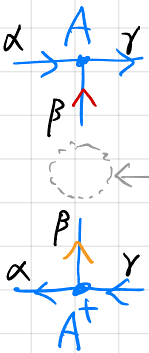
Local operators

Remind:



$|\gamma\rangle = |\beta\rangle|\alpha\rangle A^{\alpha\,\beta}_{\gamma}$

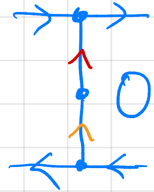
"ket tensor"

$\langle\gamma| = (A^{\alpha\,\beta}_{\gamma})^{*}\langle\alpha|\langle\beta|$

"bra tensor"

Place for operators acting on the physical Hilbert space

$\langle\psi|\hat{O}|\psi\rangle$



Particle creation, annihilation, spin, ...:

leg order convention

Spins

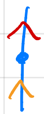
$$\hat{S}_z|S_z\rangle = S_z|S_z\rangle$$

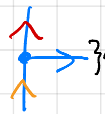
$$\hat{S}_+|S_z\rangle = \sqrt{(S-S_z)(S+S_z+1)}\;|S_z+1\rangle$$

$$\hat{S}_+ = \hat{S}_x + i\hat{S}_y = \hat{S}_-^\dagger$$

Ex) $S=1/2$:

$$\begin{array}{c|cc} & \langle\uparrow| & \langle\downarrow| \\ \hline |\uparrow\rangle & 1/2 & \\ |\downarrow\rangle & & -1/2 \end{array} \qquad \begin{array}{c|cc} & \langle\uparrow| & \langle\downarrow| \\ \hline |\uparrow\rangle & & 1 \\ |\downarrow\rangle & & \end{array}$$





encodes how the action of operator changes the state

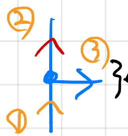
Ex) $S=1$:

$$\begin{array}{c|ccc} & \langle 1| & \langle 0| & \langle -1| \\ \hline |S_z=+1\rangle & +1 & & \\ |S_z=0\rangle & & 0 & \\ |S_z=-1\rangle & & & -1 \end{array} \qquad \begin{array}{c|ccc} & \langle 1| & \langle 0| & \langle -1| \\ \hline |S_z=+1\rangle & & \sqrt{2} & \\ |S_z=0\rangle & & & \sqrt{2} \\ |S_z=-1\rangle & & & \end{array}$$

Why rank-3?
$\Rightarrow$ Symmetry!

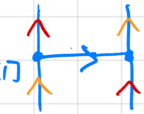
Spinor representation of $\vec{\hat{S}}$:

$$\begin{pmatrix} \hat{S}_+/\sqrt{2} \\ \hat{S}_z \\ \hat{S}_-/\sqrt{2} \end{pmatrix}$$



+, z, −
dimension = 3

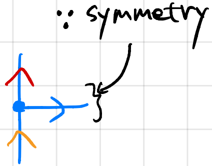
$$\vec{\hat{S}}_1 \cdot \vec{\hat{S}}_2 \doteq S_{[1]} \quad S^\dagger_{[2]}$$



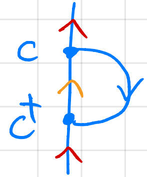
$$\begin{pmatrix} \frac{\hat{S}_{2-}}{\sqrt{2}} & \hat{S}_{2z} & \frac{\hat{S}_{2+}}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \hat{S}_{1+}/\sqrt{2} \\ \hat{S}_{1z} \\ \hat{S}_{1-}/\sqrt{2} \end{pmatrix} = \hat{S}_{1x}\hat{S}_{2x} + \hat{S}_{1y}\hat{S}_{2y} + \hat{S}_{1z}\hat{S}_{2z}$$

Spinless fermions

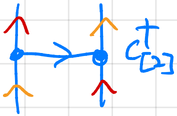
$$\hat{c} \doteq \begin{array}{c|cc} & \langle 0| & \langle 1| \\ \hline |0\rangle & & 1 \\ |1\rangle & & \end{array}$$



$$\hat{n} = \hat{c}^\dagger \hat{c} \doteq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{array}{c|cc} & \langle 0| & \langle 1| \\ \hline |0\rangle & 0 & 0 \\ |1\rangle & 0 & 1 \end{array}$$



$$\hat{c}_2^\dagger \hat{c}_1 \overset{?}{\doteq} c_{[1]} \quad c^\dagger_{[2]}$$



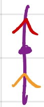
No! $[\hat{S}_{1\alpha}, \hat{S}_{2\beta}]_- = 0$, but $[c_1, c_2^\dagger]_+ = 0$

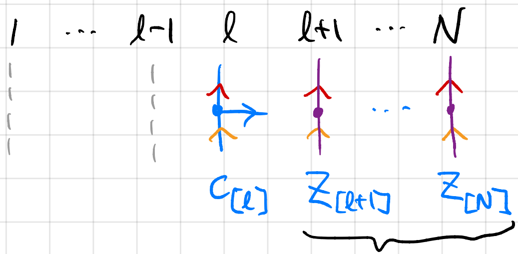
$$\hat{c}_\ell |n_N\rangle |n_{N-1}\rangle \cdots |n_{\ell+1}\rangle |n_\ell\rangle |n_{\ell-1}\rangle \cdots |n_1\rangle$$

$$= (-1)^{n_N} \cdots (-1)^{n_{\ell+1}} |n_N\rangle |n_{N-1}\rangle \cdots |n_{\ell+1}\rangle \left(\hat{c}_\ell |n_\ell\rangle\right) |n_{\ell-1}\rangle \cdots |n_1\rangle$$

Fermionic sign operator:

$$Z = (-1)^n = \begin{array}{c|cc} & \langle 0| & \langle 1| \\ \hline |0\rangle & 1 & \\ |1\rangle & & -1 \end{array}$$

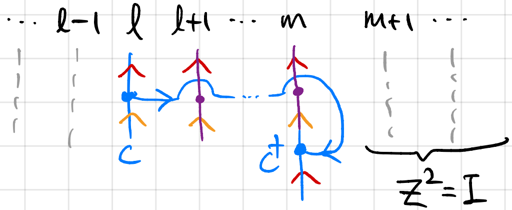




"Z string" or "Jordan-Wigner string"

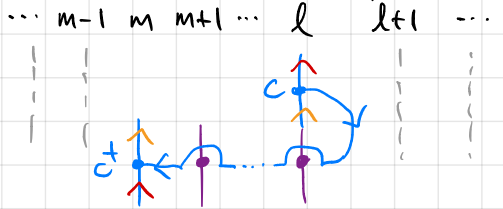
$$c_m^\dagger c_\ell \doteq \begin{cases} m > \ell: \\ \ell > m: \end{cases}$$

$m > \ell$:



$Z^2 = I$

$\ell > m$:



Spinful fermions

$$\{ |0\rangle,\ |\uparrow\rangle = c_\uparrow^\dagger |0\rangle,\ |\downarrow\rangle = c_\downarrow^\dagger |0\rangle,\ |2\rangle = c_\downarrow^\dagger c_\uparrow^\dagger |0\rangle \}$$

$$c_\uparrow \doteq \begin{array}{c} \\ |0\rangle \\ |\uparrow\rangle \\ |\downarrow\rangle \\ |2\rangle \end{array} \begin{array}{c} \begin{array}{cccc} \langle 0| & \langle \uparrow| & \langle \downarrow| & \langle 2| \end{array} \\ \begin{pmatrix} & 1 & & \\ & & & \\ & & & -1 \\ & & & \end{pmatrix} \end{array} \qquad c_\downarrow \doteq \begin{array}{c} \\ |0\rangle \\ |\uparrow\rangle \\ |\downarrow\rangle \\ |2\rangle \end{array} \begin{array}{c} \begin{array}{cccc} \langle 0| & \langle \uparrow| & \langle \downarrow| & \langle 2| \end{array} \\ \begin{pmatrix} & & 1 & \\ & & & 1 \\ & & & \\ & & & \end{pmatrix} \end{array}$$

$$n_\uparrow = c_\uparrow^\dagger c_\uparrow = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \qquad n_\downarrow = c_\downarrow^\dagger c_\downarrow = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad Z = (-1)^{n_\uparrow + n_\downarrow} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$