COMP 7180
Quantitative Methods for Data Analytics and Artificial Intelligence

Lecture 3: Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

- Matrix-Vector multiplication: **Ax**.
  - Almost all vectors change direction when they are multiplied by a matrix **A**.

- Eigenvectors (Eigen is a German word meaning “characteristic”)
  - There are certain exceptional vectors **x** whose direction is the same as **Ax**.
  - Multiplied by matrix **A** does not change the direction of these vectors.
  - These vectors are “**eigenvectors**” of **A**.

- Eigenvalues
  - For those eigenvectors, multiplied by matrix **A** is equal to multiplied by a number.
  - Therefore, $\mathbf{Ax} = \lambda\mathbf{x}$.
  - The number λ is an eigenvalue of **A**.

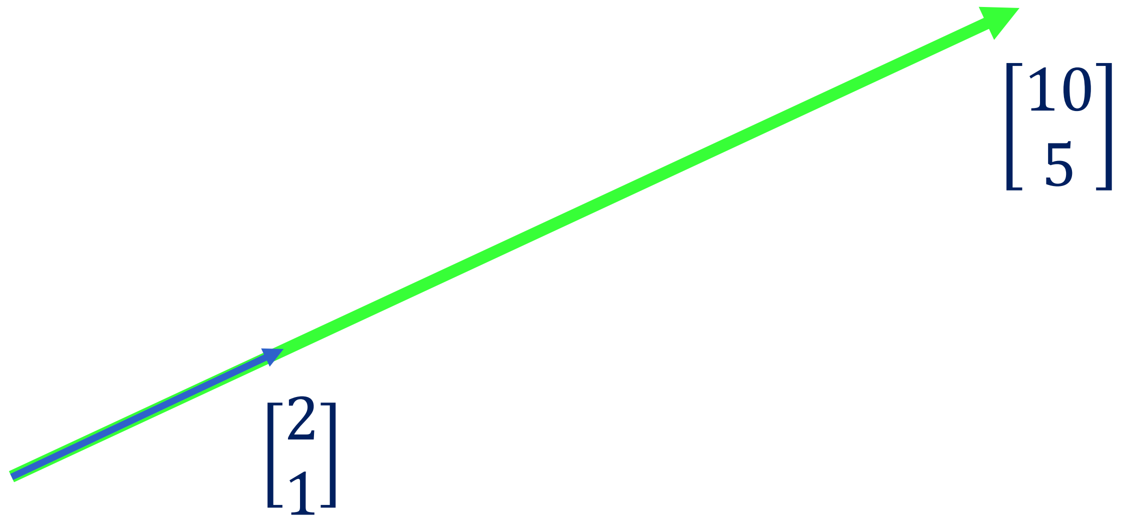
Eigenvalues and Eigenvectors

- Eigenvalue equation $\mathbf{Ax} = \lambda\mathbf{x}$

- The eigenvalue λ tells whether the eigenvector **x** is stretched or shrunk or reversed or left unchanged when it is multiplied by matrix **A**.

$$\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\mathbf{Ax} = \lambda\mathbf{x}$$



Eigenvalues and Eigenvectors – Special Matrix

- If **A** is identity matrix, every vector has **Ax** = **x**. All vectors are eigenvectors of **I**. And all eigenvalues are 1.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Computing eigenvalues, eigenvectors

Preliminary – Conditions of invertibility:

1. $Ax = b$ has a unique solution for every b.
2. $Ax = 0$ has only the solution $x = 0$.
3. The row/columns of A are linearly independent.
4. The determinant of A is not zero.
5. Zero is not an eigenvalue of A.

Computing eigenvalues, eigenvectors

- How to compute eigenvalues, eigenvectors based on eigenvalue equation $\mathbf{Ax} = \lambda\mathbf{x}$
- Let rewrite $\mathbf{Ax} = \lambda\mathbf{x}$ as $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$
  - The matrix $\mathbf{A} - \lambda\mathbf{I}$ times the eigenvector **x** is the zero vector.
  - We are **NOT** interested in the trivial solution **x** = **0**.
- Obtain eigenvalues first
  - Since **x** ≠ **0**, this requires matrix $\mathbf{A} - \lambda\mathbf{I}$ is not invertible.
  - Therefore, $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. This equation only involves λ not **x**.
- For each eigenvalue λ, solve $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x}$ to find an eigenvector **x**.

Example of Computing eigenvalues, eigenvectors

- Let us find the eigenvalues and eigenvectors of the following 2*2 matrix.

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

- Find $\mathbf{A} - \lambda \mathrm{I}$ by subtract λ from the diagonal of $\mathbf{A}$

$$\mathbf{A} - \lambda \mathbf{I} = \begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}$$

- Obtain eigenvalues by solving $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$

$$\mathbf{A} - \lambda \mathbf{I} = \begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} = (4-\lambda)(3-\lambda) - 2 * 1 = 0$$

$$\Rightarrow \quad 10 - 7\lambda + \lambda^2 = (2-\lambda)(5-\lambda) = 0 \quad \Rightarrow \quad \lambda_1 = 2 \text{ and } \lambda_2 = 5$$

Example of Computing eigenvalues, eigenvectors

- Now find the eigenvectors by solving $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = 0$ separately for $\lambda_1 = 2$ and $\lambda_2 = 5$.
- Let denote $\mathbf{x}$ as $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(1) For $\lambda_1 = 2$, we obtain

$$(\mathbf{A}-2\mathbf{I})\mathbf{x} = \mathbf{0} \Rightarrow \begin{bmatrix} 4-2 & 2 \\ 1 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} 2x_1 + 2x_2 &= 0 \\ x_1 + x_2 &= 0 \end{aligned}$$

This means any vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where $x_2 = -x_1$, such as $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$(or $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$) is an eigenvector of **A** with eigenvalue 2.

Example of Computing eigenvalues, eigenvectors

- Now find the eigenvectors by solving $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$ separately for $\lambda_1 = 2$ and $\lambda_2 = 5$.
- Let denote $\mathbf{x}$ as $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(2) For $\lambda_2 = 5$, we obtain

$$(\mathbf{A} - 5\mathbf{I})\mathbf{x} = \mathbf{0} \Rightarrow \begin{bmatrix} 4-5 & 2 \\ 1 & 3-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -x_1 + 2x_2 &= 0 \\ x_1 - 2x_2 &= 0 \end{aligned}$$

This means any vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where $x_2 = \frac{x_1}{2}$, such as $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$(or $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$) is an eigenvector of **A** with eigenvalue 5.

Non-uniqueness of eigenvectors

- In previous example, we saw both $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}4\\2\end{bmatrix}$ are eigenvectors of **A** with eigenvalue 5. There is a whole line of eigenvectors – any nonzero multiple of $\begin{bmatrix}2\\1\end{bmatrix}$ is an eigenvector.
- **Non-uniqueness of eigenvectors.** If **x** is an eigenvector of **A** with eigenvalue λ, then for any nonzero number *c* it holds that *c***x** is an eigenvector of **A** with the same eigenvalue:

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \qquad\qquad \mathbf{A}(c\mathbf{x}) = \mathrm{c}\mathbf{A}\mathbf{x} = \mathrm{c}\lambda\mathbf{x} = \lambda(c\mathbf{x})$$

Example of Computing eigenvalues, eigenvectors

- In previous example, we obtained the eigenvalues and eigenvectors for $\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$, which are $\lambda_1 = 2$ with eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\lambda_2 = 5$ with eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

- How about the eigenvalues, eigenvectors for **A** + 3**I**, $\mathbf{A}^2$?

Eigenvalues, Eigenvectors for **A** + 3**I**

- $\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$, let denote $\mathbf{B} = \mathbf{A} + 3\mathbf{I} = \begin{bmatrix} 7 & 2 \\ 1 & 6 \end{bmatrix}$
- Obtain eigenvalues by solving $\det(\mathbf{B} - \lambda\mathbf{I}) = 0$

$$\mathbf{B} - \lambda\mathbf{I} = \begin{bmatrix} 7-\lambda & 2 \\ 1 & 6-\lambda \end{bmatrix} = (7-\lambda)(6-\lambda) - 2 * 1 = 0$$

$$\Rightarrow \quad 40 - 13\lambda + \lambda^2 = (5-\lambda)(8-\lambda) = 0 \quad \Rightarrow \quad \lambda_1 = 5 \text{ and } \lambda_2 = 8$$

- When $\lambda_1 = 5$, $\begin{bmatrix} 7-5 & 2 \\ 1 & 6-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 2x_1 + 2x_2 = 0 \\ x_1 + x_2 = 0 \end{matrix} \Rightarrow$ Eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

- When $\lambda_2 = 8$, $\begin{bmatrix} 7-8 & 2 \\ 1 & 6-8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -x_1 + 2x_2 = 0 \\ x_1 - 2x_2 = 0 \end{matrix} \Rightarrow$ Eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Eigenvalues, Eigenvectors for **A** + 3**I**

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

$\lambda_1 = 2$ Eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 5$ Eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\mathbf{B} = \mathbf{A} + 3\mathbf{I} = \begin{bmatrix} 7 & 2 \\ 1 & 6 \end{bmatrix}$$

$\lambda_1 = 5$ Eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 8$ Eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

- The eigenvectors of **A** and **A**+3**I** are the same.
- The eigenvalues of **A**+3**I** are the eigenvalues of **A** plus 3.
- Why?
  - If $\mathbf{Ax} = \lambda\mathbf{x}$, then $(\mathbf{A} + c\mathbf{I})\mathbf{x} = (\lambda + \mathrm{c})\mathbf{x}$ for the same **x**.

Eigenvalues, Eigenvectors for $\mathbf{A}^2$

- $\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$, let denote $\mathbf{B} = \mathbf{A}^2 = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 14 \\ 7 & 11 \end{bmatrix}$
- Obtain eigenvalues by solving $\det(\mathbf{B} - \lambda\mathbf{I}) = 0$

$$\mathbf{B} - \lambda\mathbf{I} = \begin{bmatrix} 18-\lambda & 14 \\ 7 & 11-\lambda \end{bmatrix} = (18-\lambda)(11-\lambda) - 14 * 7 = 0$$

$$\Rightarrow \quad 100 - 29\lambda + \lambda^2 = (4-\lambda)(25-\lambda) = 0 \quad \Rightarrow \quad \lambda_1 = 4 \text{ and } \lambda_2 = 25$$

- When $\lambda_1 = 4$, $\begin{bmatrix} 18-4 & 14 \\ 7 & 11-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \end{matrix} \Rightarrow$ Eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

- When $\lambda_2 = 25$, $\begin{bmatrix} -7 & 14 \\ 7 & -14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -x_1 + 2x_2 = 0 \\ x_1 - 2x_2 = 0 \end{matrix} \Rightarrow$ Eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Eigenvalues, Eigenvectors for $\mathbf{A}^2$

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

$\lambda_1 = 2$ Eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 5$ Eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\mathbf{B} = \mathbf{A}^2 = \begin{bmatrix} 18 & 14 \\ 7 & 11 \end{bmatrix}$$

$\lambda_1 = 4$ Eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 25$ Eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

- The eigenvectors of $\mathbf{A}$ and $\mathbf{A}^2$ are the same.
- The eigenvalues of $\mathbf{A}^2$ are the square of the eigenvalues of $\mathbf{A}$.
- Why?
  - If $\mathbf{Ax} = \lambda\mathbf{x}$, then $\mathbf{A}^2\mathbf{x} = \mathbf{A}\lambda\mathbf{x} = \lambda\mathbf{Ax} = \lambda^2\mathbf{x}$ for the same $\mathbf{x}$.

Other Useful Facts for Eigenvalues: Sum of Eigenvalues

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

$$\lambda_1 = 2$$

$$\lambda_2 = 5$$

$$\mathbf{B} = \mathbf{A} + 3\mathbf{I} = \begin{bmatrix} 7 & 2 \\ 1 & 6 \end{bmatrix}$$

$$\lambda_1 = 5$$

$$\lambda_2 = 8$$

$$\mathbf{B} = \mathbf{A}^2 = \begin{bmatrix} 18 & 14 \\ 7 & 11 \end{bmatrix}$$

$$\lambda_1 = 4$$

$$\lambda_2 = 25$$

- The sum of the *n* eigenvalues equals the sum of the *n* diagonal entries.
- The sum of the entries along the main diagonal of **A** is called the ***trace*** of **A**.

$$A = \left(a_{ij}\right) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$tr\left(A\right) = \sum_{i=1}^{n} a_{ii}$$

Other Useful Facts for Eigenvalues: Product of Eigenvalues

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

$$\lambda_1 = 2$$

$$\lambda_2 = 5$$

$$\lambda_1 \lambda_2 = 10$$
$$\det(\mathbf{A}) = 10$$

$$\mathbf{B} = \mathbf{A} + 3\mathbf{I} = \begin{bmatrix} 7 & 2 \\ 1 & 6 \end{bmatrix}$$

$$\lambda_1 = 5$$

$$\lambda_2 = 8$$

$$\lambda_1 \lambda_2 = 40$$
$$\det(\mathbf{A}) = 40$$

$$\mathbf{B} = \mathbf{A}^2 = \begin{bmatrix} 18 & 14 \\ 7 & 11 \end{bmatrix}$$

$$\lambda_1 = 4$$

$$\lambda_2 = 25$$

$$\lambda_1 \lambda_2 = 100$$
$$\det(\mathbf{A}) = 100$$

- The product of the *n* eigenvalues equals the **determinant**.

Eigen-Decomposition of **A**

- Suppose the n by n matrix **A** has n linearly independent eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_n$. Put them into the columns of an ***eigenvector matrix* X.** Then

$$\mathbf{AX} = \mathbf{A}\begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{x}_1 & \ldots & \lambda_n\mathbf{x}_n \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

***eigenvector matrix* X** ***eigenvalue matrix* Λ**

Eigen-Decomposition of **A**

- $\mathbf{AX} = \mathbf{X\Lambda}$
- The matrix **X** has an inverse, because its columns (the eigenvectors of **A**) were assumed to be linearly independent. Therefore

$$\mathbf{AX} = \mathbf{X\Lambda} \Rightarrow \mathbf{AXX^{-1}} = \mathbf{X\Lambda X^{-1}} \Rightarrow \mathbf{A} = \mathbf{X\Lambda X^{-1}}$$

- **X** is the eigenvector matrix and its columns are eigenvectors of **A**. $\mathbf{\Lambda}$ is the eigenvalue matrix whose diagonal entries are eigenvalues of **A**.
- $\mathbf{A} = \mathbf{X\Lambda X^{-1}}$ is the eigen-decomposition of **A**.

Computing $\boldsymbol{A}^k$ easily using eigen-decomposition

- Eigen decomposition $\boldsymbol{A} = \mathbf{X\Lambda X^{-1}}$
- The k-th power of **A** can be computed as

$\mathbf{X}^{-1}\mathbf{X} = \mathbf{I}$

$$\begin{aligned}\boldsymbol{A}^k &= (\mathbf{X\Lambda X^{-1}})(\mathbf{X\Lambda X^{-1}})(\mathbf{X\Lambda X^{-1}})\cdots(\mathbf{X\Lambda X^{-1}}) \\ &= \mathbf{X\Lambda}^k\mathbf{X^{-1}}\end{aligned}$$

- E.g., $\boldsymbol{A} = \begin{bmatrix}1 & 5\\0 & 6\end{bmatrix} = \overset{\mathbf{X}}{\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}}\overset{\mathbf{\Lambda}}{\begin{bmatrix}1 & 0\\0 & 6\end{bmatrix}}\overset{\mathbf{X}^{-1}}{\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}}$

$$\boldsymbol{A}^k = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 6^k\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}$$

- With k = 1 we get **A**. With k = 0 we get $\mathbf{A}^0 = \mathbf{I}$. With $k = -1$, we get $\boldsymbol{A}^{-1}$.
- $\boldsymbol{A}^2 = \begin{bmatrix}1 & 35\\0 & 36\end{bmatrix}$ fits the formula when k = 2.

Eigen-decomposition of a symmetric matrix **S**

- For a symmetric matrix, transposing **S** to $\mathbf{S}^{\mathrm{T}}$ produces no change. Then $\mathbf{S}^{\mathrm{T}}$ equals **S**. Its (*j*, *i*) entry across the main diagonal equals its (*i*, *j*) entry.
- E.g., $\boldsymbol{S} = \begin{bmatrix} 1 & 5 \\ 5 & 6 \end{bmatrix}$, $\boldsymbol{S} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ are symmetric matrices.
- Symmetric matrices **S** are the most important matrices in the theory of linear algebra and also in applications (We will discuss one important application in machine learning (i.e., PCA) later).
- <u>What is special about $\boldsymbol{S}\mathbf{x} = \lambda\mathbf{x}$ when **S** is symmetric?</u>

Eigen-decomposition of a symmetric matrix **S**

- Eigen decomposition of a matrix $\boldsymbol{S} = \mathbf{X\Lambda X}^{-1}$.
- Transpose of **S:** $\boldsymbol{S}^{\mathrm{T}} = (\mathbf{X\Lambda X}^{-1})^{T} = (\mathbf{X}^{-1})^{T}\mathbf{\Lambda}\mathbf{X}^{T}$
- **S** is a symmetric matrix: $\boldsymbol{S} = \boldsymbol{S}^{\mathrm{T}}$. To satisfied it, we can choose $\mathbf{X}^{-1} = \mathbf{X}^{\mathrm{T}}$.
- Then $\mathbf{X}^{\mathrm{T}}\mathbf{X} = \mathbf{X}^{-1}\mathbf{X} = \mathbf{I}.$ The eigenvectors are chosen orthonormal: Each eigenvector in **X** orthogonal to other eigenvectors and the length of each eigenvector is 1.
- The special form of eigen-decomposition $\boldsymbol{A} = \mathbf{X\Lambda X}^{-1}$ for symmetric matrices is

$$\boldsymbol{S} = \mathbf{Q\Lambda Q}^{-1} = \mathbf{Q\Lambda Q}^{T} \text{ with } \mathbf{Q}^{-1} = \mathbf{Q}^{T}.$$

Columns of **Q** are orthonormal eigenvectors of **S**.

Example of Eigen-decomposition of a symmetric matrix **S**

- Eigen-decomposition of $\boldsymbol{S} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.
- Obtain eigenvalues of **S** by solving $\det(\boldsymbol{S} - \lambda \mathbf{I}) = \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}\right) = 0.$
  - $(1-\lambda)(4-\lambda) - 2 * 2 = \lambda^2 - 5\lambda = 0$
  - $\lambda_1 = 0$ and $\lambda_2 = 5$
- When $\lambda_1 = 0$, $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $x_1^2 + x_2^2 = 1$ ➡ Eigenvector $\begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix}$
- When $\lambda_1 = 5$, $\begin{bmatrix} 1-5 & 2 \\ 2 & 4-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $x_1^2 + x_2^2 = 1$ ➡ Eigenvector $\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$

Example of Eigen-decomposition of a symmetric matrix **S**

- $\boldsymbol{S} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$

  $\mathbf{Q}$ $\quad$ $\boldsymbol{\Lambda}$ $\quad$ $\mathbf{Q}^{-1}$ (or $\mathbf{Q}^{T}$)

- The eigenvectors are orthonormal: $\mathbf{Q}^{T}\mathbf{Q} = \mathbf{I}$

Useful Fact: If the matrix A is symmetric and the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$, with corresponding eigenvectors $\vec{x}_1, \ldots, \vec{x}_n$

$$\text{i.e. } A\vec{x}_i = \lambda_i \vec{x}_i$$

If $\lambda_i \neq \lambda_j$ then $\vec{x}_i' \vec{x}_j = 0$

Useful Fact: If the matrix A is symmetric and the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$, with corresponding eigenvectors $\vec{x}_1, \ldots, \vec{x}_n$

$$\text{i.e. } A\vec{x}_i = \lambda_i \vec{x}_i$$

If $\lambda_i \neq \lambda_j$ then $\vec{x}_i' \vec{x}_j = 0$

Why? Note $\vec{x}_j' A \vec{x}_i = \lambda_i \vec{x}_j' \vec{x}_i$

and $\vec{x}_i' A \vec{x}_j = \lambda_j \vec{x}_i' \vec{x}_j$

$$0 = \left(\lambda_i - \lambda_j\right) \vec{x}_i' \vec{x}_j$$

hence $\vec{x}_i' \vec{x}_j = 0$

Exercise

Given the $n \times k$ matrix $\mathbf{A}$ and the $k \times n$ matrix $\mathbf{B}$:

1. Use an example to show that $\mathbf{AB} \neq \mathbf{BA}$ even if $n = k$.

Exercise

Given the $n \times k$ matrix **A** and the $k \times n$ matrix **B**:

1. Use an example to show that $\mathbf{AB} \neq \mathbf{BA}$ even if $n = k$.

Solution:

Let $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then we have $\mathbf{AB} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and $\mathbf{BA} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

Obviously we have $\mathbf{AB} \neq \mathbf{BA}$.

Exercise

Given the $n \times k$ matrix **A** and the $k \times n$ matrix **B**:

2. When $n \neq k$, do we have trace(**AB**) = trace(**BA**)? Why?

Exercise

Given the $n \times k$ matrix **A** and the $k \times n$ matrix **B**:

2. When $n \neq k$, do we have trace(**AB**) = trace(**BA**)? Why?

Solution: by definition

$$\begin{aligned} trace(AB) &= (AB)_{11} + (AB)_{22} + \cdots + (AB)_{nn} \\ &= a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1k}b_{k1} \\ &+ a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2k}b_{k2} \\ &+ \vdots \\ &+ a_{n1}b_{1n} + a_{n2}b_{2n} + \cdots + a_{nk}b_{kn} \end{aligned}$$

if you view the sum according to the columns, then you see that it is the $trace(BA)$.
therefore,

$$trace(AB) = trace(BA).$$

Exercise

Given the $n \times k$ matrix $\mathbf{A}$ and the $k \times n$ matrix $\mathbf{B}$:

3. What is the relationship between eigenvalues of **AB** and eigenvalues of **BA**? What is the relationship between eigenvectors of **AB** and eigenvectors of **BA**?

Exercise

Given the $n \times k$ matrix **A** and the $k \times n$ matrix **B**:

3. What is the relationship between eigenvalues of **AB** and eigenvalues of **BA**? What is the relationship between eigenvectors of **AB** and eigenvectors of **BA**?

Solution:

Suppose that λ and x are the eigenvalue and eigenvector of matrix (**AB**), then by definition we have (**AB**)x = λx.

Then we have **B**(**AB**)x = **B**λx, which indicates that (**BA**)**B**x = λ**B**x, i.e., λ is also the eigenvalue of matrix (**BA**), and the corresponding eigenvector is **B**x.

References and Acknowledgement

- https://math.stackexchange.com/questions/1314142/trace-of-ab-trace-of-ba