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Induction formal proof

5.4.2

Suppose $b_1, b_2, b_3, ...$ is the sequence defined as follow:

$$b_1 = 4,\ b_2 = 12$$
$$b_k = b_{k-2} + b_{k-1} \quad \text{for all integer } k \geq 3$$

Prove that b_n is divisible by 4 for all integers $n \geq 1$

Theorem:
With the sequence defined as above, b_n is divisible by 4 for all integers $n \geq 1$.
Proof:
By definition of the sequence:

$$\begin{aligned}
&b_1 = 4\\
&b_2 = 12\\
&b_3 = b_{3-2} + b_{3-1} = b_1 + b_2\\
&b_4 = b_{4-2} + b_{4-1} = b_2 + b_3\\
&...
\end{aligned}$$

Set the statement: $F(n)$ is b_n is divisible by 4.
We uses strong mathematical induction to prove that for every integer $n \in \mathbb{Z}, n \geq 1$, $F(n)$ is true.
Basis:
To show that $F(1), F(2)$ are true, b_1, b_2 must be divisible by 4, by definition of divisible:

$$\text{If } a, b \in \mathbb{Z} \text{ and } a \neq 0,\ a|b \text{ if there is an integer } c \in \mathbb{Z} \text{ such that } a \cdot c = b.$$

Since $b_1 = 4$, and $b_2 = 12$, we have the following:

$$\frac{b_1}{4} = 1 \in \mathbb{Z}$$
$$\frac{b_2}{4} = 3 \in \mathbb{Z}$$

Where for b_1, $b_1 = b, c = 1, a = 4$, where $b_1|4$, b_1 is divisible by 4.
Also for b_2, $b_2 = b, c = 3, a = 4$, where $b_2|4$, b_2 is divisible by 4.
Both $F(1), F(2)$ are true, therefore the basis are true.

Induction:
Suppose integer $k \in \mathbb{Z}, k \geq 3$, if $F(i)$ is true for integer $i \in \mathbb{Z}, 1 \leq i \leq k$, then $F(k+1)$ is true.
The induction hypothesis states that: $k \in \mathbb{Z}, k \geq 3$, and b_k is divisible by 4 for all integers $i \in \mathbb{Z}, 1 \leq i \leq k$.
We will show that $F(k+1)$ is true, b_{k+1} is divisible by 4.
By definition, $k \geq 3$, which implies that $k+1 \geq 3$ and $k-1 \geq 2$, so we have:

$$b_{k+1} = b_{k-1} + b_k$$

By our inductive hypothesis, b_{k-1} and b_k are divisible by 4 since $k \leq k$, $k-1 \leq k$. By definition of divisibility, we have the following:

$$b_k = 4 \cdot x \;\; x \in \mathbb{Z}$$

$$b_{k-1} = 4 \cdot y \;\; y \in \mathbb{Z}$$

Substituting b_k, b_{k-1} into b_{k+1}, we have:

$$\begin{aligned} b_{k+1} &= b_{k-1} + b_k \\ b_{k+1} &= 4x + 4y && \text{Substituting} \\ b_{k+1} &= 4(x+y) && \text{Factoring} \end{aligned}$$

By integer addition, we have integer $z \in \mathbb{Z}, z = x + y$
Substituting t into b_{k+1}, we have:

$$b_{k+1} = 4z$$

Recall the definition of divisibility, b_{k+1} is divisible by 4.
Hence $F(n)$ is true for $n = k+1$. Induction is true.
Since both the basis and induction are true, therefore the original statement:

$$b_1 = 4,\; b_2 = 12$$

$$b_k = b_{k-2} + b_{k-1} \quad \text{for all integer } k \geq 3$$

b_n is divisible by 4 for all integers $n \geq 1$ is true.
QED