

# Singular value decomposition (SVD)

$$M = U \Sigma V^T$$

$\uparrow$   
 size:  $m \times n$

$U = [\vec{u}_1 \vec{u}_2 \dots]$   
 $\vec{u}_i^T \vec{u}_j = \delta_{ij}$   
 Left singular vectors (complex)

$\Sigma = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_n & \\ 0 & & \ddots \end{pmatrix}$   
 $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$   
 Singular values (real)

$V = [\vec{v}_1 \vec{v}_2 \dots]$   
 $\vec{v}_i^T \vec{v}_j = \delta_{ij}$   
 Right singular vectors (complex)

Full SVD:

$$\begin{array}{c}
 \begin{array}{c} n \\ \boxed{M} \\ m \end{array} = \begin{array}{c} m \\ \boxed{U} \end{array} \times \begin{array}{c} n \\ \boxed{\Sigma} \end{array} \times \begin{array}{c} n \\ \boxed{V^T} \end{array} \\
 \\
 \begin{array}{c} n \\ \boxed{M} \\ m \end{array} = \begin{array}{c} m \\ \boxed{U} \end{array} \times \begin{array}{c} n \\ \boxed{\Sigma} \end{array} \times \begin{array}{c} n \\ \boxed{V^T} \end{array}
 \end{array}$$

$$\begin{array}{l}
 UU^T = U^T U = I_{m \times m} \\
 VV^T = V^T V = I_{n \times n}
 \end{array}
 \leftarrow \text{unitary}$$

$$\Sigma_{ij} = \lambda_i \delta_{ij}$$

Thin SVD:

$$\begin{array}{c}
 \begin{array}{c} n \\ \boxed{M} \\ m \end{array} = \begin{array}{c} n \\ \boxed{U} \\ m \end{array} \times \begin{array}{c} n \\ \boxed{\Sigma} \\ n \end{array} \times \begin{array}{c} n \\ \boxed{V^T} \\ n \end{array} \\
 \\
 \begin{array}{c} n \\ \boxed{M} \\ m \end{array} = \begin{array}{c} m \\ \boxed{U} \\ m \end{array} \times \begin{array}{c} m \\ \boxed{\Sigma} \\ m \end{array} \times \begin{array}{c} n \\ \boxed{V^T} \\ m \end{array}
 \end{array}$$

$$\begin{array}{l}
 U^T U = I_{n \times n} \quad UU^T \neq I_{m \times m} \\
 U \text{ is left unitary}
 \end{array}$$

$$\begin{array}{l}
 V^T V = I_{m \times m} \quad VV^T \neq I_{n \times n} \\
 V \text{ is left unitary; } V^T \text{ is right unitary}
 \end{array}$$

Relation with the eigendecomposition:  $M = U \Sigma V^T$

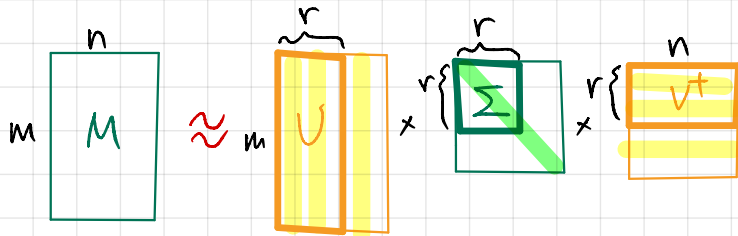
$$MM^T = U \Sigma^2 U^T$$

$\uparrow$   $\nwarrow$   
 $[\vec{u}_1 \vec{u}_2 \dots]$ : eigenvectors  
 $\begin{pmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \ddots \end{pmatrix}$   
 eigenvalues

$$M^T M = V \Sigma^2 V^T$$

$\uparrow$   
 $[\vec{v}_1 \vec{v}_2 \dots]$ : eigenvectors

Truncated SVD:



$$M \approx \tilde{M} = \sum_{i=1}^r \vec{u}_i \lambda_i \vec{v}_i^T$$

Eckart-Young theorem:  $\tilde{M}$  is the best low-rank approximation to  $M$ , given rank  $r$ ,

in terms of minimizing  $\|M - \tilde{M}\|_2$  and  $\|M - \tilde{M}\|_F$

$\uparrow$   
 spectral norm  
 = largest singular value

$\uparrow$   
 Frobenius norm

$$\|A\|_F = \sqrt{\sum_{i,j} |A_{ij}|^2} = \sqrt{\text{Tr} A^T A}$$

Computational cost:  $O(mn^2)$  (if  $m \geq n$ ) of floating-point operations

(cf.  $O(n^3)$  for the eigendecomposition)