Symmetries: Abelian and non-Abelian

App. A of [Weichselbaum2011:AP]

$$\underbrace{\hat{G}\mathcal{H}\hat{G}^{-1} = \mathcal{H} \qquad \hat{G} = \exp\Big(i\sum_\alpha a_\alpha \underbrace{\hat{S}_\alpha}_{\text{generators}}\Big)}_{\text{infinitesimal } a_\alpha\text{'s}}$$

$$[\hat{S}_\alpha, \hat{S}_\beta]_- = \underbrace{f_{\alpha\beta\gamma}}_{\text{defines Lie algebra}} \hat{S}_\gamma$$ $\Rightarrow$ matrix representation of $\{\hat{S}_\alpha\}$

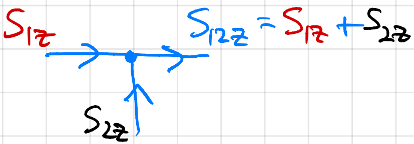
$[\hat{S}_\alpha, \hat{\mathcal{H}}] = 0$: Energy eigenstates are indexed by quantum numbers defined by $\{\hat{S}_\alpha\}$

Abelian symmetry example: U(1) spin symmetry

Just a single generator $\hat{S}_z$ (spin z) $\Rightarrow$ Smallest representation $\hat{G} = e^{i\theta}$: 1×1 unitary $\Rightarrow$ U(1)

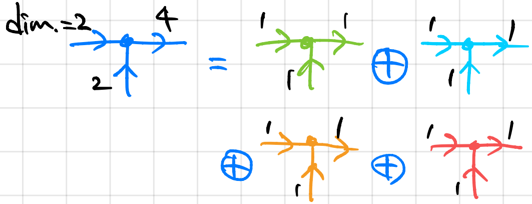
$\Rightarrow$ Quantum number: S_z

Identity merging two spin-1/2 spaces:



$$e^{iaS_{1z}} e^{iaS_{2z}} A e^{-iaS_{12z}} = A$$

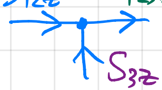
$$\begin{array}{cc|cccc}
S_{1z} & S_{2z} \backslash S_{12z} & 1 & 0 & 0 & -1 \\
\hline
1/2 & 1/2 & 1 & & & \\
-1/2 & 1/2 & & 1 & & \\
1/2 & -1/2 & & & 1 & \\
-1/2 & -1/2 & & & & 1
\end{array}$$



Block diagonal

$\Rightarrow$ Efficient representations

One more spin-1/2:

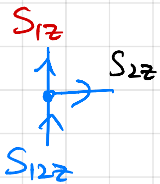


$S_{123z} = S_{12z} + S_{3z}$

S_{12z}	S_{3z}	S_{123z}: $3/2$	$1/2$	$1/2$	$-1/2$	$1/2$	$-1/2$	$-1/2$	$-3/2$
1	$1/2$	$\boxed{1}$							
0	$1/2$		1						
0	$1/2$			1					
-1	$1/2$				$\boxed{1}$				
1	$-1/2$					$\boxed{1}$			
0	$-1/2$						1		
0	$-1/2$							1	
-1	$-1/2$								$\boxed{1}$

Spin-1/2 operator:

$$\hat{\vec{S}} = \begin{pmatrix} \hat{S}_+/\sqrt{2} \\ \hat{S}_z \\ \hat{S}_-/\sqrt{2} \end{pmatrix} \qquad \begin{array}{c} S_{2z} = \\ +1 \\ 0 \\ -1 \end{array}$$



$$\hat{S}_+/\sqrt{2} \doteq \begin{array}{c|cc} & 1/2 & -1/2 \\ \hline 1/2 & & \boxed{1/\sqrt{2}} \\ -1/2 & & \end{array} \quad \to S_{2z} = +1$$

$$\hat{S}_z \doteq \begin{array}{c|cc} & 1/2 & -1/2 \\ \hline 1/2 & \boxed{1/2} & \\ -1/2 & & \boxed{-1/2} \end{array} \quad \to S_{2z} = 0$$

$$\hat{S}_-/\sqrt{2} \doteq \begin{array}{c|cc} & 1/2 & -1/2 \\ \hline 1/2 & & \\ -1/2 & \boxed{1/\sqrt{2}} & \end{array} \quad \to S_{2z} = -1$$

Non-Abelian symmetry example: SU(2) spin symmetry

Three generators: $\hat{S}_+, \hat{S}_z, \hat{S}_-$ $\Rightarrow$ Smallest matrix rep. of $\hat{G}$: SU(2) (2×2 matrix with $|\det(\cdots)| = 1$)

$$[\hat{S}_z, \hat{S}_\pm] = \pm \hat{S}_\pm$$

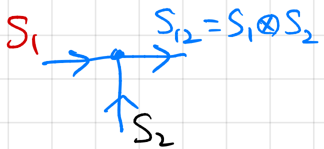
$$[\hat{S}_+, \hat{S}_-] = 2\hat{S}_z$$

$\Rightarrow$ Quantum numbers:

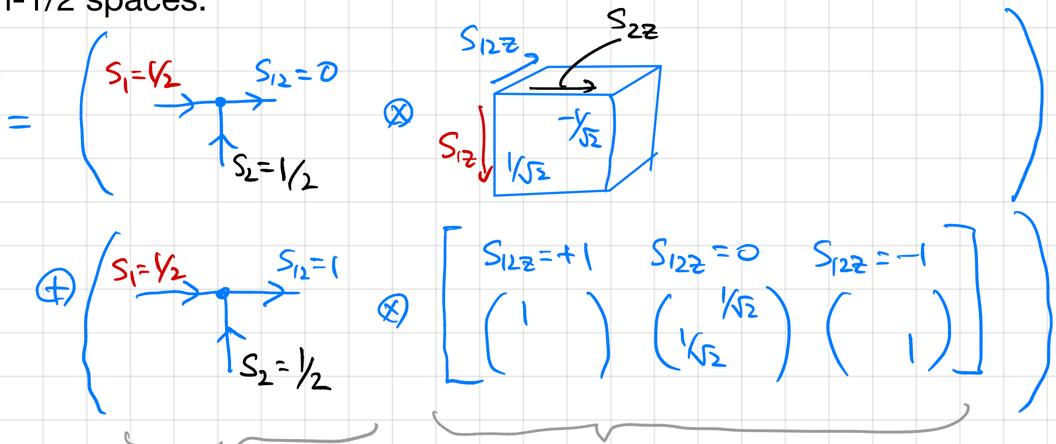
$$\hat{\vec{S}}^2 |S;S_z\rangle = S(S+1)|S;S_z\rangle \quad \text{(total spin)}$$

$$\hat{S}_z |S;S_z\rangle = S_z |S;S_z\rangle \quad \text{(spin z-component)}$$

Identity merging two spin-1/2 spaces:



$$\tfrac{1}{2} \otimes \tfrac{1}{2} = 0 \oplus 1$$

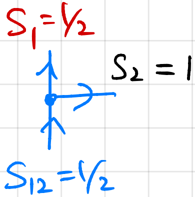


Wigner-Eckart theorem $\Rightarrow$ reduced tensors in multiplet basis $\{|s\rangle\}$ — Clebsch-Gordan coefficient tensors

$\Rightarrow$ all 1×1×1 in this example

Spin-1/2 operator:

$$\begin{array}{cc} & S_{zz}=1 \\ \hat{\vec{S}} = \begin{pmatrix} -\hat{S}_+/\sqrt{2} \\ \hat{S}_z \\ \hat{S}_-/\sqrt{2} \end{pmatrix} & \begin{array}{c} +1 \\ 0 \\ -1 \end{array} \end{array}$$



Tensor network libraries exploiting symmetries

Our bare MATLAB code: no symmetry

ITensor by E. M. Stoudenmire (Flatiron, USA) et al.: only Abelian, open source

QSpace by A. Weichselbaum (LMU Munich, Germany & Brookhaven, USA):
general non-Abelian, plan to be open-source later this year

SyTen by C. Hubig (LMU Munich & MPQ, Germany):
general non-Abelian, closed-source

only two that can use general non-Abelian symmetries