COMP 7180 Quantitative Methods for Data Analytics and Artificial Intelligence
Exercise 1 Answer

1. Let $\mathbf{v} = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 7 \\ -2 \\ 9 \end{bmatrix}$, answer the following questions:

   (a) Calculate $4\mathbf{v} + 3\mathbf{w}$

   Answer:

   $$4\mathbf{v} + 3\mathbf{w} = 4 \cdot \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix} + 3 \cdot \begin{bmatrix} 7 \\ -2 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 \times 4 + 3 \times 7 \\ 4 \times 2 + 3 \times -2 \\ 4 \times -3 + 3 \times 9 \end{bmatrix} = \begin{bmatrix} 37 \\ 2 \\ 15 \end{bmatrix} \tag{1}$$

   (b) Calculate $\mathbf{v} \cdot \mathbf{w}$

   Answer:

   $$\mathbf{v} \cdot \mathbf{w} = 4 \times 7 + 2 \times -2 + -3 \times 9 = -3 \tag{2}$$

   (c) Calculate $\|\mathbf{v}\| \cdot \|\mathbf{w}\|$

   Answer:

   $$\|\mathbf{v}\| \cdot \|\mathbf{w}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \cdot \sqrt{\mathbf{w} \cdot \mathbf{w}} = \sqrt{4^2 + 2^2 + (-3)^2} \cdot \sqrt{7^2 + (-2)^2 + 9^2} = 62.3378 \tag{3}$$

   (d) Calculate $cos\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

   Answer:

   $$cos\theta = \frac{\mathbf{v}}{\|\mathbf{v}\|} \cdot \frac{\mathbf{w}}{\|\mathbf{w}\|} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \cdot \|\mathbf{w}\|} = \frac{-3}{62.3378} = -0.0481 \tag{4}$$

2. Find the vector $\mathbf{v}$ which is orthogonal to $\mathbf{w} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$ and $\|\mathbf{v}\| = 1$.

   Answer:

Set the vector as $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.
Then, from orthogonal, we have

$$7v_1 + 2v_2 = 0 \rightarrow v_2 = \frac{-7v_1}{2} \tag{5}$$

From $\|\mathbf{v}\| = 1$, we have

$$v_1^2 + v_2^2 = v_1^2 + \frac{49v_1^2}{4} = 1 \tag{6}$$

Solving the above formulation, we have

$$v_1^2 = \frac{4}{53} \rightarrow v_1 = \pm\frac{2}{\sqrt{53}} \tag{7}$$

Then $v_2 = \frac{-7v_1}{2} = \mp\frac{7}{\sqrt{53}}$.
Therefore, $\mathbf{v} = \begin{bmatrix} \frac{2}{\sqrt{53}} \\ \frac{-7}{\sqrt{53}} \end{bmatrix}$ or $\mathbf{v} = \begin{bmatrix} \frac{-2}{\sqrt{53}} \\ \frac{7}{\sqrt{53}} \end{bmatrix}$.

3. For two vector $\mathbf{v}$ and $\mathbf{w}$, determine if $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$ is true. If yes, prove it. If not, give examples.

Answer:

$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$ is true. Proof is shown as follows.

Firstly, we have $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \cdot \|\mathbf{w}\| \cdot cos\theta$. Because $cos\theta$ can not be larger than 1. Therefore, we have

$$\|\mathbf{v} \cdot \mathbf{w}\| \leq \|\mathbf{v}\| \cdot \|\mathbf{w}\| \rightarrow \left(\sum_{\mathrm{k=1}}^{\mathrm{n}} \mathrm{v_k w_k}\right)^2 \leq \sum_{\mathrm{k=1}}^{\mathrm{n}} \mathrm{v_k^2} \sum_{\mathrm{k=1}}^{\mathrm{n}} \mathrm{w_k^2} \tag{8}$$

By applying the above inequality (CauchySchwarz inequality), we have

$$\begin{aligned} \|\mathbf{v} + \mathbf{w}\| &= \sqrt{\sum_{i=1}^{n} (v_i + w_i)^2} = \sqrt{\sum_{i=1}^{n} (v_i^2 + w_i^2 + 2v_i w_i)} \leq \sqrt{\sum_{i=1}^{n} (v_i^2 + w_i^2) + 2\sqrt{\sum_{i=1}^{n} v_i^2 \sum_{i=1}^{n} w_i^2}} \\ &= \sqrt{\sum_{i=1}^{n} v_i^2 + \sum_{i=1}^{n} w_i^2 + 2\sqrt{\sum_{i=1}^{n} v_i^2}\sqrt{\sum_{i=1}^{n} w_i^2}} = \sqrt{\sum_{i=1}^{n} v_i^2} + \sqrt{\sum_{i=1}^{n} w_i^2} = \|v\| + \|w\| \end{aligned} \tag{9}$$

4. For three martix $\mathbf{V} = \begin{bmatrix} 4 & 6 \\ 2 & 2 \end{bmatrix}$, $\mathbf{W} = \begin{bmatrix} 7 & 8 \\ -2 & 3 \end{bmatrix}$ and $\mathbf{U} = \begin{bmatrix} \mathbf{V} & \mathbf{W} \end{bmatrix}$, answer the following questions:

(a) Calculate $2\mathbf{V} + 5\mathbf{W}$

Answer:

$$2\mathbf{V} + 5\mathbf{W} = 2 \cdot \begin{bmatrix} 4 & 6 \\ 2 & 2 \end{bmatrix} + 5 \cdot \begin{bmatrix} 7 & 8 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 43 & 52 \\ -6 & 19 \end{bmatrix} \tag{10}$$

(b) Calculate $\mathbf{VU}$.

Answer:

$$\mathbf{U} = \begin{bmatrix} \mathbf{V} & \mathbf{W} \end{bmatrix} = \begin{bmatrix} 4 & 6 & 7 & 8 \\ 2 & 2 & -2 & 3 \end{bmatrix} \tag{11}$$

$$\mathbf{VU} = \begin{bmatrix} 4 & 6 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 & 6 & 7 & 8 \\ 2 & 2 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 28 & 36 & 16 & 50 \\ 12 & 16 & 10 & 22 \end{bmatrix} \tag{12}$$

(c) Calculate $2\mathbf{VU} + 5\mathbf{WU}$

Answer:

$$2\mathbf{VU} + 5\mathbf{WU} = (2\mathbf{V} + 5\mathbf{W}) \cdot \mathbf{U} = \begin{bmatrix} 43 & 52 \\ -6 & 19 \end{bmatrix} \cdot \begin{bmatrix} 4 & 6 & 7 & 8 \\ 2 & 2 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 276 & 362 & 197 & 500 \\ 14 & 2 & -80 & 9 \end{bmatrix} \tag{13}$$

5. For matrix $\mathbf{A} = \begin{bmatrix} 0.5 & 4 \\ -2 & 4 \end{bmatrix}$ vector $\mathbf{b} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$, find vector $\mathbf{x}$ that satisfies $\mathbf{Ax} = \mathbf{b}$

Answer:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} 0.4 & -0.4 \\ 0.2 & 0.05 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 1.6 \\ 1.3 \end{bmatrix} \tag{14}$$