Computable Performance Analysis of Block-Sparsity Recovery

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***Abstract*—In this paper, we employ fixed-point iteration and semidefinite programming to compute performance bounds on the basis pursuit algorithm for block-sparsity recovery. As a prerequisite for optimal sensing matrix design, computable performance bounds would open doors for wide applications in sensor arrays, MIMO radar, DNA microarrays, and many other areas where block-sparsity arises naturally.**

***Index Terms*—block-sparse signal recovery, compressive sensing, computable performance analysis, fixed-point iteration, semidefinite programming, verifiable sufficient condition**

I. Introduction

In this paper, we investigate the recovery performance for block-sparse signals. Block-sparsity arises naturally in applications such as sensor arrays, MIMO radar, multi-band signals, and DNA microarrays. A particular area that motivates this work is the application of block-sparse signal recovery in radar systems. The signals in radar applications are usually sparse because there are only a few targets to be estimated among many possibilities. However, a single target manifests itself simultaneously in the sensor domain, the frequency domain, the temporal domain, and the reflection-path domain. As a consequence, the underlying signal would be block-sparse if the radar system observes the targets from several of these domains, as is the case for MIMO radar systems [1], [2].

The applications aforementioned require a computable performance analysis of block-sparsity recovery. The sensing matrices in practical problems depend on the underlying measurement devices and the physical processes that generate the observations. Therefore, computable performance measures would open doors for wide applications. Firstly, it provides a means to pre-determine the performance of the sensing system before its implementation and the taking of measurements. In addition, in MIMO radar, sensor arrays, DNA microarrays, and MRI, we usually have the freedom to optimally design the sensing matrix. For example, in radar systems the optimal sensing matrix design is connected with optimal waveform design, a central topic of radar research. Our goodness measure and the algorithm to compute it provide a basis for performing optimal sensing matrix design.

The rest of the paper is organized as follows. In Section II, we introduce notations and the measurement model. In section III, we derive performance bounds on the basis pursuit algorithm. Section IV is devoted to the probabilistic analysis of our performance measure. In Section V, we design algorithms to verify a sufficient condition for exact block-ℓ_1 recovery in the noise-free case, and to compute the goodness measure of arbitrary sensing matrices through fixed-point iteration, bisection search, and semidefinite programming. We evaluate the algorithms' performance in Section VI. Section VII summarizes our conclusions.

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II. Notations and the Measurement Model

For any vector $\boldsymbol{x} \in \mathbb{R}^{np}$, we partition the vector into p blocks, each of length n. More precisely, we have $\boldsymbol{x} = \left[\boldsymbol{x}_1^T, \ldots, \boldsymbol{x}_p^T\right]^T$ with the ith block $\boldsymbol{x}_i \in \mathbb{R}^n$. The block-$\ell_q$ norms for this block-sparse structure are defined as:

$$\|\boldsymbol{x}\|_{\mathrm{b}q} = \left(\sum_{i=1}^{p} \|\boldsymbol{x}_i\|_2^q\right)^{1/q}, \ 1 \leq q < \infty \qquad (1)$$

$$\|\boldsymbol{x}\|_{\mathrm{b}\infty} = \max_{1\leq i\leq p} \|\boldsymbol{x}_i\|_2, \ q = \infty. \qquad (2)$$

Clearly, the block-ℓ_2 norm is the same as the ordinary ℓ_2 norm. The block support of $\boldsymbol{x} \in \mathbb{R}^{np}$, $\mathrm{bsupp}(\boldsymbol{x}) = \{i : \|\boldsymbol{x}_i\|_2 \neq 0\}$, is the index set of the non-zero blocks of $\boldsymbol{x}$. The size of the block support, denoted by the block-ℓ_0 "norm" $\|\boldsymbol{x}\|_{\mathrm{b}0}$, is the block-sparsity level of $\boldsymbol{x}$. Signals of block-sparsity level at most k are called $k-$block-sparse signals.

Suppose $\boldsymbol{x}$ is a $k-$block-sparse signal. In this paper, we observe $\boldsymbol{x}$ through the following linear model:

$$\boldsymbol{y} = A\boldsymbol{x} + \boldsymbol{w}, \qquad (3)$$

where $A \in \mathbb{R}^{m\times np}$ is the measurement/sensing matrix, $\boldsymbol{y}$ is the measurement vector, and $\boldsymbol{w}$ is noise.

Many algorithms in sparse signal recovery have been extended to recovering the block-sparse signal $\boldsymbol{x}$ from $\boldsymbol{y}$ by exploiting the block-sparsity of $\boldsymbol{x}$. In this paper, we focus on the block-sparse version of the basis pursuit algorithm [3]:

$$\text{BS-BP: } \min_{\boldsymbol{z}\in\mathbb{R}^{np}} \|\boldsymbol{z}\|_{\mathrm{b}1} \ \text{ s.t. } \|\boldsymbol{y} - A\boldsymbol{z}\|_2 \leq \varepsilon. \qquad (4)$$

III. Performance Bounds on Block-Sparsity Recovery

In this section, we derive performance bounds on the block-ℓ_∞ norm and the ℓ_2 norm of the error vectors. One distinctive feature of this work is the use of the block-ℓ_∞ norm as a performance criterion, which results in computable performance bounds. We first introduce the following definition:

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Definition 1 *For any $s \in [1, p]$ and $A \in \mathbb{R}^{m \times np}$, define*

$$\omega_2(A, s) = \min_{\boldsymbol{z}: \|\boldsymbol{z}\|_{\mathrm{b1}}/\|\boldsymbol{z}\|_{\mathrm{b}\infty} \leq s} \frac{\|A\boldsymbol{z}\|_2}{\|\boldsymbol{z}\|_{\mathrm{b}\infty}}. \quad (5)$$

The block-ℓ_∞ norm and the ℓ_2 norm of the error vectors for the BS-BP are bounded in terms of $\omega_2(A, s)$:

Theorem 1 *Suppose $\boldsymbol{x} \in \mathbb{R}^{np}$ in* (3) *is $k-$block-sparse, $\hat{\boldsymbol{x}}$ is the solution of the BS-BP* (4)*, and the noise $\boldsymbol{w}$ satisfies $\|\boldsymbol{w}\|_2 \leq \varepsilon$, then we have:*

$$\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|_{\mathrm{b}\infty} \leq \frac{2\varepsilon}{\omega_2(A, 2k)}, \quad (6)$$

$$\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|_2 \leq \frac{2\sqrt{2k}\varepsilon}{\omega_2(A, 2k)}. \quad (7)$$

One of the primary contributions of this work is the design of algorithms to efficiently compute $\omega_2(A, s)$. These algorithms provide a way to numerically assess the performance of the BS-BP according to the bounds given in Theorem 1. We note that if $S = \mathrm{bsupp}(\boldsymbol{x})$ and $\beta = \min_{i \in S} \|\boldsymbol{x}_i\|_2$, then $\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|_{\mathrm{b}\infty} < \beta/2$ implies

$$\{i : \|\hat{\boldsymbol{x}}_i\|_2 > \beta/2\} = \mathrm{bsupp}(\boldsymbol{x}), \quad (8)$$

i.e., a thresholding operator recovers the signal block-support. Therefore, the block-ℓ_∞ norm has a close relationship with the block support, which is more important than the signal component values in many applications.

IV. Probabilistic Behavior of $\omega_2(A, s)$

In this section, we analyze how good are the performance bounds in Theorem 1 for random sensing matrices. Note that a random vector $\boldsymbol{X} \in \mathbb{R}^{np}$ is called *isotropic and subgaussian* if $\mathbb{E}|\langle \boldsymbol{X}, \boldsymbol{u} \rangle|^2 = \|\boldsymbol{u}\|_2^2$ and $\mathbb{P}(|\langle \boldsymbol{X}, \boldsymbol{u} \rangle| \geq t) \leq 2\exp(-ct^2/\|\boldsymbol{u}\|_2)$ hold for any $\boldsymbol{u} \in \mathbb{R}^{np}$. Here c is a numerical constant.

Theorem 2 *Let the rows of $\sqrt{m}A$ be* i.i.d. *subgaussian and isotropic random vectors, then there exist constants c_1, c_2 such that for any $\epsilon > 0$ and $m \geq 1$ satisfying*

$$m \geq c_1 \frac{s^2 n + s^2 \log p}{\epsilon^2}, \quad (9)$$

we have

$$\mathbb{E}\ \omega_2(A, s) \geq 1 - \epsilon, \quad (10)$$

$$\mathbb{P}\{\omega_2(A, s) \geq 1 - \epsilon\} \geq 1 - \exp(-c_2 \epsilon^2 m). \quad (11)$$

Equation (9) and Theorem 1 imply that for exact signal recovery in the noise free case, we need $O(s^2(n + \log p)$ measurements for random sensing matrices. The measurement bound (9) implies that the algorithms for verifying $\omega_2 > 0$ and for computing ω_2 work for s at least up to the order $\sqrt{m/(n + \log p)}$.

We comment that sensing matrices with *i.i.d.* subgaussian and isotropic rows include the Gaussian ensemble, the Bernoulli ensemble, and normalized volume measures on various convex symmetric bodies, for example, the unit balls of ℓ_q^{np} for $2 \leq q \leq \infty$ [4].

V. Verification and Computation of ω_2

In this section, we address the computation of $\omega_2(\cdot)$.

A. Verification of $\omega_2 > 0$

A prerequisite for the bounds in Theorem 1 to be valid is the positiveness of the involved $\omega_2(\cdot)$. From Theorem 1, $\omega_2(\cdot) > 0$ implies the exact recovery of the true signal $\boldsymbol{x}$ in the noise-free case. The positiveness of $\omega_2(A, s)$ amounts to making sure $\|\boldsymbol{z}\|_{\mathrm{b1}}/\|\boldsymbol{z}\|_{\mathrm{b}\infty} \leq s$ for all $\boldsymbol{z}$ such that $A\boldsymbol{z} = 0$. Therefore, we compute

$$s^* = \min_{\boldsymbol{z}} \|\boldsymbol{z}\|_{\mathrm{b1}}/\|\boldsymbol{z}\|_{\mathrm{b}\infty} \text{ s.t. } A\boldsymbol{z} = 0. \quad (12)$$

Then, when $s < s^*$, we have $\omega_2(A, s) > 0$. The following theorem presents an optimization procedure that computes a lower bound on s^*.

Proposition 1 *The reciprocal of the optimal value of the following optimization, denoted by s_*,*

$$\max_i \min_{P_i} \max_j \|\delta_{ij} \mathrm{I}_n - P_i^T A_j\|_2 \quad (13)$$

is a lower bound on s^*. Here P is a matrix variable of the same size as A, $\delta_{ij} = 1$ for $i = j$ and 0 otherwise, and $P = [P_1, \ldots, P_p]$, $A = [A_1, \ldots, A_p]$ with P_i and A_j having n columns each.

Because $s_* < s^*$, the condition $s < s_*$ is a sufficient condition for $\omega_2 > 0$ and for the uniqueness and exactness of block-sparse recovery in the noise free case. To get s_*, for each i, we need to solve

$$\min_{P_i} \max_j \|\delta_{ij} \mathrm{I}_n - P_i^T A_j\|_2, \quad (14)$$

which is equivalent with a semidefinite program:

$$\min_{P_i, t} t \text{ s.t. } \begin{bmatrix} t\mathrm{I}_n & \delta_{ij}\mathrm{I}_n - P_i^T A_j \\ \delta_{ij} I_n - A_j^T P_i & t\mathrm{I}_n \end{bmatrix} \succeq 0. \quad (15)$$

Small instances of (15) can be solved using CVX [5].

B. Fixed-Point Iteration for Computing $\omega_2(\cdot)$

We present a general fixed-point procedure to compute ω_2. We rewrite the optimization problem defining ω_2 as

$$\frac{1}{\omega_2(A, s)} = \max_{\boldsymbol{z}} \|\boldsymbol{z}\|_{\mathrm{b}\infty} \text{ s.t. } \|A\boldsymbol{z}\|_2 \leq 1, \frac{\|\boldsymbol{z}\|_{\mathrm{b1}}}{\|\boldsymbol{z}\|_{\mathrm{b}\infty}} \leq s. \quad (16)$$

For any $s \in (1, s^*)$, we define a function over $[0, \infty)$ parameterized by s as:

$$f_s(\eta) = \max_{\boldsymbol{z}} \{\|\boldsymbol{z}\|_{\mathrm{b}\infty} : \|A\boldsymbol{z}\|_2 \leq 1, \|\boldsymbol{z}\|_{\mathrm{b1}} \leq s\eta\}. \quad (17)$$

We basically replaced the $\|\boldsymbol{z}\|_{\mathrm{b}\infty}$ in the denominator of the fractional constraint in (16) with η. It turns out that the unique positive fixed point of $f_s(\eta)$ is exactly $1/\omega_2(A, s)$ as shown by the following proposition.

Proposition 2 *The function $f_s(\eta)$ has the following properties:*

1) *$f_s(\eta)$ is continuous and strictly increasing in η;*

2) $f_s(\eta)$ *has a unique positive fixed point* $\eta^* = f_s(\eta^*)$ *that is equal to* $1/\omega_2(A, s)$*;*
3) *For* $\eta \in (0, \eta^*)$*, we have* $f_s(\eta) > \eta$*; and for* $\eta \in (\eta^*, \infty)$*, we have* $f_s(\eta) < \eta$*;*
4) *For any* $\epsilon > 0$*, there exists* $\rho_1(\epsilon) > 1$ *such that* $f_s(\eta) > \rho_1(\epsilon)\eta$ *as long as* $0 < \eta < (1-\epsilon)\eta^*$*; and there exists* $\rho_2(\epsilon) < 1$ *such that* $f_s(\eta) < \rho_2(\epsilon)\eta$ *as long as* $\eta > (1+\epsilon)\eta^*$*.*

We have transformed the problem of computing $\omega_2(A, s)$ into one of finding the positive fixed point of a scalar function $f_s(\eta)$. The property 4) of Proposition 2 states that we could start with any η_0 and use the iteration

$$\eta_{t+1} = f_s(\eta_t), t = 0, 1, \cdots \tag{18}$$

to find the positive fixed point η^*. In addition, if we start from two initial points, one less than η^* and one greater than η^*, then the gap between the generated sequences indicates how close we are from the fixed point η^*. Property 3) also suggests finding η^* by bisection search.

C. Relaxation of the Subproblem

Unfortunately, except when $n = 1$ and the signal is real, *i.e.,* the real sparse case, it is not easy to compute $f_s(\eta)$ according to (17). In the following theorem, we present a relaxation of the subproblem

$$\max_{\boldsymbol{z}} \|\boldsymbol{z}\|_{\mathrm{b}\infty} \text{ s.t. } \|A\boldsymbol{z}\|_2 \leq 1, \|\boldsymbol{z}\|_{\mathrm{b}1} \leq s\eta \tag{19}$$

by computing an upper bound on $f_s(\eta)$.

Proposition 3 $f_s(\eta)$ *is bounded as*

$$f_s(\eta) \leq \max_i \min_{P_i} \max_j s\eta \|\delta_{ij}\mathrm{I}_n - P_i^T A_j\|_2 + \|P_i\|_2. \tag{20}$$

For each $i = 1, \ldots, p$, the optimization problem

$$\min_{P_i} \max_j s\eta \|\delta_{ij}\mathrm{I}_n - P_i^T A_j\|_2 + \|P_i\|_2 \tag{21}$$

can be solved using semidefinite programming:

$$\begin{aligned}
&\min_{P_i, t_0, t_1} s\eta t_0 + t_1 \text{ s.t.} \\
&\begin{bmatrix} t_0 \mathrm{I}_n & \delta_{ij}\mathrm{I}_n - P_i^T A_j \\ \delta_{ij}\mathrm{I}_n - A_j^T P_i & t_0 \mathrm{I}_n \end{bmatrix} \succeq 0, j = 1, \ldots, p; \\
&\begin{bmatrix} t_1 \mathrm{I}_m & P_i \\ P_i^T & t_1 \mathrm{I}_n \end{bmatrix} \succeq 0.
\end{aligned} \tag{22}$$

D. Fixed-Point Iteration for Computing a Lower Bound on ω_2

Although Proposition 3 provides ways to efficiently compute upper bounds on the subproblem (19) for fixed η, it is not obvious whether we could use it to compute an upper on the positive fixed point of $f_s(\eta)$, or $1/\omega_2(A, s)$. We show in this subsection that another iterative procedure can compute such upper bounds.

To this end, we define functions $g_{s,i}(\eta)$ and $g_s(\eta)$ over $[0, \infty)$ parameterized by s for $s \in (1, s_*)$,

$$\begin{aligned}
g_{s,i}(\eta) &= \min_{P_i} s\eta \left(\max_j \|\delta_{ij}\mathrm{I}_n - P_i^T A_j\|_2 \right) + \|P_i\|_2, \\
g_s(\eta) &= \max_i g_{s,i}(\eta),
\end{aligned} \tag{23}$$

whose properties are established in the following proposition.

Proposition 4 *The functions* $g_{s,i}(\eta)$ *and* $g_s(\eta)$ *have the following properties:*

1) $g_{s,i}(\eta)$ *and* $g_s(\eta)$ *are continuous, strictly increasing in* η*, and* $g_{s,i}(\eta)$ *is concave for every* i*;*
2) $g_{s,i}$ *and* $g_s(\eta)$ *have unique positive fixed points* $\eta_i^* = g_{s,i}(\eta_i^*)$ *and* $\eta^* = g_s(\eta^*)$*, respectively; and* $\eta^* = \max_i \eta_i^*$*;*
3) *For* $\eta \in (0, \eta^*)$*, we have* $g_s(\eta) > \eta$*; and for* $\eta \in (\eta^*, \infty)$*, we have* $g_s(\eta) < \eta$*; the same statement holds also for* $g_{s,i}(\eta)$*.*
4) *For any* $\epsilon > 0$*, there exists* $\rho_1(\epsilon) > 1$ *such that* $g_s(\eta) > \rho_1(\epsilon)\eta$ *as long as* $0 < \eta \leq (1-\epsilon)\eta^*$*; and there exists* $\rho_2(\epsilon) < 1$ *such that* $g_s(\eta) < \rho_2(\epsilon)\eta$ *as long as* $\eta > (1+\epsilon)\eta^*$*.*

A consequence of Propositions 4 is the following:

Theorem 3 *Suppose* η^* *is the unique fixed point of* $g_s(\eta)$*, then we have*

$$\eta^* \geq \frac{1}{\omega_2(A, s)}. \tag{24}$$

Proposition 4 implies three ways to compute the fixed point η^* for $g_s(\eta)$.

1) **Naive Fixed-Point Iteration:** Property 4) of Proposition 4 suggests that the fixed point iteration
$$\eta_{t+1} = g_s(\eta_t), t = 0, 1, \ldots \tag{25}$$
starting from any initial point η_0 converges to η^*, no matter $\eta_0 < \eta^*$ or $\eta_0 > \eta^*$.
2) **Bisection:** The bisection approach is motivated by property 3) of Proposition 4. Starting from an initial interval $(\eta_{\mathrm{L}}, \eta_{\mathrm{U}})$ that contains η^*, we compute $g_s(\eta_{\mathrm{M}})$ with $\eta_{\mathrm{M}} = (\eta_{\mathrm{L}} + \eta_{\mathrm{U}})/2$. As a consequence of property 3), we set $\eta_{\mathrm{L}} = g_s(\eta_{\mathrm{M}})$ if $g_s(\eta_{\mathrm{M}}) > \eta_{\mathrm{M}}$ and we set $\eta_{\mathrm{U}} = g_s(\eta_{\mathrm{M}})$ if $g_s(\eta_{\mathrm{M}}) < \eta_{\mathrm{M}}$. Half the length of the interval is an upper bound on the gap between η_{M} and η^*, resulting an accurate stopping criterion.
3) **Fixed-Point Iteration + Bisection:** The third approach relies heavily on the representation $g_s(\eta) = \max_i g_{s,i}(\eta)$ and $\eta^* = \max_i \eta_i^*$. Starting from an initial interval $(\eta_{\mathrm{L}0}, \eta_{\mathrm{U}})$ and the index set $\mathcal{I}_0 = \{1, \ldots, p\}$, we pick any $i_0 \in \mathcal{I}_0$ and use a bisection method with starting interval $(\eta_{\mathrm{L}0}, \eta_{\mathrm{U}})$ to find the positive fixed point $\eta_{i_0}^*$ of $g_{s,i_0}(\eta)$. For any $i \in \mathcal{I}_0/i_0$, $g_{s,i}(\eta_{i_0}^*) \leq \eta_{i_0}^*$ implies that the fixed point η_i^* of $g_{s,i}(\eta)$ is less than or equal to $\eta_{i_0}^*$ according to the continuity of $g_{s,i}(\eta)$ and the uniqueness of its positive fixed point. As a

consequence, we remove this i from the index set $\mathcal{I}_0$. We denote $\mathcal{I}_1$ as the index set after all such is removed, *i.e.,* $\mathcal{I}_1 = \mathcal{I}_0/\{i : g_{s,i}(\eta^*_{i_0}) \leq \eta^*_{i_0}\}$. We also set $\eta_{L1} = \eta^*_{i_0}$ as $\eta^* \geq \eta^*_{i_0}$. Next we test the $i_1 \in \mathcal{I}_1$ with the *largest* $g_{s,i}(\eta^*_{i_0})$ and construct $\mathcal{I}_2$ and η_{L2} in a similar manner. We repeat the process until the index set $\mathcal{I}_t$ is empty. The η^*_i found at the last step is the maximal η^*_i, which is equal to η^*.

In many cases, we do not need to compute $g_s(\eta)$ exactly, but to verify, *e.g.*, $g_s(\eta) > \eta$ and $g_s(\eta) < \eta$. Note that there is an asymmetry between these two verifications: verifying $g_s(\eta) < \eta$ needs to compute all $g_{s,i}(\eta)$ while verifying $g_s(\eta) > \eta$ is terminated as long as we find one $g_{s,i}(\eta) > \eta$. This asymmetry can be exploited to improve the efficiency of the proposed algorithms.

VI. Preliminary Numerical Experiments

In this section, we present preliminary numerical results that assess the performance of the algorithms for verifying $\omega_2(A, s) > 0$ and computing $\omega_2(A, s)$. We also compare the error bounds based on $\omega_2(A, s)$ with the bounds based on the block RIP [3]. The involved semidefinite programs are solved using CVX.

We test the algorithms on Gaussian random matrices with columns normalized to have unit lengths. We first present the values of s_* computed by (13), $k_* = \lfloor s_*/2 \rfloor$, and compare them with the corresponding quantities when A is seen as the sensing matrix for the sparse model without knowing the block-sparsity structure [6], [7]. Keeping in mind that the true sparsity level in the block-sparse model is nk, where k is the block sparsity level, we note the nk_* in the fourth column for the block-sparse model is indeed much greater than the k_* in the sixth column for the sparse model, implying exploiting the block-sparsity structure is advantageous.

TABLE I
Comparison of the sparsity level bounds on the block-sparse model and the sparse model for a Gaussian matrix $A \in \mathbb{R}^{m\times np}$ with $n = 4, p = 60$.

m	Block Sparse Model			Sparse Model	
	s_*	k_*	nk_*	s_*	k_*
72	3.96	1	4	6.12	3
96	4.87	2	8	7.55	3
120	5.94	2	8	9.54	4
144	7.14	3	12	11.96	5
168	8.60	4	16	14.66	7
192	11.02	5	20	18.41	9

In the next experiment, we compare our recovery error bounds (7) on the BS-BP based on $\omega_2(A, s)$ with those based on the block RIP. The block RIP bounds given in [3] is valid only when the block RIP $\delta_{2k}(A) < \sqrt{2} - 1$. The block RIP is computed using Monte Carlo simulations. Without loss of generality, we set $\varepsilon = 1$.

In Tables II, we present the values of $\omega_2(A, 2k)$ and $\delta_{2k}(A)$ computed for a Gaussian matrix $A \in \mathbb{R}^{m\times np}$ with $n = 4$ and $p = 60$. We note that in all the considered cases, $\delta_{2k}(A) > \sqrt{2} - 1$, and the block RIP based bound does not apply at all. In contrast, the ω_2 based bound (7) is valid as long as $k \leq k_*$.

TABLE II
$\omega_2(A, 2k)$ and $\delta_{2k}(A)$ computed for a Gaussian matrix $A \in \mathbb{R}^{m\times np}$ with $n = 4$ and $p = 60$.

	m	72	96	120	144	168	192
	s_*	3.88	4.78	5.89	7.02	8.30	10.80
k	k_*	1	2	2	3	4	5
1	$\omega_2(A, 2k)$	0.45	0.53	0.57	0.62	0.65	0.67
	$\delta_{2k}(A)$	0.90	0.79	0.66	0.58	0.55	0.51
2	$\omega_2(A, 2k)$		0.13	0.25	0.33	0.39	0.43
	$\delta_{2k}(A)$		1.08	0.98	0.96	0.84	0.75
3	$\omega_2(A, 2k)$				0.11	0.18	0.25
	$\delta_{2k}(A)$				1.12	1.01	0.93
4	$\omega_2(A, 2k)$					0.02	0.12
	$\delta_{2k}(A)$					1.26	1.07
5	$\omega_2(A, 2k)$						0.03
	$\delta_{2k}(A)$						1.28

VII. Conclusions

In this paper, we analyzed the performance of the block-sparse basis pursuit algorithm using the block-$\ell_{\mathrm{b}\infty}$ norm of the errors as a performance criterion. A goodness measure of the sensing matrices was defined using optimization procedures. We used the goodness measure to derive upper bounds on the block-$\ell_{\mathrm{b}\infty}$ norm and the ℓ_2 norm of the reconstruction errors for the block-sparse basis pursuit. Efficient algorithms based on fixed-point iteration, bisection, and semidefinite programming were implemented to solve the optimization procedures defining the goodness measure. We expect that the goodness measure will be useful in comparing different sensing systems and recovery algorithms, as well as in designing optimal sensing matrices. In future work, we will use these computable performance bounds to optimally design transmitting waveforms for compressive sensing MIMO radar.

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