Singular value decomposition (SVD)

$$M = U \Sigma V^\dagger$$

size: $m \times n$

$$U = [\vec{u}_1 \ \vec{u}_2 \ \cdots]$$

$$\vec{u}_i^\dagger \vec{u}_j = \delta_{ij}$$

Left singular vectors (complex)

$$\Sigma = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \end{pmatrix}$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$$

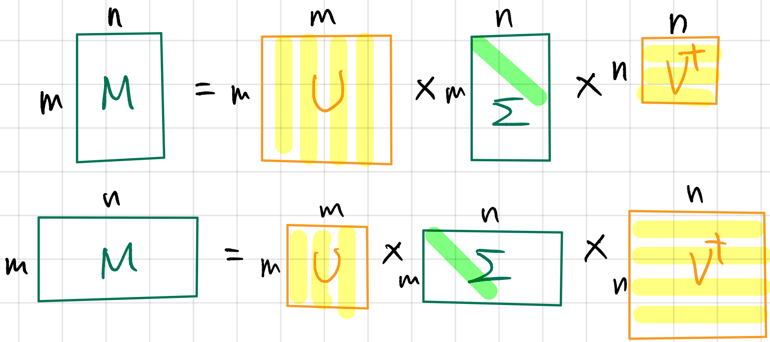
Singular values (real)

$$V = [\vec{v}_1 \ \vec{v}_2 \ \cdots]$$

$$\vec{v}_i^\dagger \vec{v}_j = \delta_{ij}$$

Right singular vectors (complex)

Full SVD:



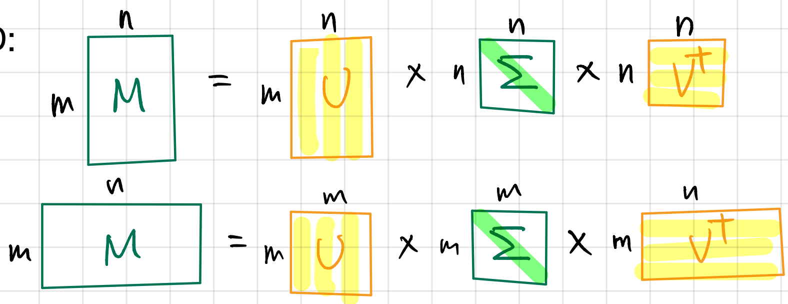
$UU^\dagger = U^\dagger U = I_{m\times m}$

$VV^\dagger = V^\dagger V = I_{n\times n}$

unitary

$\Sigma_{ij} = \lambda_i \delta_{ij}$

Thin SVD:



$U^\dagger U = I_{n\times n}$ $\quad UU^\dagger \neq I_{m\times m}$

U is left unitary

$V^\dagger V = I_{m\times m}$ $\quad VV^\dagger \neq I_{n\times n}$

V is left unitary; $V^\dagger$ is right unitary

Relation with the eigendecomposition: $M = U \Sigma V^\dagger$

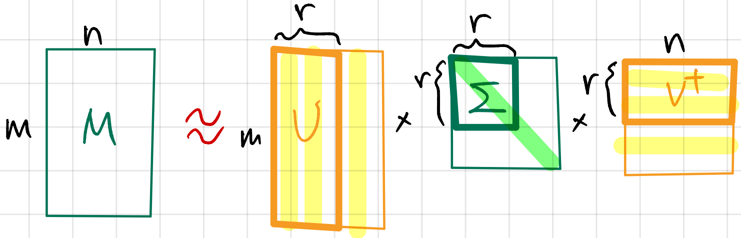
$$MM^\dagger = U \Sigma^2 U^\dagger \qquad \Sigma^2 = \begin{pmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \ddots \end{pmatrix} \qquad M^\dagger M = V \Sigma^2 V^\dagger$$

$U = [\vec{u}_1\ \vec{u}_2 \cdots]$: eigenvectors

eigenvalues

$V = [\vec{v}_1\ \vec{v}_2 \cdots]$: eigenvectors

Truncated SVD:



$$M \approx \tilde{M} = \sum_{i=1}^{r} \vec{u}_i \lambda_i \vec{v}_i^\dagger$$

Eckart-Young theorem: $\tilde{M}$ is the best low-rank approximation to M, given rank r, in terms of minimizing $\| M - \tilde{M} \|_2$ and $\| M - \tilde{M} \|_F$

$\| M - \tilde{M} \|_2$: spectral norm = largest singular value

$\| M - \tilde{M} \|_F$: Frobenius norm $\|A\|_F = \sqrt{\sum_{ij} |A_{ij}|^2} = \sqrt{\mathrm{Tr}\, A^\dagger A}$

Computational cost: $O(mn^2)$ (if $m \geq n$) of floating-point operations

(cf. $O(n^3)$ for the eigendecomposition)