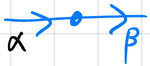
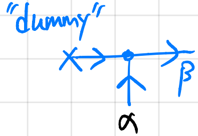
Unitaries and isometries

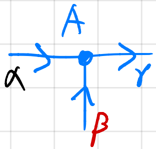
Identity as a rank-2 tensor:



$$|\beta\rangle = |\alpha\rangle = |\alpha\rangle \underbrace{A^{\alpha}{}_{\beta}}_{=\delta_{\alpha\beta}}$$



Identity as a rank-3 tensor:

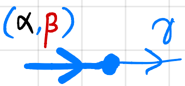


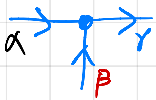
$$|\gamma\rangle = |\beta\rangle|\alpha\rangle = |\beta\rangle|\alpha\rangle \underbrace{A^{\alpha\beta}{}_{\gamma}}_{=\delta_{(\alpha,\beta),\gamma}}$$

$$\{|\gamma\rangle\} = \{|\beta\rangle\} \otimes \{|\alpha\rangle\}$$

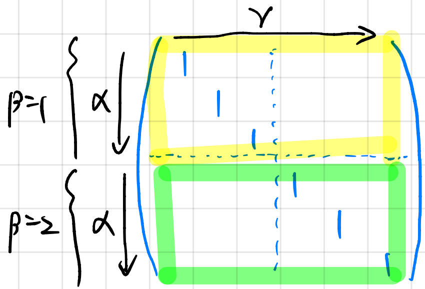
$$d_\gamma = d_\alpha d_\beta$$

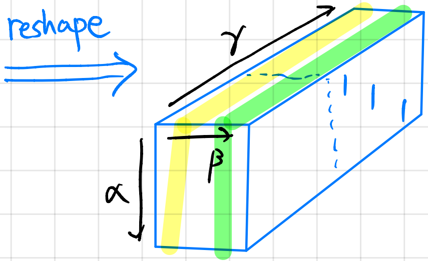
Ex) $d_\alpha = 3$, $d_\beta = 2$, $d_\gamma = 6$

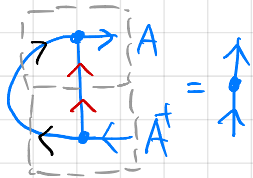




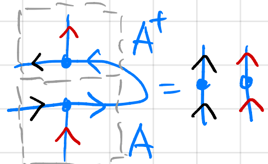
$I_{6\times 6} =$



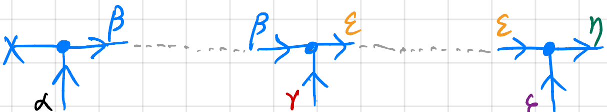




$$A^\dagger A = I_{d_\gamma \times d_\gamma}$$

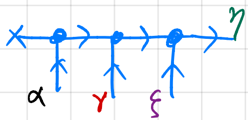


$$AA^\dagger = I_{d_\alpha \times d_\alpha} \otimes I_{d_\beta \times d_\beta}$$



$$A_{[1]}{}^{\alpha}{}_{\beta} = \delta_{\alpha\beta} \qquad A_{[2]}{}^{\beta\gamma}{}_{\varepsilon} = \delta_{(\beta,\gamma),\varepsilon} \qquad A_{[3]}{}^{\varepsilon\xi}{}_{\eta} = \delta_{(\varepsilon,\xi),\eta}$$

$=$

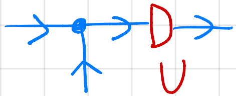


$$\delta_{(\alpha,\gamma,\xi),\eta}$$

$\leftarrow d_\alpha \times d_\gamma \times d_\xi \times d_\alpha d_\gamma d_\xi$

Exponential wall!

What if truncation happens?

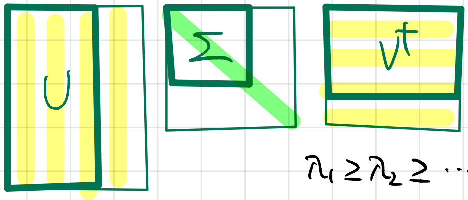


left unitary: $U^\dagger U = I$ $\quad U^\dagger U \neq I$

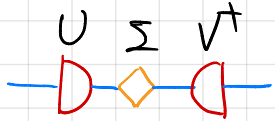
cf. Thin (truncated) SVD:

$$M = U \Sigma V^\dagger$$

$U^\dagger U = I \qquad U^\dagger U \neq I$



$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$

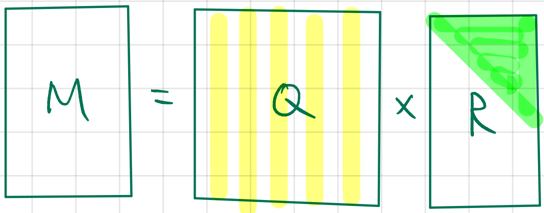


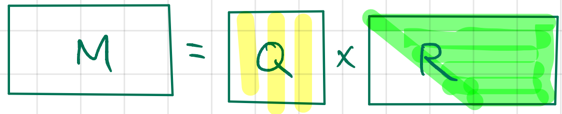
Full QR decomp.:

$$M = QR$$

$QQ^\dagger = Q^\dagger Q = I$

$R_{ij} = 0$ if $i > j$

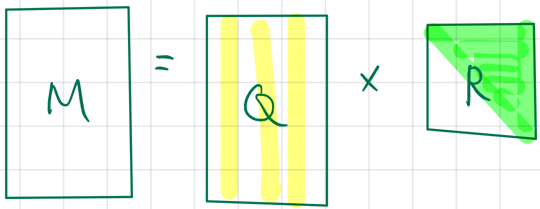




Gram-Schmidt process

Thin QR decomp.:

$Q^\dagger Q = I \qquad QQ^\dagger \neq I$

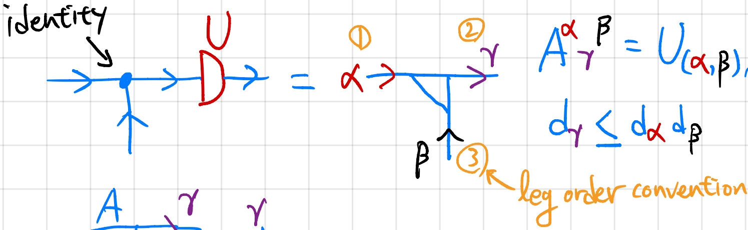


There is no “truncated” QR decomposition!

Left-normalized / left unitary / left isometry

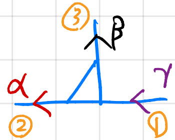
norm-preserving: $\|Uv\| = \|v\|$

$v^\dagger \underbrace{U^\dagger U}_{=I} v = v^\dagger v$

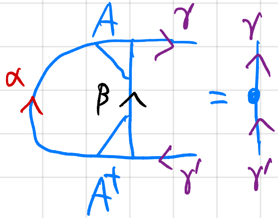


$A^{\alpha\beta}_{\gamma} = U_{(\alpha,\beta),\gamma}$

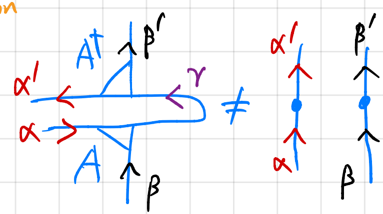
$d_\gamma \le d_\alpha d_\beta$



$A^{\dagger\gamma}_{\alpha\beta} = U^\dagger_{\gamma,(\alpha,\beta)} = U^*_{(\alpha,\beta),\gamma}$



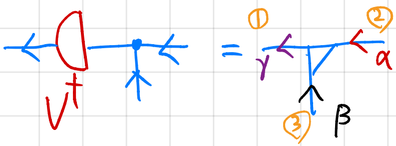
$A^\dagger A = U^\dagger U = I_{d_\gamma \times d_\gamma}$



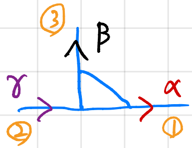
$AA^\dagger \neq I_{d_\alpha d_\beta \times d_\alpha d_\beta}$

if $d_\gamma \lneq d_\alpha d_\beta$

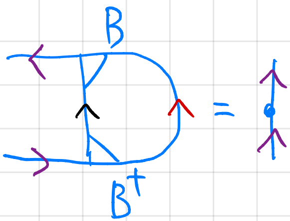
Right-normalized / right unitary / right isometry



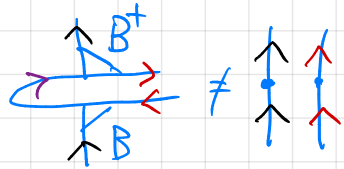
$B_\gamma^{\alpha\beta} = V^\dagger_{\gamma,(\alpha,\beta)}$



$B^{\dagger\alpha}_{\gamma\beta} = V_{(\alpha,\beta),\gamma}$



$BB^\dagger = V^\dagger V = I_{d_\gamma \times d_\gamma}$



$B^\dagger B \neq I_{d_\alpha d_\beta \times d_\alpha d_\beta}$

if $d_\gamma \lneq d_\alpha d_\beta$