Super-Resolution in SAR Imaging: Analysis With the Atomic Norm

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***Abstract*—In this paper, we investigate the synthetic aperture radar (SAR) imaging problem via sparse atomic norm reconstruction. A stepped-frequency radar operation and a side looking radar is assumed. However, the analysis presented is general, and extensions to other SAR types, scenes and other radar waveforms are straightforward. The atomic norm is formulated and employed as a penalizer for SAR denoising and reconstruction, as a convex problem. The target positions are readily estimated through the peaks of the dual polynomial. Furthermore, due to strong duality in our convex formulation, we show that the target positions are obtained from either the primal or dual problem solution. Through simulations, we demonstrate the advantages of our approach when compared with traditional back-projection imaging and recent sparse reconstruction techniques.**

***Keywords*—Synthetic Aperture Radar Imaging, Atomic Norm, Target Localization, Stepped-Frequency Radar, Semidefinite Programming**

I. Introduction

In the ever changing modern battlefield, intelligence surveillance, and reconnaissance centric technologies are extremely useful. Synthetic aperture radar (SAR) is such a technology which enables imaging wide scenes from a long range by synthetically generating an aperture through radar motion.

Synthetic aperture and stepped-frequency radars have been widely used in a variety of civil and military applications [1, 2], landmine detection [3] and geo imaging environments [4]. In this paper, we consider a multiple target imaging scene using stepped-frequency measurements from synthetic aperture radar (SAR). Target estimation or localization is one of the fundamental problems in radar signal processing. In a SAR setting, the conventional imaging method based on back-projection directly uses the measurements to create the target image by applying matched filtering with the impulse response of the data acquisition process to form the images. The peaks of this image can be used to estimate the target locations. Alternatively, recent developments in compressive sensing [5–7] inspire forming the radar image by solving an inverse sparse problem either through convex optimization or greedy algorithms [8].

The general approach for estimation using sparse modeling is to first divide the target space in crossrange and downrange uniformly to reduce the continuous parameter space into a finite set of grid points. Then one constructs a dictionary based on these grid points and formulates the radar imaging problem as a dictionary selection problem, which can be solved via a variety of sparse recovery algorithms. Options for such algorithms include the least absolute shrinkage and selection operator (Lasso) [9] and matching pursuit (MP) [10]. However, due to the discretization process, these methods suffer from several severe drawbacks. Firstly, the dictionary will spread out the energy of the target return if its position does not fall onto a regular grid, creating smearing and possible false alarms in the SAR images. Secondly, as one increases the number of grid points, performance guarantees through standard sparse recovery analysis fail since the dictionary becomes highly coherent. Lastly, finer grid points lead to high computational complexity and numerical instability issues, both of which are non-starters in practical SAR.

Chandrasekaran et.al. [11] propose to use the *atomic norm*, induced by the convex hull of the atoms, as the general convex penalty function for linear inverse problems. The atomic norm generalizes the l_1 norm for sparse recovery problems and more importantly, it provides a framework to handle a dictionary with an infinite number of atoms. In the case of line spectrum estimation (or super-resolution [12] if we exchange the time and frequency domains), the atomic norm minimization approach achieves nearly optimal recovery performance [13, 14] and recovers the frequencies exactly under moderate conditions when there is no noise [15]. In this case, the atomic norm minimization can be rewritten as a semidefinite program, which can be solved efficiently using off-the-shelf solvers such as SDPT3 [16] and SeDuMi [17]. Alternatively, some non-convex approaches considering the off-grid problem in line spectrum estimation are presented in [18, 19].

In this paper, we propose a convex relaxation approach to target estimation for stepped-frequency radar. In consideration of noise in real applications, we apply the atomic norm penalty function and formulate the denoising problem as *atomic norm soft thresholding* (AST), which also provides an efficient method to localize the targets without having to discretize the target space. Moreover, we show that AST can be approximately solved using semidefinite programming. We compare our proposed approach with classical radar imaging algorithms including back-projection and Lasso. The outline of this paper is as follows. In Section II, the main problem is illustrated. Our main approaches to denoise the radar signal and estimate the target locations are in Section III. Section IV presents a computational method to solve the AST. Section V

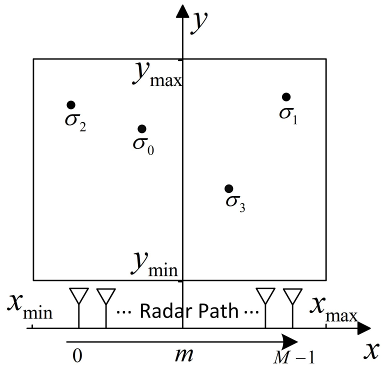

Fig. 1. Illustration of the scene layout.

presents some simulations to support our proposed methods and Section VI concludes the paper.

II. Problem Setup

Let $[N]$ denote the set $\{0, 1, \ldots, N-1\}$ for any natural number $N \in \mathbb{N}$. An M-element synthetic linear aperture (located along the x-axis) is used to transmit waveforms and receive the return signals. Let (x_m^a, y_m^a) denote the position of the m-th antenna for all $m \in [M]$. We assume that each transceiver receives a stepped-frequency signal consisting of N frequencies equispaced over the band $[F_0, F_{N-1}]$; that is

$$F_n = F_0 + n\Delta F, \quad n \in [N],$$

where $\Delta F := \frac{F_{N-1} - F_0}{N-1}$ is the frequency step size. Further, we assume that the measurements at each transceiver do not have any interfering component from the other transceivers.

Suppose there are K point targets in the field $\mathbb{I}$. Without loss of generality, we denote such region by

$$\mathbb{I} = \{(x, y) : x_{\min} \le x \le x_{\max}, y_{\min} \le y \le y_{\max}\}. \quad (1)$$

Let (x_k^t, y_k^t) denote the position of the k-th target for all $k \in [K]$. The antennas are parallel to the x-axis with $y_m^a = 0$ for all $m \in [M]$.

The measurement observed by the m-th antenna corresponding to the n-th frequency can be expressed as

$$\widetilde{\boldsymbol{z}}_m[n] := \boldsymbol{z}_m^\natural[n] + \boldsymbol{w}_m[n],$$

where

$$\boldsymbol{z}_m^\natural[n] = \sum_{k=0}^{K-1} \sigma_k e^{-j2\pi F_n \tau_m(x_k^t, y_k^t)} \quad (2)$$

is the cumulative return from all the targets and $\boldsymbol{w}_m[n]$ is the additive noise. Here, σ_k is the complex reflectivity of the k-th target (we assume the target reflectivity is independent of frequency), and $\tau_m(x_k^t, y_k^t)$ is the two-way travel time between the k-th point target and the m-th antenna. Figure 1 illustrates the geometric model of the radar system. If we assume the scattering and refraction are negligible, $\tau_m(x_k^t, y_k^t) = 2\|(x_k^t, y_k^t) - (x_m^a, y_m^a)\|_2/c$, where the constant c is the speed of light in meters per second.

Without loss of generality, we assume $F_0 = 0$. We then rewrite the target return in (2) as

$$\begin{aligned}\boldsymbol{z}_m^\natural[n] &= \sum_{k=0}^{K-1} \sigma_k e^{-j2\pi n \Delta F \tau_m(x_k^t, y_k^t)} \\ &= \sum_{k=0}^{K-1} \sigma_k e^{-j2\pi n \theta_m(x_k^t, y_k^t)},\end{aligned}$$

with $\theta_m(x_k^t, y_k^t) = \Delta F \tau_m(x_k^t, y_k^t), \ \forall m \in [M], n \in [N]$. Arrange the set of measurements $\{\widetilde{\boldsymbol{z}}_m\}_{m\in[M]}$ and target return $\{\boldsymbol{z}_m^\natural\}_{m\in[M]}$ into $N \times M$ matrices as

$$\begin{aligned}\widetilde{\boldsymbol{Z}} &= [\widetilde{\boldsymbol{z}}_0 \ \widetilde{\boldsymbol{z}}_1 \ \cdots \ \widetilde{\boldsymbol{z}}_{M-1}], \\ \boldsymbol{Z}^\natural &= [\boldsymbol{z}_0^\natural \ \boldsymbol{z}_1^\natural \ \cdots \ \boldsymbol{z}_{M-1}^\natural].\end{aligned}$$

Note that the frequency step size ΔF determines the size of the region $\mathbb{I}$ that the radar system can image. To be precise, we consider the region $\mathbb{I}$ such that

$$\max_{(x,y)\in\mathbb{I}} \theta_m(x, y) - \min_{(x,y)\in\mathbb{I}} \theta_m(x, y) < 1$$

for all $m \in [M]$.

We use $\Omega = \left\{(x_0^t, y_0^t), \ldots, (x_{K-1}^t, y_{K-1}^t)\right\}$ to denote the unknown set of target positions. Our goal is to estimate the potential target locations from the measurements $\widetilde{\boldsymbol{Z}}$.

III. Our Approach

A. Atoms and Atomic Norm

We can model the target return $\boldsymbol{Z}^\natural$ as a sparse combination of observations due to single point targets. Define the matrix $\boldsymbol{A}(x, y) \in \mathbb{C}^{N\times M}$ for any $(x, y) \in \mathbb{I}$ with elements

$$[\boldsymbol{A}(x, y)][n, m] = e^{-j2\pi n \theta_m(x,y)}, \forall \ m \in [M], n \in [N]$$

where

$$\theta_m(x, y) = \Delta F \tau_m(x, y) = 2\Delta F \|(x, y) - (x_m^a, y_m^a)\|_2/c$$

represents the two-way travel time between the the m-th antenna and the point at position (x, y). Then, we rewrite the target return as follows:

$$\boldsymbol{Z}^\natural = \sum_{k=0}^{K-1} \sigma_k \boldsymbol{A}(x_k^t, y_k^t).$$

Thus the target return $\boldsymbol{Z}^\natural$ can be viewed as a sparse combination of elements from the atomic set

$$\mathcal{A} = \{\boldsymbol{A}(x, y) \in \mathbb{C}^{N\times M} : (x, y) \in \mathbb{I}\}.$$

The atomic set $\mathcal{A}$ can be viewed as an infinite dictionary parameterized by the continuous variables (x, y).

For any $\boldsymbol{Z} \in \mathbb{C}^{N\times M}$, we define its atomic norm with respect to $\mathcal{A}$ by

$$\|\boldsymbol{Z}\|_{\mathcal{A}} = \inf_{(x_k, y_k)\in\mathbb{I}} \left\{ \sum_k |c_k| : \boldsymbol{Z} = \sum_k c_k \boldsymbol{A}(x_k, y_k) \right\},$$

which is the gauge function associated with the convex hull of $\mathcal{A}$. Define the inner product as $\langle \boldsymbol{Q}, \boldsymbol{Z} \rangle = \mathrm{trace}(\boldsymbol{Z}^H \boldsymbol{Q})$.

Let $\langle \boldsymbol{Q}, \boldsymbol{Z} \rangle_{\mathbb{R}} = \mathrm{Re}\left(\langle \boldsymbol{Q}, \boldsymbol{Z} \rangle\right)$ be the real inner product. By definition, the dual norm of $\|\cdot\|_{\mathcal{A}}$ is

$$\|\boldsymbol{Q}\|_{\mathcal{A}}^* = \sup_{\boldsymbol{A} \in \mathcal{A}} \langle \boldsymbol{Q}, \boldsymbol{A} \rangle_{\mathbb{R}} .$$

B. Atomic Norm Soft Thresholding (AST)

We obtain an estimate $\widehat{\boldsymbol{Z}}$ that solves the atomic norm soft thresholding (AST):

$$\underset{\boldsymbol{Z}}{\text{minimize}}\, \lambda \|\boldsymbol{Z}\|_{\mathcal{A}} + \frac{1}{2} \left\| \widetilde{\boldsymbol{Z}} - \boldsymbol{Z} \right\|_F^2 , \tag{3}$$

where τ is an appropriately chosen regularization parameter. In [13], the authors show that the choice of τ is governed by the noise model and, if $\boldsymbol{w}_m$ follows an i.i.d. Gaussian distribution with mean 0 and variance ϑ^2, a good choice for λ is

$$\vartheta(1 + \frac{1}{\log(NM)})\sqrt{NM \log(NM) + NM \log(4\pi \log(NM))}.$$

C. Localizing the Target Position using the Dual Problem

The dual problem of AST is given by

$$\begin{aligned} &\underset{\boldsymbol{Q}}{\text{maximize}}\, \frac{1}{2}\|\widetilde{\boldsymbol{Z}}\|_F^2 - \frac{1}{2}\|\widetilde{\boldsymbol{Z}} - \lambda \boldsymbol{Q}\|_F^2 \\ &\text{subject to } \ \|\boldsymbol{Q}\|_{\mathcal{A}}^* \le 1 \end{aligned} \tag{4}$$

The optimal solution to (4) is denoted by $\widehat{\boldsymbol{Q}}$, which can be used to localize the target positions. To be precise, consider the dual polynomial

$$\widehat{q}(x, y) = \left\langle \widehat{\boldsymbol{Q}}, \boldsymbol{A}(x, y) \right\rangle . \tag{5}$$

The target positions can be obtained by finding the peaks of $|\widehat{q}(x, y)|$:

$$\widehat{\Omega} = \left\{(x, y) : |\widehat{q}(x, y)| = 1\right\} .$$

We note that strong duality holds and the primal solution $\widehat{\boldsymbol{Z}}$ and the dual solution $\widehat{\boldsymbol{Q}}$ satisfy

$$\widetilde{\boldsymbol{Z}} = \widehat{\boldsymbol{Z}} + \lambda \widehat{\boldsymbol{Q}}.$$

This indicates that we can obtain the dual optimal solution $\widehat{\boldsymbol{Q}}$ for free (i.e., $\widehat{\boldsymbol{Q}} = \left(\widetilde{\boldsymbol{Z}} - \widehat{\boldsymbol{Z}}\right)/\lambda$) when solving the primal problem (3).

IV. Computational Method

In this section, we present a computational scheme that approximately solves the AST (3).

The atomic norm $\|\boldsymbol{Z}\|_{\mathcal{A}}$ is approximately equivalent to the optimal value of the following problem:

$$\begin{aligned} &\underset{\boldsymbol{U}, \gamma, \widetilde{u}}{\text{minimize}}\ \ \frac{1}{2}\gamma + \frac{1}{2MN} \sum_{m=0}^{M-1} \text{trace}(\text{Toep}(\boldsymbol{u}_m)), \\ &\text{subject to} \\ &\begin{bmatrix} \text{Toep}(\boldsymbol{u}_m) & \boldsymbol{z}_m \\ \boldsymbol{z}_m^H & \gamma \end{bmatrix} \succeq 0, \forall m \in [M], \\ &\boldsymbol{u}_0[0] = \boldsymbol{u}_1[0] = \cdots = \boldsymbol{u}_{M-1}[0] = \widetilde{u}, \end{aligned} \tag{6}$$

where $\boldsymbol{u}_m$ denotes the m-th column of $\boldsymbol{U}$, $\boldsymbol{u}_m[0]$ represents the first element of $\boldsymbol{u}_m$ and $\text{Toep}(\boldsymbol{u}_m)$ denotes a Hermitian Toeplitz matrix with $\boldsymbol{u}_m$ as its first row. Denote the optimal value of (6) by $\text{SDP}(\boldsymbol{Z})$. One can show that $\text{SDP}(\boldsymbol{Z})$ is a lower bound of $\|\boldsymbol{Z}\|_{\mathcal{A}}$, i.e., $\text{SDP}(\boldsymbol{Z}) \le \|\boldsymbol{Z}\|_{\mathcal{A}}$.

Thus the AST problem (3) can be approximately rewritten as a semidefinite program

$$\begin{aligned} &\underset{\boldsymbol{U}, \gamma, \widetilde{u}}{\text{minimize}}\ \ \lambda \left(\frac{1}{2}\gamma + \frac{1}{2}\widetilde{\mu}\right) + \frac{1}{2} \left\| \widetilde{\boldsymbol{Z}} - \boldsymbol{Z} \right\|_F^2 , \\ &\text{subject to} \\ &\begin{bmatrix} \text{Toep}(\boldsymbol{u}_m) & \boldsymbol{z}_m \\ \boldsymbol{z}_m^H & \gamma \end{bmatrix} \succeq 0, \forall m \in [M], \\ &\boldsymbol{u}_0[0] = \boldsymbol{u}_1[0] = \cdots = \boldsymbol{u}_{M-1}[0] = \widetilde{u}, \end{aligned} \tag{7}$$

The SDP (7) can be solved efficiently using off-the-shelf solvers such as SDPT3 [16].

V. Simulations

We present synthetic experiments to support the proposed approach and to test the computational method. We consider a $4\text{m} \times 5.5\text{m}$ target space centered at $(0\text{m}, 4.75\text{m})$, i.e.,

$$\mathbb{I} = [-2\text{m} : 2\text{m}] \times [2\text{m} : 7.5\text{m}].$$

A 5-element synthetic linear aperture (located along the x-axis) with interelement spacing of 1m is used. A stepped-frequency signal consisting of $N = 50$ frequencies from -580 MHz to 580 MHz with 20 MHz frequency steps is utilized to obtain measurements.

We compare the radar imaging performance of AST, the discretized Lasso and the back-projection method. We solve (7) with CVX [20] coupled with SDPT3. We divide the target space $\mathbb{I}$ into a grid of $L_x \times L_y$ pixels with $L_x = 50$ and $L_y = 50$. We arrange the pixels of the image into an $L_x L_y \times 1$ vector $\boldsymbol{\alpha}$. Let (x_q, y_q) denote the location of the q-th pixel and let $\boldsymbol{\theta}_q$ be the vectorization of the atom $\boldsymbol{A}(x_q, y_q)$, i.e., $\theta_q = \text{vec}(\boldsymbol{A}(x_q, y_q))$. We concatenate the $\boldsymbol{\theta}_q$ into a matrix as

$$\boldsymbol{\Theta} := [\boldsymbol{\theta}_0 \ \ \boldsymbol{\theta}_1 \ \ \cdots \ \ \boldsymbol{\theta}_{L_x L_y - 1}].$$

The Lasso is a regularized l_1 minimization problem

$$\underset{\boldsymbol{\alpha}}{\text{minimize}}\, \lambda \|\boldsymbol{\alpha}\|_1 + \frac{1}{2}\|\text{vec}(\widetilde{\boldsymbol{Z}}) - \boldsymbol{\Theta}\boldsymbol{\alpha}\|_2^2, \tag{8}$$

which is a discretized version of AST (3). The vector $\boldsymbol{\alpha}$ indicates the location of the targets: the value of q-th pixel equals the target reflectivity if there is a target at this pixel, and otherwise it is 0. We solve Lasso with CVX software.

We simulate four point targets located at $(-0.69\text{m}, 5.31\text{m})$, $(0.31\text{m}, 5.81\text{m})$, $(-1.33\text{m}, 2.28\text{m})$ and $(-0.65\text{m}, 6.64\text{m})$ with relative complex refractivities of $-0.77 + 1.47j$, $0.41 - 0.04j$, $-0.04 - 0.42j$ and $0.80 - 0.73j$, respectively. The actual target positions are depicted in Figure 2 (a). The measurement $\widetilde{\boldsymbol{z}}_m$ is produced by adding complex white noise $\boldsymbol{w}_m$ with mean zero and standard deviation $\vartheta = 0.1$. Figure 2 (b) shows the radar image reconstruction using back-projection, and Figure 2 (c) shows the recovered target positions (i.e., the solution to (8)) by solving Lasso. Finally, the recovered target positions by the proposed method are illustrated in Figure 2 (d). As can been

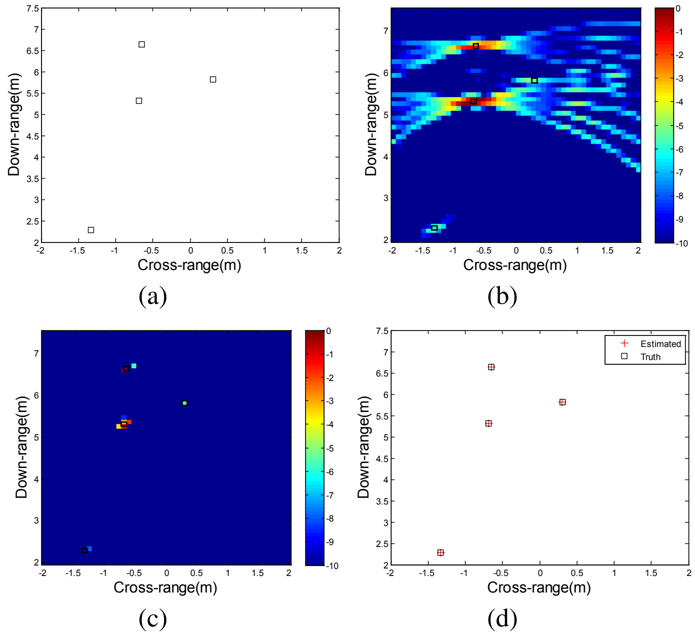

Fig. 2. Reconstruction results with different methods: (a) ground truth; (b) back-projection (the maximum intensity is normalized to 0dB); (c) Lasso (the maximum intensity is normalized to 0dB); (d) proposed approach. The black squares represent the original target positions.

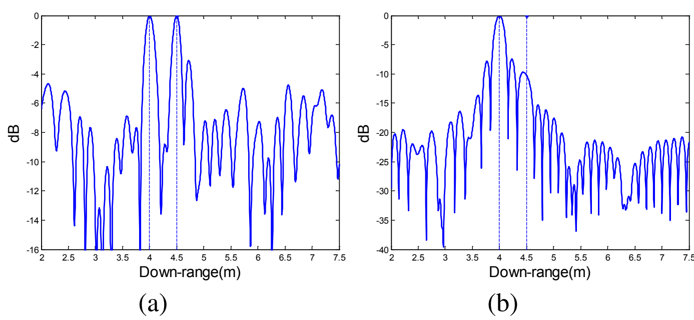

Fig. 3. Dual polynomials with different methods: (a) proposed approach; (b) back-projection (the maximum intensity is normalized to 0dB). The dashed blue lines represent the original target positions.

seen, the proposed method can correctly find all the targets and their locations.

To better illustrate the advantage of the proposed method against back-projection, we simulate two closely spaced point targets located at $(3\text{m}, 4\text{m})$ and $(3\text{m}, 4.5\text{m})$, with complex reflectivities of 10 and 1, respectively. Figure 3 (a) shows the dual polynomial $\widehat{q}(x, y)$, defined in (5), at $x = 3\text{m}$. Figure 3 (b) shows the polynomial

$$\overline{q}(x, y) := \left\langle \widetilde{\boldsymbol{Z}}, \boldsymbol{A}(x, y) \right\rangle$$

at $x = 3\text{m}$. As can been observed, the targets can be localized by identifying points where the dual polynomial $\widehat{q}(x, y)$ has modulus one. In contrast, we may fail to localize the targets by finding the peaks of the polynomial $\overline{q}(x, y)$ because of the slow decay of the Dirichlet kernel.

VI. Conclusions

Motivated by recent work on atomic norms, we denoise the stepped-frequency radar measurements in SAR by solving the AST and thereby providing estimates of the target positions. This is achieved by finding the peaks of the dual polynomial, which can be easily obtained from the optimal solution of AST. Compared to classical SAR imaging based on back-projection, the proposed method can estimate the targets with higher resolution. Unlike radar imaging based on traditional sparse recovery solvers, which work on a finite set of grid points, the atomic norm formulation works in the continuous domain and therefore alleviates the basis mismatch issue. A computational method based on semidefinite programming was derived to solve AST. Numerical experiments demonstartes that the proposed method outperforms the other two approaches.

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