COMP 7180 Quantitative Methods for Data Analytics and Artificial Intelligence
Exercise 2 Answer

1. Given matrices $\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 7 & -9 \\ -2 & 6 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 4 \\ 6 & 8 \\ -8 & 3 \end{bmatrix}$, answer the following questions:

   (a) Calculate $\mathbf{A}^T$

   Answer:

$$\mathbf{A}^T = \begin{bmatrix} 5 & 7 & -2 \\ 2 & -9 & 6 \end{bmatrix} \tag{1}$$

   (b) Calculate $\mathbf{A}^T\mathbf{B}$

   Answer:

$$\mathbf{A}^T\mathbf{B} = \begin{bmatrix} 5 & 7 & -2 \\ 2 & -9 & 6 \end{bmatrix} \cdot \begin{bmatrix} 4 & 4 \\ 6 & 8 \\ -8 & 3 \end{bmatrix} = \begin{bmatrix} 78 & 70 \\ -94 & -46 \end{bmatrix} \tag{2}$$

   (c) Calculate $\mathbf{B}^T\mathbf{A}$

   Answer:

$$\mathbf{B}^T\mathbf{A} = (\mathbf{A}^T\mathbf{B})^T = \begin{bmatrix} 78 & -94 \\ 70 & -46 \end{bmatrix} \tag{3}$$

2. Determine whether the following column vectors are linearly independent. If not, calculate the rank r and give an example of one vector that can be represented by other vectors.

   (a) $\begin{bmatrix} 2 & 9 & 5 & 4 \\ 3 & 2 & 7 & 6 \\ -4 & 8 & 3 & -8 \end{bmatrix}$

   Answer:

It is not linearly independent, the rank $r = 3$. $\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 \\ 6 \\ -8 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 8 & 8 & 9 & 2 \\ 0 & 3 & 1 & 1 & 3 \\ 0 & 0 & 5 & 2 & 8 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}$

Answer:

It is not linearly independent, the rank $r = 4$. $\begin{bmatrix} 9 \\ 1 \\ 2 \\ 0 \end{bmatrix} = 0.4 \begin{bmatrix} 8 \\ 1 \\ 5 \\ 0 \end{bmatrix} + 0.2 \begin{bmatrix} 8 \\ 3 \\ 0 \\ 0 \end{bmatrix} + 1.4 \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 8 & 7 & 0 & 1 & 3 \\ 8 & 1 & 3 & 9 & 10 \\ 4 & 7 & 0 & 7 & 0 \end{bmatrix}$

Answer:

It is not linearly independent, the rank $r = 3$. $\begin{bmatrix} 1 \\ 9 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 7 \end{bmatrix} - 2 \begin{bmatrix} 3 \\ 10 \\ 0 \end{bmatrix} + \frac{28}{3} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$

3. Given matrix $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$ and vectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are independent vectors. Let vectors $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$,

$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, $||\mathbf{b}|| = ||\mathbf{c}||$ and $\mathbf{b} \cdot \mathbf{c} = -||\mathbf{b}||^2$. Are vectors $\mathbf{Ab}$ and $\mathbf{Ac}$ independent?

Answer:

$\mathbf{Ab}$ and $\mathbf{Ac}$ are linear dependent.

From $||\mathbf{b}|| = ||\mathbf{c}||$ and $\mathbf{b} \cdot \mathbf{c} = -||\mathbf{b}||^2$, we have

$$\mathbf{b} \cdot \mathbf{c} = ||\mathbf{b}|| \cdot ||\mathbf{c}|| \cdot cos\theta = ||\mathbf{b}|| \cdot ||\mathbf{c}|| \cdot cos\theta = ||\mathbf{b}||^2 \cdot cos\theta = -||\mathbf{b}||^2 \tag{4}$$

Therefore, $cos\theta = -1$, the direction between vector $\mathbf{b}$ and $\mathbf{c}$ is $180°$ (π). As a result, $\mathbf{b} + \mathbf{c} = \mathbf{0}$, where $\mathbf{0}$ is vector with all element is equal to 0.

Therefore,

$$\mathbf{Ab} + \mathbf{Ac} = \mathbf{A} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{A} \cdot \mathbf{0} = \mathbf{0} \tag{5}$$

4. Project the vector $\mathbf{b}$ on to the line through vector $\mathbf{a}$:

(a) $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

Answer:

$$p = \frac{a^T b}{a^T a} a = \frac{8}{8} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \tag{6}$$

(b) $\mathbf{b} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

Answer:

$$p = \frac{a^T b}{a^T a} a = \frac{12}{45} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{8}{5} \end{bmatrix} \tag{7}$$

5. Given matrix $\mathbf{A} = \begin{bmatrix} -6 & 1 \\ 6 & -6 \end{bmatrix}$ and vector $\mathbf{b} = \begin{bmatrix} -20 \\ -21 \end{bmatrix}$. Calculate vector $\mathbf{x}$ that minimize $||\mathbf{Ax} - \mathbf{b}||^2$.

Answer:

$$
\begin{aligned}
\hat{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b} &= \left(\begin{bmatrix} -6 & 1 \\ 6 & -6 \end{bmatrix}^T \begin{bmatrix} -6 & 1 \\ 6 & -6 \end{bmatrix}\right)^{-1} \begin{bmatrix} -6 & 1 \\ 6 & -6 \end{bmatrix}^T \begin{bmatrix} -20 \\ -21 \end{bmatrix} \\
&= \left(\begin{bmatrix} -6 & 6 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} -6 & 1 \\ 6 & -6 \end{bmatrix}\right)^{-1} \begin{bmatrix} -6 & 6 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} -20 \\ -21 \end{bmatrix} = \begin{bmatrix} 72 & -42 \\ -42 & 37 \end{bmatrix}^{-1} \begin{bmatrix} -6 & 6 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} -20 \\ -21 \end{bmatrix} \\
&= \begin{bmatrix} 0.0411 & 0.0467 \\ 0.0467 & 0.0800 \end{bmatrix} \begin{bmatrix} -6 & 6 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} -20 \\ -21 \end{bmatrix} = \begin{bmatrix} -0.2 & -0.0333 \\ -0.2 & -0.2 \end{bmatrix} \begin{bmatrix} -20 \\ -21 \end{bmatrix} = \begin{bmatrix} 4.7 \\ 8.2 \end{bmatrix}
\end{aligned} \tag{8}
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