Convex Blind Deconvolution with Random Masks

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Abstract: We solve blind deconvolution problems where one signal is modulated by multiple random masks using nuclear norm minimization. Theoretical analysis shows the number of masks for successful recovery scales as poly-logarithm of the problem dimension.
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1. Introduction

Blind deconvolution is a fundamental signal processing problem that finds numerous applications in image processing, communication, astronomy, and microscopy. The problem falls into the category of nonlinear inverse problems and is ill-posed without prior information or multiple measurements. As a consequence, most existing methods for blind deconvolution suffer from convergence issues and lack theoretical guarantees. One exception is the work in [1], where the authors mitigate the ill-posedness by assuming the signals live in known subspaces and recast the deconvolution problem as low-rank matrix recovery from linear measurements. This latter problem is then relaxed into a nuclear norm minimization program which is solved efficiently with guaranteed success.

Subspace information for the unknown signals is not always available in practice. In this work, we propose to make blind deconvolution well-posed by diversifying measurements. More explicitly, we modulate one of the signals by multiple known masks and observe their convolutions with the other signal. We motivate this observation model by two examples from communication and photography. In the first example, the transmitter sends a message through an unknown channel and the receiver observes the output of the channel as a convolution. Without further knowledge of the channel and the message, there is no way for the receiver to recover the transmitted message. Instead of relying upon prior knowledge, the transmitter can choose to modulate the message using pre-agreed masks and send the modulated messages through the channel. The receiver then recovers both the message and the channel response based on the received signals and knowledge of the masks. Modulation is the preferred approach in scenarios where prior information is hard to obtain or might become obsolete as time goes.

In the second example, we aim to recover a scene from motion blurred images. The motion blur kernel can be modulated by a binary mask through fluttering the shutter of the camera. This coded exposure technique gives state-of-the-art performance in motion deblurring when the blurring kernel is known [2], which is not a reasonable assumption in most cases. We diversify observations by either using a camera array with different fluttering patterns for each individual camera, or taking multiple shots with a single camera when the motion remains unchanged or is periodic.

2. Problem Setup

We set up the problem we aim to solve in this section. Suppose we have two signals x and z. For ease of presentation, we assume both signals are of the same size and live in $\mathbb{R}^n$. In many cases, one signal is the impulse response of a linear system and the other is an input signal to the system. We have available m masks $\{b^i, i = 1, \ldots, m\} \subset \mathbb{R}^n$ that we apply to the signal z to obtain m convolution results:

$$y^i = x \circledast (b^i \odot z), i = 1, \ldots, m \tag{1}$$

where $\circledast$ represents circular convolution, and $\odot$ denotes elementwise/Hadamard product. Note that we have used circular convolution for ease of derivation. Since linear convolution can be obtained through zero padding with a minor increase in problem dimension, the main result carries over to linear convolutions.

3. Convex Blind Deconvolution

The measurements in (1) are quadratic in x and z, but linear in the products $x_j z_k$. This observation motivates us to write the measurements y^i_j as linear functions of the rank-1 matrix xz^T. Indeed, algebraic manipulation shows that

$$y^i_j = \text{trace}((xz^T)^T H^j F \text{diag}(b^i)), \tag{2}$$

where H^j is the matrix that, when multiplied by a column vector from right, circularly shifts that vector down jth times, and F is the matrix that flips a vector upside down. Therefore, finding x and z satisfying (1) is equivalent to

$$\underset{M}{\text{minimize}} \ \text{rank}(M)$$
$$\text{subject to } y^i_j = \text{trace}(M^T H^j F \, \text{diag}(b^i)), j = 1, \ldots, n; i = 1, \ldots, m. \tag{3}$$

The formulation (3) is not readily solvable due to the non-convex rank function. We replace the objective function with the convex nuclear norm, which was shown to be a powerful relaxation for the rank function [3,4]:

$$\underset{M}{\text{minimize}} \ \|M\|_*$$
$$\text{subject to } y^i_j = \text{trace}(M^T (H^j F \, \text{diag}(b^i))), j = 1, \ldots, n; i = 1, \ldots, m. \tag{4}$$

Due to its convexity, the formulation (4) has no non-global local minima and many efficient and scalable algorithms have been designed to solve nuclear norm minimization problems.

3.1. What Signals Can be Blindly Deconvolved?

A natural question is in what circumstances the convex relaxation (4) would successfully return $M^* = xz^T$. To build intuition, we start with a discussion of what signals can be blindly deconvoled using the proposed approach. There is no hope that the optimization (4) will be able to deconvolve all signal pairs from much fewer than n masked convolution results. In the extreme case when x is the all-one vector, for each mask b, the convolution result is a multiple of the all-one vector since $y_j = [x \circledast (b \odot z)]_j = \sum_k b_k z_k, \forall j$. So we need to make some diversity assumption on x:

Definition For any $x \in \mathbb{R}^n$, denote the circulant matrix $D(x) = \sum_j H^{jT} xx^T H^j$ whose first row is the circular autocorrelation function of x. We define

$$\mu = \mu(x) = \frac{\|D(x)\|}{\|x\|_2^2} = \frac{n\|U_n x\|_\infty^2}{\|x\|^2}, \tag{5}$$

where U_n is the unitary Fourier transform matrix.

The parameter μ measures the decaying speed of the (circular) autocorrelation function of x. Faster decaying speed of the autocorrelation implies a smaller μ and fewer masks we need in order to perform blind deconvolution using (4). Typically, μ depends on n, e.g., $\mu = \log^\gamma(n)$ for some $\gamma \geqslant 0$. The maximum $\mu = n$ is obtained when x is the all-one vector in $\mathbb{R}^n$. The following lemma shows that random Gaussian signals have $\mu = \log(n)$ with high probability.

Lemma 3.1 *If $x \sim N(0, \sigma^2 I_n)$, then $\mu(x) \leq C \log(n)$ with high probability for some numerical constant C.*

3.2. Main Result

We now present our major theorem for independent, identically distributed (i.i.d.) binary masks, *i.e.*, all entries of the masks assume values $\{\pm 1\}$ independently with probability $\frac{1}{2}$.

Theorem 3.2 *For i.i.d. binary masks, if the number of masks*

$$m \geq C(\beta)\mu \log^3(n)$$

for some constant $C(\beta)$ depending on β, then with probability at least $1 - O\left(\frac{1}{n^\beta}\right)$, the optimization (4) will recover $M^* = xz^T$ exactly. The signals x and z can then be recovered by singular value decomposition up to global scaling.

4. Numerical Experiments

In this section, we perform numerical simulations to test the performance of (4). In the first experiment, we test the success rate of (4) when applied to one-dimensional signals. For $n = 128$, and each $m \in \{4, 6, \ldots, 20\}$, we generate $T = 50$ signal pairs x and z both following Gaussian distributions. For each instance, optimization (4) was solved using the Burer and Monteiro nonliner solver [1,5]. We declare success if both recovered signals have an ℓ_2 error less than 10^{-4}. The rate of success is plotted in Figure 1. We get decent recovery rate even with 4 masks and 100% recovery when we have more than 12 masks.

In the second experiment, we test the efficacy of the proposed algorithm for motion deblurring. For the 18×28 motion blur kernel shown in Figure 2a, we mask and convolve it with an unknown image to obtain 20 blurred images of size 256×256 in Figure 2b. We then run our blind deconvolution algorithm on these blurred images to successfully recover the original image shown in Figure 2c. Note in this case, the motion blur kernel has a $\mu \approx 96$, while a randomly generated kernel of the same size has a $\mu \approx 6$.
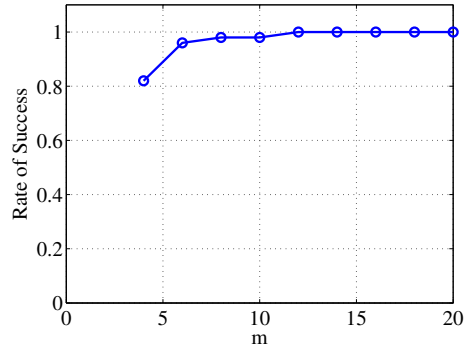
Fig. 1: Rate of success as a function the number of masks.

(a) Blur kernel   (b) Motion blurred images   (c) Recovered image

Fig. 2: Motion deblur using random masks.

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