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Analysis formal proof

Proof that $n!$ is $\Omega(2^n)$. Recall that $n! = \prod_{i=1}^{n} i$ and $2^n = \prod_{i=1}^{n} 2$

Theorem:
$n!$ is $\Omega(2^n)$.

Proof:
By definition of Big Ω notation, $f(x)$ is $\Omega(g(x))$ if and only if there exist a positive real number A and a nonnegative real number a such that:

$$A|g(x)| \leq |f(x)| \text{ for all real numbers } x > a.$$

Set x as an integer, $x \in \mathbb{Z}$.
Set the $f(x) = n!$ and $g(x) = 2^n$.
By definition of $n!$, $n! = \prod_{i=1}^{n}$, we have the following:

$$f(x) = n! = 1 \cdot 2 \cdot 3 \cdot ... \cdot n \tag{1}$$

Also, by the definition of 2^n, we have:

$$g(x) = 2^n = 2 \cdot 2 \cdot 2 \cdot ... \cdot 2 \ \ (n \text{ of 2s}) \tag{2}$$

When $x = 3$, we have $|f(3)|, |g(3)|$ equal the following:

$$|f(3)| = 3! = 3 \cdot 2 \cdot 1 = 6 \tag{3}$$

$$|g(3)| = 2^3 = 2 \cdot 2 \cdot 2 = 8 \tag{4}$$

Which we have $f(3) < g(3)$.
When $x > 3$, we plug $x = 4$ into $f(x), g(x)$.

$$|f(4)| = 4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24 \tag{5}$$

$$|g(4)| = 2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16 \tag{6}$$

Which we have $f(4) > g(4)$.
Set integer $i \in \mathbb{Z}, i > 3$. We substituting i into our $f(x), g(x)$, we have the following:

$$\begin{aligned} |f(i)| &= i! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot ... \cdot i \\ &= 1 \cdot (2+0) \cdot (2+1) \cdot (2+2) \cdot (2+3) \cdot ... \cdot (2+(i-2)) \end{aligned}$$

$$|g(x)| = 2^i = 2 \cdot 2 \cdot 2 \cdot ... \cdot 2 \qquad\qquad i \text{ of 2s}$$

In both $f(i), g(i)$, we have i terms. In $f(i)$, all of the terms are positive, and except the first two terms, all of the terms are greater than 2. Therefore the product of all the terms in $f(i)$ is greater

than the result of $g(i)$. Therefore, $|f(i)| > |g(x)|$ when $i \in \mathbb{Z}, i > 3$.
Set integer $A \in \mathbb{Z}$, and $A = 1$. We multiply A with $|g(x)|$, we have:

$$A|g(x)| = 1 \cdot 2^x = 2^x$$

Therefore, $1|g(x)| \leq |f(x)|$ for all real numbers $x > 3$.
Therefore, $n!$ is $\Omega(2^n)$ when $A = 1$ and $a = 3$
QED