COMP 7180 Quantitative Methods for Data Analytics and Artificial Intelligence
Exercise 3 Answer

1. Calculate the eigenvector and eigenvalue of following matrices:

   (a) $\begin{bmatrix} 5 & -1 \\ 0 & 3 \end{bmatrix}$

   Answer:

   eigenvector $\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$, eigenvalue $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$.

   (b) $\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$

   Answer:

   eigenvector $\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$, eigenvalue $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

   (c) $\begin{bmatrix} 5 & -1 \\ 2 & 8 \end{bmatrix}$

   Answer:

   $\begin{bmatrix} 5 & -1 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} + 5\mathbf{I}$.

   eigenvector $\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$, eigenvalue $\begin{bmatrix} 7 & 0 \\ 0 & 6 \end{bmatrix}$.

2. Calculate the sum of eigenvalues of following matrix : $\begin{bmatrix} 4 & 3 & 3 \\ 6 & 10 & 2 \\ 7 & 7 & 1 \end{bmatrix}$

   Answer:

   The sum of eigenvalues is the trace of martix $tr(\begin{bmatrix} 4 & 3 & 3 \\ 6 & 10 & 2 \\ 7 & 7 & 1 \end{bmatrix}) = 4 + 10 + 1 = 15$

3. Prove "λ^{-1} is the eigenvalue of $\mathbf{A}^{-1}$", with equation $\mathbf{Ax} = \lambda\mathbf{x}$.

   Answer:

Multiply by $\mathbf{A}^{-1}$:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}^{-1}\lambda\mathbf{x} \tag{1}$$

Then, we divide the λ on both side of equation:

$$\frac{1}{\lambda}\mathbf{x} = \mathbf{A}^{-1}\mathbf{x} \tag{2}$$

4. Given a matrix $\mathbf{A} = \begin{bmatrix} -3 & 1 \\ -5 & 3 \end{bmatrix}$, calculate the $\mathbf{A}^2, \mathbf{A}^5$ and $\mathbf{A}^{20}$, respectively.

Answer:

Perform eigen-decomposition to A: eigenvector $\mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}$, eigenvalue $\mathbf{\Lambda} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$.

Then, $\mathbf{A}^k = \mathbf{X}\mathbf{\Lambda}^k\mathbf{X}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} (-2)^k & 0 \\ 0 & 2^k \end{bmatrix} \begin{bmatrix} \frac{5}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$

Therefore,

$$\mathbf{A}^2 = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} (-2)^2 & 0 \\ 0 & 2^2 \end{bmatrix} \begin{bmatrix} \frac{5}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \tag{3}$$

$$\mathbf{A}^5 = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} (-2)^5 & 0 \\ 0 & 2^5 \end{bmatrix} \begin{bmatrix} \frac{5}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} -48 & 16 \\ -80 & 48 \end{bmatrix} \tag{4}$$

$$\mathbf{A}^{20} = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} (-2)^{20} & 0 \\ 0 & 2^{20} \end{bmatrix} \begin{bmatrix} \frac{5}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1048576 & 0 \\ 0 & 1048576 \end{bmatrix} \tag{5}$$

5. Consider matrix $\mathbf{A} = \begin{bmatrix} 0.4 & 1-c \\ 0.6 & c \end{bmatrix}$. Find the eigenvalues of $\mathbf{A}$, which may represent by c. And calculate the rank of $\mathbf{A}$ when $c = 0.6$.

Answer:

Consider $det(\mathbf{A} - \lambda\mathbf{I}) = 0$, then we have

$$(0.4 - \lambda)(c - \lambda) - 0.6 \times (1 - c) = (\lambda - 1)(\lambda - (c - 0.6)) = 0 \tag{6}$$

Then, the $\lambda_1 = 1$ and $\lambda_2 = c - 0.6$.

When $c = 0.6$, $\lambda_2 = 0$, and the rank of matrix $\mathbf{A}$ is 1.

6. Find the condition of b that can ensure the symmetric matrix $\begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}$ has 1 negative eigenvalue. In

this case, determine whether this matrix can have 2 negative eigenvalue.

Answer:

Consider $det(\mathbf{A} - \lambda \mathbf{I}) = 0$, then we have

$$(1-\lambda)^2 - b^2 = 0 \rightarrow 1 - \lambda = b \text{ or } 1 - \lambda = -b \tag{7}$$

Then, we have $\lambda = 1 \pm b$. Therefore, when $b \in (-\infty, -1) \cap (1, \infty)$, we have 1 negative eigenvalue. In this case, we have determinant $1 - b^2 < 0$, which is negative. And two negative number can not have a negative product.