Lehmann representation for a non-interacting system

Consider a generic non-interacting system of fermions, whose Hamiltonian is quadratic, i.e.,

$$\hat{H} = \sum_{ij} [\boldsymbol{h}]_{ij} \hat{c}_i^\dagger \hat{c}_j, \tag{1}$$

where $\hat{c}_i^\dagger$ creates a fermionic particle in the i-th spin-orbital, and the spin-orbitals form the orthonormal basis. The single-particle Hamiltonian $\boldsymbol{h}$, which is Hermitian, can be diagonalized as

$$\boldsymbol{h} = \boldsymbol{V} \boldsymbol{\epsilon} \boldsymbol{V}^\dagger, \tag{2}$$

where $[\boldsymbol{\epsilon}]_{ij} = \epsilon_i \delta_{ij}$ is the diagonal matrix containing the single-particle energy eigenvalues ϵ_i, and $\boldsymbol{V} = (\vec{v}_1 \vec{v}_2 \cdots)$ is the unitary matrix whose column vectors are the eigenvectors of $\boldsymbol{h}$.

Since the (many-body) Hamiltonian $\hat{H}$ is quadratic, the retarded Green's function,

$$G[\hat{c}_i, \hat{c}_j^\dagger](t) = -i\theta(t) \, \mathrm{Tr}\Big(\hat{\rho} \, [\hat{c}_i(t), \hat{c}_j^\dagger]_\pm \Big) , \tag{3}$$

and its spectral function,

$$A[\hat{c}_i, \hat{c}_j^\dagger](\omega) = \frac{-1}{\pi} \mathrm{Im} \int_{-\infty}^{+\infty} \mathrm{d}t \, e^{\mathrm{i}\omega t} \, G[\hat{c}_i, \hat{c}_j^\dagger](t), \tag{4}$$

can be computed exactly.

(a) Evaluate the spectral function $A[\hat{c}_i, \hat{c}_j^\dagger](\omega)$ by using the Lehmann representation.

(b) The local spectral function is the case of $i = j$ in which the two defining operators $\hat{c}_i$ and $\hat{c}_i^\dagger$ act on the same spin-orbital. Explain why the local spectral function can be interpreted as the local density of states.