数据科学与工程算法基础 习题12

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必要性

假设 f 是一个子模函数

对于 $S \subseteq T \subseteq V, C \subseteq V \backslash T$

有 $f(S \cup C \cup T) + f((S \cup C) \cap T) \leq f(S \cup C) + f(T)$

又 $S \cup C \cup T = T \cup C, (S \cup C) \cap T = S$

因此 $f(T \cup C) + f(S) \leq f(S \cup C) + f(T)$

即 $f(S \cup C) - f(S) \geq f(T \cup C) - f(T)$

充分性

假设 $S \subseteq T \subseteq V, C \subseteq V \backslash T$，且 $f(S \cup C) - f(S) \geq f(T \cup C) - f(T)$

则有 $f(S \cup C) - f((S \cup C) \cap T) \geq f(S \cup C \cup T) - f(T)$

即 $f(S \cup C \cup T) + f((S \cup C) \cap T) \leq f(S \cup C) + f(T)$

令 $A = S \cup C$，有 $A, T \subseteq V$

且 $f(A \cup T) + f(A \cap T) \leq f(A) + f(T)$

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第 1 次循环，$S = \varnothing$

$$\begin{aligned}&f(S) = 0\\&\Delta(A_1) = 3, \Delta(A_2) = 3, \Delta(A_3) = 3, \Delta(A_4) = 3\\&\Delta(A_5) = 3, \Delta(A_6) = 7, \Delta(A_7) = 2, \Delta(A_8) = 3\end{aligned}$$

选取 A_6，S 更新为 $\varnothing \cup A_6 = A_6 = \{c, d, g, h, k, l\}$

第 2 次循环

$$\begin{aligned}&f(S) = 6\\&\Delta(A_1) = 1, \Delta(A_2) = 2, \Delta(A_3) = 2, \Delta(A_4) = 3\\&\Delta(A_5) = 2, \Delta(A_7) = 1, \Delta(A_8) = 3\end{aligned}$$

选取 A_4，S 更新为 $A_6 \cup A_4 = \{a, c, d, e, g, h, i, k, l\}$

第 3 次循环

$$\begin{aligned}&f(S) = 9\\&\Delta(A_1) = 1, \Delta(A_2) = 1, \Delta(A_3) = 1, \Delta(A_5) = 1\\&\Delta(A_7) = 0, \Delta(A_8) = 0\end{aligned}$$

选取 A_1，S 更新为 $A_6 \cup A_4 \cup A_1 = \{a,b,c,d,e,g,h,i,k,l\}$

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1. 取 $S, T \subseteq V$

有 $f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$

因此

$$\begin{aligned} f(\bar{S}) + f(\bar{T}) &\geq f(\bar{S} \cup \bar{T}) + f(\bar{S} \cap \bar{T}) \\ &= f(\overline{S \cap T}) + f(\overline{S \cup T}) \end{aligned}$$

即

$$\bar{f}(S) + \bar{f}(T) \geq \bar{f}(S \cup T) + \bar{f}(S \cap T)$$

因此 $\bar{f}(A) = f(\bar{A})$ 是子模函数

2. 取 $A, B \subseteq V$

由 $S \subseteq V$ 可知 $A \cap S, B \cap S \subseteq V$

$$\begin{aligned} f(A \cap S) + f(B \cap S) &\geq f((A \cap S) \cup (B \cap S)) + f(A \cap S \cap B \cap S) \\ &= f((A \cup B) \cap S) + f(A \cap B \cap S) \end{aligned}$$

即 $g(A) + g(B) \geq g(A \cup B) + g(A \cap B)$

因此 $g(A) = f(A \cap S)$ 为子模函数

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对 $\forall S, T \subseteq N$

有

$$\begin{aligned} &f(S) + f(T) - f(S \cup T) - f(S \cap T) \\ =&\sum_{i \in S} w_i + \sum_{i \in T} w_i - \sum_{i \in S \cup T} w_i - \sum_{i \in S \cap T} w_i \\ =&\left(\sum_{i \in S} w_i - \sum_{i \in S \cap T} w_i\right) - \left(\sum_{i \in S \cup T} w_i - \sum_{i \in T} w_i\right) \\ =&\sum_{i \in S \setminus T} w_i - \sum_{i \in S \setminus T} w_i \\ =&0 \end{aligned}$$

即 $f(S) + f(T) = f(S \cup T) + f(S \cap T)$

因此 $f(S)$ 是子模函数