MANIFOLD THEORY

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Submitted By : ATRAJIT SARKAR

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Supervisor : Dr Biplab Basak

Contents

1 Introduction

A topological space M which is locally Euclidean space($\mathbb{R}^n$), Hausdorff and second countable, is called an n-manifold corresponding to the value n.
We want to study such spaces in terms of simplicial complex structure which is relatively simple structured topological spaces and easy to visualize.

1.1 Preliminaries

1.1.1 Definitions:

1. **Manifold:**
   Let (X,τ) be a topological space .Such that X is a Hausdorff space , X is second countable , X is locally euclidean of dimension 'n'.

2. **Simplexes and Simplicial Complex:**
   Definition 7.1.1 Let $\mathcal{E}$ be any normed affine space, say $\mathcal{E} = \mathbb{E}^m$ with its usual Euclidean norm. Given any $n+1$ affinely independent points $a_0, \ldots, a_n$ in $\mathcal{E}$, the n-simplex (or simplex) σ defined by $a_0, \ldots, a_n$ is the convex hull of the points $a_0, \ldots, a_n$, that is, the set of all convex combinations $\lambda_0 a_0 + \cdots + \lambda_n a_n$, where $\lambda_0 + \cdots + \lambda_n = 1$ and $\lambda_i \geq 0$ for all $i, 0 \leq i \leq n$.

   Definition 7.1.2 A simplicial complex in $\mathbb{E}^m$ (for short, a complex in $\mathbb{E}^m$) is a set K consisting of a (finite or infinite) set of simplices in $\mathbb{E}^m$ satisfying the following conditions:

   (1) Every face of a simplex in K also belongs to K.

   (2) For any two simplices σ_1 and σ_2 in K, if $\sigma_1 \cap \sigma_2 \neq \emptyset$, then $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.

   Every k-simplex, $\sigma \in K$, is called a k-face (or face) of K. A 0 -face $\{v\}$ is called a vertex and a 1 -face is called an edge. The dimension of the simplicial complex K is the maximum of the dimensions of all simplices in K.

3. **Principal and Free face of a simplex**
   Let $\sigma^n \in K$ where K is a Simplicial Complex , $\delta^{n-1} \in K$ then σ is a Principal face if it is not a proper face of any simplex in K. In a sense , it is a maximal face and it is not properly contained in any other face.

   Free Face
   δ is a free face if it is not a proper face of any simplex in K other than σ.

1.1.2 Collapsing:

To discuss spines, we need to define precisely collapsing. We start with the definition of an elementary simplicial collapse.
Let K be a simplicial complex, and let σ,$\delta^{n-1} \in K$ be two open simplices such that σ is the principal face and δ is a free face of it.

1. **Elementary Simplicial Collapse**
   The transition from **K** to $K - (\sigma \cup \delta)$ is called an elementary simplicial collapse.

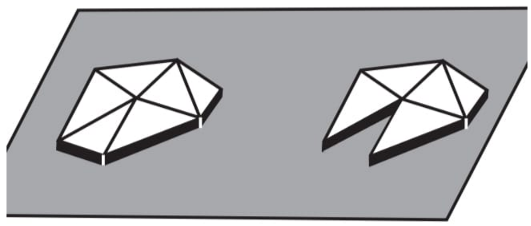


Fig. 1.1. Elementary simplicial collapse

Figure 1: SimplicialCollapse

2. Ployhedral Collapse:
   A polyhedron P collapses to a sub polyhedron Q (P $\searrow$ Q) if for some triangulation (K, L) of the pair (P, Q) the complex K collapses onto L by a sequence of elementary simplicial collapses.
   for example , any n-dimensional cell B^n collapses to any (n-1) dimensional face $B^{n-1} \subset \partial B^n$.

   The **Transition** from P to Q is called elementary Polyhedral Collapse.

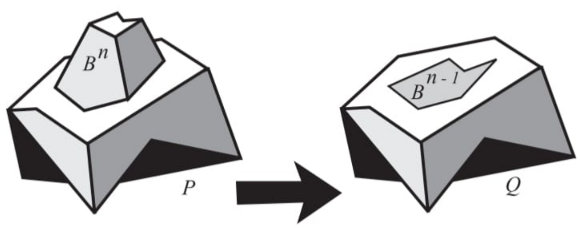


Fig. 1.2. Elementary polyhedral collapse

Figure 2: Polyhedral Collapse

Note:
It is easy to see that an elementary simplicial collapse is a special case of an elementary polyhedral collapse. Likewise, it is possible to choose a triangulation of the ball Bn such that the collapse of B^n onto its face B^{n-1} can be expressed as a sequence of elementary simplicial collapses. It follows that the same is true for any elementary polyhedral collapse.
By a simplicial collapse of a simplicial complex K onto its subcomplex L we mean any sequence of elementary simplicial collapses transforming K into L. Similarly, a polyhedral collapse is a sequence of elementary polyhedral collapses.

2 Spines

2.1 Definition:

Let M be a compact connected 3-manifold with a boundary. A sub polyhedron $P \subset M$ is called a spine of M if $M \searrow P$, i.e., M collapses to P. By a spine of a closed connected 3-manifold M, we mean a spine of $M \backslash IntB^3$ where B^3 is a 3-ball in M. By the spine of a disconnected 3-manifold, we mean the union of spines of its connected components.

2.1.1 Remark:

A simple argument shows that any compact triangulated 3-manifold M always possesses spine of dimension ≤ 2. Indeed, let M collapse to a subcomplex K. If K contains a 3-simplex, then K contains a 3-simplex with a free face, so the collapsing can be continued.

2.1.2 Remark:

Let M be a closed connected 3-manifold then P is a spine of M if is a spine of some bounded 3-manifold M^c obtained by deleting one or more disjoint open 3-balls from M.

Explanation:
In this case, we can recover M from spine P as,

1. First "thicken" P to recover a bounded 3-manifold M^c.
2. Fill in the holes in M^c by attaching cones on all the boundary components, where M^c is the bounded 3-manifold obtained by deleting the small open nbd bdd by all the vertex link in τ: triangulation of M.

   i.e. M^c is obtained from τ by truncating the vertices of all the tetrahedra in τ.

3. Each truncated tetrahedron is homeomorphic to a nbd of a butterfly, so P must be a spine of M^c and hence M.

It is often convenient to view 3-manifolds as mapping cylinders over their spines and as a regular neighborhood of the spines. The following theorem justifies this point of view. We first recall the definition of a mapping cylinder.

Mapping Cylinder:

Let $f : X \to Y$ be a map between topological spaces. The mapping cylinder C_f is defined as $Y \cup (X \times [0,1])/\sim$, where the equivalence relation is generated by identifications $(x,1) \sim f(x)$. If Y is a point, then C_f is called the cone over X.

Fig. 1.3. The mapping cylinder and the cone

Figure 3: Mapping cylinder and cone

2.1.3 Theorem:

[3, Theorem 1.1.7]
The following conditions on a compact sub polyhedron $P \subset IntM$ of a compact 3-manifold with boundary are equivalent:

(a) P is a spine of M.

(b) M is homeomorphic to a regular neighborhood of P in M.

(c) M is homeomorphic to the mapping cylinder of a map $f : \partial M \to P$.

(d) the manifold $M \backslash P$ is homeomorphic to $\partial M \times [0,1)$

Explanation:

Now, let us understand what this theorem tells us and how from this theorem we can recover a manifold from its spine.

Regular Neighborhood:

Let $P \subset M$ a n-manifold. If there is a neighborhood of P in M which is homeomorphic to a mapping cylinder $f : Y \to P$ s.t. the embedding $C_f \to M$ embeds Y as a smooth submanifold of M for some space Y, then the image of the embedding $C_f \to M$ is called the regular neighborhood of P in M.

So, we have due to the above theorem that if P is a spine of M then M is homeomorphic to a regular neighborhood of P in M, and not only that this regular neighborhood can be taken as the mapping cylinder $f : \partial M \to P$.

So, given a spine P, we can make a mapping cylinder C_f via $f : Y \to P$. Now, make the homeomorphism class of C_f viz. $[C_f]$. Now, select a subclass from this for which the homeomorphism $C_f \to M$ embeds Y into M smoothly. This sub-collection is the required collection of all manifolds whose spine is P due to $(b) \implies (a)$ from the above theorem.

Note:

Spine is a useful tool to study manifold but the problem with them -

1. Several manifolds may have the same spine.
2. Given manifold may have several different spines.

   (2D example):- Consider 2d Annulus and Mobius strip both of these two manifolds collapse to a circle, but are not homeomorphic to each other.

   (3D example):- A solid Klein bottle and a solid torus have circles as spines.

   Therefore to eliminate these problems we will study special spines.

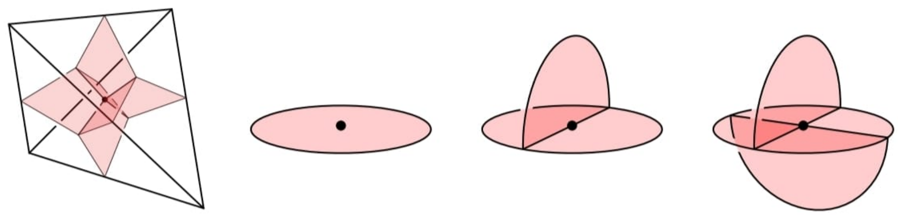
2.2 Polyhedra

Idea:
Each connected compact 3-manifold with a boundary has a standard spine and if two compact 3-manifolds have homeomorphic standard spines then they are homeomorphic.

2.2.1 Simple Polyhedra:

Definition: A compact polyhedron P is called simple if the link of each of its points $x \in P$ is homeomorphic to one of the following 1D polyhedra,

1. A circle (such a point is called a non-singular point).
2. A circle with a diameter (such an x is a triple point)
3. a circle with three radii (such a point is called a true vertex.)



(A) A butterfly inside of a tetrahedron.

(B) Three different types of point on a spine. From left to right: a non-singular point, a triple point, and a vertex.

FIGURE 16. Spines.

Figure 4: Neighbourhood of simple polyhedron

Note:
The polyhedron with true vertex singularity is called a butterfly.

Butterfly Polyhedron:

We can visualize it as 4 segments having a common endpoint and six wings. Each wing spans two segments and each pair of segments is spanned by exactly one wing.
Each butterfly has four edges emanating out from a single central vertex.

1. Consider this vertex as dual to a tetrahedron.
2. Four edges as dual to the four triangular faces of the tetrahedron.
3. To each pair of edges in the butterfly is a 2D wing which is dual to a tetrahedron edge.

2.2.2 Singular Graph

Definition:
The set of singular points of a simple polyhedron is called its singular graph and is denoted by SP.

To describe the structure of a simple polyhedron in detail we need to consider the definition of stratum.

- *2-stratum:* A 2D component which is a connected component of a set of non-singular points.
- *1-stratum:* Stratum of dimension 1 is a set of open or closed triple lines.

- *0-stratum:* This consists of true vertices.

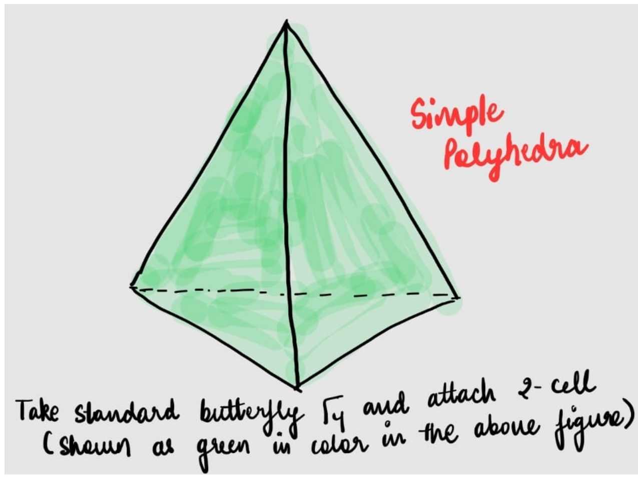


Figure 5: simple polyhedron

Here is an example of singular graph

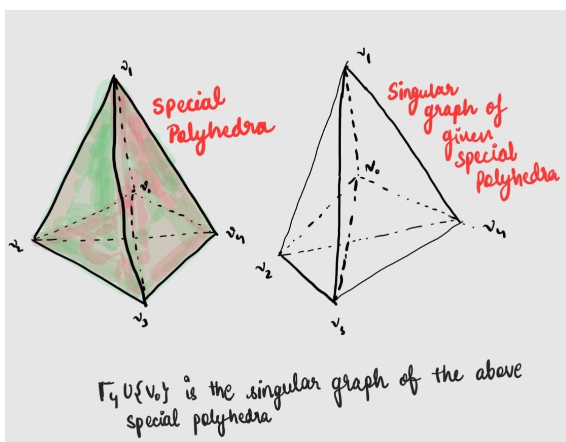


Figure 6: Singular Graph

2.2.3 Special Polyhedra

Definition: A simple polyhedra is called special if —

1. It contains at least one vertex.
2. each of its 1-stratum of P is an open 1-cell.
3. each 2-stratum of P is an open 2-cell.

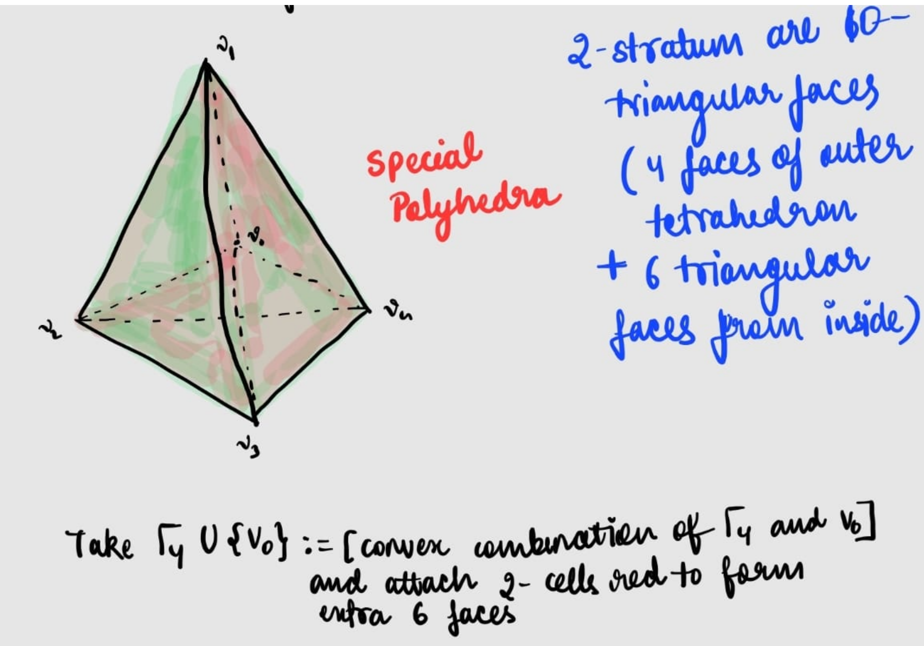


Figure 7: special polyhedron

Special/Simple spine of a Manifold: A spine of a 3-manifold is called special or simple if it is a simple or special polyhedron, respectively.

One example of the 'special spines of the 3-ball is shown in the figure. Bing's House with Two Rooms. It is known that any homeomorphism between special spines can be extended to homeomorphism between corresponding manifolds. It means that a special spine P of a 3-manifold M may serve as a presentation of M. Moreover, M can be reconstructed from a regular neighborhood $N(SP)$ of the singular graph SP of P: Starting from $N(SP)$, one can easily reconstruct P by attaching 2-cells to all the circles in $\partial N(SP)$, and then reconstruct M from P. If M is orientable, then $N(SP)$ can be embedded into $\mathbb{R}^3$. This gives us a very convenient way of presenting 3-manifolds.

Let us describe the collapse of the 3-ball onto Bing's House. First, we collapse the 3-ball onto a cube B contained in it. Next, we penetrate through the upper tube into the lower room and exhaust the room's interior, keeping the quadrilateral membrane fixed. Finally, we do the same with the upper room.

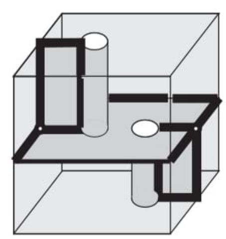


Figure 8: Bings House , special spines of three ball

Now, we dive into a useful theorem:

Theorem:
Any compact 3-manifold possesses a special spine.

Proof : Its proof requires further topics like handle decomposition and Arch construction which as of now, we haven't discussed yet .[3, Theorem 1.1.13,page 6]

2.3 Special Polyhedra and Singular Triangulation

This section brings together two dual ways of presenting 3-manifolds: special spines and triangulation.

2.3.1 Tiangulation:

Definition: A triangulation τ is a collection of n tetrahedrons such that each of the 4n triangular faces is affinely identified with one of the other triangular faces.

We can think of this as 'n' tetrahedrons have been 'glued together' along their faces to form a 3d complex.

Mathematical Model: Let $\mathcal{D} = \{\Delta_1, \Delta_2, ..., \Delta_n\}$ be a finite set of disjoint tetrahedrons and a finite set $\phi = \{\phi_1, \phi_2, ..., \phi_{2n}\}$ of affine homeomorphisms between the 2n triangular faces of the tetrahedra such that every face has a unique counterpart.

This means that a set of all faces of the tetrahedrons should be divided into pairs and the faces of every pair should be related by an affine homeomorphism.

We will denote this as pair $(\mathcal{D}, \phi)$ as "face identification scheme."

The resulting polyhedra will be called quotient space and denoted by $\hat{M}(\mathcal{D}, \phi)$.

We call τ a 3-manifold if its underlying topological space is a 3-manifold.

2.3.2 Singular Triangulation:

Depending upon the face identification scheme that makes up τ, there are two possibilities for an edge $'e'$.

1. If e is identified with itself in reverse, then every small neighborhood of the midpoint of e will be bounded by a projective plane.
   In this case, the midpoint of e is a singularity and τ is not a 3-manifold triangulation.

2. Otherwise every interior point of e will have a small neighborhood bounded by a sphere, in which case, the edge doesn't produce a singularity.

   Now, what happens at a vertex, after the identification of faces in pairs,
   Draw a small Δ near each tetrahedron vertex, if this Δ globally glued up to a sphere then in the identification space the sphere formed by the Δ bounds a ball, and the vertex in the identification space looks like the center of a ball.[1, sec 2.1,page 3]

Now, what happens at a vertex, after the identification of faces in pairs,
Draw a small Δ near each tetrahedron vertex, if this Δ globally glued up to a sphere then in the identification space the sphere formed by the Δ bounds a ball, and the vertex in the identification space looks like the center of a ball. In this case, the vertex is called 'material.'

Remark:, after gluing all the triangles together to form a closed surface called the link of v, if the link of v is a sphere then v is not a singularity and v ia called a material vertex otherwise v is called as an ideal vertex.[see fig. 9]
In this case, the ideal vertex can appear as a sphere with at least one handle or crosscrap, and then the sphere near this vertex looks like a cone on this surface.[see fig. 10]

- The situation at the midpoint of edges arises from the issue that an edge may be identified with itself but in the opposite direction.
- In this case a small neighborhood of the midpoint of such an edge is bounded by a projective plane. If this happens we can turn the midpoint

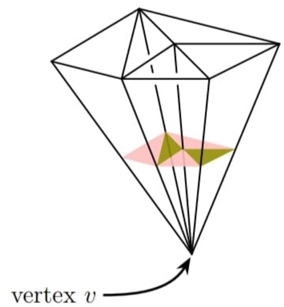
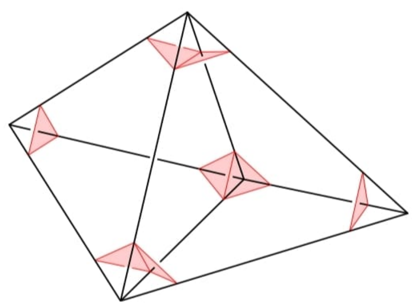


Figure 9: Ideal vertex



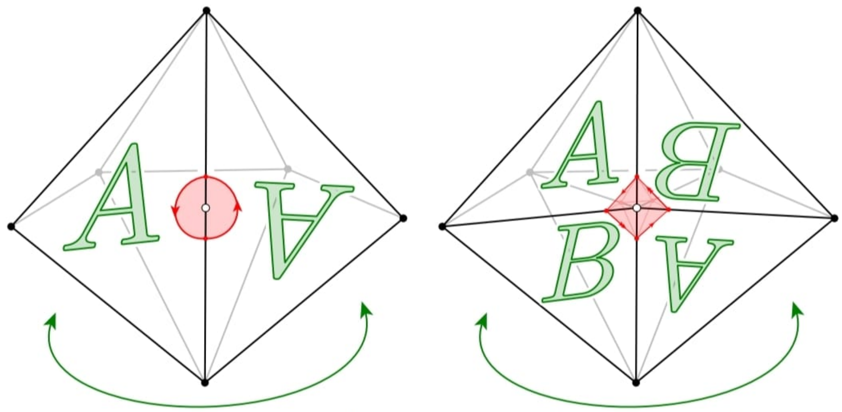
. Small triangles near the vertices of the tetrahedra glue up to form a surface.

Figure 10: neighbourhood of vertices in a Tetrahedron

into an ideal vertex by subdividing the tetrahedron.

Hence, we will assume that when we identify the faces of a collection of tetrahedrons, then the resulting space looks like 3D Euclidean space except possibly at the vertices.

- If all vertices are material, we have a 3D manifold and if there is at least one ideal vertex, we have a 3D pseudo manifold.
- In each case, the space also comes with a triangulation, namely the collection of tetrahedrons and face pairing with the property that the only nonmanifold points are at vertices.
- In computational topology, we allow triangulation in which edges or vertices can be joined together such triangulation can describe such topolog-



(A) A projective plane bounds the neighbourhood of the midpoint of an edge.

(B) After subdivision we get an ideal vertex whose small triangles glue up to produce the projective plane.

Figure 11: Projective plane surrounding the midpoint of an edge

ical structure using remarkably few tetrahedra.

e.g. 3-sphere can be built from just one tetrahedron.

- **One-vertex triangulation:** This is an n-tetrahedron triangulation where 4n tetrahedron vertices are identified to make a single vertex.

2.3.3 Singular Manifold

Definition(Singular 3-manifold):
A compact polyhedron Q is called a singular 3-manifold if the $linkF_x = lk(x, Q)$ of every point $x \in Q$ is a closed connected surface.

It follows from the definition that every point $x \in Q$ has a conic regular neighborhood $N(x) \approx Con(F_x)$, where $Con(F_x)$ is the cone with the vertex x. If F_X is not a 2-sphere, x is called singular. Since all other points of $Con(F_x)$ are nonsingular, Q contains only finitely many singular points. All other points of Q have ball neighborhoods.

Remark: . A simple way to construct an example of a singular 3- manifold is to take a genuine 3-manifold M with boundary and add cones over all the boundary components, see Fig.7.

Note: It is readily seen that any singular 3-manifold Q can be obtained

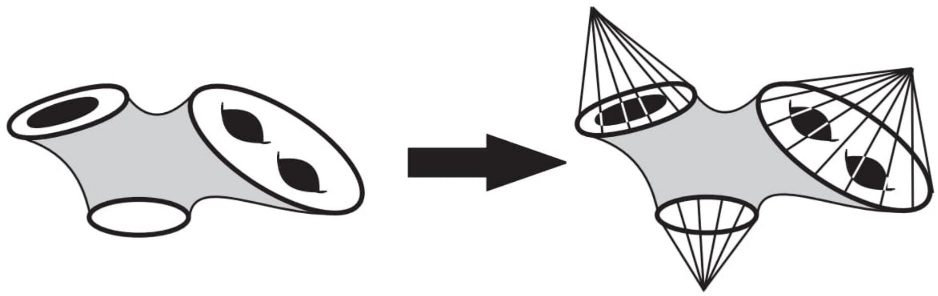


Fig. 1.10. Singular manifold

Figure 12: Singular Manifold

in this way. An easy calculation of the Euler Characteristic of Q shos that $\chi(Q) = \chi(M) + \sum_i (1 - \chi(F_i))$, where F_i are the components of ∂M. Since $\chi(M) = \frac{1}{2}\chi(\partial M) = \frac{1}{2}\sum_i \chi(F_i)$, we have $\chi(Q) = \frac{1}{2}\sum_i (2 - \chi(F_i))$. It follows that Q is a genuine manifold iff $2 - \chi(F_i) = 0$ for all i, i.e., if $\chi(Q) = 0$.Indeed, since $\chi(F_i) \leq 2$ for all i, $\frac{1}{2}\sum_i \chi(F_i) = 0$ iff all the summands are zeros, i.e., all F_i are 2-spheres.

Proposition: The quotient space $\hat{M} = \hat{M}(\mathcal{D}, \Phi)$ of any identification scheme $(\mathcal{D}, \Phi)$ is a singular manifold. $\hat{M}$ is a genuine manifold iff $\chi(\hat{M}) = 0$.

pf: If a point $x \in \hat{M}$ corresponds to a point in the interior of a tetrahedron or a pair of points on the faces, then the existence of a ball neighborhood of the point x in Mˆ is obvious. Suppose that x comes from identifying points $x_1, x_2, ...x_n$ which either are vertices of tetrahedra or lie inside edges. The link of each point xi in the corresponding tetrahedron is a polygon: A biangle, if x_i lies inside an edge, and a triangle if it is a vertex. The link of the corresponding point of $\hat{M}$ is obtained from these polygons by pairwise identifications of their edges. It follows that it is a closed-connected surface.

3 Complexity of 3-mannifold

3.0.1 Motivaton:

Let M be set of all 3-manifolds.
We want to classify the set $\mathcal{M}$ i.e. give an equivalence class on the set $\mathcal{M}$ to classify all 3-manifolds up to an equivalence relation. For, this we can use a tool called a measure of "**Complexity**". This just comes from the following idea called triangulation.

3.0.2 Triangulation:

Triangulation is a process of dividing the parent manifold M into smaller pieces of simplices i.e. viewing the manifold as a simplicial complex such that each simplicial complex can be glued to another simplicial complex along at most one common face.
Now, there is a theorem on 3-manifolds that every compact 3-manifold can be triangulated. That's why our project today is on 3-manifolds.
There is even a way to minimize the number of simplices needed to triangulate any 3-manifold called "$\Delta-$ **Complex**".

3.0.3 Δ-Complex:

In the process of making simplicial complexes, we now allow gluing simplices along their common faces and also gluing one simplex to itself along any face of it. Then using quotient map $p : \Delta_n \to \tilde{\Delta}_n$ (where $\tilde{\Delta}_n$ is a $\Delta - Complex$) we can easily apply theories developed through triangulation using simplicial complex to triangulation using Δ-Complex.

3.0.4 Complexity of a Manifold:

Let $c : \mathcal{M} \to \mathbb{Z}$ is a function with following properties,

- c is a non-negative function
- c is additive, i.e. $c(M_1 \# M_2) = c(M_1) + c(M_2)$
- for any $k \in \mathbb{Z}$, $\exists$ only finitely many compact irreducible manifolds $M \in \mathcal{M}$ with $c(M) = k \ \forall M \in \mathcal{M}$ (**Irriducible Manifold** is a manifold $M \in \mathcal{M}$ such that $\nexists D^3 \neq M', M'' \in \mathcal{M}$ such that $M' \# M'' = M$
- $c(M)$ is relatively easy to calculate.

3.0.5 Examples of Complexity Measures:

We can use the following decomposition of our $M \in \mathcal{M}$ to measure complexity of a manifold,[2, See]

- **Heegaard Genus:** Minimal genus over all Heegaard decompositon of that manifold.
- **#of simplicial complexes:** Minimal no of simplicial complexes required for triangulation of a given $M \in \mathcal{M}$.
- **Crossing no.:** Minimal crossing no. in a surgery representation of a given manifold.

3.0.6 Handle Body:

If $(W, \partial W)$ is a n-manifold with boundary and $S^{r-1} \times D^{n-r} \subset \partial W$ is an embedding. Now, the n-dim. manifold with boundary $(W', \partial W')$ to be obtained from $(W, \partial W)$ by attaching r handles i.e.
if $(W', \partial W') = (W \cup (D^r \times D^{n-r})), (\partial W - (S^{r-1} \times D^{n-r}) \cup (D^r \times S^{n-r-1}))$, where $\partial W'$ is obtained from ∂W through surgery. Now, the handle body is the decomposition of a given manifold in the above way starting from a basic manifold viz. 3 ball.[4, see]

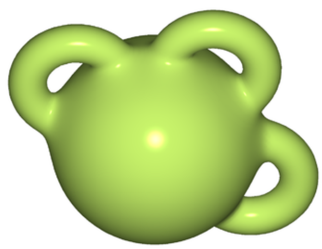
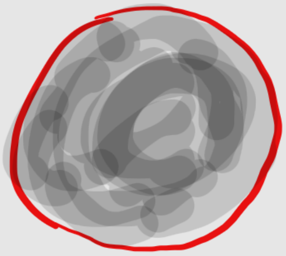


Figure 13: A 3-ball with 3 1-handle attached

Let $W =$  $\cong D^3.$

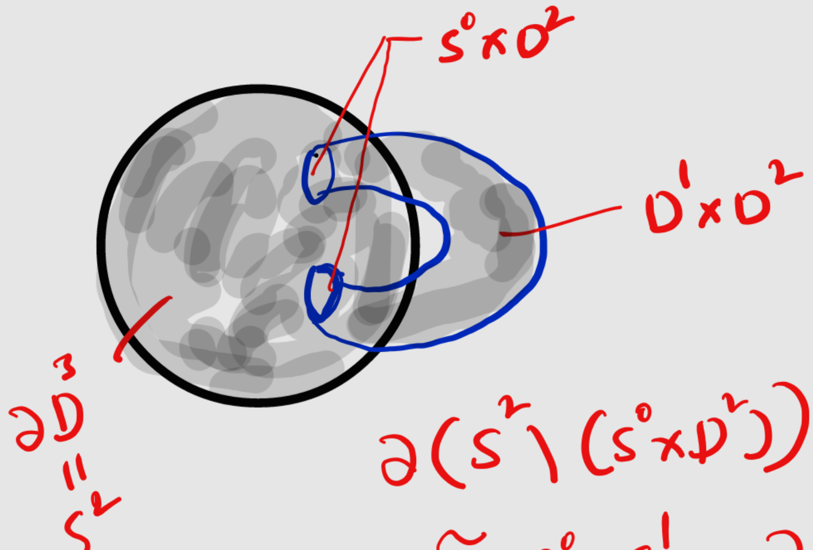
Take $D^1 \times D^{3-1} = D^1 \times D^2 \cong$ 

Note: $S^0 \times D^2 \cong$   $\subseteq \partial(D^3)$ (smoothly)

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \cong S^2$

So, we set $(W', \partial W') = \left(D^3 \cup (D^1 \times D^2), \left(S^2 - (S^0 \times D^2)\right) \cup (D^1 \times S^1)\right)$

basically this tells us that,



$$\partial\left(S^2 \setminus (S^0 \times D^2)\right) \underset{\varphi}{\cong} S^0 \times S^1 = \partial(D^1 \times S^1)$$
$$= S^0 \times S^1$$

So, $\left(S^2 \setminus (S^0 \times D^2)\right) \sqcup_{\varphi} (D^1 \times S^1)$ is valid.

3.0.7 Surgery:

For an n-manifold M, where $n = p + q$. We define p surgery as:

$$M' := (M - (im(\phi))^o) \cup_{\phi|_{S^p \times S^{n-p-1}}} (D^{p+1} \times S^{n-p-1})$$

where $\phi : S^p \times D^{n-p} \to \mathcal{M}$ is an embedding. And also note that M' is well defined as $\partial(S^p \times D^{n-p}) = S^p \times S^{n-p-1} = \partial(D^{p+1} \times S^{n-p-1})$.[5, see]

3.0.8 Heegaard Handlebody Decomposition:

Let V, W are handle bodies of genus g and let f be a orientation reversing homomorphism $f : \partial V \to \partial W$ Gluing $V$to W along f gives $M = V \cup_f W$ a 3-manifold M. And it follows from a deep result due to Moise, that every compact orientable 3-manifold can be obtained through
HEEGAARD HANDLEBODY DECOMPOSITION.[6, see]

3.0.9 A particular Method for computing complexity:

Denote $\Delta^{(1)}$ as the complete graph with 4 vertices. Clearly, $\Delta^{(1)} \cong$ 1-skeleton of the standard 3-simplex.

Some useful definitions:

- *Almost Simple Polyhedron*: A compact 2D polyhedron P is called almost simple if the links of each of its points can be embedded in $\Delta^{(1)}$. The points whose links are homeomorphic to $\Delta^{(1)}$ are said to be vertices of P.

  E.G.: We take any point inside $\Delta^{(1)}$ say v and take $C(\Delta^{(1)}, v)$: convex hull of the set $\Delta^{(1)}$ and point v.

- *Complexity:* The complexity $c(M)$ of a compact 3 manifold equals k if M possesses an almost simple spine with K genuine vertices and has no almost simple spine with a smaller no. of vertices.

  E.G.: Consider $C(\Delta^{(1)}, v_1, v_2, ..., v_k)$ and attach solid balls to each faces this way: The resulting manifold M has an almost simple spine $C(\Delta^{(1)}, v_1, v_2, ..., v_k)$ and $\nexists$ any almost simple spine of M with smaller number of vertices and hence $c(M) = k$.

  - See figure 6, a manifold with an almost simple spine has complexity 4.
  - See the figure below, The manifold with this spine has complexity 1.

- *Simple Polyhedron:* A compact polyhedron P is called simple if the link of each of its points is homeomorphic to one of the following 1D polyhedron:

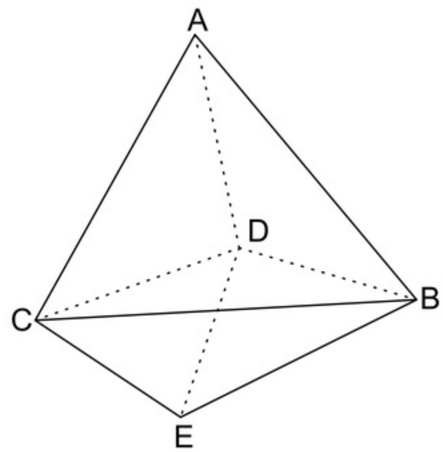


Figure 14: Almost simple polyhedron

1. a Circle
2. a Circle with a diameter
3. a Circle with three radii.

e.g.: A tetrahedron without an interior. It has all its points with links homeomorphic to a circle.

The points whose links are homomorphic to circles are called non-singular points and points whose links are of the form 2,3 are called singular points.

- *Singular Graph:* The set of singular points of a simple polyhedron is called its singular graph and is denoted by SP.

  e.g.:

  To describe the structure of a simple polyhedron in detail we need to consider the definition of stratum.

- *2-stratum:* A 2D component which is a connected component of a set of non-singular points.
- *1-stratum:* Stratum of dimension 1 is a set of open or closed triple lines.
- *0-stratum:* This consists of true vertices.
- *Special Polyhedra:* A simple polyhedra is called special if —

  1. each of its 1-stratum of P is an open 1-cell.

2. each 2-stratum of P is an open 2-cell.

- *Special/Simple spine of a Manifold:* A spine of a 3-manifold is called special or simple if it is a simple or special polyhedron, respectively.

  Two examples of special spines of the 3-ball are shown in the figure. Bing's House with Two Rooms and Abalone (a marine mollusk with an oval, somewhat spiral shell). It is known that any homeomorphism between special spines can be extended to homeomorphism between corresponding manifolds. It means that a special spine P of a 3-manifold M may serve as a presentation of M. Moreover, M can be reconstructed from a regular neighborhood $N(SP)$ of the singular graph SP of P: Starting from $N(SP)$, one can easily reconstruct P by attaching 2-cells to all the circles in $\partial N(SP)$, and then reconstruct M from P. If M is orientable, then $N(SP)$ can be embedded into $\mathbb{R}^3$. This gives us a very convenient way of presenting 3-manifolds.

4 Computing Complexity of 3-manifold

4.1 Some Useful propositions:

In this section, we will compute the complexity of some orientable 3-manifolds according to the method discussed in the above sections. Towards this, we need some results and propositions so that we can use them as methods for computing complexity.
In general, calculating the complexity $c(M)$ is very difficult. Let us start with a simpler problem of estimating $c(M)$. To do that it suffices to construct an almost simple spine P of M and calculate the number of true vertices of P. This will serve as an upper bound for complexity. Since almost simple spines can be easily constructed from practically any representation of manifold, the estimation problem does not give rise to any difficulties. Let us describe several estimates of complexity based on the different presentations of 3-manifolds. It is convenient to start with an observation that removing an open ball does not affect the complexity.

Proposition 4.1.1. *Suppose that B is a 3-ball in a 3-manifold M. Then $c(M) = c(M \setminus intB)$*

proof. If M is closed, then $c(M) = c(M \setminus intB)$ since M and $M \setminus intB$ have the same spines by definition of the spine of a closed manifold. Let $\partial M \neq \phi$, and let P be an almost simple spine of $M \setminus IntB$ possessing $c(M \setminus IntB)$ true vertices. Denote by C the connected component of the space $M \setminus P$ containing B.Since M is not closed there exists a 2-component α of P that seperates C from another component of $M \setminus P$. Removing an open 2-disc from α and collapsing yields an almost simple spine $P_1 \subset P$ of M. The number of true vertices of P_1 is no greater than that of P, since puncturing α and collapsing results in no new true vertices. Therefore, $c(M) \leq c(M \setminus IntB)$.
To prove the converse inequality, consider an almost simple spine P_1 of M with $c(M)$ true vertices. Let us take a 2-sphere S in M such that $S \cap P_1 = \phi$. Join S to P_1 by an arc ℓ that has no common points with $P_1 \cup S$ except the endpoints. $P = P_1 \cup S \cup \ell$ is an almost simple spine of $M \setminus IntB$. New true vertices do not arise. It follows that $c(M) \geq c(M \setminus IntB)$.

Remark 4.1.1. *It follows that closed 3-manifold M possesses a special spine with n true vertices iff it can be obtained by pasting together n tetrahedra. We will define, therefore, the complexity as the minimal number of tetrahedra that is sufficient to obtain M.*

Proposition 4.1.2. *Suppose $M = H_1 \cup H_2$ is a Heegaard splitting of a closed 3-manifold M such that the meridians of handle body H_1 intersect the ones of H_2 transversally at n points. Suppose also that the closure of one of the components into which the meridians of H_1, H_2 decomposes the Heegaard surface $\partial H_1 = \partial H_2$ contains m such points. then $c(M) \leq n - m$.*

proof: Denote by P the union o the Heegard surafce $F = \partial H_1 = \partial H_2$ with the meridional discs of the two handlebodies. Then P is a simple polyhedron whose true vertices are the crossing points of the meridians. Since the complement of P in M consists of two open 3-balls, P is a spine of M punctured twice. Removing from P the 2-component $\alpha \subset F$ whose closure contains m true vertices, we fuse the balls and get an almost simple spine of M which has $n-m$ true vertices, since the vertices in the closure of α will cease to be true vertices.

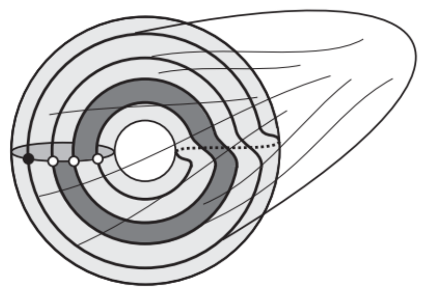


Figure 15: Special spine of $L_{4,1}$ obtained from the standard Heegaard diagram of $L_{4,1}$

Proposition 4.1.3. *Suppose $\tilde{M}$ is a k-fold covering space of a 3-manifold M. Then $c(\tilde{M}) \leq kc(M)$*

Proof: Let P be an almost simple spine of M having $c(M)$ true vertices. Consider the almost simple polyhedron $\tilde{P} = p^{-1}(P)$, where $p : \tilde{M} \to M$ is the covering map. Since the degree of covering is k, the polyhedra $\tilde{P}$ has $kc(M)$. If $\partial M \neq \phi$, then $\tilde{P}$ is an almost simple spine of $\tilde{M}$, since the collapse of M onto P can be lifted to a collapse of $\tilde{M}$ onto $\tilde{P}$. Therefore, $c(\tilde{M}) \leq kc(M)$.
If M is closed, $\tilde{P}$ is a spine of the manifold $\tilde{M} \setminus p^{-1}(V)$, where V is an open 3-ball in M. the inverse image $p^{-1}(V)$ consists of k open 3-balls, hence, we have $c(\tilde{M}) = c(\tilde{M} \setminus p^{-1}(V)) \leq kc(M)$.

Remark 4.1.2. *If M in the above proof is closed, then one can get an almost spine of $\tilde{M}$ by puncturing those 2-components of $\tilde{P}$ that separate different balls in $p^{-1}(V)$. To fuse k balls, we must make $k-1$ punctures, and each of them decreases the total number of true vertices by the number of true vertices in the boundary of the 2-component we are piercing through. Thus, as a rule,$c(\tilde{M})$ is significantly less than $kc(M)$.*

Now we turn our attention to link components and surgery presentations of 3-manifolds.

Proposition 4.1.4. *Suppose a link $L \subset S^3$ is given by a projection $\bar{L}$ with n crossing points so that there is an overpass of degree k and an independent underpass of degree m. Then the complexity of the complement space $C(L)$ of L is no greater than $4(n-m-k-2)$.*

proof:

Remark 4.1.3. *It can be shown that if the projection $\bar{L}$ has $n \geq 6$ crossings, then one can always find an overpass and independent underpass satisfying $k + m \geq 2$. The complexity of $C(L)$ can then be estimated by $4n - 16$. If there are no independent overpasses and underpasses, then one can use dependent ones or puncture a 2-component that lies on S^2 and separates the balls B_1, B_2. The number of disappearing true vertices in this case may be smaller, since the same true vertex may be taken into account twice.*

Proposition 4.1.5. *Suppose M is obtained by Dehn surgery along a knot K with framing s such that the projection $\bar{K}$ of K has $n \geq 1$ crossing points. Then $c(M) \leq 5n + |s - w(\bar{K})|$.*

proof:

4.2 Properties of Complexity:

4.2.1 Converting Almost Simple Spines into Special ones

We have already stated the advantages of using almost simple spines, yet there are important downsides too. In general, almost simple spines determine 3-manifolds in a nonunique way, and cannot be represented by regular neighborhoods of their singular graphs alone. Since special spines, as has been mentioned before, are free from such liability, we would like to go from almost simple polyhedra to special ones whenever possible. So the question is: when is it possible? We shall study it in this section.
Let P be an almost simple spine of a 3-manifold M that is not a special one. Then P either possesses a 1-dimensional part or has 2 components not homeomorphic to a disc. We aim to transform P into a special spine of M without increasing the number of true vertices. In general, this is impossible.
For example, if M is reducible or has a compressible boundary, any minimal almost simple spine of M must contain a 1-dimensional part. Nevertheless, in some cases it is possible. To give an exact formulation, we need to recall a few notions of 3-manifold topology.

Definition 4.2.1. *A 3-manifold M is called irreducible, if every 2-sphere in M bounds a 3-ball.*
If M is reducible, then either it can be decomposed into nontrivial connected sum, or is one of the manifolds $S^2 \times S^1$, $S^2 \tilde{\times} S^1$.
Recall that a compact surface F in a 3-manifold M is called proper, if $F \cap \partial M = \partial F$.

Definition 4.2.2. *A 3-manifold M is boundary irreducible, if for every proper disc $D \subset M$ the curve bounds a disc in ∂M.*

Definition 4.2.3. *Let M be an irreducibla boundary irreducible 3-manifold. A proper annulus $A \subset M$ is called inessential, if either it is parallel to an annulus in ∂M, or the core circle of A is contractible in M(in second case A can*

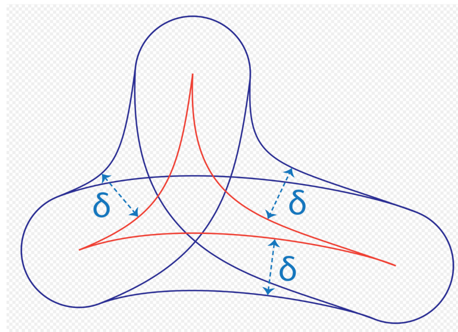


Figure 16: δ-slim condition

be viewed s a tube possessing a meridional disc). Otherwise A is called essential.

4.2.2 The Finiteness Property

Theorem 4.2.1. *For any integer k, there exists only a finite number of distinct compact irreducible boundary irreducible 3-manifolds that contain no essential annuli and have complexity k.*

Corollary 4.2.1. *For any integer k, there exists only a finite number of distinct closed orientable irreducible 3-manifolds of complexity k.*

Recall 4.2.1. *Recall that a compact 3-manifold M is hyperbolic if Int M admits a complete hyperbolic metric space of finite volume.*

Recall 4.2.2. *Hyperbolic Metric space is a δ-hyperbolic space which is a geodesic metric space all of whose geodesic triangles are δ-slim. See here:[7], For further reading see here:* `https://math.iisc.ac.in/~gadgil/expos/hyp.pdf`

δ-slim condition: Let $\delta > 0$. A geodesic triangle in a metric space X is said to be -slim each of its sides is contained in the δ-neighbourhood of the union of the other two sides. A geodesic space X is said to be δ-hyperbolic if every triangle in X is δ-slim.
Geodesic space and geodesic Triangle: If any two points of a space $x, y \in X$ are end points of a geodesic segment $[x, y]$(an isometric image of a compact subinterval $[a, b] \subset \mathbb{R}$).
A geodesic triangle with vertices $x, y, z \in X$ is the union of geodesic segments $[x, y], [y, z], [z, x]$(where $[p, q]$ denotes a segment with endpoints p and q).

It is known that any hyperbolic 3-manifold is irreducible, has incompressible boundary, and contains no essential annuli.

Corollary 4.2.2. *For any integer k, there exists only a finite number of distinct orientable hyperbolic 3-manifolds of complexity k.*

4.2.3 The Additivity Property

Recall that the connected sum $M1\#M2$ of two compact 3-manifolds $M1, M2$ is defined as the manifold $(M_1 \setminus IntB_1) \cup_h (M_2 \setminus IntB_2)$, where $B_1 \subset IntM_1$, $B_2 \subset IntM_2$ are 3-balls, and h is a homeomorphism between their boundaries. If the manifolds are orientable, their connected sum may depend on the choice of h. In this case, $M_1\#M_2$ will denote any of the two possible connected sums. Alternatively, one can use signs and write $M_1\#(\pm M_2)$.
To define the boundary connected sum, consider two discs $D_1 \subset \partial M_1$, $D_2 \subset \partial M_2$ in the boundaries of two 3-manifolds. Glue M_1 and M_2 together by identifying the discs along a homeomorphism $h : D_1 \to D_2$. Equivalently, one can attach an index 1 handle to $M_1 \cup M_2$ such that the base of the handle coincides with $D_1 \cup D_2$. The manifold M thus obtained is called the boundary connected sum of $M1, M2$ and is denoted by $M_1 \amalg M_2$. Of course, M depends on the choice of the discs (if at least one of the manifolds has a disconnected boundary) and on the choice of h (homeomorphisms that differ by a reflection may produce different results). Thus the notation $M_1 \amalg M_2$ is slightly ambiguous, like the notation for the connected sum. When shall use it to mean that $M_1 \amalg M_2$ is one of the manifolds that can be obtained by the above gluing.

Theorem 4.2.2. *For any 3-manifolds M_1, M_2 we have:*

1. $c(M_1\#M_2) = c(M_1) + c(M_2)$
2. $c(M_1 \amalg M_2) = c(M_1) + c(M_2)$

proof: We begin by noticing that the first conclusion of the theorem follows from the second one. To see that, we choose 3-balls $V_1 \subset IntM_1, V_2 \subset IntM_2$, and $V_3 \subset Int(M_1\#M_2)$ are homeomorphic, where the index 1 handle realizing the boundary connected sum is chosen so that it joins ∂V_1 and ∂V_2. Assuming (2) and using proposition 4.1.1, we have: $c(M_1\#M_2) = c((M_1\#M_2) \setminus V_3) = c(M_1 \setminus IntV_1) + c(M_2 \setminus IntV_2) = c(M_1) + c(M_2)$.
Let us prove the second conclusion. The inequality $c(M_1)+c(M_2) \geq c(M_1 \amalg M_2)$ is obvious, since if we join minimal almost simple spines of M_1, M_2 by an arc, we get an almost simple spine of $M_1 \amalg M_2$ having $c(M_1) + c(M_2)$ true vertices. The proof of the inverse inequality is based on Haken's theory of normal surfaces.

4.2.4 Closed Manifolds of Small Complexity

4.2.5 Enumertion Procedure

It follows from the finiteness property that for any k there exist finitely many closed orientable irreducible 3-manifolds of complexity k. The question is: how many? The constructive proof of Theorem 2.1.1 allows us to organize a computer enumeration of special spines with k true vertices. Of course, the list of corresponding 3-manifolds can contain duplicates and nonorientable, nonclosed, or reducible manifolds. All such manifolds must be removed. Let us briefly describe the enumeration results in historical order. First, Matveev and Savvateev tabulated closed irreducible orientable manifolds up to complexity 5. The manifolds were listed with the help of a computer and recognized manually. This was the first paper on computer tabulation of 3-manifolds. It contained all the basic elements of the corresponding theory, which much later have been rediscovered by various mathematicians. This table was extended to the level of complexity 6. Ovchinnikov used the same approach in composing the table of complexity 7. The manifolds were still manually recognized, although by an improved method (distinguishing and using elementary blocks). Later Martelli wrote a computer program that is based on the same principle but tabulates 3-manifolds in two steps. First, it enumerates some special building blocks (bricks), and only then assembles bricks into 3-manifolds. An interesting relative version of the complexity theory serves as a theoretical background for the program. Let us present the results of these enumeration processes for k 7.

Theorem 4.2.3. *The number $n_c(k)$ of closed orientable irreducible 3-manifolds of complexity k for $k \leq 7$ is given by the following table:*

k	0	1	2	3	4	5	6	7
$n_c(k)$	3	2	4	7	14	31	74	175

Closed orientable irreducible 3-manifolds of complexity 0 are the following ones: the sphere S^3, the projective space RP^3, and the lense space $L_{3,1}$. Their almost simple spines without true vertices were described earlier. The complexity $S^2 \times S^1$ is also equal to 0, but this manifold is reducible. Closed orientable irreducible 3-manifolds of complexity 1 are lens spaces $L_{4,1}$ and $L_{5,2}$ There are four 3-manifolds of complexity 2. They are the lens spaces $L_{5,1}, L_{7,2}, L_{8,3}$ and the manifold S^3/Q_8 where $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is the quaternion unit group(the action of Q_8 on S^3 is linear.)

Recall 4.2.3. ***Lense Spaces***:*Let m, l_i for $i = 1, \ldots, d$ be natural numbers such that $(l_i, m) = 1$ for all i. The lens space $L(m; l_1, \ldots, l_d)$ is defined to be the orbit space of the free action of the cyclic group $\mathbb{Z}_m$ on the sphere $S^{2d-1} = S(\mathbb{C}^d)$ given by the formula*

$$(z_1, \ldots, z_d) \mapsto (z_1 \cdot e^{2\pi i l_1/m}, \ldots, z_d \cdot e^{2\pi i l_d/m}).$$

Basically, $L(m; l_1, l_2, \cdots, l_d) = S^{2d-1}/\mathbb{Z}_m$ with the above action.

4.2.6 Idea of Computer Programme to find out complexity

Let us give a nonformal description of the computer program that was used for creating the table up to complexity 7. The computer enumerates all the regular graphs of degree 4 with a given number of vertices. The graphs may be considered workpieces for singular graphs of special spines. For each graph, the computer lists all possible gluings together of butterflies that are taken instead of true vertices. Note that if the graph has k vertices, then there are 2k edges, and thus potentially 6^{2k} different gluings of the triodes[See the following theorem]. Not all of them produce spines of orientable manifolds: we may get a special polyhedron that is not a spine or is a spine of a nonorientable manifold.

To avoid this, we supply each copy of E with an orientation (in an appropriate sense) and use orientation-reversing identifications of the triodes. This leaves us with no more than $2^{k-1}3^k$ spines of orientable manifolds. One may decrease this number by selecting spines of closed manifolds, but it remains too large. The problem is that we get a list of spines, while it is a list of manifolds we are interested in (as we know, any 3-manifold has many different special spines). Also, some manifolds from the list thus created would be reducible. A natural idea to obtain a list of manifolds that do not contain duplicates and reducible manifolds consists in considering minimal spines, i.e., spines of minimal complexity. Unfortunately, there are no general criteria for minimality. The good news here is that there are a lot of partial criteria for nonminimality. In Sect. 2.3.2 we present two of them that appeared to be sufficient for casting out all reducible manifolds and almost all duplicates up to k = 6.
The completion of the table of closed orientable irreducible 3-manifolds up to complexity 6 was made by hand. It was a big job indeed: for each pair of spines that had passed the minimality tests one must decide whether or not they determine homeomorphic manifolds. In practice we calculated their invariants: homology groups and, in worst cases, fundamental groups [83,91]. Later, after Turaev–Viro invariants had been discovered, we used them to verify the table. If the invariants did not help to distinguish the manifolds, we tried to transform one spine into the other by different moves that preserve the manifold. In all cases, a definitive answer was obtained.

We point out that the Turaev–Viro invariants are extremely powerful for distinguishing 3-manifolds. In particular, invariants of order ≤ 7 distinguish all orientable closed irreducible 3-manifolds up to complexity 6 having the same homology groups. The only exceptions are lens spaces, for which there is no need to apply these invariants.

Theorem 4.2.4. *For any integer k, there exists only a finite number of special spines with k true vertices. All of them can be constructed algorithmically.*

proof: We will construct a finite set of special polyhedra that a fortiori contains all special spines with k true vertices. First, one should enumerate all regular graphs of degree 4 with k true vertices. Clearly, there is only a finite number of them. Given a regular graph, we replace each true vertex v with a copy of the butterfly E that presents a typical neighborhood of a true vertex in a simple polyhedron. Neighborhoods in ∂E of triple points of ∂E (we will call them triodes) correspond to edges having an endpoint at v.See Fig. 17 the triodes are shown by fat lines. For each edge e, we glue together the triodes that correspond to endpoints of e via a homeomorphism between them. It can be done in six different ways (up to isotopy). We get a simple polyhedron P with a boundary. Attaching 2-discs to the circles in ∂P, we get a special polyhedron. Since at each step we have had only a finite number of choices, this method produces a finite set of special polyhedra. Not all of them are thickenable. Nevertheless, the set contains all special spines with k true vertices.

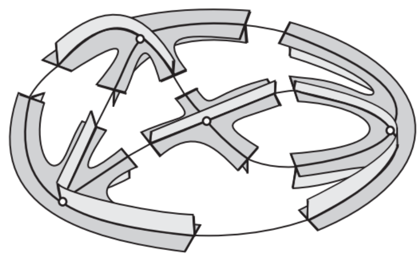


Figure 17: A decomposition of N(SP) into copies of E

5 Appendix

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