COMP 7180 Quantitative Methods for Data Analytics and Artificial Intelligence
Exercise 11 Answer

1. Determine whether the following statement is Ture.
   (a) In MLE, the probability is calculated with the given parameter θ.
   (b) The likelihood function to determine the probability of flipping heads with a biased coin if we see 3 heads out of 5 flips is a polynomial of degree 5.

   Answer:

   (a) **FALSE**. The likelihood is calculated that the parameter θ is a given value by calculating the probability of the outcomes given the outcome.
   (b) **True**. If p is the probability of flipping heads, then the likelihood function is $p^3(1-p)^2$.

2. There is a bag with 12 red and blue balls. Each time, we take out one ball and put it back. After three times, we get "Blue, Blue, Red". What is the maximum likelihood for the number of blue balls in bag?

   Answer:

   Let n be the number of blue balls. Then the likelihood is given by $\frac{n}{12} \times \frac{n}{12} \times \frac{12-n}{12}$. Taking the derivative and setting equal to 0, and we have $n = 8$.

3. We have a coin that you think is biased. We flip it 4 times and get the sequence H H H T (H is heat, T is tail). What is the maximum likelihood estimate for the probability of getting heads?

   Answer:

   Let p be the probability of getting heads. Then the probability of getting H H H T is $p^3(1-p)$. Taking the derivative and setting equal to zero gives $3p^2 - 4p^3 = 0 \rightarrow p = \frac{3}{4}$

4. We have a bag with 5 red and blue balls. We pull out a ball and it is red. We put it back and add 3 blue balls and pull out another ball, which is blue. What is the maximum likelihood for the original number of blue balls.

   Answer:

   Let n be the given number of blue balls. Then the probability for picking a red is $1 - \frac{n}{5}$. After, the probability of getting a blue ball after adding 3 blue balls is $\frac{n+3}{8}$. So, we want to maximize

$$\frac{5-n}{5} \times \frac{n+3}{8} = \frac{-n^2+2n+15}{40} \tag{1}$$

   Taking the derivative and setting it equal to 0, we get that $2n = 2$ or $n = 1$ is the maximum likelihood.

5. Consider the exponential distribution $f(x|\theta) = \theta e^{-\theta x}$ where $\theta > 0$. We have a random independent sample $x_1, x_2, \cdots, x_n$. The mean of this distribution is $\mu = \frac{1}{\theta}$.
   Find the maximum likelihood estimators of the mean μ and θ.

   Answer:

$$f(x_1, x_2, \cdots, x_n) = \theta^n exp(-\theta \sum_{i=1}^{n} x_i) \tag{2}$$

Therefore, we have

$$\log(f(x_1, x_2, \cdots, x_n)) = n \log(\theta) - \theta \sum_{i=1}^{n} x_i \tag{3}$$

Take derivative with respect to θ and set it to zero to find the maximum:

$$\frac{n}{\hat{\theta}} - \sum_{i=1}^{n} x_i = 0 \tag{4}$$

Therefore, the MLE for θ is

$$\hat{\theta} = \left(\frac{1}{n} \sum_{i=1}^{n} x_i \right)^{-1} = \frac{1}{\bar{X}} \tag{5}$$

where $\bar{X}$ is the mean value of $x_1, x_2, \cdots, x_n$.

Since MLE's satisfy the principle of functional in-variance, the MLE of $\mu = \frac{1}{\theta}$ is

$$\hat{\mu} = \bar{X} \tag{6}$$

6. Consider the geometric density $f(x|p) = p(1-p)^x$ where $x = 0, 1, 2, \cdots, n$. We have a random independent samples $x_1, x_2, \cdots, x_n$. Find the maximum likelihood estimator of the mean and p.

   Answer:

$$f(x_1, x_2, \cdots, x_n) = p^n(1-p)^{\sum_{i=1}^{n} x_i} \tag{7}$$

$$\log(f(x_1, x_2, \cdots, x_n)) = n\log(p) + \sum_{i=1}^{n} x_i \log(1-p) \tag{8}$$

Take derivative with respect to p and set it to zero to find the maximum:

$$n\frac{1}{\hat{p}} - \sum_{i=1}^{n} x_i \frac{1}{1-\hat{p}} = 0 \tag{9}$$

Solving for $\hat{p}$, we find the MLE for p is

$$\hat{p} = \frac{1}{1+\bar{X}} \tag{10}$$

The mean of the geometric distribution is given by $\mu = \frac{(1-p)}{p}$. So by functional invariance, the MLE for the mean is

$$\hat{\mu} = \frac{1-\hat{p}}{\hat{p}} = \bar{X} \tag{11}$$