COMP 7180 Quantitative Methods for Data Analytics and Artificial Intelligence
Assignment 1 Answer

1. **(10 Marks)**

   (a) (4 Marks) Given vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 9 \\ 0 \\ 3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} -28 \\ 35 \\ 22 \end{bmatrix}$. Please calculate a, b, c that satisfy equation $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{x}$, and write down the calculation details.

   (b) (6 Marks) Construct 2 vectors $\mathbf{u}$ and $\mathbf{v}$ with the last four numbers of your student ID. $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, where a, b are the fifth and sixth numbers of your ID. $\mathbf{v} = \begin{bmatrix} c \\ d \end{bmatrix}$, where c, d are the seventh and eighth numbers of your ID. (For student ID: 23456789, we have $\mathbf{u} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$). Calculate $cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, and write down the calculation details.

Answer:

(a) We solve the formulation:

$$
\begin{aligned}
a + 2b + 9c &= -28 \\
7a + 2b + 0c &= 35 \\
3a + 4b + 3c &= 22
\end{aligned} \tag{1}
$$

And we will have the solution:

$$a = 3, b = 7, c = -5 \tag{2}$$

(b)

$$cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|} \tag{3}$$

2. **(14 Marks)** Given matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$. Proof the following multiplication laws of matrix:

   (a) (6 Marks) $\mathbf{AB} \neq \mathbf{BA}$.

   (b) (8 Marks) $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$.

Answer:

(a) There could be many way to proof the inequality, two examples are shown as follows.

i) Let $\mathbf{A}$ be an $m \times n$ matrix, $\mathbf{B}$ an $n \times p$ matrix, and $m \neq p$. Then the multiplication of $\mathbf{AB}$ is valid, and $\mathbf{BA}$ is not valid.

ii) Give an example of $\mathbf{A}$ and $\mathbf{B}$, such as setting $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then, the results of $\mathbf{AB}$ and $\mathbf{BA}$ are different.

(b) Let $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$ matrixs, $\mathbf{C}$ an $n \times p$ matrix.

The (i, j) entry of $(\mathbf{A}+\mathbf{B})\mathbf{C}$ is

$$\begin{aligned} &(a_{i1}+b_{i1})c_{1j} + (a_{i2}+b_{i2})c_{2j} + \cdots + (a_{in}+b_{nj})c_{nj} \\ =&(a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}) + (a_{i1}c_{1j} + a_{i2}c_{2j} + \cdots + a_{in}c_{nj}) \end{aligned} \tag{4}$$

which is the (i, j) entry of $\mathbf{AB}+\mathbf{AC}$.

3. **(16 Marks)** Construct 2 vectors $\mathbf{u}$ and $\mathbf{v}$ with your student ID. $\mathbf{u} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, where a, b, c, d are the first four numbers of your ID. $\mathbf{v} = \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix}$, where e, f, g, h are the last four numbers of your ID. (For student ID: 23456789, we have $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 6 \\ 7 \\ 8 \\ 9 \end{bmatrix}$). We have vector $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

   (a) (6 Marks) Write down one vector $\mathbf{a}$, which is in the space that is spanned by vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$. And prove that vector $\mathbf{a}$ is in the space that is spanned by vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

   (b) (10 Marks) Write down one vector $\mathbf{b}$, which is **not** in the space that is spanned by vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Find the projection point p of vector $\mathbf{b}$ onto the space that is spanned by vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$, and write down the calculation details.

Answer:

(a) Any $\mathbf{a}$ that satisfies $\mathbf{a} = c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w}$.

(b) Any $\mathbf{b}$ that does not satisfy $\mathbf{b} = c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w}$.

Most of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linear independent. We can consider them as the base vectors of the space that they span (a 3-dimensional space). (If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linear dependent, then the space that they span is a plane or a line.)

Then, we consider $\mathbf{M} = [\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$ as the space that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ span, and $\hat{\mathbf{x}} = [x_1\ x_2\ x_3]$.

All we need is the geometrical fact that the line from $\mathbf{b}$ to the closest point $p = \mathbf{M}\hat{\mathbf{x}}$ is orthogonal to the space $\mathbf{M}$:

$$\mathbf{M}^T(\mathbf{b} - \mathbf{M}\hat{\mathbf{x}}) = 0 \text{ or } \mathbf{M}^T\mathbf{M}\hat{\mathbf{x}} = \mathbf{M}^T\mathbf{b} \tag{5}$$

The solution of above formulation is

$$\hat{\mathbf{x}} = (\mathbf{M}^T\mathbf{M})^{-1}\mathbf{M}^T\mathbf{b}. \tag{6}$$

The projection of $\mathbf{b}$ onto the subspace is

$$p = \mathbf{M}\mathbf{x} = \mathbf{M}(\mathbf{M}^{\mathbf{T}}\mathbf{M})^{-1}\mathbf{M}^T\mathbf{b} \tag{7}$$

Note: Compare with projection onto a line $\mathbf{a}$, when $\mathbf{M}$ has only one column : $\mathbf{M}(\mathbf{M}^{\mathbf{T}}\mathbf{M})$ is $\mathbf{a}^T\mathbf{a}$.

4. **(14 Marks)** Supposing 3 measurements b_1, b_2, b_3 are marked:

$$b = 0 \text{ at } t = 3 \text{ , } b = 2 \text{ at } t = 9 \text{ , } b = 5 \text{ at } t = 38 \tag{8}$$

(a) (6 Marks) Find the closest straight line $b = Dt$, and write down the calculation details.

(b) (8 Marks) Find the closest parabola $b = C + Dt + Et^2$, and write down the calculation details.

Answer:

(a) Write down the data

$$\mathbf{A} = \begin{bmatrix} 3 \\ 9 \\ 38 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}, \mathbf{A}^T\mathbf{A} = 1534, \mathbf{A}^T\mathbf{b} = 208 \tag{9}$$

Then solving $\mathbf{A}^T\mathbf{A}D = \mathbf{A}^T\mathbf{b}$, we have $D = \frac{208}{1534} = \frac{104}{767}$

The best line is $b = Dt = \frac{208}{1534}t = \frac{104}{767}t$.

(b) Construct vector $\mathbf{x} = \begin{bmatrix} C \\ D \\ E \end{bmatrix}$. And we construct matrix $\mathbf{A}$ and vector $\mathbf{b}$ as

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 9 \\ 1 & 9 & 81 \\ 1 & 38 & 1444 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}. \tag{10}$$

Then solving $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$, we have

$$\begin{aligned} \begin{bmatrix} 1 & 3 & 9 \\ 1 & 9 & 81 \\ 1 & 38 & 1444 \end{bmatrix}^\top \begin{bmatrix} 1 & 3 & 9 \\ 1 & 9 & 81 \\ 1 & 38 & 1444 \end{bmatrix} \mathbf{x} &= \begin{bmatrix} 1 & 3 & 9 \\ 1 & 9 & 81 \\ 1 & 38 & 1444 \end{bmatrix}^\top \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} \\ \rightarrow \mathbf{x} = (\mathbf{A^T A})^{-1}\mathbf{A}^T\mathbf{b} = \begin{bmatrix} -1.1773399 \\ 0.41215107 \\ -0.00656814 \end{bmatrix} &= \begin{bmatrix} C \\ D \\ E \end{bmatrix} \end{aligned} \tag{11}$$

5. **(9 Marks)** Calculate the eigenvalue of following matrix.

   (a) (3 Marks) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$.

   (b) (3 Marks) $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$.

   (c) (3 Marks) $\mathbf{C} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$.

Answer:

Consider 3×3 matrix A

$$|A| = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{12}$$

To calculate the eigenvalue λ, we need to solve the problem:

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0 \tag{13}$$

(a) Calculate $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$. The solutions are $\lambda =$ 1,4.

Therefore, the eigenvalues of $\mathbf{A}$ are 1, 4.

(b) Calculate $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$. The solutions are $\lambda = -\sqrt{2}$, $\sqrt{2}$.

Therefore, the eigenvalues of $\mathbf{B}$ is $-\sqrt{2}$, $\sqrt{2}$.

(c) Calculate $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$. The solutions are $\lambda = 10, 0$.

Therefore, the eigenvalues of $\mathbf{C}$ is 10,0.

6. **(14 Marks)** Consider a 3×3 matrix $\mathbf{A}$ with eigenvalues $0, 3, 8$. Calculate the following questions, and write down the calculation details.

   (a) (6 Marks) The rank of matrix $\mathbf{A}$.

   (b) (8 Marks) The eigenvalue of $(\mathbf{A}^3 + \mathbf{I})^{-1}$.

Answer:

(a) The rank is 2.

The rank is at most 2, because 0 is an eigenvalue of $\mathbf{A}$.

The rank is not 0, because $\mathbf{A}$ has more than one nonzero eigenvalue.

The rank is not 1, because a rank-1 matrix has only one nonzero eigenvalue.

Thus, the rank is 2.

(b) The eigenvalue of $(\mathbf{A}^3 + \mathbf{I})^{-1}$ is $1, \frac{1}{28}, \frac{1}{513}$.

Because $\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$, we have $\mathbf{A^3 x} = \lambda \mathbf{A^2 x} = \cdots = \lambda^3 \mathbf{x}$.

Then, we have $(\mathbf{A}^3 + \mathbf{I})\mathbf{x} = (\lambda^3 + 1)\mathbf{x}$

Then, we have $\mathbf{x} = (\mathbf{A}^3 + \mathbf{I})^{-1}(\mathbf{A}^3 + \mathbf{I})\mathbf{x} = (\mathbf{A}^3 + \mathbf{I})^{-1}(\lambda^3 + 1)\mathbf{x}$. Then, we divide the $(\lambda^3 + 1)$ on both side of equation:$\frac{1}{(\lambda^3+1)}\mathbf{x} = (\mathbf{A}^3 + \mathbf{I})^{-1}\mathbf{x}$.

Therefore, the eigenvalue of $(\mathbf{A}^3 + \mathbf{I})^{-1}$ is $\frac{1}{(\lambda^3+1)}$, which is $\frac{1}{0^5+1} = 1, \frac{1}{3^3+1} = \frac{1}{28}, \frac{1}{8^3+1} = \frac{1}{513}$.

7. **(10 Marks)** Performe SVD to matrix $\mathbf{A}$, and we have $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^\top$. There are r singular values of matrix $\mathbf{A}$, which are $\sigma_1, \sigma_2, \cdots, \sigma_r$. Prove that: The eigenvalue of matrix $\mathbf{A}^\top\mathbf{A}$ is the square of singular value $\sigma_1^2, \sigma_2^2, \cdots, \sigma_r^2$.

Answer:

According to SVD, we have $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^\top$. Then, we have

$$\mathbf{A}^\top\mathbf{A} = (\mathbf{U}\Sigma\mathbf{V}^\top)^\top(\mathbf{U}\Sigma\mathbf{V}^\top) = \mathbf{V}\Sigma^\top\Sigma\mathbf{V}^\top \tag{14}$$

As only the elements on the diagonal of matrix Σ are not 0, $\Sigma^\top\Sigma$ is a diagonal matrix where the elements on the diagonal are $\sigma_1^2, \sigma_2^2, \cdots, \sigma_r^2$.
Set λ is the eigenvalue of matrix $\mathbf{A}^\top\mathbf{A}$, and $\mathbf{x}$ is the corresponding eigenvector, we have

$$\mathbf{A}^\top\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{15}$$

Because $\mathbf{A}^\top\mathbf{A} = \mathbf{V}\Sigma^\top\Sigma\mathbf{V}^\top$, we have

$$\mathbf{V}\Sigma^\top\Sigma\mathbf{V}^\top\mathbf{x} = \lambda\mathbf{x} \tag{16}$$

Let $\mathbf{y} = \mathbf{V}^\top\mathbf{x}$, then we have

$$\begin{aligned} &\mathbf{V}\Sigma^\top\Sigma\mathbf{V}^\top\mathbf{x} = \mathbf{V}\Sigma^\top\Sigma y \\ &x = (\mathbf{V}^\top)^{-1}\mathbf{y} \end{aligned} \tag{17}$$

As $\mathbf{V}$ is orthogonal matrix, we have $(\mathbf{V}^\top)^{-1} = \mathbf{V}$, and we have

$$\begin{aligned} &\mathbf{V}\Sigma^\top\Sigma y = \lambda(\mathbf{V}^\top)^{-1}\mathbf{y} = \lambda\mathbf{V}\mathbf{y} \\ &\Sigma^\top\Sigma y = \lambda\mathbf{y} \end{aligned} \tag{18}$$

Because $\Sigma^\top\Sigma$ is a diagonal matrix where the elements on the diagonal are $\sigma_1^2, \sigma_2^2, \cdots, \sigma_r^2$. Therefore, λ can be $\sigma_1^2, \sigma_2^2, \cdots, \sigma_r^2$. Thus, the eigenvalue of matrix $\mathbf{A}^\top\mathbf{A}$ is the square of singular value $\sigma_1^2, \sigma_2^2, \cdots, \sigma_r^2$.

8. **(13 Marks)** Construct 4 vectors $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$, where the number $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ are picked from your student ID. (For student ID: 23456789, we have $\mathbf{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}, \mathbf{d} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$ or $\mathbf{a} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \mathbf{d} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$.) And we have $\mathbf{e} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{f} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

   (a) (6 Marks) Perform PCA with the data $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}$, write down the calculation details, and write down the largest principal component.

   (b) (2 Marks) Visualize these 6 vectors as data points. And divide these 6 vectors into 2 classes, each class contains 3 vectors. The vectors in each class are picked by yourself. (For example, we could have class 1 $(\mathbf{a}, \mathbf{b}, \mathbf{c})$, class 2 $(\mathbf{d}, \mathbf{e}, \mathbf{f})$, or class 1 $(\mathbf{a}, \mathbf{c}, \mathbf{e})$, class 2 $(\mathbf{b}, \mathbf{d}, \mathbf{f})$). Ensure that these two classes are linearly separable.

   (c) (5 Marks) Perform LDA with the data you obtain in question 8(b), write down the projection vector $\mathbf{w}$, project your data in the subspace, and write down the calculation details.

Answer:

(a)
- Center the data $\mathbf{X}_c$.
- Compute the covariance matrix $\mathbf{S}$ using the centered data as $\mathbf{S} = \frac{1}{n}\mathbf{X}_c\mathbf{X}_c^T$.
- Do an eigen decomposition of the covariance matrix $\mathbf{S}$.
- Take first leading eigenvectors $\mathbf{u}_1$ with largest eigenvalue λ_1. (**The largest principal component**)
- The final one-dimensional representation of data is obtained by $\mathbf{Z} = \mathbf{u}_1^T\mathbf{X}_c$.

(b) Any separation that are linear separable.

(c)
- Calculate the mean of two classes μ_1, μ_2,
- Calculate the covariance matrix of two classes $\mathbf{S}_i = \sum_j^n (\mathbf{x}_j - \mu_i)(\mathbf{x}_j - \mu_i)^T (i = 1, 2)$.
- Calculate the within-class scatter matrix $\mathbf{S}_W = \mathbf{S}_1 + \mathbf{S}_2$.
- Calculate the between-class matrix $\mathbf{S}_B = (\mu_1 - \mu_2)(\mu_1 - \mu_2)^T$.
- Perform eigen decomposition to matrix $\mathbf{S}_W^{-1}\mathbf{S}_B$.
- Take the eigenvector $\mathbf{w}$ corresponding to the largest eigenvalue.
- The projected data is $\mathbf{z}_i = \mathbf{w}^T\mathbf{x}_i$, where $\mathbf{x}_i$ is $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}$, respectively.