Some Exercises

- Suppose you solve **Ax** = **b** for three special right side vectors **b** (Here **A** is a 3*3 matrix, and **x** and **b** are 3*1 vectors):
- $\mathbf{A}\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$ and $\mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{A}\mathbf{x}_3 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$.
- The three solutions are $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{x}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.
- What is the inverse of **A**?

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- The three solutions are $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{x}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.
- What is the inverse of **A**?

- Solution:
- $\mathbf{A}[\mathbf{x}_2\ \ \mathbf{x}_3\ \ \mathbf{x}_1] = [\mathbf{Ax}_2\ \ \mathbf{Ax}_3\ \ \mathbf{Ax}_1] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.
- $\mathbf{A}\begin{bmatrix} 0 & 3 & 3 \\ 2 & 0 & 2 \\ 3 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow \mathbf{A}\begin{bmatrix} 0 & 1 & 1 \\ \frac{2}{3} & 0 & \frac{2}{3} \\ 1 & \frac{1}{3} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{A}^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ \frac{2}{3} & 0 & \frac{2}{3} \\ 1 & \frac{1}{3} & 1 \end{bmatrix}$

- Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be two $n \times n$ projection matrices.
- (a) What are the eigenvalues of $\mathbf{P}_1$ and $\mathbf{P}_2$?
- (b) Do we have $\mathbf{P}_1(\mathbf{P}_1 - \mathbf{P}_2)^2 = (\mathbf{P}_1 - \mathbf{P}_2)^2\mathbf{P}_1$?

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- Solution: $\mathbf{P}_1$ and $\mathbf{P}_2$ are projection matrices, so we have $\mathbf{P}_1^2 = \mathbf{P}_1$ and $\mathbf{P}_2^2 = \mathbf{P}_2$.
- (a) Assume that $\mathbf{P}_1$ has the eigenvalues α, then $\mathbf{P}_1\mathbf{x} = \alpha\mathbf{x}$.
- For $\mathbf{P}_1^2$, we have $\mathbf{P}_1^2\mathbf{x} = \mathbf{P}_1\mathbf{P}_1\mathbf{x} = \mathbf{P}_1\alpha\mathbf{x} = \alpha\mathbf{P}_1\mathbf{x} = \alpha*\alpha\mathbf{x} = \alpha^2\mathbf{x}$
- Since $\mathbf{P}_1^2 = \mathbf{P}_1$, we have $\alpha^2 = \alpha \Rightarrow \alpha(\alpha - 1) = 0 \Rightarrow \alpha = 0 \text{ or } \alpha = 1$.
- For $\mathbf{P}_2$, it is the same.

- (b)
- $\mathbf{P}_1(\mathbf{P}_1 - \mathbf{P}_2)^2 = \mathbf{P}_1(\mathbf{P}_1 - \mathbf{P}_2)(\mathbf{P}_1 - \mathbf{P}_2) = \mathbf{P}_1(\mathbf{P}_1^2 - \mathbf{P}_1\mathbf{P}_2 - \mathbf{P}_2\mathbf{P}_1 + \mathbf{P}_2^2) = \mathbf{P}_1^3 - \mathbf{P}_1^2\mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_2\mathbf{P}_1 + \mathbf{P}_1\mathbf{P}_2^2$

$= \mathbf{P}_1^3 - \mathbf{P}_1\mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_2\mathbf{P}_1 + \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1^3 - \mathbf{P}_1\mathbf{P}_2\mathbf{P}_1$

- $(\mathbf{P}_1 - \mathbf{P}_2)^2\mathbf{P}_1 = (\mathbf{P}_1 - \mathbf{P}_2)(\mathbf{P}_1 - \mathbf{P}_2)\mathbf{P}_1 = (\mathbf{P}_1^2 - \mathbf{P}_1\mathbf{P}_2 - \mathbf{P}_2\mathbf{P}_1 + \mathbf{P}_2^2)\mathbf{P}_1 = \mathbf{P}_1^3 - \mathbf{P}_1\mathbf{P}_2\mathbf{P}_1 - \mathbf{P}_2\mathbf{P}_1^2 + \mathbf{P}_2^2\mathbf{P}_1$

$= \mathbf{P}_1^3 - \mathbf{P}_1\mathbf{P}_2\mathbf{P}_1 - \mathbf{P}_2\mathbf{P}_1 + \mathbf{P}_2\mathbf{P}_1 = \mathbf{P}_1^3 - \mathbf{P}_1\mathbf{P}_2\mathbf{P}_1$

So, yes, we have $\mathbf{P}_1(\mathbf{P}_1 - \mathbf{P}_2)^2 = (\mathbf{P}_1 - \mathbf{P}_2)^2\mathbf{P}_1$