Tensor network diagrams

NB: Rank of matrix = # of non-zero singular values

Tensor = multi-dimensional array

Rank (order, degree) of a tensor = # of dimensions = # of indices = # of legs

Dimension of a leg = size of the tensor along the corresponding dimension

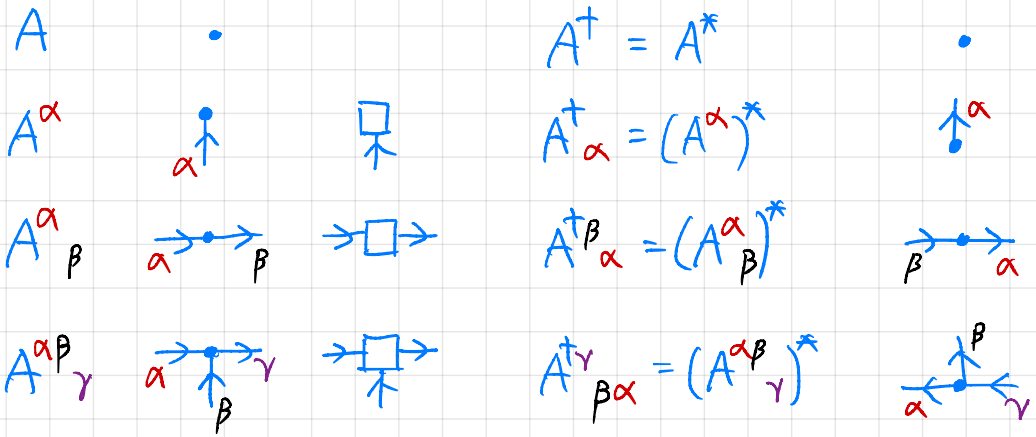
Hermitian conjugates:

Rank 0: scalar A $\quad A^\dagger = A^*$

Rank 1: vector A^α $\quad A^\dagger{}_\alpha = (A^\alpha)^*$

Rank 2: matrix $A^\alpha{}_\beta$ $\quad A^{\dagger\beta}{}_\alpha = (A^\alpha{}_\beta)^*$

Rank-3 tensor $A^{\alpha\beta}{}_\gamma$ $\quad A^{\dagger\gamma}{}_{\beta\alpha} = (A^{\alpha\beta}{}_\gamma)^*$



Covariant notation: transformation within each "leg space" differs with super- vs subscript, visualized by leg directions

$\Rightarrow$ In TN: $|\alpha\rangle = |\beta\rangle U^\beta{}_\alpha$ (ket) vs. $\langle\alpha| = U^{\dagger\alpha}{}_\beta \langle\beta|$ (bra)

$U^\beta{}_\alpha = \langle\beta|\alpha\rangle$ $\quad U^{\dagger\alpha}{}_\beta = \langle\alpha|\beta\rangle$

(sum over repeated indices)

Example: two spins

Consider spin 1/2 (or qubit): $\{|\uparrow\rangle, |\downarrow\rangle\}$ $(\{|0\rangle, |1\rangle\})$

index = 1 2 (1 2)

Tensor network representation of two spins:

S_{1z} S_{2z} (put bases backwards)

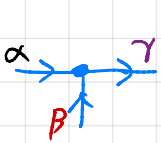
index =

$|S=1, S_z=1\rangle = |\uparrow\rangle|\uparrow\rangle$ ①

$|S=1, S_z=0\rangle = \frac{1}{\sqrt{2}}(|\downarrow\rangle|\uparrow\rangle + |\uparrow\rangle|\downarrow\rangle)$ ②

$|S=1, S_z=-1\rangle = |\downarrow\rangle|\downarrow\rangle$ ③

$|S=0, S_z=0\rangle = \frac{1}{\sqrt{2}}(|\downarrow\rangle|\uparrow\rangle - |\uparrow\rangle|\downarrow\rangle)$ ④



$|\gamma\rangle = A^{\alpha\beta}{}_{\gamma}|\beta\rangle|\alpha\rangle$

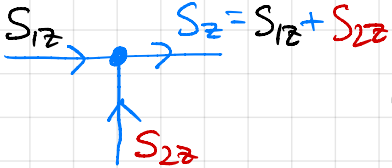
$A^{\alpha\beta}{}_{\gamma} = \langle\alpha|\langle\beta|\gamma\rangle$

$A^{11}{}_{1} = A^{22}{}_{3} = 1$

$A^{12}{}_{2} = A^{21}{}_{2} = A^{12}{}_{4} = -A^{21}{}_{4} = 1/\sqrt{2}$

$A^{\alpha\beta}{}_{\gamma} = 0$ otherwise

If we exploit the conservation of S_z :



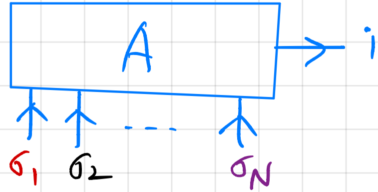
"Kirchhoff's law"

Example: spineless fermions on N sites

$$|E_i\rangle = |n_1, n_2, \cdots, n_N\rangle A^{n_1 n_2 \cdots n_N}_{;i}$$

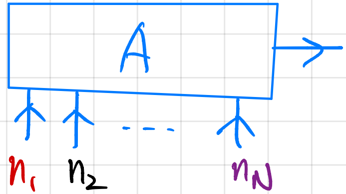
$$|n_1, n_2, \cdots, n_N\rangle = |n_N\rangle \cdots |n_2\rangle |n_1\rangle = (c_N^{\dagger})^{n_N} \cdots (c_2^{\dagger})^{n_2} (c_1^{\dagger})^{n_1} |0\rangle$$

(put bases backwards)



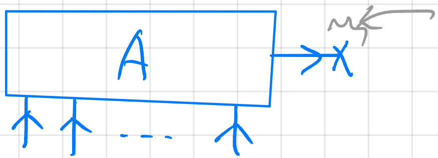
$(\sigma_\ell = n_\ell + 1 \ : \text{index})$

If we exploit the particle number conservation:



$$n_{tot} = n_1 + n_2 + \cdots + n_N$$

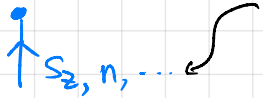
Projection onto a specific state:



“dummy” leg with singleton dimension

Rank-1 tensors do not appear in practice

Physical reason: conservation laws



can describe trivial cases only

Computational reason: vectors are “thin” matrices

```
>> A = ones(1,3);
>> whos A
  Name        Size              Bytes  Class     Attributes

  A           1x3                  24  double
```

```
>> A = rand(10,1);
>> whos A
  Name        Size              Bytes  Class     Attributes

  A           10x1                 80  double
```

Why some of literature don’t specify arrows?

If symmetries (conservation laws) are not used (e.g., our coding materials based on the bare MATLAB), arrow directions are mere bookkeeping.
If a tensor network library exploits symmetries (e.g., QSpace), arrow directions should be incorporated into the data structure.
In any case, it’s a good practice to specify arrows for identifying underlying quantum mechanical structure!