) I and III (c) II and III (d) I, II and III

Solution: (a). (I) $(n + k)^m = n^k + c1 * n^{k-1} + \ldots k^m = \Theta(n^k)$ and (II) $2^{n+1} = 2 * 2^n = O(2^n)$

Problem-52 Consider the following functions:

$f(n) = 2^n$ $g(n) = n!$ $h(n) = n^{logn}$

Which of the following statements about the asymptotic behaviour of $f(n)$, $g(n)$, and $h(n)$ is true?

(A) $f(n) = O(g(n))$; $g(n) = O(h(n))$ (B) $f(n) = \Omega(g(n))$; $g(n) = O(h(n))$

(C) $g(n) = O(f(n))$; $h(n) = O(f(n))$ (D) $h(n) = O(f(n))$; $g(n) = \Omega(f(n))$

Solution: (D). According to rate of growths: h(n) < f(n) < g(n) (g(n) is asymptotically greater than f(n) and f(n) is asymptotically greater than h(n)). We can easily see above order by taking logarithms of the given 3 functions: $lognlogn < n < log(n!)$. Note that, $log(n!) = O(nlogn)$.

Problem-53 Consider the following segment of C-code:

```
int j=1, n;
while (j <=n)
        j = j*2;
```

The number of comparisons made in the execution of the loop for any $n > 0$ is:

(A) ceil(log_2^n)+ 1 (B) n (C) ceil(log_2^n) (D) floor(log_2^n) + 1

Solution: (a). Let us assume that the loop executes k times. After k^{th} step the value of j is 2^k. Taking logarithms on both sides gives $k = log_2^n$. Since we are doing one more comparison for exiting from loop, the answer is ceil(log_2^n)+ 1.

Problem-54 Consider the following C code segment. Let T(n) denotes the number of times the for loop is executed by the program on input n. Which of the following is TRUE?

```
int IsPrime(int n){
 for(int i=2;i<=sqrt(n);i++)
 if(n%i == 0)
 {       printf("Not Prime\n"); return 0;    }
 return 1;
}
```

(A) T(n) = O($\sqrt{n}$) and T(n) = $\Omega(\sqrt{n})$ (B) T(n) = O($\sqrt{n}$) and T(n) = $\Omega(1)$

(C) T(n) = O(n) and T(n) = $\Omega(\sqrt{n})$ (D) None of the above

Solution: (B). Big O notation describes the tight upper bound and Big Omega notation describes the tight lower bound for an algorithm. The *for* loop in the question is run maximum $\sqrt{n}$ times and minimum 1 time. Therefore, T(n) = O($\sqrt{n}$) and T(n) = $\Omega(1)$.

Problem-55 In the following C function, let $n \geq m$. How many recursive calls are made by this function?

```
int gcd(n,m){
        if (n%m ==0) return m;
        n = n%m;
        return gcd(m,n);
}
```

(A) $\Theta(log_2^n)$ (B) $\Omega(n)$ (C) $\Theta(log_2 log_2^n)$ (D) $\Theta(n)$

Solution: No option is correct. Big O notation describes the tight upper bound and Big Omega notation describes the tight lower bound for an algorithm. For $m = 2$ and for all $n = 2^i$, running time is O(1) which contradicts every option.

Problem-56 Suppose $T(n) = 2T(n/2) + n$, T(0)=T(1)=1. Which one of the following is FALSE?

(A) $T(n) = O(n^2)$ (B) $T(n) = \Theta(nlogn)$ (C) $T(n) = \Omega(n^2)$ (D) $T(n) = O(nlogn)$

Solution: (C). Big O notation describes the tight upper bound and Big Omega notation describes the tight lower bound for an algorithm. Based on master theorem, we get $T(n) = \Theta(nlogn)$. This indicates that tight lower bound and tight upper bound are same. That means, O($nlogn$) and $\Omega(nlogn)$ are correct for given recurrence. So option (C) is wrong.

Problem-57 Find the complexity of the below function.

```
function(int n) {
  for (int i = 0; i<n; i++)
    for(int j=i; j<i*i; j++)
      if (j %i == 0){
        for (int k = 0; k < j; k++)
```



```
          printf(" * ");
      }
  }
```

Solution:

```
function(int n) {
  for (int i = 0; i<n; i++)              // Executes n times
    for(int j=i; j<i*i; j++)             // Executes n*n times
      if (j %i == 0){
        for (int k = 0; k < j; k++) // Executes j times = (n*n) times
          printf(" * ");
      }
}
```

Time Complexity: O(n^5).

Chapter-2

RECURSION AND BACKTRACKING

2.1 Introduction

In this chapter, we will look at one of the important topic "*recursion*" which will be used in almost every chapter and also its relative "*backtracking*".

2.2 What is Recursion?

Any function which calls itself is called *recursive*. A recursive method solves a problem by calling a copy of itself to work on a smaller problem. This is called the recursion step. The recursion step can result in many more such recursive calls. It is important to ensure that the recursion terminates. Each time the function calls itself with a slightly simpler version of the original problem. The sequence of smaller problems must eventually converge on the base case.

2.3 Why Recursion?

Recursion is a useful technique borrowed from mathematics. Recursive code is generally shorter and easier to write than iterative code. Generally, loops are turned into recursive functions when they are compiled or interpreted. Recursion is most useful for tasks that can be defined in terms of similar subtasks. For example, sort, search, and traversal problems often have simple recursive solutions.

2.4 Format of a Recursive Function

A recursive function performs a task in part by calling itself to perform the subtasks. At some point, the function encounters a subtask that it can perform without calling itself. This case, where the function does not recur, is called the *base case*, the former, where the function calls itself to perform a subtask, is referred to as the *recursive case*. We can write all recursive functions using the format:

```
if(test for the base case)
        return some base case value
else if(test for another base case)
        return some other base case value
// the recursive case
else
        return (some work and then a recursive call)
```

As an example consider the factorial function: $n!$ is the product of all integers between n and 1. Definition of recursive factorial looks like:

$$n! = 1, \quad \text{if } n = 0$$
$$n! = n * (n-1)! \quad \text{if } n > 0$$

This definition can easily be converted to recursive implementation. Here the problem is determining the value of $n!$, and the subproblem is determining the value of $(n-1)!$. In the recursive case, when n is greater than 1, the function

calls itself to determine the value of $(n-1)!$ and multiplies that with n. In the base case, when n is 0 or 1, the function simply returns 1. This looks like the following:

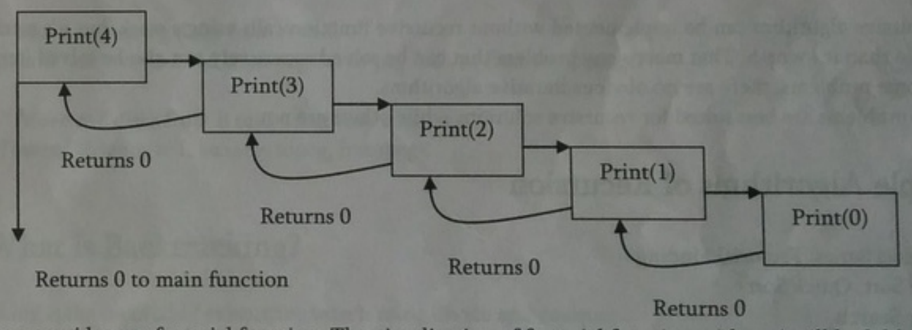
```
// calculates factorial of a positive integer
int Fact(int n) {
        // base cases: fact of 0 or 1 is 1
        if(n == 1) return 1;
        else if(n == 0) return 1;
        // recursive case: multiply n by (n - 1) factorial
        else    return n*Fact(n-1);
}
```

2.5 Recursion and Memory (Visualization)

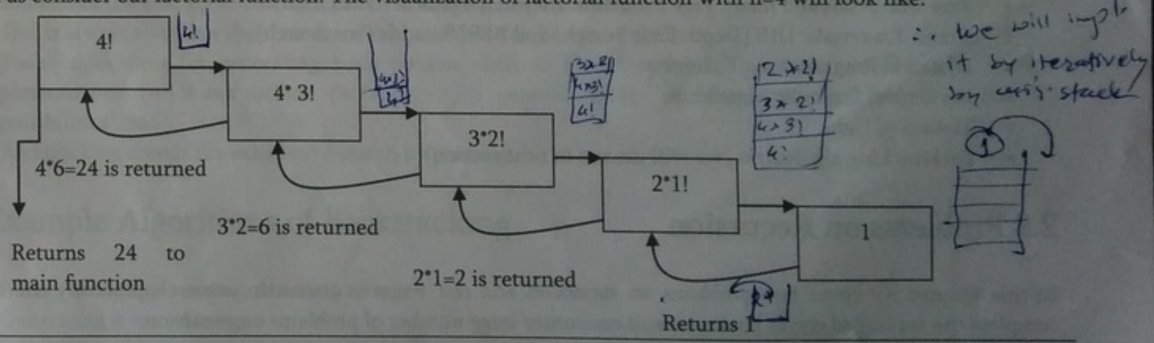
Each recursive call makes a new copy of that method (actually only the variables) in memory. Once a method ends (i.e. returns some data), the copy of that returning method is removed from memory. The recursive solutions look simple but visualization and tracing takes times. For better understanding, let us consider the following example.

```
int Print(int n) { //print numbers 1 to n backward
        if( n = = 0) // this is the terminating base case
                return 0;
        else {  printf ("%d",n);
                return Print(n-1); // recursive call to itself again
        }
}
```

For this example, if we call the print function with n=4, visually our memory assignments may look like:



Now, let us consider our factorial function. The visualization of factorial function with n=4 will look like:



2.6 Recursion versus Iteration

While discussing recursion the basic question which comes into mind is, which way is better? – Iteration or recursion? Answer to this question depends on what we are trying to do. Generally, a recursive approach mirrors the problem that we are trying to solve. A recursive approach makes it simpler to solve a problem which may not have the most obvious of answers. But, recursion adds overhead for each recursive call (needs space on the stack frame).

Recursion

- Terminates when a base case is reached.
- Each recursive call requires extra space on the stack frame (i.e. memory).
- If we get infinite recursion, the program may run out of memory and gives stack overflow.
- Solutions to some problems are easier to formulate recursively.

Iteration

- Terminates when a condition is proven to be false.
- Each iteration does not require any extra space.
- An infinite loop could loop forever since there is no extra memory being created.
- Iterative solutions to a problem may not always be as obvious as a recursive solution.

2.7 Notes on Recursion

- Recursive algorithms have two types of cases, recursive cases and base cases.
- Every recursive function case must terminate at base case.
- Generally iterative solutions are more efficient than recursive solutions [due to the overhead of function calls].
- A recursive algorithm can be implemented without recursive function calls using a stack, but it's usually more trouble than it's worth. That means any problem that can be solved recursively can also be solved iteratively.
- For some problems, there are no obvious iterative algorithms.
- Some problems are best suited for recursive solutions while others are not.

2.8 Example Algorithms of Recursion

- Fibonacci Series, Factorial Finding
- Merge Sort, Quick Sort
- Binary Search
- Tree Traversals and many Tree Problems: InOrder, PreOrder PostOrder
- Graph Traversals: DFS [Depth First Search] and BFS [Breadth First Search]
- Dynamic Programming Examples
- Divide and Conquer Algorithms
- Towers of Hanoi
- Backtracking algorithms [we will discuss in next section]

2.9 Problems on Recursion

In this chapter we cover few problems on recursion and rest we will discuss in other chapters. By the time you complete the reading of entire book you will encounter huge number of problems on recursion.

Problem-1 Discuss Towers of Hanoi puzzle.

Solution: The Tower of Hanoi is a mathematical puzzle. It consists of three rods (or pegs or towers), and a number of disks of different sizes which can slide onto any rod. The puzzle starts with the disks on one rod in ascending order of size, the smallest at the top, thus making a conical shape. The objective of the puzzle is to move the entire stack to another rod, satisfying the following rules:

- Only one disk may be moved at a time.
- Each move consists of taking the upper disk from one of the rods and sliding it onto another rod, on top of the other disks that may already be present on that rod.
- No disk may be placed on top of a smaller disk.

Algorithm

- Move the top $n-1$ disks from *Source* to *Auxiliary* tower,
- Move the n^{th} disk from *Source* to *Destination* tower,
- Move the $n-1$disks from Auxiliary tower to *Destination* tower.
- Transferring the top $n-1$ disks from *Source* to *Auxiliary* tower can again be thought as a fresh problem and can be solved in the same manner. Once we solve *Tower of Hanoi* with three disks, we can solve it with any number of disks with the above algorithm.

```
void TowersOfHanoi(int n, char frompeg, char topeg, char auxpeg) {
        /* If only 1 disk, make the move and return */
        if(n==1) {
                printf("Move disk 1 from peg %c to peg %c",frompeg, topeg);
                return;
        }
        /* Move top n-1 disks from A to B, using C as auxiliary */
        TowersOfHanoi(n-1, frompeg, auxpeg, topeg);

        /* Move remaining disks from A to C */
        printf("\nMove disk %d from peg %c to peg %c", n, frompeg, topeg);

        /* Move n-1 disks from B to C using A as auxiliary */
        TowersOfHanoi(n-1, auxpeg, topeg, frompeg);
}
```

2.10 What is Backtracking?

Backtracking is the method of exhaustive search using divide and conquer.

- Sometimes the best algorithm for a problem is to try all possibilities.
- This is always slow, but there are standard tools that can be used to help.
- Tools: algorithms for generating basic objects, such as binary strings [2^n possibilities for n-bit string], permutations [$n!$], combinations [$n!/r!\,(n-r)!$], general strings [k −ary strings of length n has k^n possibilities], etc...
- Backtracking speeds the exhaustive search by pruning.

2.11 Example Algorithms of Backtracking

- Binary Strings: generating all binary strings
- Generating k −ary Strings

- The Knapsack Problem
- Generalized Strings
- Hamiltonian Cycles [refer *Graphs* chapter]
- Graph Coloring Problem

2.12 Problems on Backtracking

Problem-2 Generate all the strings of n bits. Assume $A[0..n-1]$ is an array of size n.

Solution:

```
void Binary(int n) {
        if(n < 1 )
            printf("%s", A);              //Assume array A is a global variable
        else {   A[n-1] = 0;
                 Binary(n - 1);
                 A[n-1] = 1;
                 Binary(n - 1);
        }
}
```

Let $T(n)$ be the running time of $binary(n)$. Assume function $printf$ takes time O(1).

$$T(n) = \begin{cases} c, & \text{if } n < 0 \\ 2T(n-1) + d, & \text{otherwise} \end{cases}$$

Using Subtraction and Conquer Master theorem we get, $T(n)$ =O(2^n). This means the algorithm for generating bit-strings is optimal.

Problem-3 Generate all the strings of length n drawn from $0 \ldots k-1$.

Solution: Let us assume we keep current k-ary string in an array $A[0..n-1]$. Call function $k\text{-}string(n, k)$:

```
void k-string(int n, int k) {
        //process all k-ary strings of length m
        if(n < 1 )
            printf("%s",A);               //Assume array A is a global variable
        else {   for (int j = 0 ; j < k ; j++) {
                        A[n-1] = j;
                        k-string(n- 1, k);
                 }
        }
}
```

Let $T(n)$ be the running time of $k-string(n)$. Then,

$$T(n) = \begin{cases} c, & if\ n < 0 \\ kT(n-1) + d, & otherwise \end{cases}$$

Using Subtraction and Conquer Master theorem we get, $T(n)$ =O(k^n).

Note: For more problems, refer *String Algorithms* chapter.

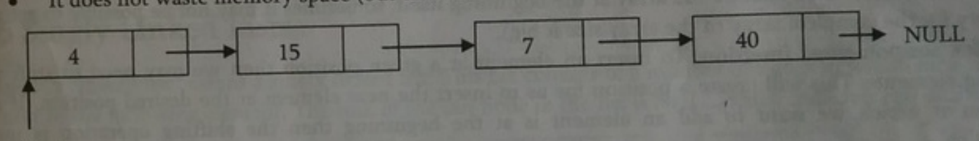
LINKED LISTS

Chapter-3

3.1 What is a Linked List?

Linked list is a data structure used for storing collections of data. Linked list has the following properties.

- Successive elements are connected by pointers
- Last element points to NULL
- Can grow or shrink in size during execution of a program
- Can be made just as long as required (until systems memory exhausts)
- Can be made just as long as required
- It does not waste memory space (but takes some extra memory for pointers)



3.2 Linked Lists ADT

The following operations make linked lists an ADT.

Main Linked Lists Operations

- Insert: inserts an element into the list
- Delete: removes and returns the specified position element from the list

Auxiliary Linked Lists Operations

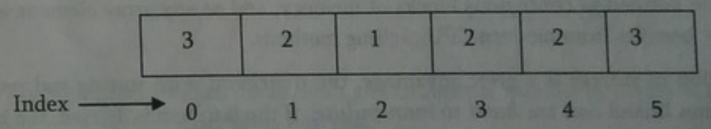
- Delete List: removes all elements of the list (disposes the list)
- Count: returns the number of elements in the list
- Find n^{th} node from the end of the list etc…

3.3 Why Linked Lists?

There are many other data structures which do the same thing as that of linked lists. Before discussing linked lists it is important to understand the difference between linked lists and arrays. Both linked lists and arrays are used to store collections of data. Since both are used for the same purpose, we need to differentiate the usage of them. That means in which cases *arrays* are suitable and in which cases *linked lists* are suitable.

3.4 Arrays Overview

One memory block is allocated for the entire array to hold the elements of the array. The array elements can be accessed in a constant time by using the index of the particular element as the subscript.



Why Constant Time for Accessing Array Elements?

To access an array element, address of an element is computed as an offset from the base address of the array and one multiplication is needed to compute what is supposed to be added to the base address to get the memory address of the element. First the size of an element of that data type is calculated and then it is multiplied with the index of the element to get the value to be added to the base address.

This process takes one multiplication and one addition. Since these two operations take constant time, we can say the array access can be performed in constant time.

Advantages of Arrays

- Simple and easy to use
- Faster access to the elements (constant access)

Disadvantages of Arrays

- **Fixed size:** The size of the array is static (specify the array size before using it).
- **One block allocation:** To allocate the array at the beginning itself, sometimes it may not be possible to get the memory for the complete array (if the array size is big).
- **Complex position-based insertion:** To insert an element at a given position then we may need to shift the existing elements. This will create a position for us to insert the new element at the desired position. If the position at which we want to add an element is at the beginning then the shifting operation is more expensive.

Dynamic Arrays

Dynamic array (also called as *growable array*, *resizable array*, *dynamic table*, or *array list*) is a random access, variable-size list data structure that allows elements to be added or removed.

One simple way of implementing dynamic arrays is, initially start with some fixed size array. As soon as that array becomes full, create the new array of size double than the original array. Similarly, reduce the array size to half if the elements in the array are less than half.

Note: We will see the implementation for *dynamic arrays* in *Stacks*, *Queues* and *Hashing* chapters.

Advantages of Linked Lists

Linked lists have advantages and disadvantages. The advantage of linked lists is that they can be *expanded* in constant time. To create an array we must allocate memory for a certain number of elements. To add more elements to the array then we must create a new array and copy the old array into the new array. This can take lot of time.

We can prevent this by allocating lots of space initially but then you might allocate more than you need and wasting memory. With a linked list we can start with space for just one element allocated and *add* on new elements easily without the need to do any copying and reallocating.

Issues with Linked Lists (Disadvantages)

There are a number of issues in linked lists. The main disadvantage of linked lists is *access time* to individual elements. Array is random-access, which means it takes O(1) to access any element in the array. Linked lists takes $O(n)$ for access to an element in the list in the worst case. Another advantage of arrays in access time is special *locality* in memory. Arrays are defined as contiguous blocks of memory, and so any array element will be physically near its neighbors. This greatly benefits from modern CPU caching methods.

Although the dynamic allocation of storage is a great advantage, the *overhead* with storing and retrieving data can make a big difference. Sometimes linked lists are *hard* to *manipulate*. If the last item is deleted, the last but one must

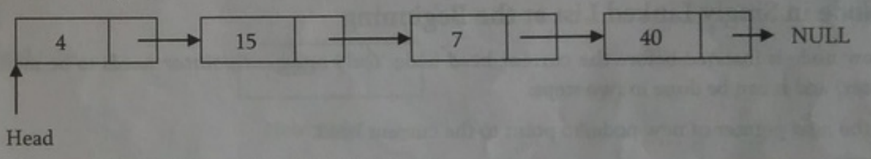
now have its pointer changed to hold a NULL reference. This requires that the list is traversed to find the last but one link, and its pointer set to a NULL reference. Finally, linked lists wastes memory in terms of extra reference points.

3.5 Comparison of Linked Lists with Arrays & Dynamic Arrays

Parameter	Linked list	Array	Dynamic array
Indexing	$O(n)$	$O(1)$	$O(1)$
Insertion/deletion at beginning	$O(1)$	$O(n)$, if array is not full (for shifting the elements)	$O(n)$
Insertion at ending	$O(n)$	$O(1)$, if array is not full	$O(1)$, if array is not full $O(n)$, if array is full
Deletion at ending	$O(n)$	$O(1)$	$O(n)$
Insertion in middle	$O(n)$	$O(n)$, if array is not full (for shifting the elements)	$O(n)$
Deletion in middle	$O(n)$	$O(n)$, if array is not full (for shifting the elements)	$O(n)$
Wasted space	$O(n)$	0	$O(n)$

3.6 Singly Linked Lists

Generally "linked list" means a singly linked list. This list consists of a number of nodes in which each node has a *next* pointer to the following element. The link of the last node in the list is NULL which indicates end of the list.



Following is a type declaration for a linked list of integers:

```
struct ListNode {
        int data;
        struct ListNode *next;
};
```

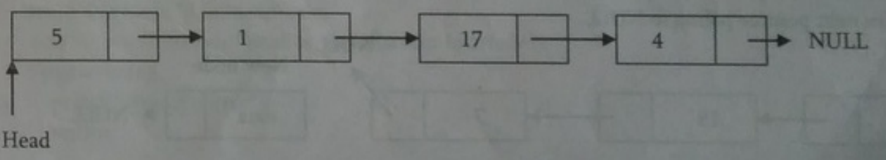
Basic Operations on a List

- Traversing the list
- Inserting an item in the list
- Deleting an item from the list

Traversing the Linked List

Let us assume that the *head* points to the first node of the list. To traverse the list we do the following.

- Follow the pointers.
- Display the contents of the nodes (or count) as they are traversed.
- Stop when the next pointer points to NULL.



The ListLength() function takes a linked list as input and counts the number of nodes in the list. Below function can be used for printing the list data with extra print function.

```
int ListLength(struct ListNode *head) {
        struct ListNode *current = head;
        int count = 0;
        while (current != NULL) {
                count++;
                current = current→next;
        }
        return count;
}
```

Time Complexity: O(n), for scanning the list of size n. Space Complexity: O(1), for creating temporary variable.

Singly Linked List Insertion

Insertion into a singly-linked list has three cases:

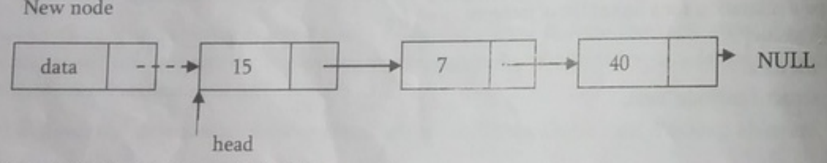
- Inserting a new node before the head (at the beginning)
- Inserting a new node after the tail (at the end of the list)
- Inserting a new node at the middle of the list (random location)

Note: To insert an element in the linked list at some position p, assume that after inserting the element the position of this new node is p.

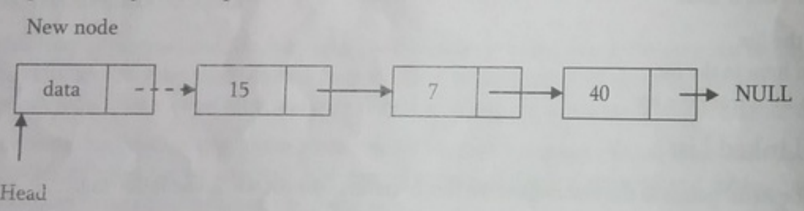
Inserting a Node in Singly Linked List at the Beginning

In this case, a new node is inserted before the current head node. *Only one next pointer* needs to be modified (new node's next pointer) and it can be done in two steps:

- Update the next pointer of new node, to point to the current head.



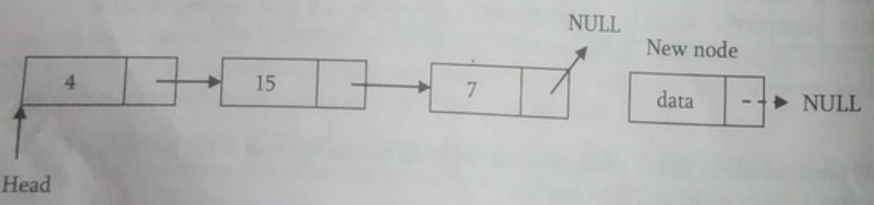
- Update head pointer to point to the new node.



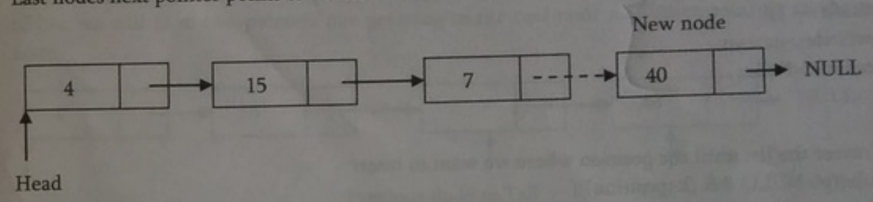
Inserting a Node in Singly Linked List at the Ending

In this case, we need to modify *two next pointers* (last nodes next pointer and new nodes next pointer).

- New nodes next pointer points to NULL.



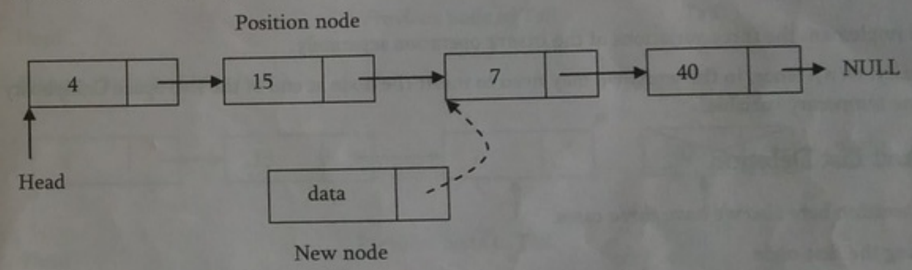
- Last nodes next pointer points to the new node.



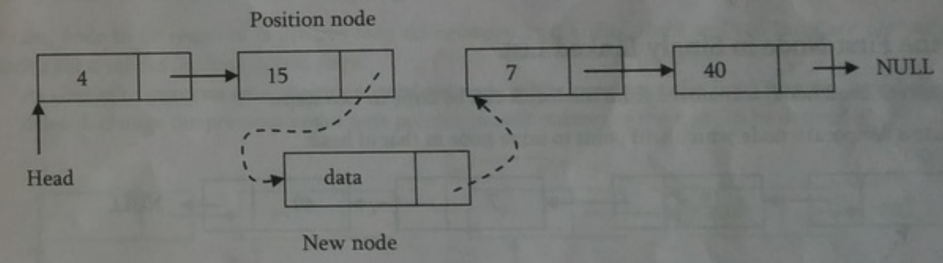
Inserting a Node in Singly Linked List at the Middle

Let us assume that we are given a position where we want to insert the new node. In this case also, we need to modify two next pointers.

- If we want to add an element at position 3 then we stop at position 2. That means we traverse 2 nodes and insert the new node. For simplicity let us assume that second node is called *position* node. New node points to the next node of the position where we want to add this node.



- Position nodes next pointer now points to the new node.



Let us write the code for all these three cases. We must update the first element pointer in the calling function, not just in the called function. For this reason we need to send double pointer. The following code inserts a node in the singly linked list.

```
void InsertInLinkedList(struct ListNode **head,int data,int position) {
  int k=1;
  struct ListNode *p,*q,*newNode;
  newNode = (ListNode *)malloc(sizeof(struct ListNode));
  if(!newNode){
    printf("MemoryError");
    return;
  }
  newNode->data=data;
  p=*head;
```