AKLT state

The AKLT state is a translationally invariant matrix product state in which the same rank-3 tensor B is repeated. Here we consider a chain of length L with periodic boundary conditions. In this case, the AKLT state $|\psi\rangle$ is written as

$$\begin{gathered} |\psi\rangle = \sum_{\sigma_1,\cdots,\sigma_L} |\sigma_L \sigma_{L-1} \cdots \sigma_2 \sigma_1\rangle \mathrm{Tr}[B^{\sigma_L} B^{\sigma_{L-1}} \cdots B^{\sigma_2} B^{\sigma_1}], \\ B^1 = \sqrt{\frac{2}{3}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B^2 = \sqrt{\frac{1}{3}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B^3 = \sqrt{\frac{2}{3}} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \end{gathered} \tag{1}$$

where $\sigma = 1, 2, 3$ are the indices for the $S_z = +1, 0, -1$ states at each chain site, respectively.

(a) Verify that the tensor B is both left- and right-normalized.

(b) Compute the transfer operator $T^{(\alpha,\alpha')}{}_{(\beta,\beta')} = \sum_\sigma B^{\dagger\beta'}{}_{\alpha'\sigma} B^{\alpha}{}_{\beta}{}^{\sigma}$ without local operators. Verify that the eigenvalues of T are $(1, -1/3, -1/3, -1/3)$. Note that the arrows for the left and right legs of $B^\dagger$, indexed by α' and β', respectively, are implicitly flipped.

(c) A transfer operator involving a local operator $\hat{O}$ acting on the physical legs of B and $B^\dagger$ is defined as

$$[T_{\hat{O}}]^{(\alpha,\alpha')}{}_{(\beta,\beta')} = \sum_{\sigma,\sigma'} B^{\dagger\beta'}{}_{\alpha'\sigma'} [\hat{O}]^{\sigma}{}_{\sigma'} B^{\alpha}{}_{\beta}{}^{\sigma}. \tag{2}$$

Obtain the transfer operators for $\hat{O} = \hat{S}_z$ and for $\hat{O} = \exp(\mathrm{i}\pi\hat{S}_z)$.

(d) Derive the asymptotic (i.e., $\lim_{|m-n|\to\infty} \lim_{L\to\infty}$) behaviors of

$$\begin{gathered} \chi_{zz}(m-n) = \langle\psi|\hat{S}_{z[m]}\hat{S}_{z[n]}|\psi\rangle, \\ \chi_{\mathrm{string}}(m-n) = \langle\psi|\hat{S}_{z[m]} e^{\mathrm{i}\pi\hat{S}_{z[m-1]}} e^{\mathrm{i}\pi\hat{S}_{z[m-2]}} \cdots e^{\mathrm{i}\pi\hat{S}_{z[n+2]}} e^{\mathrm{i}\pi\hat{S}_{z[n+1]}} \hat{S}_{z[n]}|\psi\rangle. \end{gathered} \tag{3}$$

Check whether you get $\chi_{zz} \sim e^{-|m-n|/\xi}$ with $\xi = 1/\log 3$ and $\chi_{\mathrm{string}} = -4/9$.