SEMESTER 1 Quiz 2, 2024-2025

Course Code: COMP7180 Section Number: 1 Time Allowed: 1.5 Hours
Course Title: Quantitative Methods for Data Analytics and Artificial Intelligence
Total No. of Pages: 1 Paper Type: A

Problem 1: Convex Sets and Convex Functions (25 Marks)

1.1 Determine whether or not the set $S = \{(x, y) \in \mathbb{R}^2 : 1 \leq x \leq 3\}$ is convex.
(Hint: Definition of Convex Set) (10 Marks)

Solution: Consider two points (x_1, y_1) and (x_2, y_2) such that $1 \leq x_1, x_2 \leq 3$. A point on the line segment is given by:

$$x = \lambda x_1 + (1 - \lambda)x_2$$

Since $1 \leq x_1 \leq 3$ and $1 \leq x_2 \leq 3$, then $x = \lambda x_1 + (1 - \lambda)x_2$ for any $\lambda \in [0,1]$. Thus, the set S is convex.

1.2 Determine whether or not the function $f(x, y) = 2x^2 + 3xy + 2y^2$ is convex.
(Hint: Take the 2nd-order derivatives for the Hessian matrix) (15 Marks)

Solution: To determine if a function is convex, we examine the Hessian matrix H, which is

$$H = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$$

Since the eigenvalues of H are $\lambda = 7$ and $\lambda = 1$, both of which are greater than 0, the Hessian matrix H is positive definite. Therefore, the function is convex.

Problem 2: Lagrangian Multiplier Method (20 Marks)

2.1 Minimize the objective function $f(x, y) = x^2 + y^2$ subject to the constraint $g(x, y) : x + y = 4$.

Solution: Define the Lagrangian function

$$\mathscr{L}(x, y, \lambda) = x^2 + y^2 + \lambda(x + y - 4)$$

Take partial derivatives and set them to zero:

$$\frac{\partial \mathscr{L}}{\partial x} = 2x + \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = 2y + \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x + y - 4 = 0$$

By resolving the above equations, we obtain the minimum point $(x, y) = (2,2)$, which corresponds to the minimum value of $f(x, y)$ is 8.

Problem 3: Conjugate Function (20 Marks)

3.1 Find the conjugate function $f^*(y)$ of $f(x) = \frac{1}{2}x^2 + 3x + 4$.

Solution: The conjugate function $f^*(y) = \sup_x\{xy - f(x)\}$. Find the derivative of $xy - f(x)$ w.r.t. x and set it to 0, we get $x = y - 3$.

Substitute $x = y - 3$ back into $f^*(y)$, we finally attain the conjugate function as:

$$f^*(y) = \frac{1}{2}y^2 - 3y + \frac{1}{2}.$$

Problem 4: Probability (35 Marks)

4.1 Consider two binary variables, X and Y, having a joint distribution given by

$x \backslash y$	0	1
0	1/3	1/3
1	0	1/3

Compute the probabilities P(X=0), P(Y=1), P(Y=1|X=0) and P(X=0|Y=1). (20 Marks)

Solution:

$$P(X = 0) = P(X = 0, Y = 0) + P(X = 0, Y = 1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$P(Y = 1) = P(X = 0, Y = 1) + P(X = 1, Y = 1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$P(Y=1 \mid X=0) = \frac{P(X=0, Y=1)}{P(X=0)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

$$P(X=0 \mid Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

4.2 Suppose the two variables, X and Z, are statistically independent. Show that the variance of their sum satisfies:

$$Var(X+Z) = Var(X) + Var(Z).$$ (15 Marks)

Solution:

For independent random variables, the variance of their sum is equal to the sum of their variances. This is because:

$$\mathrm{Var}(X+Z) = \mathrm{Var}(X) + \mathrm{Var}(Z) + 2\,\mathrm{Cov}(X,Z)$$

Since X and Z are independent, $\mathrm{Cov}(X,Z) = 0$, hence:

$$\mathrm{Var}(X+Z) = \mathrm{Var}(X) + \mathrm{Var}(Z).$$