COMP 7180 Quantitative Methods for Data Analytics and Artificial Intelligence
Assignment 2

Note:

1. Instruction of assignment submission:
    (a) Write all your answers clearly using Microsoft Word/Latex;
    (b) For students who use Microsoft Word, please "Insert → Equation" to write all the formulations;
    (c) Name your document using the following format: COMP7180_Assignment_2_StudentID_StudentName.doc or COMP7180_Assignment_2_StudentID_StudentName.pdf;
    (d) Submit the document on Moodle;
    (e) Taking pictures/photos of handwritten manuscript won't b accepted and will be given Zero Mark!
2. The submission deadline is 2024 November 30 17:00 PM
3. This is an individual work. Plagiarism is strictly forbidden.

Problem 1. (**15 Marks**) Consider whether the following functions are convex function, and write down the reason.

(a) (**5 Marks**) $f(x) = e^x - 1$

(b) (**5 Marks**) $f(x) = \sum_{i=1}^{n} \alpha_i x_{[i]}$, where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0$ and $x_{[i]}$ is the i^{th} largest component of x. (Hint: $g(x) = \sum_{j=1}^{k} x_{[j]}$ is convex.)

(c) (**5 Marks**) $f(x) = -\log(-\log(\sum_{i=1}^{m} e^{a_i x + b_i}))$, where $\sum_{i=1}^{m} e^{a_i x + b_i} < 1$. (Hint: $\log(\sum_{i=1}^{n} e^{y_i})$ is convex.)

Problem 2. (**10 Marks**) Let $f(x) = \frac{1}{2}x^2$ be a convex function. Find its conjugate function $f^*(y)$ using the Lagrange multiplier method.

Problem 3. (**10 Marks**)
Minimizing the objective function $f(x, y) = (x - 2)^2 + (y - 3)^2$ subject to the constraint $x^2 + y^2 = 16$. (Hint: Lagrange multiplier method)

Problem 4. (**15 Marks**)
Determine whether the function $f(x, y) = x^3 + xy + y^3 - 3x - 6y$ is convex, concave, or neither by analyzing its Hessian matrix. If neither, find a point where it is not convex.

Problem 5. (**10 Marks**) Derive the dual norm of l_2-norm (Euclidean norm):

$$\langle \mathbf{y} \rangle_* = \sup\{\mathbf{y}^T\mathbf{x} : \mathbf{x}^T\mathbf{x} \leq 1\},$$

using the Lagrangian multiplier method.

Problem 6. (**15 Marks**)
Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a distribution with probability density function

$$f(x, \alpha) = \alpha^{-2} x e^{-\frac{x}{\alpha}}, \; x > 0, \alpha > 0$$

A dataset is given as

$$x_1 = 0.25, \; x_2 = 0.75, \; x_3 = 1.5, \; x_4 = 2.5, \; x_5 = 2$$

(a) (**9 Marks**) Obtain the maximum likelihood estimator of α, $\hat{\alpha}$. Write down the calculation in detail. (The final result should be a function of the unobserved x_i and the sample size n.)

(b) (**6 Marks**) Calculate the estimate with the given dataset. Write down the calculation in detail. (The final result should be a single number for this dataset.)

Problem 7. (**10 Marks**) Consider two Variables X and Y with joint distribution $P(X, Y)$. Prove that the following two results:

(a) (**5 Marks**) $E[X] = E_Y[E_X[X \mid Y]]$

(b) **(5 Marks**) $Var[X] = E_Y[Var_X[X \mid Y]] + Var_Y[E_X[X \mid Y]]$

Problem 8. (**15 Marks**) There are four dice in a drawer: one tetrahedron (4-sides), one hexahedron (i.e., cube, 6-sides), and two octahedral (8-sides). Your friend secretly grabs one of the four dice at random. Let S be the number of sides on the chosen die.

Now your friend rolls the chosen die and without showing the result to you. Let R be the result of the roll.

Solve the following questions and write down the calculation in detail.

(a) (**5 Marks**) What is the probability mass function (PMF) of S?

(b) (**5 Marks**) Use Bayes rule to find $P(S = k \mid R = 3)$ for $k = 4,6,8$. Which die is most likely if $R = 3$? Terminology: The PMF of "S given $R = 3$."

(c) (**5 Marks**) Which die is most likely if $R = 6$? Hint: You can either repeat the computation in (b), or you can reason based on your result in (b).