COMP 7180 Quantitative Methods for Data Analytics and Artificial Intelligence
Exercise 6 Answer

1. Are the following sets convex? Give a brief justification for each of the following cases:
   (a) C = $\left\{\mathbf{x} \in \mathbb{R}^n | \mathbf{a}^\mathbf{T}\mathbf{x} \geq \mathbf{b} \text{ or } \|\mathbf{x}-\mathbf{a}\| \leq \varepsilon\right\}$
   (b) C = $\left\{\mathbf{x} \in \mathbb{R}^n | \mathbf{x}^\mathbf{T}\mathbf{y} \geq 1 \text{ for all } \mathbf{y} \in S\right\}$
   (c) C = $\{(x, y) \in \mathbb{R}^2 | y \geq x^2\}$

   Answer:

   (a) No in general. The union of two convex sets may not be convex.
   (b) C is the intersection of hyperplanes (half-space) $\mathbf{x}^\mathbf{T}\mathbf{y} \geq 1$ parametrized by $\mathbf{y} \in S$ (even if S is not convex), so it is convex.
   (c) Take any two points $(x_1, y_1) \in C$ and $(x_2, y_2) \in C$. We need to prove that

   $$(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \in C \rightarrow \lambda y_1 + (1-\lambda)y_2 \geq (\lambda x_1 + (1-\lambda)x_2)^2 \tag{1}$$

   for $0 \geq \lambda \geq 1$. And we have

   $$(\lambda x_1 + (1-\lambda)x_2)^2 \leq \lambda x_1^2 + (1-\lambda)x_2^2 \leq \lambda y_1 + (1-\lambda)y_2 \tag{2}$$

   where the first inequality is from the convexity of x^2, and the second inequality is from the definition of C.

2. Prove that any locally minimum point of a convex function is globally minimum.

   Answer:

   Suppose x is locally minimum (around a ball of radius R) and y is minimum with $f_0(y) < f_0(x)$. We will show this cannot be.
   Just take the segment from x to y: $z = \theta y + (1-\theta)x$. Obviously, the function is strictly decreasing along the segment since $f_0(y) < f_0(x)$:

   $$\theta f_0(y) + (1-\theta)f_0(x) < f_0(x) \quad \theta \in [0,1] \tag{3}$$

   Using now the convexity of the function, we can write

   $$f_0(\theta y + (1-\theta)x) < f_0(x) \quad \theta \in [0,1] \tag{4}$$

   Finally, just choose θ sufficiently small such that the point z is in the ball of local minimum of x, arriving at a contradiction.

3. Consider whether the following functions are convex function:

   (a) $f(x) = ax + b$, where $a, b \in \mathbf{R}$

   (b) $f(x) = x^p$, where $x > 0$, and $p \geq 1$ or $p \leq 0$

   (c) $f(x) = x^p$, where $x > 0$, and $0 < p < 1$.

   (d) $f(x) = x \log x$, for $x > 0$.

   (e) $f(x) = \log x$, for $x > 0$.

   Answer:

   (a) Yes. The second order of $f(x)$ is 0.

   (b) Yes. The second order of $f(x)$ is $(p^2 - p)x^{p-2}$. When $p \geq 1$ or $p \leq 0$, we have $(p^2 - p) \geq 0$.

   (c) No. The second order of $f(x)$ is $(p^2 - p)x^{p-2}$.When $0 < p < 1$, we have $(p^2 - p) < 0$.

   (d) Yes. The second order of $f(x)$ is $\frac{1}{x} > 0$, for $x > 0$.

   (e) No. The second order of $f(x)$ is $-\frac{1}{x^2} < 0$, for $x > 0$.