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Convergence of a Class of Multi-Agent Systems in Probabilistic Framework

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Abstract— **In this paper, we will study how locally interacting agents lead to synchronization of the overall system for a basic class of multi-agent systems that are described by a simplification of the well-known Vicsek Model. This model looks simple, but the rigorous theoretical analysis appears to be quite complicated, because there are strong nonlinear interactions among the update laws of the agents' positions and headings. In fact, most of the existing theoretical analyzes hinge on certain connectivity conditions on the global behavior of the agents' trajectories (or on the neighborhood graphs of the underlying dynamical systems), which are quite hard to verify in general. In this paper, by working in a probabilistic framework, we will give a complete and rigorous proof for the following fact observed in simulation: the overall multi-agent system will synchronize with large probability for large population. The proof is carried out by analyzing both the dynamical properties of the nonlinear system evolution and the asymptotic properties of the spectrum of random geometric graphs.**

I. INTRODUCTION

Multi-agent systems arise from diverse fields in natural and artificial systems, which have attracted much attention from researchers in recent years (see, e.g., [1]-[11]). A basic and interesting problem is to understand how locally interacting systems can spontaneous generate various kinds of collective behaviors, such as synchronization, whirlpool, *etc.*, without central control and global information exchange.

Typical examples of multi-agent systems include animal aggregations such as flocks, schools, herds and bacteria colonies. Biologists have given detailed descriptions and discussions on the mechanisms of flying, swimming and migrating in these aggregations. In many cases, agents have the tendency to move as other agents do in their neighborhood. Inspired more or less by this, Vicsek *et al.* proposed a model to simulate and explain the clustering, transportation and phase transition in nonequilibrium systems [5][6]. The model consists of finite agents (particles, animals, robots, etc.) on the plane, each of which moves with constant velocity. At each time step, a given agent assumes the average direction of agents' motions in its neighborhood of radius r. Through simulation, Vicsek *et al.* explained the kinetic phase transition exhibited in the model by the spontaneous symmetry breaking of the rotational symmetry. The Vicsek Model can also be viewed as a special case of the well-known Boid Model introduced by Reynolds in 1987 [7], where the purpose was to simulate the behavior in flocks of flying birds and schools of fishes. Agents in the Boid Model obey three rules in their movement: Collision Avoidance, Velocity Matching and Flock Centering. All of these three rules are local ones, which means that each agent adjusts its behavior based on the behaviors of the agents in its neighborhood.

Inspired by both the nature phenomena and the computer simulations, scientists have kept trying to give rigorous theoretical foundations and explanations. The most notable attempt has been made recently by Jadbabaie *et al.*[8], where the heading update laws in the Vicsek Model are linearilized, and it was shown that the headings of all agents will converge to a common one provided that the neighborhood graphs of the underlying system are jointly connected with sufficient frequency. Some related results may also been found in an earlier paper [12], but in a different context. These results provide preliminary theoretical explanations of the phenomena observed in simulations. However, even for the seemingly simple model studied in Jadbabaie *et al.*[8], a complete theoretical analysis is still lacking. The main reason is that all the conditions in the existing theoretical analysis are imposed on the "closed-loop" graphs, which are resulted from the iteration of the system dynamics, and should be determined by both the initial states and model parameters. These results have not given any clue to how the neighboring graphs evolve, and how to verify the required connectivity conditions. It is worth mentioning that if the local rules are modified to weighted but global ones along the way, for example, suggested in [11], a complete theoretical analysis can be given with the convergence conditions imposed on system initial states and model parameters only [11]. The first complete result which guarantees the synchronization of the original Vicsek Model by imposing conditions only on the system initial states and model parameters seems to have been given in [26], but these conditions are still not satisfactory in the sense that they may not be valid for large population. Nevertheless, the results in [8], [9], [11], [26] all suggest that the connectivity of the dynamical neighbor graphs resulted from the system iteration is crucial to synchronization.

In this paper, we will take a somewhat different perspective to introduce a probabilistic framework for investigating the convergence of multi-agent systems described by the linearized Vicsek Model as studied in Jadbabaie *et al.*[8]. We will first give a detailed analysis of both the dynamical properties of the nonlinear system evolution and the asymptotic properties of the spectrum of random geometric graphs, and

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then demonstrate that this class of multi-agent systems will synchronize with large probability for any given interaction radius r and motion speed v, whenever the population size is large enough.

II. The Main Results

We first introduce the model to be studied in the paper, assuming that the readers are familiar with some basic concepts in graph theory[13]-[20],[21],[22].

The original Vicsek Model[5] consists of n agents on the plane, which are labeled by $1, 2, \cdots, n$. At any time t, each agent moves with a constant absolute velocity v, and assumes the average direction of motion of agents in its neighborhood with radius r at time $t-1$. The intrinsic nonlinearity in the moving direction iteration rule makes the theoretical analysis quite complicated. [8] proposed the linearized Vicsek Model, which will be considered in this paper. At time t, the neighborhood set of any agent k is defined as

$$N_k(t) = \{j : \|\boldsymbol{x}_j(t) - \boldsymbol{x}_k(t)\| < r\}, \tag{1}$$

where $\boldsymbol{x}_k(t) \in \mathbb{R}^2$ denotes the location of the kth agent at time t, and r is the interaction radius. Obviously, if we denote $\boldsymbol{x}(t) = col(\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), ..., \boldsymbol{x}_n(t))$, then the graph induced by the neighborhood relationship is a geometric graph $G(\boldsymbol{x}(t), r)$, which can be abbreviated as $G(t)$. Quantities related to $G(t)$ may also be functions of time t.

The position iteration rule of any agent $k (1 \le k \le n)$ is simply given by

$$\boldsymbol{x}_k(t) = \boldsymbol{x}_k(t-1) + v\tilde{\boldsymbol{s}}(\theta_k(t)), \tag{2}$$

where $\tilde{\boldsymbol{s}}(\theta) \triangleq col(\cos\theta, \sin\theta)$. In the linearized Vicsek Model, the moving direction iteration rule expressed in terms of the angles between the velocity vectors and a fixed direction is given by

$$\theta_k(t) = \frac{1}{n_k(t-1)} \sum_{j \in N_k(t-1)} \theta_j(t-1), \quad t \ge 1, \tag{3}$$

where $n_k(t)$ denotes the number of elements in $N_k(t)$.

Now, let $\boldsymbol{\theta}(t) = col(\theta_1(t), \theta_2(t), ..., \theta_n(t))$, $\boldsymbol{s}(\boldsymbol{\theta}(t)) = col(\tilde{\boldsymbol{s}}(\theta_1(t)), \tilde{\boldsymbol{s}}(\theta_2(t)), ..., \tilde{\boldsymbol{s}}(\theta_n(t)))$, then the iteration rules (2) and (3) of the linearized Vicsek Model can be rewritten as

$$\begin{cases} \boldsymbol{\theta}(t) = P(t-1)\boldsymbol{\theta}(t-1) \\ \boldsymbol{x}(t) = \boldsymbol{x}(t-1) + v\boldsymbol{s}(\boldsymbol{\theta}(t)) \end{cases}, \tag{4}$$

where $P(t)$ is the average matrix of the graph $G(t)$ with $\{P(t)\}_{kj} = 1/n_k(t)$ if $j \in N_k(t)$ and 0 otherwise.

It is easy to see that for any fixed model parameters v and r, the graph sequence $\{G(t), t \ge 0\}$ is totally determined by the initial states $\boldsymbol{\theta}(0)$ and $\boldsymbol{x}(0)$. Our main question is: Under what conditions the angles $\{\theta_k(t), 1 \le k \le n\}$ will converge to a common value $\bar{\bar{\theta}}$, i.e., $\boldsymbol{\theta}(t) \to \bar{\bar{\theta}}\mathbf{1}_n$ $(t \to \infty)$ with $\mathbf{1}_n = (1, 1, \cdots, 1)^T$. When this happens, we call the system (4) converges, or synchronizes. In this paper, we will consider the above model in the following probabilistic framework: In the system (4), assume that the initial positions $\{\boldsymbol{x}_j(0),\ 1 \le j \le n\}$ are i.i.d. random vectors uniformly distributed on the unit square $\mathcal{S}$, the initial angles $\{\theta_j(0),\ 1 \le j \le n\}$ are i.i.d. random variables uniformly distributed in the interval $(-\pi, \pi]$, and the initial positions and initial angles are mutually independent. The underlying probability space will be denoted by (P, Ω) with the element of Ω given by ω. Under these hypotheses, the initial graph $G(\boldsymbol{x}(0), r)$ is a random geometric graph. The main result of this paper is as follows:

Theorem 1: Consider the above probabilistic framework for the multi-agent system described by (1),(2) and (3). Then for any given speed $v > 0$, any radius $r > 0$ and all large population size n, the system will synchronize on a set with probability not less than $1 - O(1/n^{g_n})$, where $g_n = n/(\log n)^{1+\epsilon}, \forall \epsilon > 0$.

The main task of the following three sections is to provide a proof of the theorem.

III. Analysis of System Dynamics

In order to prove the main result of this paper, the analysis of the dynamics of system (4) is necessary. In this section, a key lemma will first be provided to give an estimation of the convergence rate when the closed-loop graphs undergo small changes. Denote $\delta(\boldsymbol{\theta}) = \max_{1 \leqslant j \leqslant n} \theta_j - \min_{1 \leqslant j \leqslant n} \theta_j$, then we have

$$\delta(\boldsymbol{\theta}) \le 2 \max_{1 \le j \le n} |\theta_j|, \text{ and } \delta(\boldsymbol{\theta}) \le \sqrt{2}\|\boldsymbol{\theta}\|. \tag{5}$$

Lemma 1: Let $\{G(t), t \geqslant t_0\}$ be a sequence of time-varying undirected graphs with average matrix $\{P(t), t \ge t_0\}$ and let $\{\boldsymbol{\theta}(t), t \ge t_0\}$ be recursively defined by

$$\boldsymbol{\theta}(t+1) = P(t)\boldsymbol{\theta}(t).$$

If there exists an undirected graph G with average matrix P such that $\|P(t) - P\| \le \varepsilon$, for some $\varepsilon > 0$, then

$$\delta(\boldsymbol{\theta}(t)) \le \sqrt{2}\kappa\big(\bar{\lambda} + \kappa\varepsilon\big)^{t-t_0} \|\boldsymbol{\theta}(t_0)\|, \quad t \ge t_0,$$

where $\bar{\lambda}$ is the spectral gap of G, and κ denotes the square root of the ratio of the maximum degree to the minimum degree of G, *i.e.*, $\sqrt{d_{\max}/d_{\min}}$.

In the following analysis, the number n which denotes the number of vertexes of a graph or the number of agents in the model is taken as a variable, and we will analyze the asymptotic properties of the Laplacian for large n .

Lemma 2: Let $\mathcal{L}$ be the normalized Laplacian matrix of a geometric graph $G(\boldsymbol{x}, r)$, and $\widehat{G}$ be another graph formed by changing the neighborhood of $G(\boldsymbol{x}, r)$. If the number of points changed in the neighborhood of the kth $(1 \le k \le n)$ node satisfies $R_k \le R_{\max}$, and there exist two positive constants $\alpha \in (0, 1)$ and $\beta \ge 1$ such that for large n, $R_{\max} \le \alpha d_{\min}(1 + o(1))$ and $d_{\max} \le \beta d_{\min}(1 + o(1))$,then the corresponding normalized Laplacian matrix $\widehat{\mathcal{L}}$ and average matrix $\widehat{P}$ satisfy

$$\begin{aligned}\|\mathcal{L}-\widehat{\mathcal{L}}\| &\leq 2\Big[1+\frac{\beta+\alpha}{(1-\alpha)^2}\Big]\frac{R_{\max}}{d_{\min}}(1+o(1)),\\ \|P-\widehat{P}\| &\leq \frac{1+\beta}{1-\alpha}\cdot\frac{R_{\max}}{d_{\min}}(1+o(1)).\end{aligned}$$

Lemma 2 could be proven by exploring the special structures of the Laplacian matrices of geometric graphs.

Combining the above lemmas, we can obtain a sufficient condition guaranteeing the synchronization of the system (4). For simplicity of notations, we will omit the subscript 0 in all the variables corresponding to the initial graph $G(\boldsymbol{x}(0),r)$. For any node j, we introduce the following ring,

$$\mathcal{R}_j \triangleq \{\boldsymbol{x}\in\mathbb{R}^2 : (1-\eta)r \leq \|\boldsymbol{x}-\boldsymbol{x}_j(0)\| \leq (1+\eta)r\},$$

where $0<\eta<1$ is any given positive number.

Proposition 1: For the linearized Vicsek Model (4), if the number of agents is sufficiently large and the following three conditions are satisfied, then the systems will synchronize asymptotically:

i) For any node j, the number of nodes within the ring $\mathcal{R}_j$ has an upper bound $R_{\max}$, which satisfies

$$R_{\max} \leq \alpha d_{\min}(1+o(1)), \quad d_{\max} \leq \beta d_{\min}(1+o(1)), \tag{6}$$

where $0<\alpha<1$ and $\beta\geq 1$ are constants.

ii) The spectral gap $\bar{\lambda}$ of the initial graph $G(\boldsymbol{x}(0),r)$ satisfies

$$\bar{\lambda}+\varepsilon < 1, \tag{7}$$

where $\varepsilon \triangleq 2\big[1+(\beta+\alpha)/(1-\alpha)^2\big]\sqrt{\beta}R_{\max}/d_{\min}$.

iii) The speed v satisfies the following inequality

$$\frac{v\delta(\boldsymbol{\theta}(1))}{1-(\bar{\lambda}+\varepsilon)}\Big(2+\log\frac{\sqrt{2\beta}\|\boldsymbol{\theta}(1)\|}{\delta(\boldsymbol{\theta}(1))}\Big) \leq \eta r. \tag{8}$$

Proof: We only need to prove the following claim: at any time t, the number of neighbors of any agent j in the graph $G(t)$ which are different from those in $G(0)$ does not exceed $R_{\max}$. Because if this is true, according to Lemma 2, we get for n large enough

$$\begin{aligned}&\lambda_1\big(\mathcal{L}(t)\big) \geq \lambda_1\big(\mathcal{L}(0)\big) - \big\|\mathcal{L}(t)-\mathcal{L}(0)\big\|\\ \geq\ &\lambda_1\big(\mathcal{L}(0)\big) - 2\Big[1+\frac{\beta+\alpha}{(1-\alpha)^2}\Big]\frac{R_{\max}}{d_{\min}}(1+o(1)) > 0.\end{aligned}$$

Therefore, the graph $G(t)$ is connected, and Theorem 1 in [8] guarantees the convergence of the linearized Vicsek Model (4).

Next we prove the above claim by induction. At $t=0$, the claim is obviously true.

Suppose the claim is valid for $s<t$. As a result of Lemma 2, we get $\|P(s)-P(0)\| \leq \varepsilon/\sqrt{\beta},\ \forall\ s<t$. Hence, by Lemma 1, when n is large enough it is true that for arbitrary $s\leq t$,

$$\delta(\boldsymbol{\theta}(s)) \leq \sqrt{2\beta}(\bar{\lambda}+\varepsilon)^{s-1}\|\boldsymbol{\theta}(1)\|.$$

By this and Condition ii), we can calculate the maximal distance between any two agents in motion as follows : First of all, for arbitrary $1\leq j\neq k\leq n$

$$\begin{aligned}&\|\boldsymbol{x}_j(t)-\boldsymbol{x}_k(t)\|\\ \leq\ &\|\boldsymbol{x}_j(t-1)-\boldsymbol{x}_k(t-1)\| + v|\ 2\sin(\frac{\theta_j(t)-\theta_k(t)}{2})|\\ \leq\ &\|\boldsymbol{x}_j(t-1)-\boldsymbol{x}_k(t-1)\| + v\delta(\boldsymbol{\theta}(t))\\ \leq\ &\|\boldsymbol{x}_j(0)-\boldsymbol{x}_k(0)\| + v\sum_{s=1}^{t}\delta(\boldsymbol{\theta}(s)).\end{aligned} \tag{9}$$

Similarly, we can get

$$\|\boldsymbol{x}_j(0)-\boldsymbol{x}_k(0)\| \leq \|\boldsymbol{x}_j(t)-\boldsymbol{x}_k(t)\| + v\sum_{s=1}^{t}\delta(\boldsymbol{\theta}(s)). \tag{10}$$

Now, let us denote $s_0=\min\{s : \sqrt{2\beta}\ (\bar{\lambda}+\varepsilon)^{s-1}\|\boldsymbol{\theta}(1)\| \leq \delta(\boldsymbol{\theta}(1))\}$, then

$$s_0 = \Big\lceil \frac{\log\frac{\delta(\boldsymbol{\theta}(1))}{\sqrt{2\beta}\|\boldsymbol{\theta}(1)\|}}{\log(\bar{\lambda}+\varepsilon)}+1\Big\rceil \leq \frac{\log\frac{\delta(\boldsymbol{\theta}(1))}{\sqrt{2\beta}\|\boldsymbol{\theta}(1)\|}}{\log(\bar{\lambda}+\varepsilon)}+2.$$

Hence, we obtain

$$\begin{aligned}&v\sum_{s=1}^{t}\delta(\boldsymbol{\theta}(s)) = v\Big(\sum_{s=1}^{s_0-1}\delta(\boldsymbol{\theta}(s)) + \sum_{s=s_0}^{t}\delta(\boldsymbol{\theta}(s))\Big)\\ <\ &v(s_0-1)\delta(\boldsymbol{\theta}(1))\\ &+v\sqrt{2\beta}(\bar{\lambda}+\varepsilon)^{s_0-1}\|\boldsymbol{\theta}(1)\|\sum_{s=s_0}^{t}(\bar{\lambda}+\varepsilon)^{s-s_0}\\ \leq\ &v\delta(\boldsymbol{\theta}(1))\Big(\frac{\log\frac{\delta(\boldsymbol{\theta}(1))}{\sqrt{2\beta}\|\boldsymbol{\theta}(1)\|}}{\log(\bar{\lambda}+\varepsilon)}+1+\frac{1}{1-(\bar{\lambda}+\varepsilon)}\Big)\\ \leq\ &\frac{v\delta(\boldsymbol{\theta}(1))}{1-(\bar{\lambda}+\varepsilon)}\Big(2+\log\frac{\sqrt{2\beta}\|\boldsymbol{\theta}(1)\|}{\delta(\boldsymbol{\theta}(1))}\Big)\\ \leq\ &\eta r,\end{aligned}$$

where for the last but one inequality we have used the following simple facts that $\log\big[\delta(\boldsymbol{\theta}(1))/(\sqrt{2\beta}\|\boldsymbol{\theta}(1)\|)\big]<0$ and $\log x\leq x-1,\ \forall\ 0<x<1$.

According to this and the inequality (9), we conclude that if $\|\boldsymbol{x}_j(0)-\boldsymbol{x}_k(0)\| \leq (1-\eta)r$, then $\|\boldsymbol{x}_j(t)-\boldsymbol{x}_k(t)\| < r$; Otherwise if $\|\boldsymbol{x}_j(0)-\boldsymbol{x}_k(0)\| \geq (1+\eta)r$, then by (10), $\|\boldsymbol{x}_j(t)-\boldsymbol{x}_k(t)\| \geq r$. Hence at time t the number of neighbors changed for any agent j cannot exceed the number of agents in the ring $\mathcal{R}_j$ at time 0, hence cannot exceed $R_{\max}$. This completes the induction argument. ■

It is worth noting that all the conditions in the above proposition are imposed on the model parameters and the initial conditions. The task of the next section is to show how these conditions can be satisfied by analyzing random geometric graphs.

IV. Estimation for the Characteristics of Random Geometric Graph

Throughout the sequel, we denote $\{a_n, g_n, n\in\mathbb{N}\}$ as positive sequences satisfying

$$\sqrt{\log n/n} \ll a_n \ll 1 \ll g_n \ll n{a_n}^2/\log n,$$

where by definition $a_n \ll b_n$ means that $\lim_{n\to\infty} a_n/b_n = 0$ for any positive sequences $\{a_n, b_n, n \in \mathbb{N}\}$.

Let us partition the unit square $\mathcal{S}$ into $M_n = \lceil 1/a_n \rceil^2$ equal-size squares with the length of each side being $a_n(1+o(1))$, where $\lceil x \rceil$ is the smallest integer not less than x. Furthermore, we number these small squares from left to right and from top to bottom. This idea of tessellation and the following lemma are both inspired by [22].

Now, we place n agents independently on $\mathcal{S}$ according to the uniform distribution with their positions denoted by $\boldsymbol{X} = col(\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n)$. Denote by N_j the number of agents that fall into the jth small square. The following lemma gives a uniform estimation for N_j, which could be proven by using the well-known Chernoff Bound [23].

Lemma 3:

$$Pr\Big\{N_j = na_n^2(1+o(1)), 1 \le j \le M_n\Big\} = 1 - O(1/n^{g_n}). \tag{11}$$

Denote the set $B(a_n) = \{\omega \in \Omega : N_j = na_n^2(1+o(1)), 1 \le j \le M_n\}$. The following analysis is carried out on this set. By lemma 3, it is easy to prove the following lemma, which asserts that for certain simple figures, the number of points contained in it is proportional to its area.

Lemma 4: For random geometric graph $G(\boldsymbol{X}, r)$ in $\mathbb{R}^2$, given $\omega \in B(a_n)$, suppose that one of the following three figures intersects with the unit square $\mathcal{S}$ with an area A of the intersecting part and a length L of the arc in $\mathcal{S}$,

i) Rectangle $\{\boldsymbol{x} = (x^1, x^2) \in \mathbb{R}^2 : |x^1 - x_0^1| < a, |x^2 - x^0| < b\}$;

ii) Disk $\{\boldsymbol{x} \in \mathbb{R}^2 : \| \boldsymbol{x} - \boldsymbol{X}_j \| < r\}$;

iii) Ring $\{\boldsymbol{x} \in \mathbb{R}^2 : (1-\eta)r \le \| \boldsymbol{x} - \boldsymbol{X}_j \| \le (1+\eta)r\}$, where $\boldsymbol{x}_0 = (x_0^1, x_0^2)$ is a fixed point, and a, b and $0 < \eta < 1$ are positive constants, j is an arbitrary vertex in $G(\boldsymbol{X}, r)$ with $\boldsymbol{X}_j$ as its position(random variable), then the number of vertexes in the intersection part is

$$M_d = nA(1+o(1)).$$

According to Lemma 4, we could give an estimate of $d_{\min}, d_{\max}$ and $R_{\max}$ in the following theorem, the proof of which is omitted.

Theorem 2: For random geometric graph $G(\boldsymbol{X}, r)$, on the set $B(a_n)$, we have for n sufficiently large:

i)For $0 < r < \frac{1}{2}$,

$$d_{\min} = n\pi r^2/4(1+o(1)), \quad d_{\max} = n\pi r^2(1+o(1)); \tag{12}$$

ii) For $r \ge \frac{1}{2}$,

$$n\pi/64(1+o(1)) \le d_{\min} \le d_{\max} \le n. \tag{13}$$

iii) Denote by R_j the number of vertexes in the intersection part of the ring $\mathcal{R}_j = \{\boldsymbol{x} \in \mathbb{R}^2 : (1-\eta)r \le \|\boldsymbol{x} - \boldsymbol{X}_j\| \le (1+\eta)r\}$ with the unit square $\mathcal{S}$, where $R_{\max} = \max_j R_j$, then

$$R_{\max} \le 4n\pi\eta r^2(1+o(1)). \tag{14}$$

Remark 1: i) In Proposition 1, β can take the value 4 when $0 < r < 1/2$, and when $r > 1/2$, β can take the value $64/\pi$.

ii) When $0 < r < 1/2$, $R_{\max}/d_{\min} = 16\eta(1+o(1))$; and when $r \ge 1/2$, $R_{\max}/d_{\min} \le 256r^2\eta(1+o(1))$. No matter in which case, we can always pick η so small that $\alpha = 3/4$ in Proposition 1, hence,

$$\begin{aligned} &\varepsilon \\ = &\begin{cases} 308R_{\max}/d_{\min}, & 0 < r < 1/2; \\ (208/\sqrt{\pi} + 2^{14}/\pi^{3/2})R_{\max}/d_{\min}, & r \ge 1/2. \end{cases} \\ \le &\begin{cases} 308 \cdot 16\eta(1+o(1)), & 0 < r < \frac{1}{2}; \\ (\frac{208}{\sqrt{\pi}} + \frac{2^{14}}{\pi^{3/2}}) \cdot 256r^2\eta(1+o(1)), & r \ge 1/2. \end{cases} \end{aligned}$$

Due to the importance of $\bar{\lambda}$ in Proposition 1, we will give an estimation for it, part of which is to calculate λ_{n-1} which in turn depends on the following lemma whose proof is given by Prof. Feng TIAN and Dr. Mei LU (see [25] for details).

Lemma 5: Let triangles be extracted from a complete graph K^n in such a way that every time one triangle is extracted with its three edges deleted while three vertexes remain. Then there exits an algorithm such that the number of residual edges at each vertex is no more than three.

Proposition 2: For random geometric graph $G(\boldsymbol{X}, r)$, on the set $B(a_n)$, we have for n sufficiently large

$$\lambda_{n-1} \le 2\big[1 - \frac{1}{4(1+2\sqrt{3})^2}(1+o(1))\big].$$

Proof: Given $\omega \in B(a_n)$, firstly we split $\mathcal{S}$ into $M = \lceil \sqrt{3}/r \rceil^2$ equal-size small squares numbered by $1 \le k \le M$ with side length b satisfying $r/(r+\sqrt{3}) \le b = \sqrt{1/M} \le \sqrt{3}r/3 < \sqrt{2}r/2$. Then any two vertexes in each small square have a distance less than r, making them linked by an edge, which implies that all the vertexes in the small square and edges among them form a clique. Since the area of each small square satisfies $r/(\sqrt{3}+r)^2 \le A = b^2 \le r^2/3$, according to Lemma 4, we know that the number of vertexes M_d in the small square satisfies $nr^2/(\sqrt{3}+r)^2(1+o(1)) \le M_d = nb^2(1+o(1)) \le nr^2/3(1+o(1))$.

Suppose the triangles extracted from the clique in the kth small square according to the algorithm in Lemma 5 form a set Δ_k, the elements of which take the form $G_{\Delta_k} = \{(x,y),(y,z),(z,x)\}$, where x, y, z lie in the kth small square. Let $\Delta = \bigcup_{k=1}^{M} \Delta_k, \Delta_e = \{(x,y) \in G_\Delta : G_\Delta \in \Delta\}$, and $\Delta_e^c = E\{G(X,r)\} - \Delta_e$. For each vertex j which lies in the kth small square and thus the neighborhood disk entirely contains it, there are $d_j - 1$ edges linking to it except the self-loop one, hence at least M_d edges among them belong to Δ_e. Therefore, for vertex j, the ratio of the number of edges in Δ_e to the total number of edges linking to j except the self-loop one satisfies

$$\begin{aligned} &M_d/(d_j - 1) \ge nb^2/d_{\max}(1+o(1)) \\ \ge &\begin{cases} 1/[\pi(1/2+\sqrt{3})^2](1+o(1)), & 0 < r < 1/2; \\ 1/(1+2\sqrt{3})^2(1+o(1)), & r \ge 1/2. \end{cases} \end{aligned}$$

Hence, for any vector $z \in \mathbb{R}^n$, when n is sufficiently large, we have,

$$\begin{aligned}
&\sum_{j\sim k}(z_j - z_k)^2 \\
=& \sum_{\{(j,k),(k,l),(l,j)\}\in\Delta} [(z_j - z_k)^2 + (z_k - z_l)^2 + (z_l - z_j)^2] \\
&+ \sum_{(j,k)\in\Delta_e^c} (z_j - z_k)^2 \\
\le& \sum_{\{(j,k),(k,l),(l,j)\}\in\Delta} 3(z_j^2 + z_k^2 + z_l^2) + \sum_{(j,k)\in\Delta_e^c} 2(z_j^2 + z_k^2) \\
=& \sum_j\Big(\sum_{k:\ (j,\ k)\in\Delta_e} 3z_j^2/2\Big) + \sum_j\Big(\sum_{k:\ (j,\ k)\in\Delta_e^c} 2z_j^2\Big) \\
=& \sum_j\Big(M_d\cdot 3z_j^2/2\Big) + \sum_j\Big((d_j - 1 - M_d)2z_j^2\Big) \\
=& \sum_j (d_j - 1)\big[M_d/(d_j-1)\cdot 3z_j^2/2\big] \\
&+ \sum_j (d_j - 1)\big[(1 - M_d/(d_j-1))2z_j^2\big] \\
\le& \sum_j d_j\big[\big(1 - M_d/(4(d_j-1))\big)2z_j^2\big] \\
\le& 2\big[1 - b^2/(4d_{\max})(1+o(1))\big]\sum_j d_j z_j^2,
\end{aligned}$$

where we have employed elementary inequality

$$(a-b)^2 + (b-c)^2 + (c-a)^2 \leqslant 3(a^2+b^2+c^2).$$

Therefore, by the Rayleigh quotient expression of λ_{n-1}, we obtain,

$$\begin{aligned}
\lambda_{n-1} &= \sup_z \frac{\sum_{j\sim k}(z_j - z_k)^2}{\sum_j z_j^2 d_j} \le 2[1 - \frac{nb^2}{4d_{\max}}(1+o(1))] \\
&\le \begin{cases} 2\Big(1 - \frac{1}{\pi(1+2\sqrt{3})^2}(1+o(1))\Big), & 0 < r < \frac{1}{2}; \\ 2\Big(1 - \frac{1}{4(1+2\sqrt{3})^2}(1+o(1))\Big), & r \ge \frac{1}{2}. \end{cases} \\
&\le 2\Big(1 - \frac{1}{4(1+2\sqrt{3})^2}(1+o(1))\Big). \quad \blacksquare
\end{aligned}$$

In the following, we will estimate λ_1.

Proposition 3: For random geometric graph $G(\boldsymbol{X}, r)$, on the set $B(a_n)$, we have for n sufficiently large,

$$\lambda_1 \ge \frac{\pi r^2}{512(r+\sqrt{6})^4}(1+o(1)). \qquad (15)$$

Proof: This proposition could be proven via a slightly improved version of Theorem 4.3 in [16]. See [25] for more details.

Combining the above propositions, we could obtain an estimation for the spectral gap $\bar{\lambda}$:

Theorem 3: For random geometric graph $G(\boldsymbol{X}, r)$, on the set $B(a_n)$, we have for n sufficiently large,

$$\bar{\lambda} \le 1 - \frac{\pi r^2}{512(r+\sqrt{6})^4}(1+o(1)). \qquad (16)$$

Remark 2: According to Theorem 3 and Remark 1, we can always take η so small that

$$\bar{\lambda} + \varepsilon \le 1 - \frac{\pi r^2}{1024(r+\sqrt{6})^4}(1+o(1)) \triangleq 1 - C_r.$$

Next we will deal with $\|\boldsymbol{\theta}(1)\|$ and $\delta(\boldsymbol{\theta}(1))$. For that we first need the following lemma which can be proven by using the Hoeffding Inequality [23]. Throughout the sequel, we will denote $h_n = na_n^2/(g_n \log n)$, which satisfies $lim_{n\to\infty} h_n = \infty$ by the choice of a_n and g_n.

Lemma 6: Denote by S_j the set of points in the jth small square and $\Theta_j = \sum_{k\in S_j}\theta_k(0), j = 1, 2, \cdots, M_n$. Then

$$\begin{aligned}
&Pr\Big\{\max_{1\le j\le M_n}|\Theta_j| \le na_n^2\pi\sqrt{2/h_n}(1+o(1))\Big|B(a_n)\Big\} \\
&= 1 - O(1/n^{g_n}).
\end{aligned}$$

Hence, if we denote the set

$$\begin{aligned}
&D(a_n, g_n, h_n) \\
=& \{\omega\in\Omega : \max_{1\le j\le M_n}|\Theta_j| \le na_n^2\pi\sqrt{2/h_n}(1+o(1))\},
\end{aligned}$$

then we get,

$$\begin{aligned}
& Pr\{D(a_n, g_n, h_n)B(a_n)\} \\
=& Pr\{D(a_n, g_n, h_n)|B(a_n)\}Pr\{B(a_n)\} \\
=& 1 - O(1/n^{g_n}).
\end{aligned}$$

Theorem 4: On the set $B(a_n)\bigcap D(a_n, g_n, h_n)$, we obtain for n sufficiently large,

$$\begin{aligned}
\|\boldsymbol{\theta}(1)\| &\le \sqrt{n}\big[\pi\sqrt{2/h_n} + 16\sqrt{2}a_n/r\big](1+o(1)), \\
\delta(\boldsymbol{\theta}(1)) &\le 2\big[\pi\sqrt{2/h_n} + 16\sqrt{2}a_n/r\big](1+o(1)).
\end{aligned}$$

Proof: Given $\omega \in B(a_n)\bigcap D(a_n, g_n, h_n)$, for any agent k, denote by J_k the index set of those small squares entirely contained in the neighborhood disk of agent k, and A the area of the intersection part between the neighborhood disk and the unit square $\mathcal{S}$, then we obtain,

$$\Big|\sum_{i\in N_k(0)}\theta_i(0)\Big| \le \sum_{j\in J_k}|\Theta_j| + \Big|\sum_{i\in N_k(0)/\bigcup_{j\in J_k}S_j}\theta_i(0)\Big|.$$

Hence, according to Lemma 6, we have

$$\begin{aligned}
&\Big|\sum_{i\in N_k(0)}\theta_i(0)\Big| \\
\le& \Big[\sum_{j\in J_k} na_n^2\pi\sqrt{\frac{2}{h_n}} + (\frac{2\pi r 2\sqrt{2}a_n}{a_n^2})\cdot(na_n^2)\Big](1+o(1)) \\
=& \frac{A}{a_n^2}\cdot na_n^2\pi\sqrt{2/h_n}(1+o(1)) + 4\sqrt{2}\pi r n a_n(1+o(1)) \\
=& n\big[\pi A\sqrt{2/h_n} + 4\sqrt{2}\pi r a_n\big](1+o(1)).
\end{aligned}$$

Hence,

$$\begin{aligned}
|\theta_k(1)| &= \frac{|\sum_{j\in N_k(0)}\theta_j(0)|}{n_k(0)} \\
&\le \big[\pi\sqrt{2/h_n} + 16\sqrt{2}a_n/r\big](1+o(1)).
\end{aligned}$$

Therefore, according to (5),

$$\|\boldsymbol{\theta}(1)\| \leq \sqrt{n}\big[\pi\sqrt{2/h_n} + 16\sqrt{2}a_n/r\big](1+o(1)),$$
$$\delta(\boldsymbol{\theta}(1)) \leq 2\big[\pi\sqrt{2/h_n} + 16\sqrt{2}a_n/r\big](1+o(1)). \quad \blacksquare$$

V. THE PROOF OF THEOREM 1

For any $\epsilon > 0$, let us take the positive sequences $\{a_n, g_n, h_n, n \in \mathbb{N}\}$ as

$$a_n = 1/(\log n)^{\epsilon/3}, \quad h_n = (\log n)^{\epsilon/3}, \quad g_n = n/(\log n)^{1+\epsilon}.$$

Then $a_n = o(1/\sqrt{h_n})$, hence we get,

$$\|\boldsymbol{\theta}(1)\| \leq \sqrt{n}\pi\sqrt{2/h_n}(1+o(1)),$$
$$\delta(\boldsymbol{\theta}(1)) \leq 2\pi\sqrt{2/h_n}(1+o(1)).$$

Then, on the set $B(a_n)\bigcap D(a_n, g_n, h_n)$ with n sufficiently large, we have by Theorem 4 and Remark 2,

$$\begin{aligned}
& v\delta(\boldsymbol{\theta}(1))/(1-(\bar{\lambda}+\varepsilon))\Big(2+\log\frac{\sqrt{2\beta}\parallel\boldsymbol{\theta}(1)\parallel}{\delta\boldsymbol{\theta}(1)}\Big) \\
\leq\; & \Big[v\delta(\boldsymbol{\theta}(1))\big(2+\log(\sqrt{2\beta}\parallel\boldsymbol{\theta}(1)\parallel)\big) \\
& \qquad -v\delta(\boldsymbol{\theta}(1))\log\delta(\boldsymbol{\theta}(1))\Big]/C_r \\
\leq\; & \Big[2v\pi\sqrt{2/h_n}\big(2+\log(2\sqrt{\beta}\pi\sqrt{n/h_n})\big)(1+o(1)) \\
& \qquad -2v\pi\sqrt{2/h_n}\log(2\pi\sqrt{2/h_n})\Big]/C_r \\
\leq\; & \frac{2v\pi\sqrt{2/h_n}}{C_r}\Big(2+\log\frac{2\sqrt{\beta}\pi\sqrt{n/h_n}}{2\pi\sqrt{2/h_n}}\Big)(1+o(1)) \\
=\; & 2v\pi\sqrt{2/h_n}\log n/C_r(1+o(1)),
\end{aligned}$$

where we have used the fact that $-x\log x$ is an increasing function of x on $(0, e^{-1})$. Thus in order to satisfy condition iii) in Proposition 1, it is sufficient to take n large enough so that

$$2v\pi\sqrt{2/h_n}\log n/C_r(1+o(1)) \leqslant \eta r,$$

which is obviously true by the choice of the sequences $\{a_n, g_n, h_n, n \in \mathbb{N}\}$ above. Thus when n is sufficiently large, the probability for convergence will be greater than or equal to $Pr\{B(a_n)D(a_n, g_n, h_n)\} = 1 - O(1/n^{g_n})$. This completes the proof.

VI. CONCLUSIONS

It is conceivable that the probabilistic methods that we used in this paper can lead to much weaker conditions than those used in the deterministic framework (see, e.g., [8][26]). The main result proved in this paper is that the multi-agent systems described by a simplified Vicsek Model will synchronize with large probability for large population and for any nonzero fixed model parameters(i.e., the radius r and the speed v). Intuitively, the same results should also be true if both r and v depend on the population size n (and hence denoted as r_n and v_n) and decrease as n increases in a suitable way. In fact, it can be shown that under the parameter condition that $v_n/r_n^5 = O(1/\log n)$, and $(\log n/n)^{1/6} = o(r_n), r_n = o(1)$, similar synchronization result can also be established [24]. To the best of our knowledge, this kind of synchronization results for multi-agent systems are established for the first time. Of course, it would be interesting to study more complicated multi-agent systems and more complicated collective behaviors in future work.

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