

OPTIMAL TIME-OF-USE ELECTRICITY PRICING USING GAME THEORY

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ABSTRACT

Typical user demands of electricity vary throughout the day, which increases the cost to utility companies and decreases the stability of the power system. Time-of-use (TOU) pricing has been proposed as a demand-side management (DSM) method to influence user demands. In this paper, we describe a new approach of optimal TOU pricing strategy based on game theory (GT-TOU). We propose models for costs due to the fluctuating user demands to the utility companies, as well as the user satisfaction measurement because of the difference between the demand and actual load. We design utility functions for the company and the user, and obtain the Nash equilibrium using backward induction and iterative methods. Numerical example shows that our method is effective in leveling the user demand by setting optimal TOU prices, in potentially increasing the profit of the utility companies and ensuring overall user benefit.

Index Terms— Time-of-use, electricity price, game theory, optimization, smart grid

1. INTRODUCTION

The fluctuation of electricity demand throughout the day has long been a problem for utility companies. During peak hours, the utility companies face significant pressure to provide users with enough electricity, and may even have to cut off the electricity supply of certain areas when the gap between demand and generation is too large; while in off-peak hours, only a small number of generators are needed to provide sufficient electricity to meet user demand, and the idle generators result in a waste of generation capacity. The utility companies wish to operate the power system on a base load, on which the system is optimized, and therefore is the most efficient. The base load is not the highest load that a unit can provide, but operation far away from base load is not cost efficient and may harm the stability of the power system. Therefore, utility companies wish the user demand to remain relatively “constant” during the day, so that they can design and build generation units according to the “constant demand”.

Time-of-use (TOU) pricing is an efficient way of demand side management (DSM), which utility companies can em-

ploy to influence user behavior. By setting different prices during the day, the utility company can encourage customers to shift their demand to off-peak hours, resulting in a more leveled demand curve. In [1], Caves *et al.* provides econometric analysis of a TOU pricing experiment in Wisconsin showing that short term electricity demand is not inelastic, and that peak and off-peak electricity are partial substitutes. Hartway *et al.* demonstrated in [2] through an experiment that TOU is profitable to a utility, and in general, the customers are satisfied with the TOU price option. In recent years, time-of-use pricing and real time pricing have attracted growing attention both in academia and in industry [3]–[6], especially with the development of smart grid, which enables the implementation of time-dependent pricing.

We propose an optimal time-of-use pricing strategy for smart grids using game theory (GT-TOU). A day is divided into N periods, and the price is optimized for each time period. The goal is to influence the user behavior through TOU pricing such that the load throughout the day is leveled. Because utility companies seek to maximize profits while users seek minimized costs and assured supply, we consider a game between utility companies and users using a multi-stage game model. In this model, the utility company sets the electricity prices, while the customers respond to the price by adjusting the amount of electricity they use. Utility functions are designed for the company and the users, in which we take into account the cost of varying demands to the utility company, and the satisfaction measurement of users. Our pricing strategy is different from the real-time pricing in [5] and [6]. As described in [5], users are not well prepared to respond to the time-varying prices. Therefore, price prediction is often required to implement real-time pricing [5], and energy management controller may be needed to help users manage their power usage [6]. In contrast, in our model users are informed of the price ahead of time, and the TOU prices remain stable during a relatively long time unless there is significant change in the characteristics of user demands or generation cost. We believe these features would make our model easier to implement in practice.

The remainder of this paper is organized as follows: Section 2 describes the notation and formation of the game model; Sections 3 and 4 solve the equilibrium of the game; a numerical example is given in Section 5; and the paper is

concluded in Section 6.

2. NOTATION AND MODEL

We divide a day into N periods, where N depends on the scenario of the application. For hourly based pricing $N = 24$. Let $\mathbf{c} = [c_1, c_2, \dots, c_N]^T$ denote the unit marginal cost of electricity, $\mathbf{p} = [p_1, p_2, \dots, p_N]^T$ denote the unit sales price of electricity, $\mathbf{g} = [g_1, g_2, \dots, g_N]^T$ denote the electricity generation, $\mathbf{d} = [d_1, d_2, \dots, d_N]^T$ denote the user demand in a constant price scenario where the constant price is p_0 , and $\mathbf{l} = [l_1, l_2, \dots, l_N]^T$ denote the actual user load in response to the time-of-use price. The subscripts denote the corresponding time period.

We model the utility function of the company u_1 as:

$$u_1 = \sum_{k=1}^N p_k l_k - \sum_{k=1}^N c_k g_k - v(\mathbf{g}) \quad (1)$$

where $v(\mathbf{g})$ corresponds to the cost caused by varying user demand during the day. We model this cost using the variance of electricity generation multiplied by a parameter μ , i.e.,

$$v(\mathbf{g}) = \mu \frac{\sum_{k=1}^N (g_k - \bar{g})^2}{N}, \quad (2)$$

where \bar{g} is the average electricity generation during the day. The cost function C of users include the cost they pay for the electricity and their satisfaction with the service, i.e.,

$$C = \sum_{k=1}^N p_k l_k + \sum_{k=1}^N s_k(l_k, d_k), \quad (3)$$

where $s_k(l_k, d_k)$ denote the user satisfaction function. The user satisfaction function quantitatively models the user satisfaction caused by the gap between demand and actual load. If the actual load is smaller than the demand, the function value is positive, meaning the users are not satisfied. The satisfaction increases faster as the actual load decreases. On the other hand, if the actual load is greater than the user demand, the function value is negative, meaning the users are satisfied. However, the increase of satisfaction slows down as the actual load continues to increase, because the users won't be "infinitely" happier when they use more electricity. When the actual load equals the user demand, the function value is zero. Therefore the satisfaction function $s_k(l_k, d_k)$ should satisfy the following conditions:

1. If $l_k = d_k$:

$$s_k(l_k, d_k) = 0.$$

2. If $l_k > d_k$:

$$s_k(l_k, d_k) < 0, \quad \frac{\partial s_k}{\partial l_k} < 0, \quad \frac{\partial^2 s_k}{\partial^2 l_k} > 0.$$

3. If $l_k < d_k$:

$$s_k(l_k, d_k) > 0, \quad \frac{\partial s_k}{\partial l_k} < 0, \quad \frac{\partial^2 s_k}{\partial^2 l_k} > 0.$$

In this paper we select $s_k(l_k, d_k)$ as:

$$s_k(l_k, d_k) = \beta_k \left[\left(\frac{l_k}{d_k} \right)^{\alpha_k} - 1 \right], \quad (4)$$

where $0 < \alpha_k < 1$ and $\beta_k < 0$. Other proper functions can also be selected based on the nature and behavior of users. The utility function of users is then the negative of their cost function C , i.e.,

$$u_2 = -C = -\sum_{k=1}^N p_k l_k - \sum_{k=1}^N s_k(l_k, d_k). \quad (5)$$

In this game model, we need to maximize the utility functions of both the company and the users. The optimization problem is described as:

$$\begin{aligned} (\mathbf{p}^*, \mathbf{g}^*) &= \arg \max_{\mathbf{p}, \mathbf{g}} u_1 = \sum_{k=1}^N [p_k l_k - c_k g_k - \frac{\mu}{N} (g_k - \bar{g})^2] \\ \mathbf{l}^* &= \arg \max_{\mathbf{l}} u_2 = -\sum_{k=1}^N \left\{ p_k l_k + \beta_k \left[\left(\frac{l_k}{d_k} \right)^{\alpha_k} - 1 \right] \right\} \\ \text{subject to} \quad &0 \leq l_k \leq g_k \leq g_{\max}, \quad k = 1, 2, \dots, N \\ &c_k \leq p_k, \quad k = 1, 2, \dots, N. \end{aligned}$$

In this preliminary work, we assume that the generation equals user load, i.e., $\mathbf{g} = \mathbf{l}$, and that the actual load is less than the generation capacity g_{\max} . Then, the problem can be simplified as:

$$\mathbf{p}^* = \arg \max_{\mathbf{p}} u_1 = \sum_{k=1}^N [p_k l_k - c_k l_k - \frac{\mu}{N} (l_k - \bar{l})^2] \quad (6)$$

$$\begin{aligned} \mathbf{l}^* &= \arg \max_{\mathbf{l}} u_2 = -\sum_{k=1}^N \left\{ p_k l_k + \beta_k \left[\left(\frac{l_k}{d_k} \right)^{\alpha_k} - 1 \right] \right\} \\ \text{subject to} \quad &c_k \leq p_k, \quad k = 1, 2, \dots, N \\ &0 \leq l_k, \quad k = 1, 2, \dots, N. \end{aligned} \quad (7)$$

3. OPTIMIZING UTILITY FUNCTIONS

Since this is a multi-stage game, we use backward induction [7] to solve for the equilibrium. The utility company takes action first by setting the electricity price, and then customers adjust the amount of electricity they use. Therefore, we first minimize u_2 with respect to $\{l_k\}_{k=1}^N$, then plug the minimizer into u_1 and optimize u_1 with respect to $\{p_k\}_{k=1}^N$.

3.1. Optimal Demand Response to Price

In order to find user's optimal demand response to the price set by utility companies, we consider the electricity prices of different time periods $\{p_k\}_{k=1}^N$ as given, and set the first order

derivatives of u_2 with respect to $\{l_k\}_{k=1}^N$ to be zero. Then we get the optimal response $\{l_k^*\}_{k=1}^N$ as:

$$l_k^* = \left(-\frac{p_k d_k}{\alpha_k \beta_k}\right)^{\frac{1}{\alpha_k - 1}} d_k, \quad k = 1, 2, \dots, N. \quad (8)$$

The form of (8) is interesting as it reminds us of the price elasticity of demand [8] in economics, which measures the percentage change in demand quantity caused by percentage change in price. Let ϵ denote the elasticity constant, d denote the demand and p denote the price. The relationship between change of price and change of demand can be described as:

$$\frac{\partial d/d}{\partial p/p} = \epsilon, \quad (9)$$

where ϵ is almost always negative, because in most cases the increase of price will result in reduced demand, and vice versa. We can also assume ϵ to be constant when the changes of d and p are not too big. After integration on both sides and simple computation, we obtain the following relationship:

$$\tilde{d} = \left(\frac{\tilde{p}}{p_0}\right)^\epsilon d_0. \quad (10)$$

d_0 and p_0 are the original demand and price, respectively, and \tilde{d} is the new demand corresponding to the new price \tilde{p} . It can be observed that (8) is in exactly the same form as (10), if we select parameters $\{\alpha_k\}_{k=1}^N$ and $\{\beta_k\}_{k=1}^N$ such that $\epsilon_k = \frac{1}{\alpha_k - 1}$ and $p_0 = -\alpha_k \beta_k / d_k$. Equation (8) can then be rewritten as:

$$l_k^* = \left(\frac{p_k}{p_0}\right)^{\epsilon_k} d_k, \quad k = 1, 2, \dots, N. \quad (11)$$

Note that d_k and l_k^* in (11) correspond to d_0 and \tilde{d} in (10), respectively. We will use (11) instead of (8) in the rest of this paper, because on the one hand, it will simplify the notation for the rest of this paper; on the other hand, the parameters have practical meanings.

3.2. Optimal Pricing Based on User Response

In section 3.1 we obtained the optimal response of users to electricity prices. In this section, we will maximize the utility function of companies by finding the optimal pricing strategy based on the user response. In order to maximize u_1 , we take derivatives of u_1 with respect to $\{p_k\}_{k=1}^N$. Note that in this case, user loads $\{l_k\}_{k=1}^N$ are functions of the prices $\{p_k\}_{k=1}^N$.

$$\frac{\partial u_1}{\partial p_k} = l_k + p_k \frac{\partial l_k}{\partial p_k} - c_k \frac{\partial l_k}{\partial p_k} - \frac{2\mu}{N} (l_k - \bar{l}) \frac{\partial l_k}{\partial p_k} \quad (12)$$

Plug (11) into (12), and let (12) equal zero, we obtain the optimal price p^* as a function of user response l^* :

$$p_k^* = \frac{\epsilon_k}{1 + \epsilon_k} c_k + \frac{2\mu}{N} \cdot \frac{\epsilon_k}{1 + \epsilon_k} (l_k^* - \bar{l}^*). \quad (13)$$

Equation (13) provides an intuitive interpretation of the optimal price p^* . If the actual load l_k at time period k is higher than the average load \bar{l} , namely $l_k - \bar{l} > 0$, then the price is raised; if the actual load at time period k is lower than average, i.e., $l_k - \bar{l} < 0$, then the price is reduced.

4. SOLVING FOR THE EQUILIBRIUM

So far we have obtained p^* and l^* as functions of l^* and p^* , respectively, as follows:

$$\begin{cases} p_k^* &= \frac{\epsilon_k}{1 + \epsilon_k} c_k + \frac{2\mu}{N} \cdot \frac{\epsilon_k}{1 + \epsilon_k} (l_k^* - \bar{l}^*) \\ l_k^* &= \left(\frac{p_k^*}{p_0}\right)^{\epsilon_k} d_k, \quad k = 1, 2, \dots, N \end{cases} \quad (14)$$

The Nash equilibrium, which is the optimal strategy pair (p^*, l^*) , can be obtained by solving (14). Since there are $2N$ equations, $\{p_k\}_{k=1}^N$ depend on all $\{l_k\}_{k=1}^N$, and $\{l_k\}_{k=1}^N$ are not linear functions of $\{p_k\}_{k=1}^N$, it is extremely difficult, if not impossible, to find a closed form solution. Therefore, we use two iterative methods to solve (14) in this section.

4.1. Newton-Raphson iteration

In order to employ the Newton-Raphson iteration method, we first reduce the number of equations by combining each equation pair. Plug (11) into (13) and then let

$$f_k(p) = p_k - \frac{\epsilon_k}{1 + \epsilon_k} c_k - \frac{2\mu}{N} \cdot \frac{\epsilon_k}{1 + \epsilon_k} \left[\left(\frac{p_k}{p_0}\right)^{\epsilon_k} d_k - \frac{1}{N} \sum_{i=1}^N \left(\frac{p_i}{p_0}\right)^{\epsilon_i} d_i \right]. \quad (15)$$

We now have only N equations $\{f_k(p) = 0\}_{k=1}^N$. Denote

$$\mathbf{f}(p) = [f_1(p), f_2(p), \dots, f_N(p)]^T, \quad (16)$$

and then we obtain $\frac{\partial \mathbf{f}(p)}{\partial p^T}$, where

$$\frac{\partial f_k(p)}{\partial p_k} = 1 - \frac{2\mu}{N} \left(1 - \frac{1}{N}\right) \frac{\epsilon_k^2}{1 + \epsilon_k} \frac{p_k^{\epsilon_k - 1}}{p_0^{\epsilon_k}} d_k, \quad (17)$$

$$\frac{\partial f_k(p)}{\partial p_i} = \frac{2\mu}{N^2} \frac{\epsilon_k \epsilon_i}{1 + \epsilon_k} \frac{p_i^{\epsilon_i - 1}}{p_0^{\epsilon_i}} d_i, \quad k \neq i. \quad (18)$$

After that, we apply Newton-Raphson iteration to solve for the optimal p^* and l^* :

1. Initialize $p^{(i)} = p^{(0)}$
2. $p_k^{(i+1)} = p^{(i)} - \left(\frac{\partial \mathbf{f}(p)}{\partial p^T}\right)^{-1} \mathbf{f}(p)$
3. Repeat 2 until $\|p^{(i)} - p^{(i-1)}\| \leq \delta_1$.

4.2. Alternating minimization

Although the Newton-Raphson method is fast in convergence, the convergence depends on the initial value and also on the parameters. When the Newton-Raphson iteration has convergence issues, we employ alternating minimization as

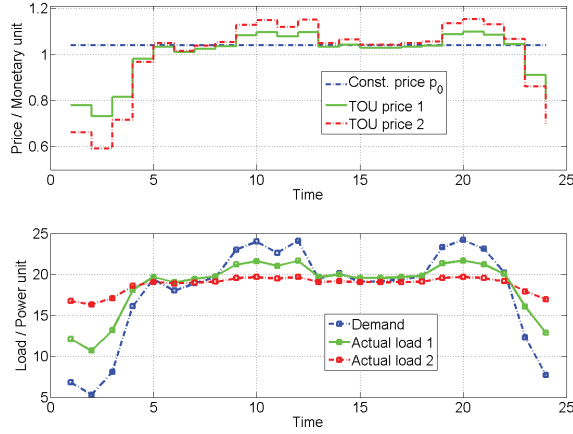


Fig. 1: Comparison of customer prices and loads between constant pricing and the proposed GT-TOU pricing.

Table 1: Comparison of company and user utility function

	Strategy	Total Load	Peak Load	Unit Price	u_1	u_2
Case 1 $\mu = 0.2$	Const.	433.50	24.20	1.040	227.46	-450.84
	GT-TOU	448.89	21.67	1.018	230.40	-435.51
Case 2 $\mu = 1.0$	Const.	433.50	24.20	1.040	200.94	-450.84
	GT-TOU	450.26	19.68	1.008	227.93	-448.91

an alternative way to solve (14), which works well for large μ .

1. For each k : Initialize $p_k^{(i)} = p_k^{(0)}$, $l_k^{(0)} = \left(\frac{p_k^{(0)}}{p_0}\right)^{\epsilon_k} d_k$,
 $\bar{l}^{(0)} = \frac{1}{N} \sum_{k=1}^N l_k^{(0)}$;
2. For each k : $l_k^{(i)} = \frac{N(1+\epsilon_k)}{2\mu\epsilon_k} \left(p_k^{(i-1)} - \frac{c_k\epsilon_k}{1+\epsilon_k}\right) + \bar{l}_k^{(i-1)}$,
 $\bar{l}^{(i)} = \frac{1}{N} \sum_{k=1}^N l_k^{(i)}$, and $p_k^{(i)} = \left(\frac{l_k^{(i)}}{d_k}\right)^{\frac{1}{\epsilon_k}} p_0$;
3. Repeat 2 until $\|l_k - l_{k-1}\| \leq \delta_2$.

5. NUMERICAL EXAMPLE

In this numerical example, we adopted hourly based pricing by dividing a day into $N = 24$ equal time periods. Provided the demands $\{d_k\}_{k=1}^N$ corresponding to constant price p_0 , we used the game based model to find the optimal pricing strategy and user load. The price elasticity ϵ was assumed to be constant during the day and equals -2.0 . We ran the simulation twice with different parameter μ selected as 0.2 and 1.0 respectively. The other parameters were set to be the same.

Fig.1 shows a comparison of prices and loads using constant and GT-TOU pricing. When using GT-TOU pricing, the price is set higher in peak hours, and lower in off-peak hours. The figure also illustrates the effect of cost due to fluctuating user demands: when μ is large, such cost is high, and thus

greater gap between peak hour price and off-peak hour price is set by utility companies to better influence user behavior. Table 1 shows comparison of utility functions using constant pricing and GT-TOU pricing, from which we conclude that when applying GT-TOU prices: (i) the peak hour load significantly decreases; (ii) the utility functions of both the company and users increase; and (iii) users pay a lower unit price for the electricity.

6. CONCLUSION

We proposed an optimal time-of-use pricing strategy for electricity using game theory. We designed utility functions for both utility companies and users, and solved for the Nash equilibrium. The Nash equilibrium provides us with optimal prices and user response. In practice, the parameters of the model can be estimated using historic data from utility companies. This model is flexible, as we can modify the utility functions according to the nature of different types of utility companies and users. Simulation results illustrate that our strategy can level user demand, increase the profits of the utility companies, and reduce bills for electricity users. The leveled user demand also helps ensure a more stable power system. We are now incorporating different types of users, and making the model of the cost to utility companies due to fluctuating user demands more realistic.

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