COMP 7180: Quantitative Methods for Data Analytics and Artificial Intelligence

Lecture 5: An Introduction to Optimization in AI and ML – Part I

J. Duchi, Introduction to Convex Optimization for Machine Learning, University of California, Berkeley, 2009.
M. P. Deisenroth, A. A. Faisal, and C. S. Ong, Mathematics for Machine Learning, 2019.
Cliff Ragsdale, Spreadsheet Modeling and Decision Analysis: A Practical Introduction to Business Analytics, 2014.

What is Optimization ?

- Finding the minimizer/maximizer of a function subject to constraints:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0, \ i = \{1, \ldots, k\} \\ & h_j(x) = 0, \ j = \{1, \ldots, l\} \end{aligned}$$

Why Optimization?

Limited Resources…

- The amount of oil we can pump out of the earth is limited.
- The amount of land available for garbage dumps and hazardous waste is limited
- Each of us has limited amount of time and money to spend
- A company has limited number of workers
- A restaurant has a limited amount of space for seating

Optimization

- Deciding how best to use the limited resources available to an individual or a business is a universal problem
- In today's competitive business environment, it is increasingly important to make sure that an individual's or a company's limited resources are used in the most efficient manner
- This involves determining how to allocate the resources in such a way to maximize profits or minimize costs
- Optimization aims to find the optimal, or the most efficient way of using limited resources to achieve the objective of an individual or a business

Applications of Optimization

- Determining Product Mix
  - Most manufacturing companies can make a variety of products.
  - However, each product usually requires different amounts of raw materials and labor.
  - Similarly, the amount of profit generated by the products varies.
  - The manager of such a company must decide how many of each product to produce in order to maximize profits or to satisfy demand at minimum cost

Applications of Optimization

- Logistics
  - Many retail companies have warehouses around the country that are responsible for keeping stores supplied with products to sell.
  - The amount of products available at the warehouses and the amount needed at each store tends to fluctuate, as does the cost of shipping or delivering products from the warehouses to the retail locations.
  - Determining the least costly method of transferring products from the warehouses to the stores can save large amounts of money

Applications of Optimization

- Financial Planning
  - The US federal government requires individuals to begin withdrawing money from individual retirement accounts and other tax-sheltered retirement programs no later than age 70.5.
  - There are various rules that must be followed to avoid paying penalty taxes on these withdrawals.
  - Most individuals want to withdraw their money in a manner that minimize the amount of taxes they must pay while still obeying the tax laws.

Features of Optimization Problems

The above examples give some ideas about optimization.

- One or more **Decisions** to be made, E.g.
  - How many of each product should be produced?
  - How much of each product should be shipped form each warehouse to the various retail locations?
  - How much money should an individual withdraw each year from various retirement accounts?

Features of Optimization Problems

- **Constraints** are often placed on the alternatives available to the decision maker. E.g.
  - A production manager is faced with a limited amounts of raw materials and labor.
  - There is a physical limitation on the amount of products a truck can carry.
  - Laws determine the minimum and maximum amounts that can be withdrawn.

Features of Optimization Problems

- The existence of some goal or **objective** when deciding actions to take. E.g.
  - The production manager can decide to produce several different mix of products that maximize the profits
  - The company want to identify the routing that minimizes the total transportation cost
  - Individuals can withdraw money from their retirement accounts with a method that minimizes their tax liability

Elements of Optimization Problems

- Decisions (or Decision Variables)
- Constraints
- Objective

Mathematical Form of Three Elements

Decisions

- The decisions in an optimization problem are often represented in a mathematical model by the symbols

$$X_1, X_2, \ldots, X_n$$

- These variables are referred as decision variables

Mathematical Form of Three Elements

Constraints

- The constraints in an optimization problem can be represented in a mathematical model in a number of ways:

A less than or equal to constraint: $f_1(X_1, X_2, \ldots, X_n) <= b$

A greater than or equal to constraint: $f_2(X_1, X_2, \ldots, X_n) >= b$

An equal to constraint: $f_3(X_1, X_2, \ldots, X_n) = b$

Mathematical Form of Three Elements

Objective

- The objective in an optimization problem is represented mathematically by an objective function in the general format:

  MAX (or MIN): $f_0(X_1, X_2, \ldots, X_n)$
- The Objective function identifies some function of the decision variables that the decision maker wants to either MAX or MIN

General Form of an Optimization Problem

$$
\begin{array}{ll}
\text{MAX (or MIN):} & f_0(X_1, X_2, \ldots, X_n) \\
\text{Subject to:} & f_1(X_1, X_2, \ldots, X_n) \;\; <= \;\; b_1 \\
 & \vdots \\
 & f_k(X_1, X_2, \ldots, X_n) \;\; >= \;\; b_k \\
 & \vdots \\
 & f_m(X_1, X_2, \ldots, X_n) \;\; = \;\; b_m
\end{array}
$$

Linear Programming

Linear Programming (LP) Problems

MAX (or MIN): $c_1X_1 + c_2X_2 + \ldots + c_nX_n$

Subject to: $a_{11}X_1 + a_{12}X_2 + \ldots + a_{1n}X_n <= b_1$

$$\vdots$$

$a_{k1}X_1 + a_{k2}X_2 + \ldots + a_{kn}X_n >= b_k$

$$\vdots$$

$a_{m1}X_1 + a_{m2}X_2 + \ldots + a_{mn}X_n = b_m$

An Example LP Problem

The Blue Ridge Hot Tubs Limited Company sells two types of hot tubs (热水浴池): Aqua-Spa & Hydro-Lux. Howie Jones, the owner and manager of the company, needs to decide how many of each type of hot tub to produce during his next production cycle.

Howie buys hot tub shells from a local supplier and adds the pump and tubing to the shells to create his hot tubs. Howie installs the same type of pump into both types of hot tubs. He will have only 200 pumps available during his next production cycle.

An Example LP Problem

From a manufacturing standpoint, the main difference between the two models of hot tubs is the amount of tubing and labor required. Each Aqua-Spa requires 9 hours of labor and 12 feet of tubing. Each Hydro-Lux requires 6 hours of labor and 16 feet of tubing.

	Aqua-Spa	Hydro-Lux
Pumps	1	1
Labor	9 hours	6 hours
Tubing	12 feet	16 feet
Unit Profit	$350	$300

An Example LP Problem

Howie expects to have 1,566 production labor hours and 2,880 feet of tubing available during the next production cycle. Howie earns a profit of $350 on each Aqua-Spa and $300 on each Hydro-Lux. He is confident that he can sell all the hot tubs he produces.

The question is, how many Aqua-Spa and Hydro-Lux should Howie produce if he wants to maximize his profits during the next production cycle?

An Example LP Problem

The Blue Ridge Hot Tubs Limited Company produces two types of hot tubs: Aqua-Spa & Hydro-Lux.

	Aqua-Spa	Hydro-Lux
Pumps	1	1
Labor	9 hours	6 hours
Tubing	12 feet	16 feet
Unit Profit	$350	$300

There are 200 pumps, 1566 hours of labor, and 2880 feet of tubing available.

5 Steps In Formulating LP Models:

1. Understand the problem.
2. Identify the decision variables.

   X_1=number of Aqua-Spas to produce

   X_2=number of Hydro-Luxes to produce
3. State the objective function as a linear combination of the decision variables.

$$\text{MAX: } 350X_1 + 300X_2$$

5 Steps In Formulating LP Models (continued)

4. State the constraints as linear combinations of the decision variables.

$$1X_1 + 1X_2 <= 200 \quad \} \text{ pumps}$$

$$9X_1 + 6X_2 <= 1566 \quad \} \text{ labor}$$

$$12X_1 + 16X_2 <= 2880 \quad \} \text{ tubing}$$

5. Identify any upper or lower bounds on the decision variables.

$$X_1 >= 0$$

$$X_2 >= 0$$

LP Model for Blue Ridge Hot Tubs

Decision Variables

X_1=number of Aqua-Spas to produce

X_2=number of Hydro-Luxes to produce

Objective Functions:

MAX: $350X_1 + 300X_2$

Constraints:

$1X_1 + 1X_2 <= 200$

$9X_1 + 6X_2 <= 1566$

$12X_1 + 16X_2 <= 2880$

$X_1 >= 0$

$X_2 >= 0$

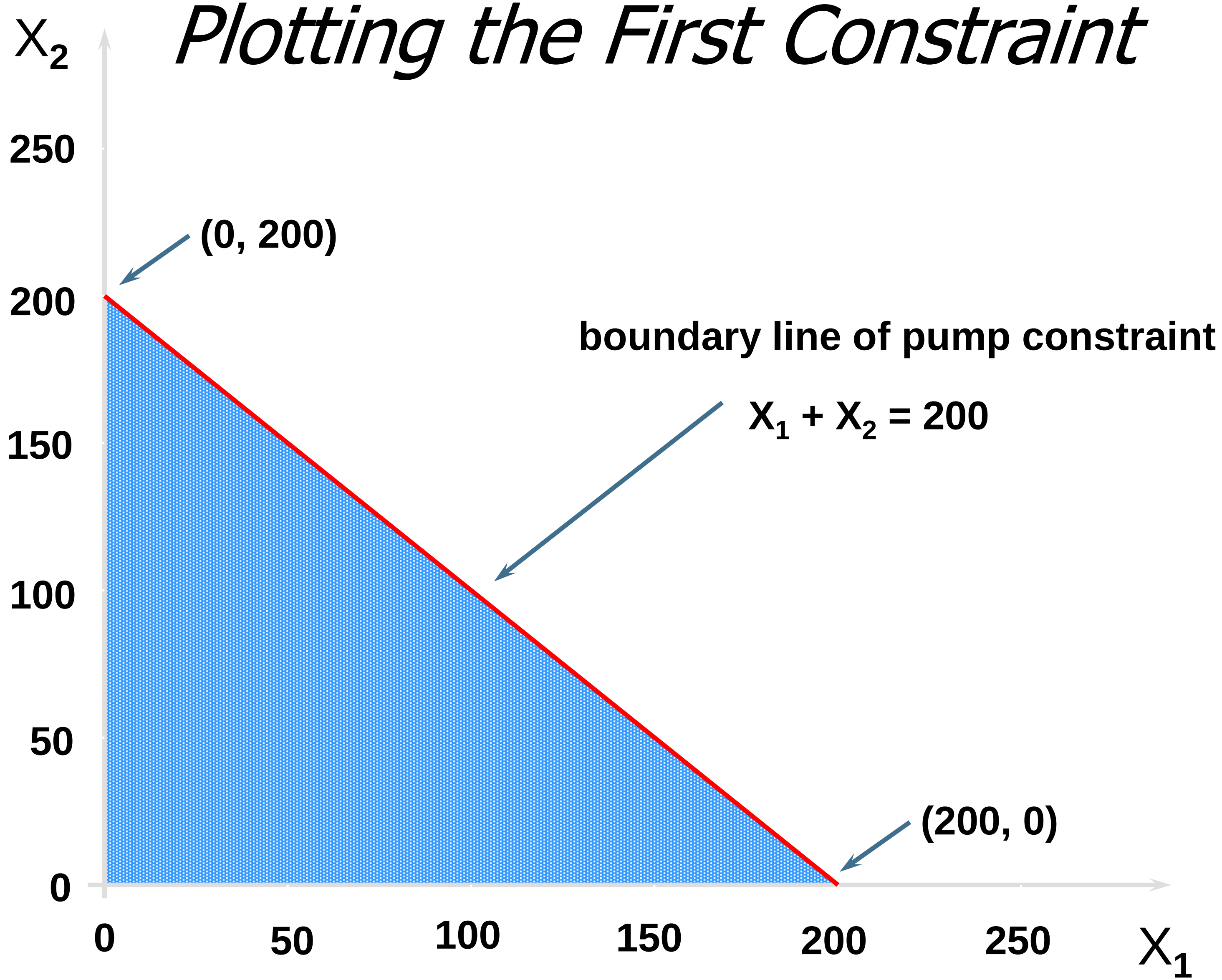
Solving LP Problems: An Intuitive Approach

- Idea: Each Aqua-Spa (X_1) generates the highest unit profit ($350), so let's make as many of them as possible!
- How many would that be?
  - Let $X_2 = 0$
    - 1st constraint: $1X_1 + 1X_2 <= 200$
    - 2nd constraint: $9X_1 + 6X2 <=1566$ or $X_1 <=174$
    - 3rd constraint: $12X_1 + 16X2 <= 2880$ or $X_1 <= 240$
- If $X_2=0$, the maximum value of X_1 is 174 and the total profit is $\$350*174 + \$300*0 = \$60,900$
- This solution is *feasible*, but is it *optimal*?

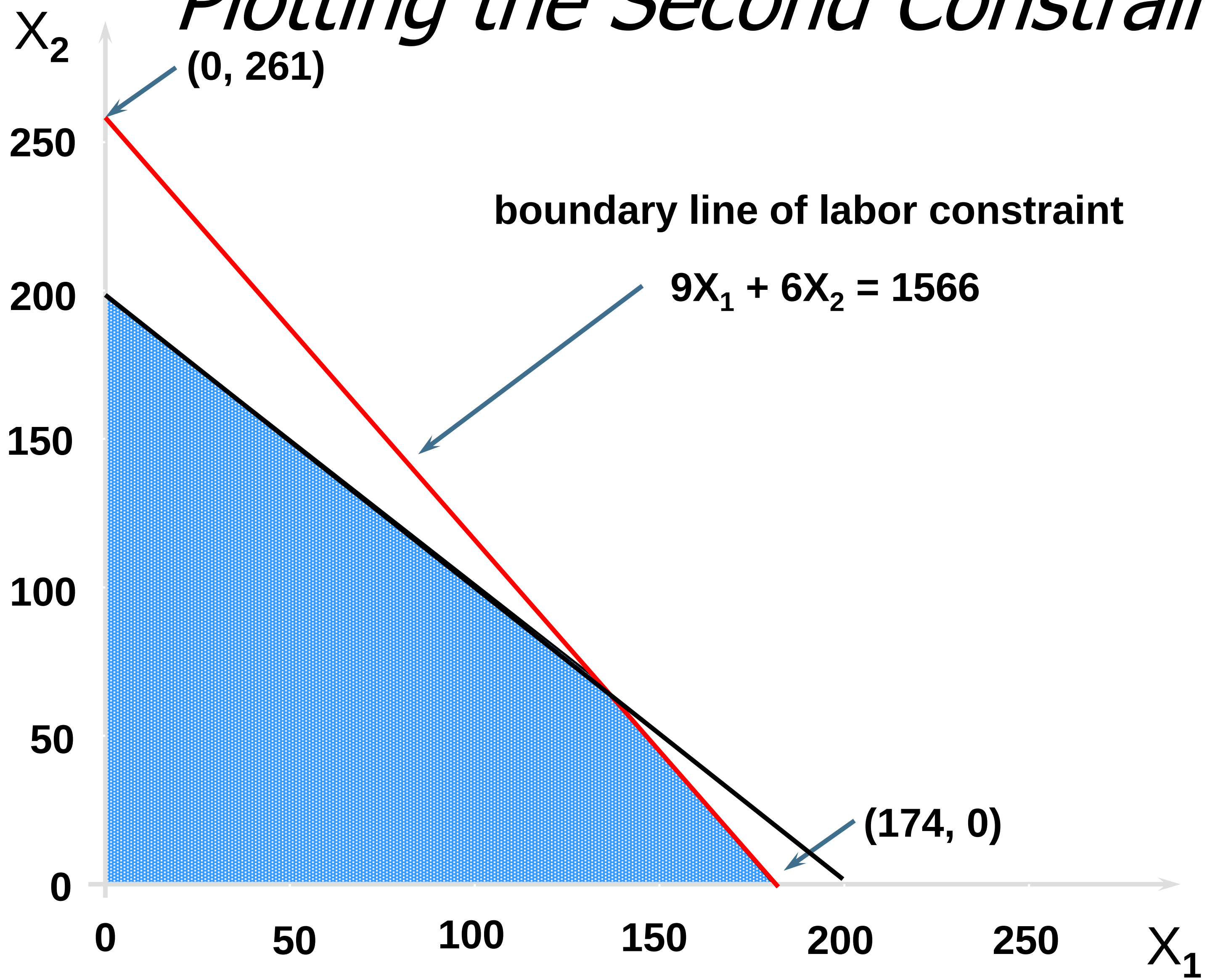
Solving LP Problems: A Graphical Approach

- The constraints of an LP problem defines its feasible region.
- The best point in the feasible region is the optimal solution to the problem.
- For LP problems with 2 variables, it is easy to plot the feasible region and find the optimal solution.

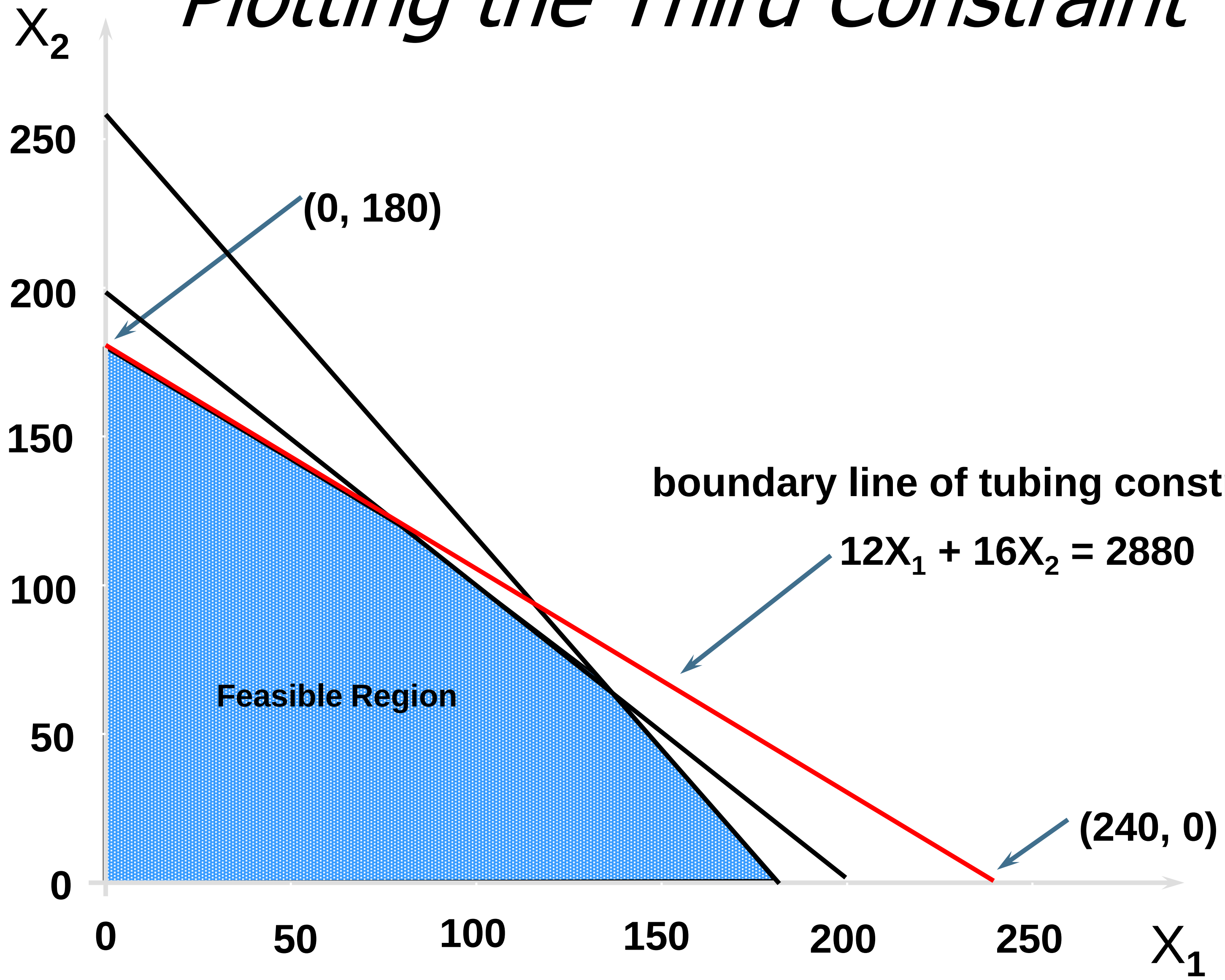
Plotting the First Constraint



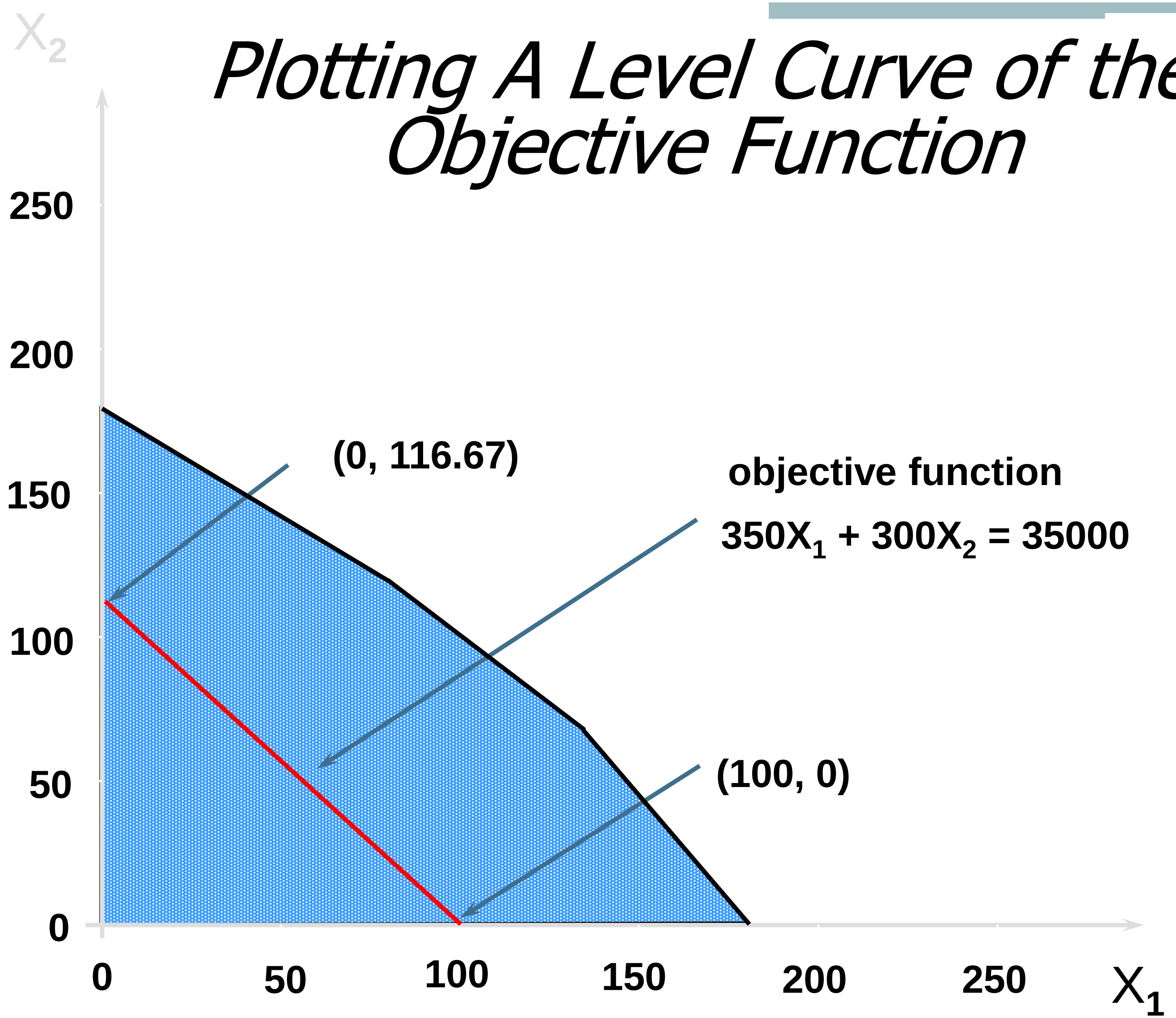
Plotting the Second Constraint



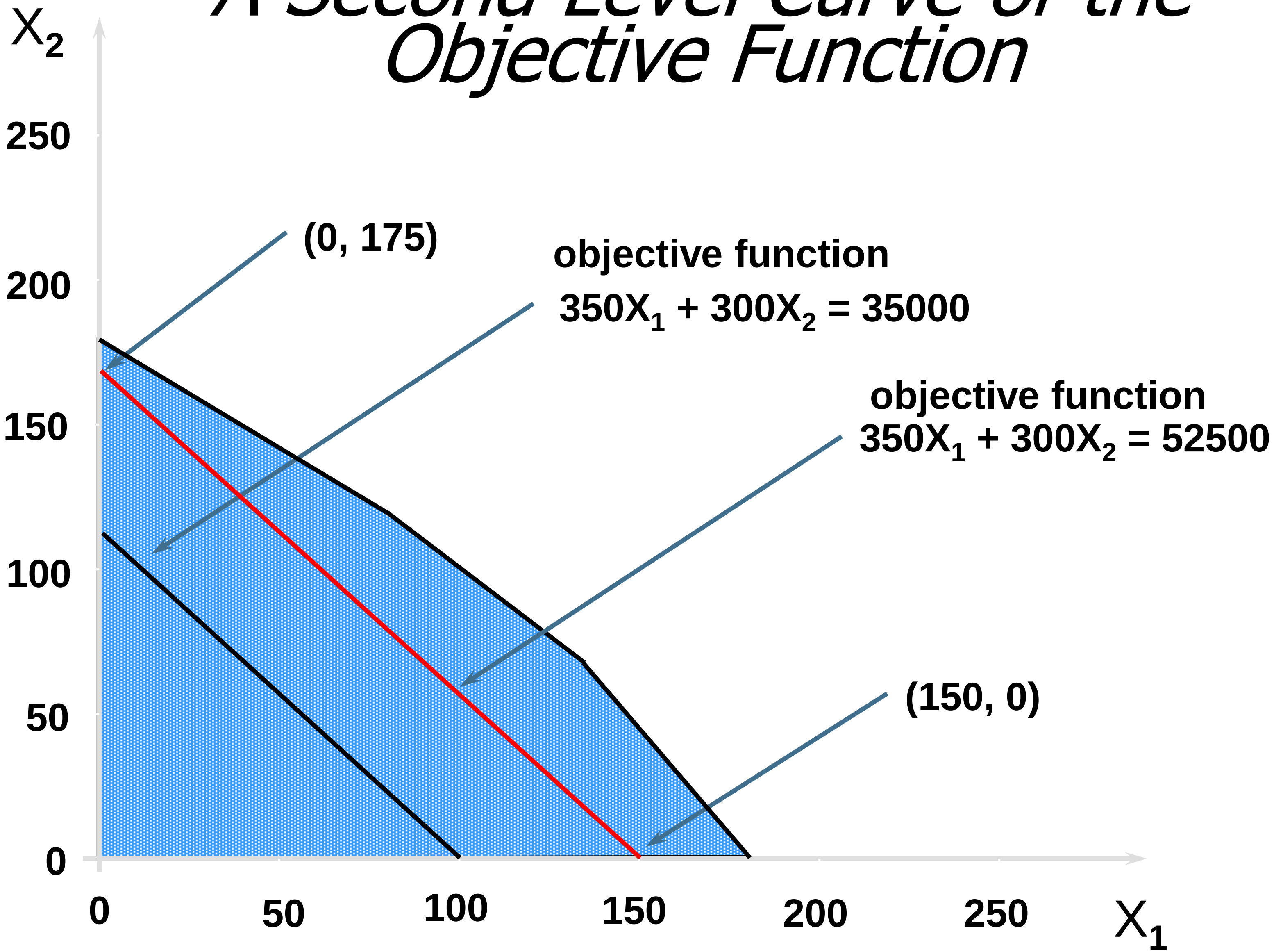
Plotting the Third Constraint



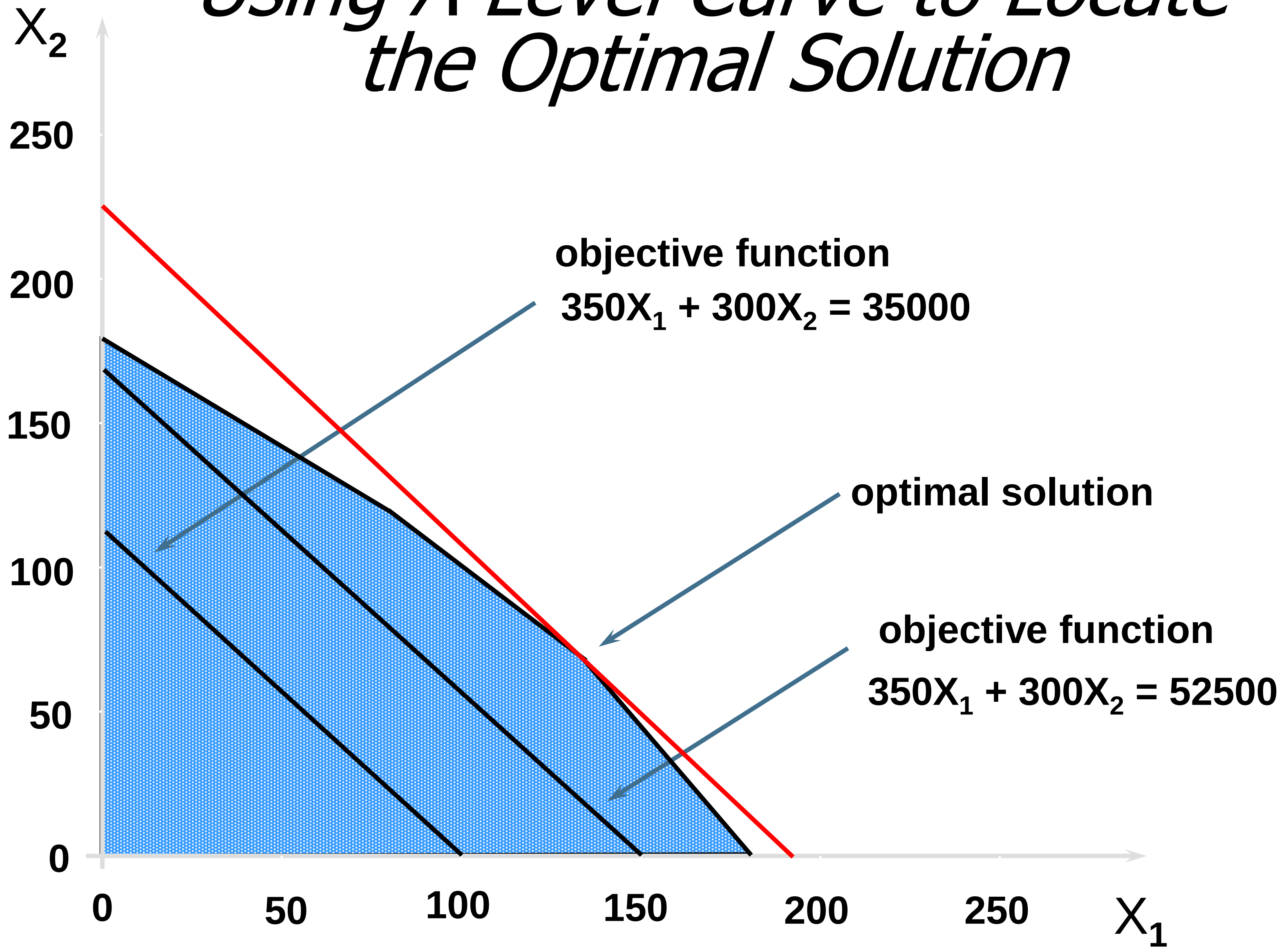
Plotting A Level Curve of the Objective Function



A Second Level Curve of the Objective Function



Using A Level Curve to Locate the Optimal Solution



Calculating the Optimal Solution

- The optimal solution occurs where the “pumps” and “labor” constraints intersect.
- This occurs where:

$$X_1 + X_2 = 200 \qquad (1)$$

and $$9X_1 + 6X_2 = 1566 \qquad (2)$$

- From (1) we have, $X_2 = 200 - X_1$ (3)
- Substituting (3) for X_2 in (2) we have,

$$9X_1 + 6\,(200 - X_1) = 1566$$

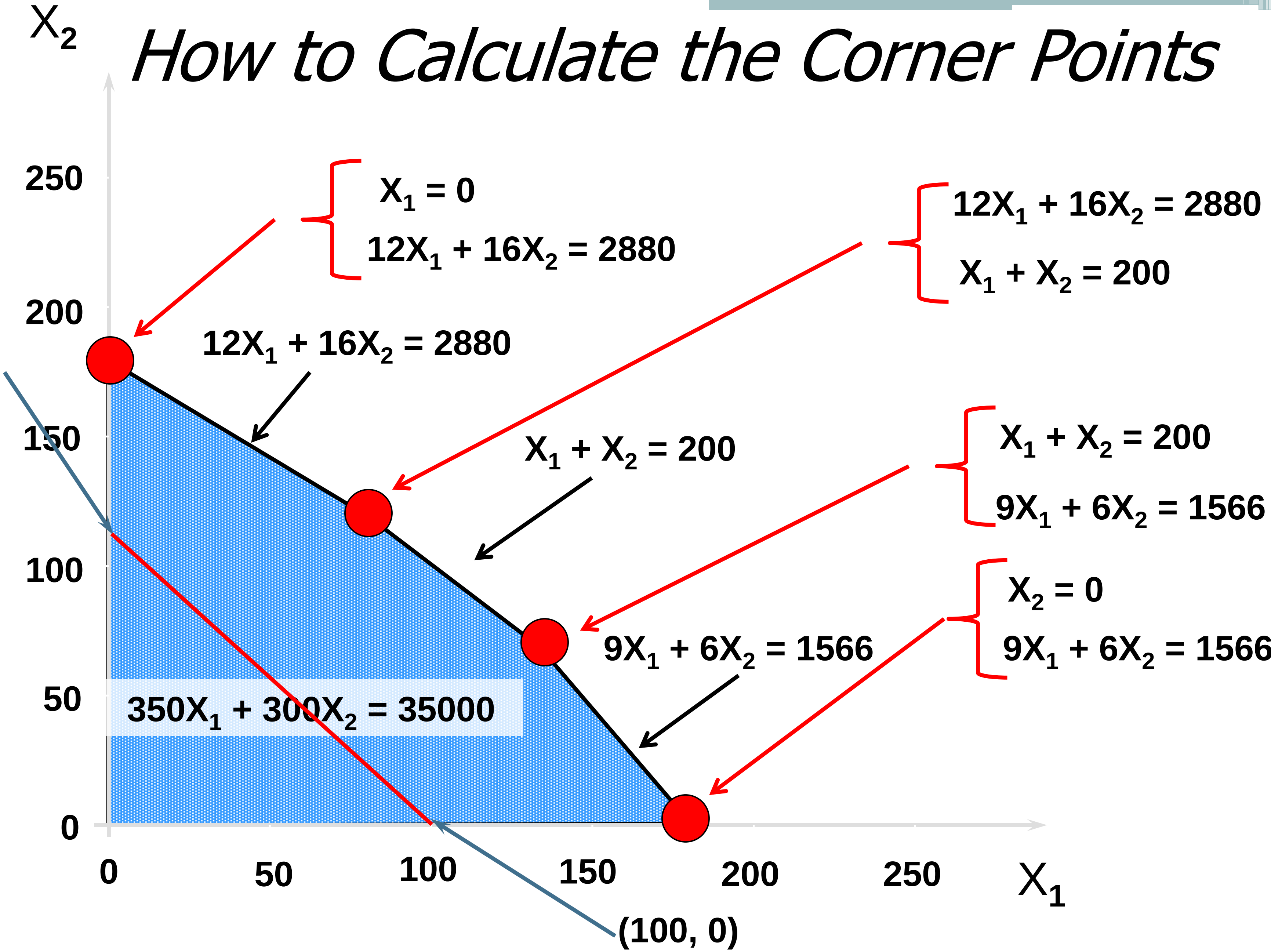
which reduces to $X_1 = 122$

- So the optimal solution is,

$$X_1 = 122,\ X_2 = 200 - X_1 = 78$$

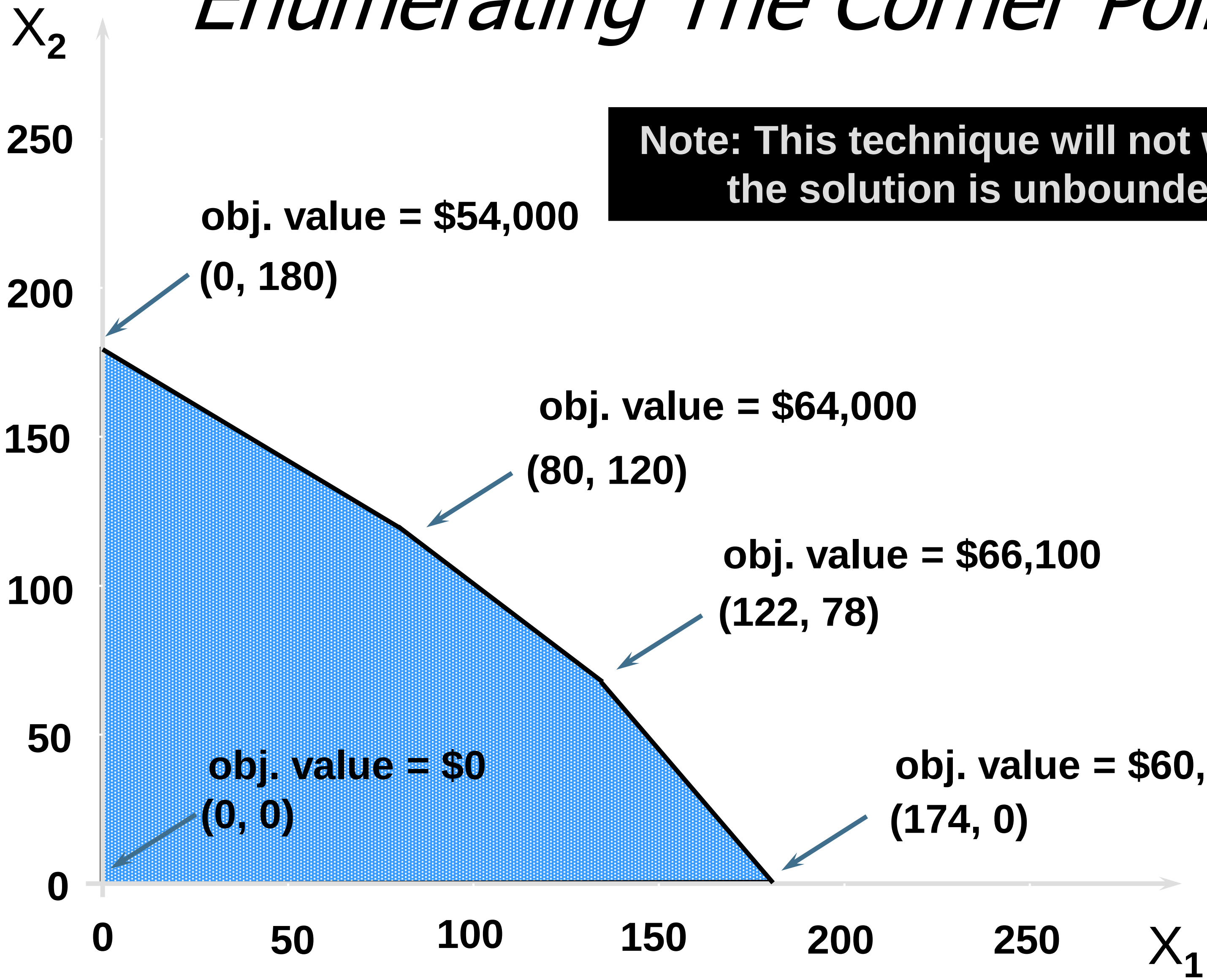
Total Profit = $\$350*122 + \$300*78 = \$66,100$

How to Calculate the Corner Points



Enumerating The Corner Points

Note: This technique will not work if the solution is unbounded.



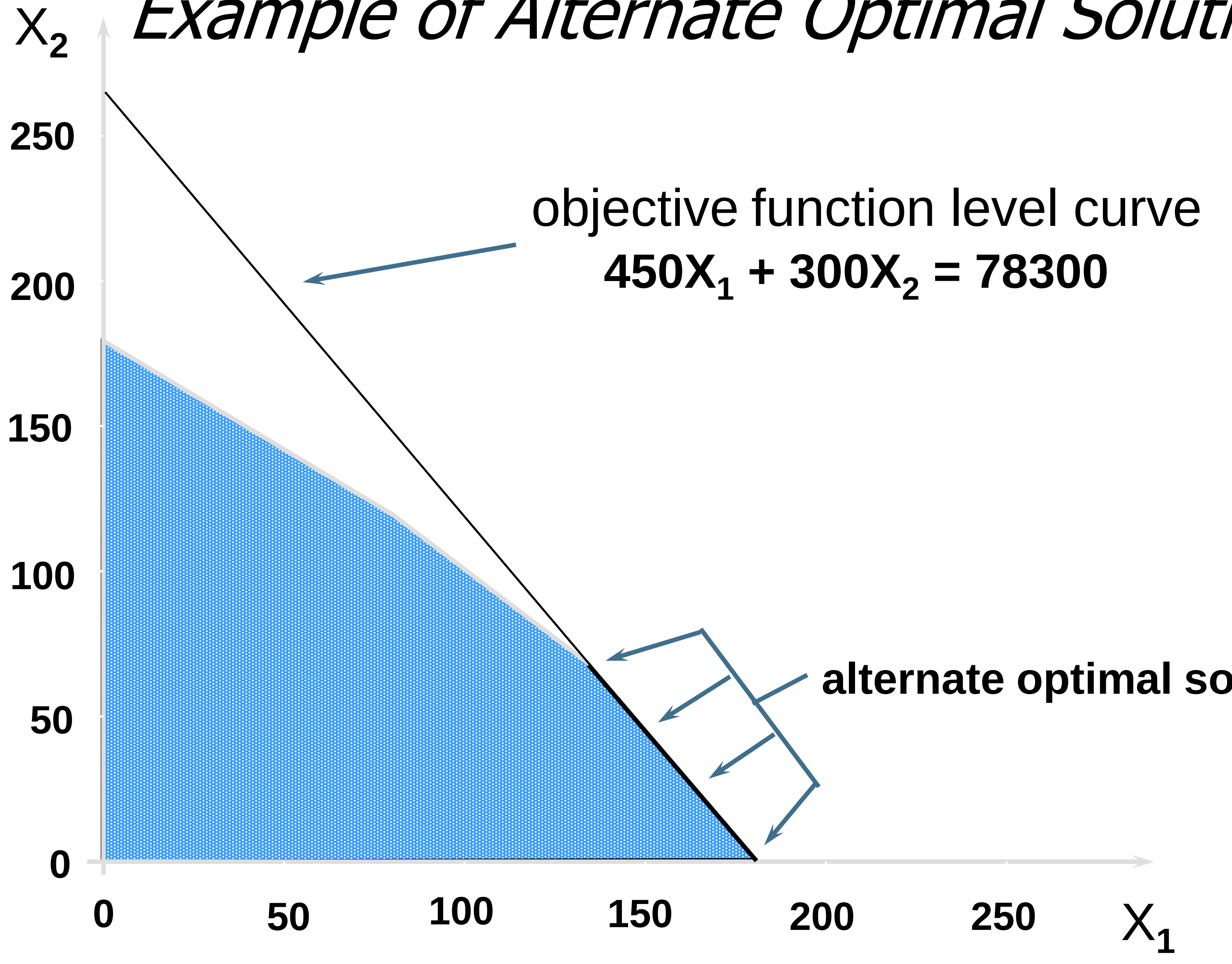
Summary of Graphical Solution to LP Problems

1. Plot the boundary line of each constraint
2. Identify the feasible region
3. Locate the optimal solution by either:
    a. Plotting level curves
    b. Enumerating the extreme points

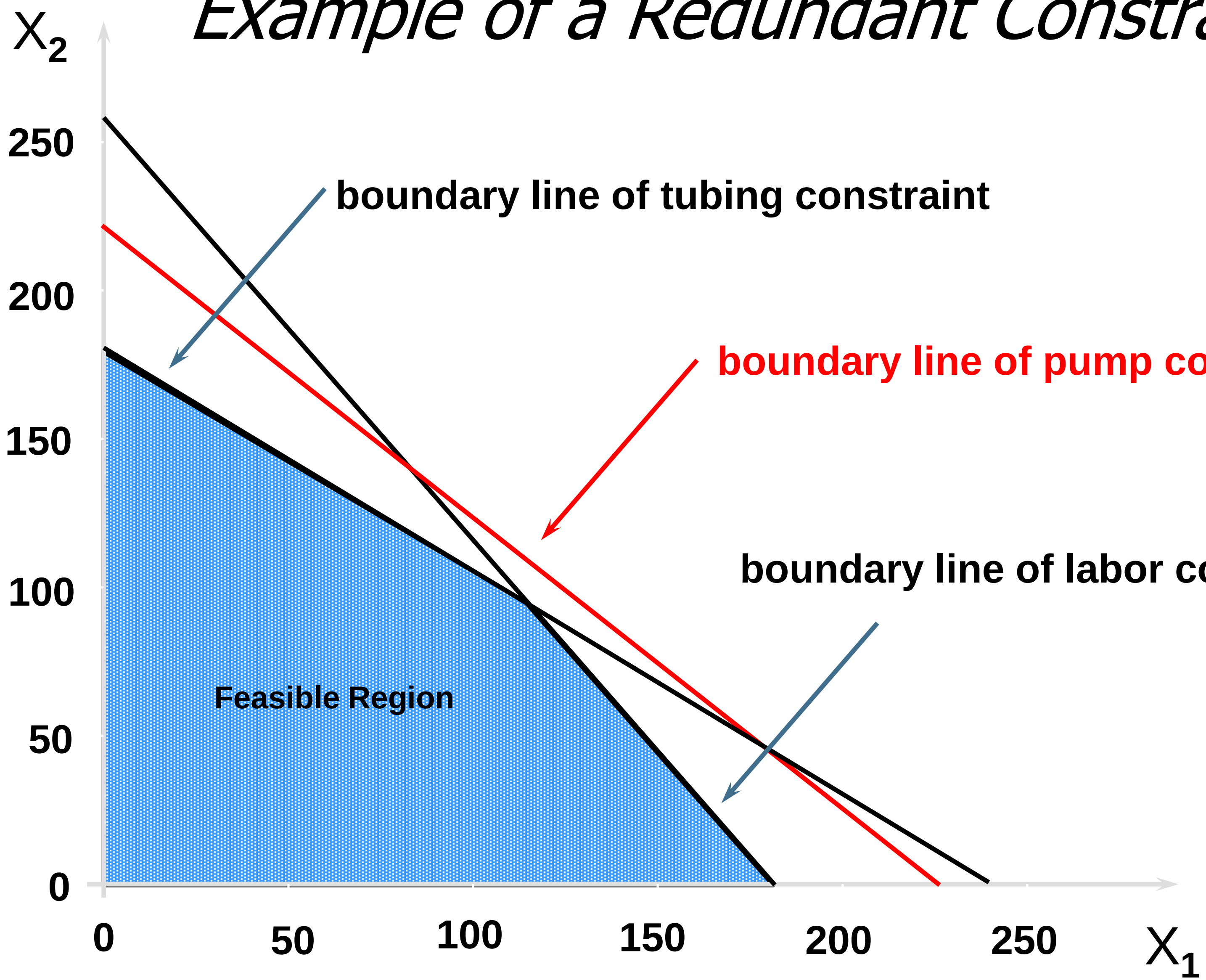
Special Conditions in LP Models

- A number of anomalies can occur in LP problems:
  - Alternate Optimal Solutions
  - Redundant Constraints
  - Unbounded Solutions
  - Infeasibility

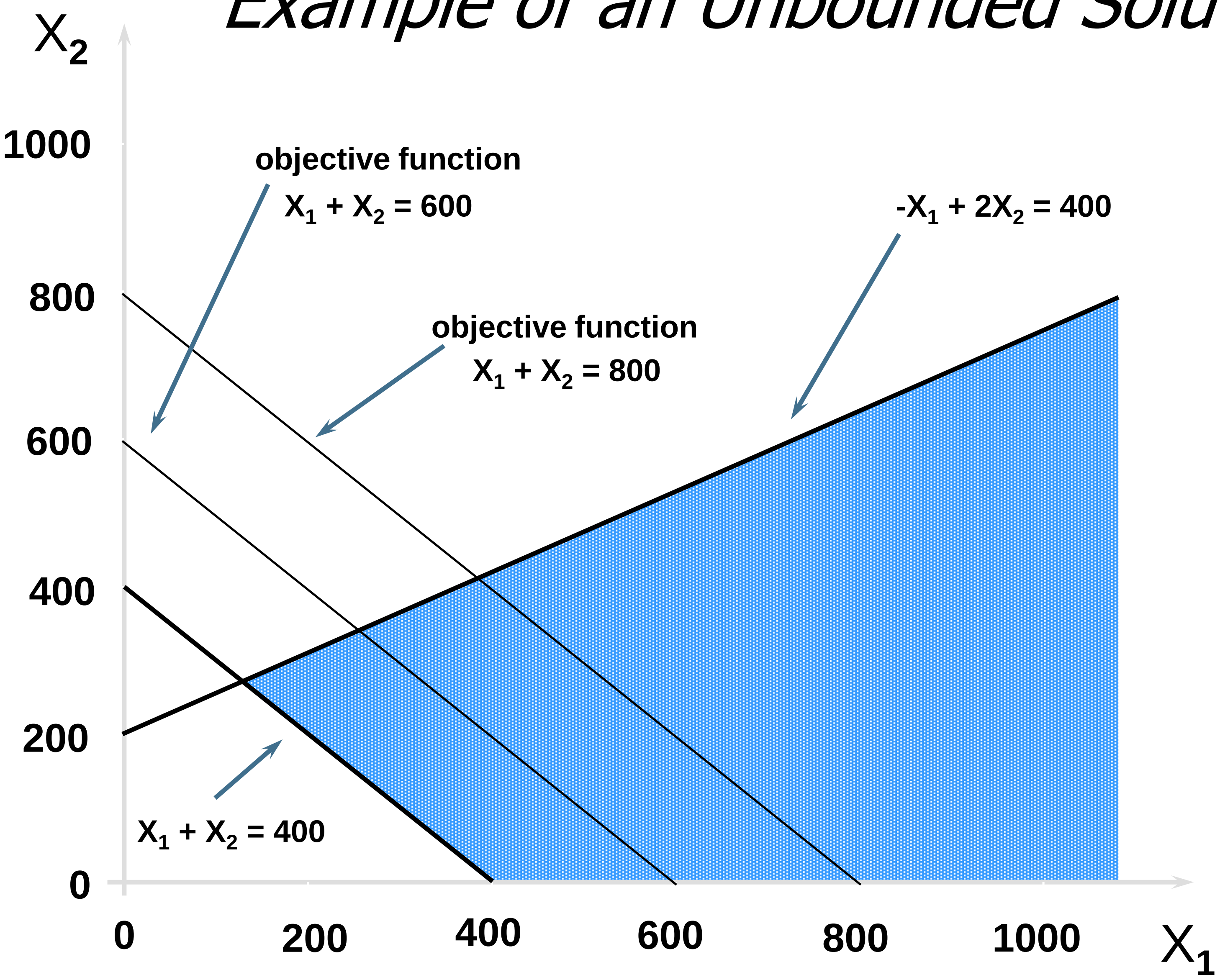
Example of Alternate Optimal Solutions



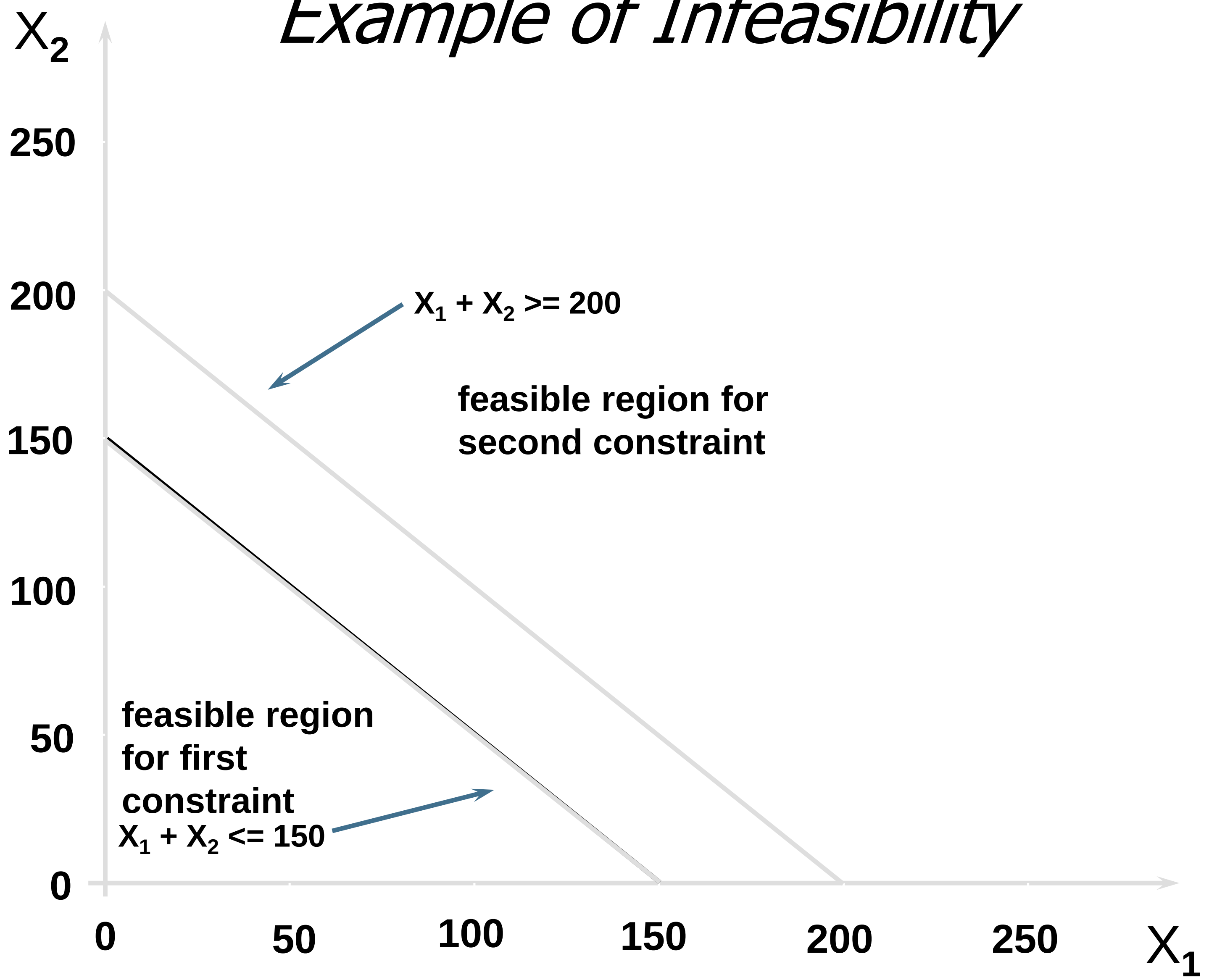
Example of a Redundant Constraint



Example of an Unbounded Solution



Example of Infeasibility



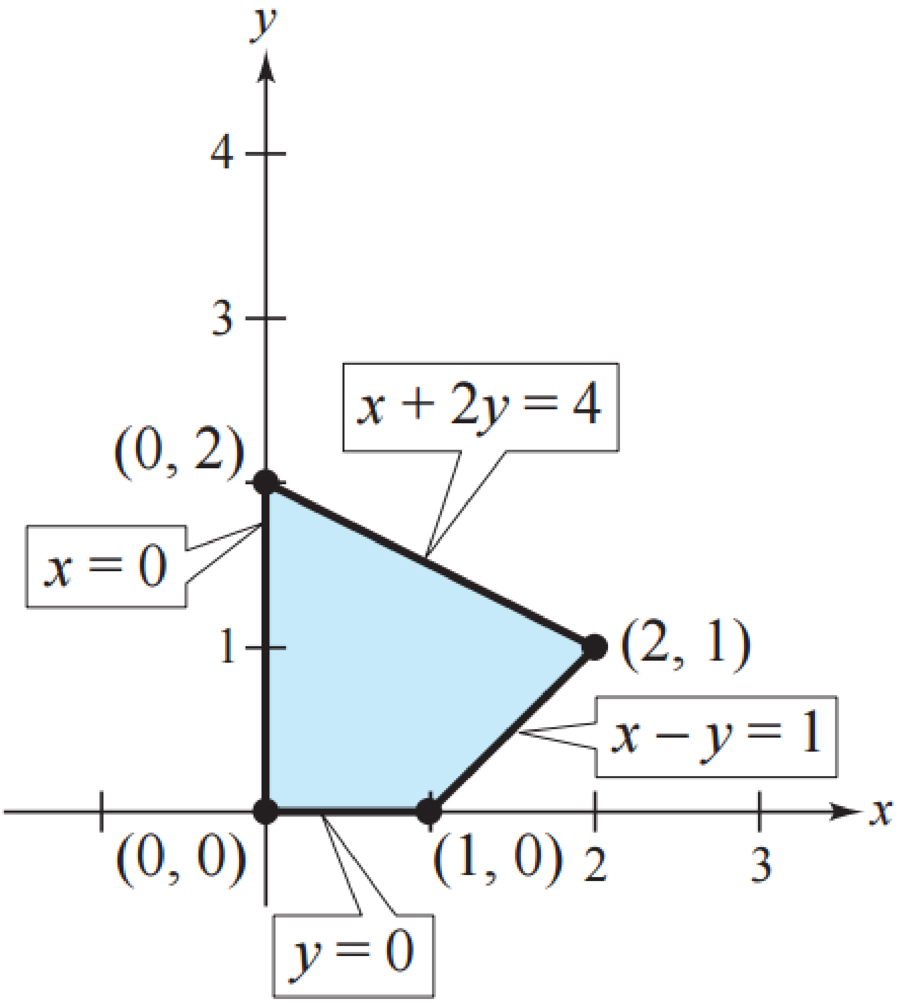
EXAMPLE 1 *Solving a Linear Programming Problem*

Find the maximum value of

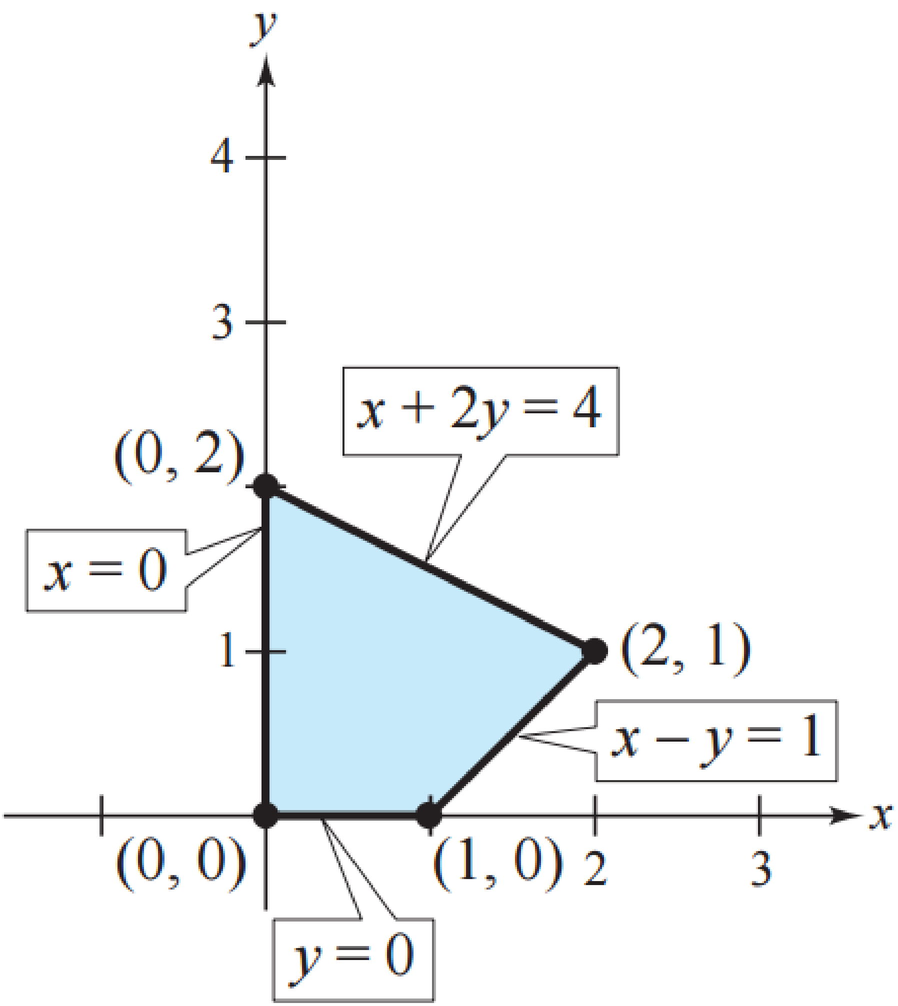
$$z = 3x + 2y \qquad \textbf{Objective function}$$

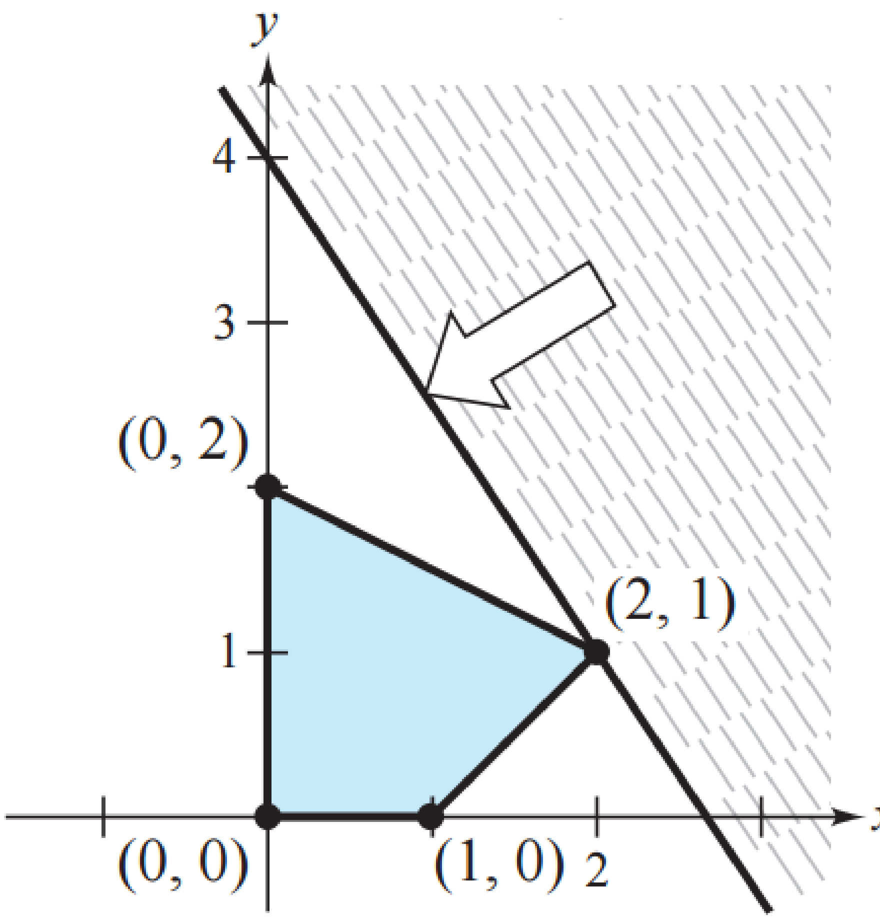
subject to the following constraints.

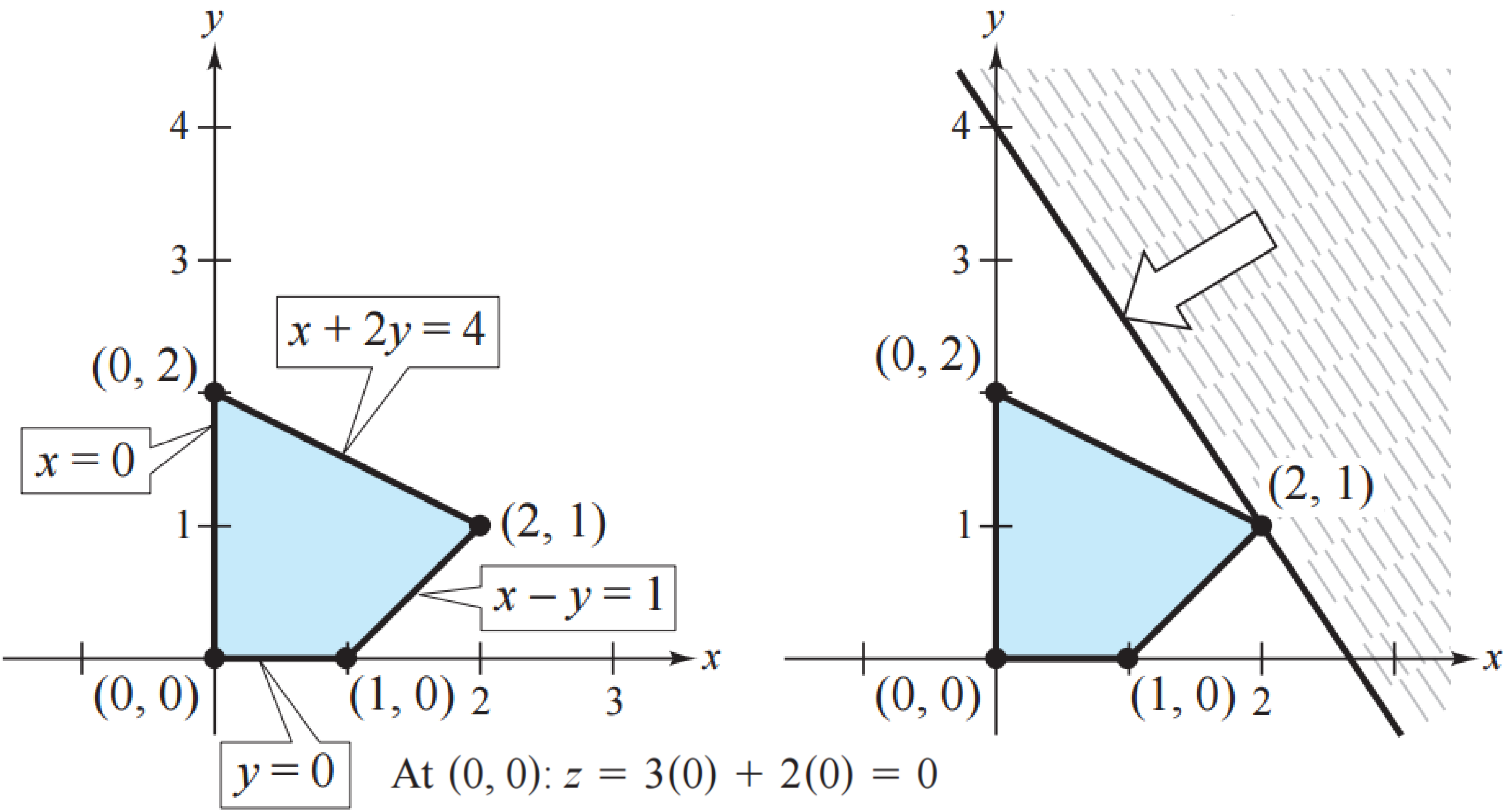
$$\left.\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 4 \\ x - y &\leq 1 \end{aligned}\right\} \quad \textbf{Constraints}$$



y
4
3
1
x
2
3
x + 2y = 4
(0, 2)
x = 0
(2, 1)
x − y = 1
(0, 0)
(1, 0)
y = 0


y
4
3
1
(0, 2)
x = 0
x + 2y = 4
(2, 1)
x − y = 1
(0, 0)
(1, 0) 2
3
x
y = 0


y
4
3
1
(0, 2)
(2, 1)
(0, 0)
(1, 0) 2
x



At $(0, 0)$: $z = 3(0) + 2(0) = 0$

At $(1, 0)$: $z = 3(1) + 2(0) = 3$

At $(2, 1)$: $z = 3(2) + 2(1) = 8$ **(Maximum value of z)**

At $(0, 2)$: $z = 3(0) + 2(2) = 4$

Exercise of Real-world Application

The liquid portion of a diet is to provide at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C daily. A cup of dietary drink X provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C. Now, suppose that dietary drink X costs \$0.12 per cup and drink Y costs \$0.15 per cup. How many cups of each drink should be consumed each day to minimize the cost and still meet the stated daily requirements?

Solution to Exercise

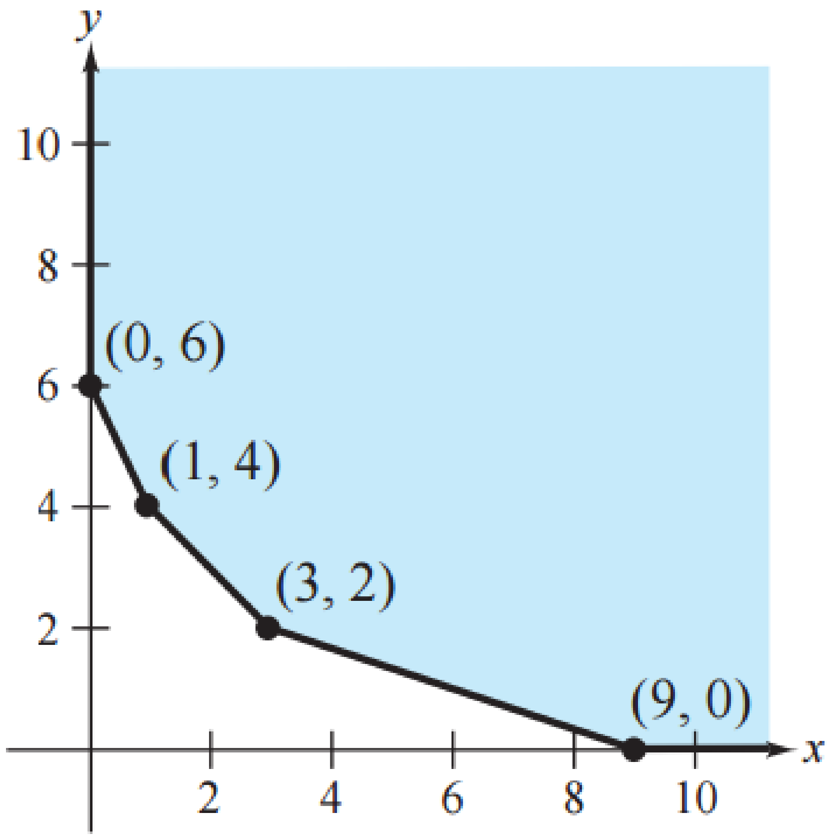
We begin by letting x be the number of cups of dietary drink X and y be the number of cups of dietary drink Y. Moreover, to meet the minimum daily requirements, the following inequalities must be satisfied.

$$\left.\begin{array}{ll} \text{For calories:} & 60x + 60y \geq 300 \\ \text{For vitamin A:} & 12x + 6y \geq 36 \\ \text{For vitamin C:} & 10x + 30y \geq 90 \\ & x \geq 0 \\ & y \geq 0 \end{array}\right\} \quad \textbf{Constraints}$$

The cost C is given by

$$C = 0.12x + 0.15y. \qquad \textbf{Objective function}$$

Solution to Exercise



At $(0, 6)$: $C = 0.12(0) + 0.15(6) = 0.90$

At $(1, 4)$: $C = 0.12(1) + 0.15(4) = 0.72$

$\boxed{\text{At } (3, 2)\text{: } C = 0.12(3) + 0.15(2) = 0.66}$

At $(9, 0)$: $C = 0.12(9) + 0.15(0) = 1.08$

References and Acknowledgement

- J. Duchi, Introduction to Convex Optimization for Machine Learning, University of California, Berkeley, 2009.

- M. P. Deisenroth, A. A. Faisal, and C. S. Ong, Mathematics for Machine Learning, 2019.

- Cliff Ragsdale, Spreadsheet Modeling and Decision Analysis: A Practical Introduction to Business Analytics, 2014.