A Study on $\mathcal{L}^p$ Spaces

B.Sc. Dissertation

by Atrajit Sarkar

Roll No. 317

Under the supervision of Dr. Arnab Jyoti Dasgupta

March 11, 2025

Acknowledgement:

Respected, sir,
I hereby acknowledge that I did all my works by my own efforts and in all the process my project guide Dr. Arnab Jyoti Dasgupta helped me a lot and provided me all the materials and software to build the project!

Used Symbols:-

- μ : Measure in general spaces.
- m_N : Measure in $\mathbb{R}^N$
- $\mathcal{C}_c(\Omega)$: Space of continuous functions on a compact support $\subset \Omega$.
- $\|\cdot\|_p$: p-norm.
- $\mathbb{R}, \mathbb{N}, \mathbb{C}$: Set of reals, Naturals, Complex numbers.
- $\int_\Omega$: Lebesgue integral over a set Ω.

Contents

1 Preliminaries

1.1 Definition of measure

• We specially define measure as a function with some domain of definition and some special properties. And according to study on the bunch of beneficial conditions allotted to measure a set which is the domain of definition of measure we see that all the conditions cannot hold simultaneously! The beneficial conditions are as follow:-

A measure is a function (denoted as μ) on the power set of a set X, is an extended real valued function on R such that

(i) $\mu(E) > 0, \forall E \in \mathcal{P}(x)$,

(ii) $\mu(\phi) = 0, and,$

(iii) μ is subadditive i.e $\mu(A) + \mu(B) \geq \mu(A \cup B) \forall A, B \in \mathcal{P}(x)$.

• Now what we do is to reform the domain of μ from the whole power set to a subset of a power set to actually reform the third condition from sub-additivity to countable additivity. And we call the subset of $\mathcal{P}(x)$ set of lebesgue measurable sets (denoted by S) and the new restricted measure to be lebesgure measure.We will use μ to denote lebesgue measure all over our discussion.

1.2 Lebesgue Measurablity

• Lebesgue measurable functions: Let (X, Σ) and (Y, T) be measurable spaces equipped with respective σ-algbras Σ and T. A function $f : X \to Y$ is said to be measurable if for every $E \in T$ the pre-image of E under f is in Σ; i.e $\forall E \in T$

$$f^{-1}(E) = \{x \in X | f(x) \in E\} \in \Sigma.$$

•Note: As we are dealing with $\mathcal{L}^p$-spaces so your function is real-valued. Hence the above definition can be modified very neatly as

$$\{x \in X | f(x) > a \ \forall a \in \mathbb{R}\} \in \Sigma.$$

1.3 p-integrablity

• Definition of p-integrablity:

Let (X, S, μ)be a measure space. Let $f : X \to \mathbb{R}$ be a measurable function. Let $1 \leq p < \infty$. We define

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{1/p}$$

and we say f is p-integrable is $\|f\|_p < +\infty$.

•Definition of essential supremum:

This is nothing but $\|f\|_\infty$.We say essential supremum exists finitly iff $\|f\|_\infty < +\infty$.

1.4 Some useful In-equalities

1.4.1 Holder's Inequality

• Definition of conjugate exponent:

p and p' are called cojugate exponents when

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

if $p = 1$ we say $q = \infty$.

• Lemma: Let $1 < p < \infty$. Let p' be it's conjugate exponent. Then, if a and b are non-negative reals, we have

$$a^{1/p} b^{1/p'} \leq \frac{a}{p} + \frac{b}{p'}.$$

• Holder's Inequality: Let $1 \leq p < \infty$ and let p' be the cojuget exponent. If f is p-integrable and g is p'-integrable (essetially bounded if $p = 1$), then

$$\int_X |fg| \, d\mu \leq \|f\|_p \|g\|_{p'}.$$

Remark: If $p = 2, p' = 2$ then the in-equality has a special name called **Cauchy-Schwartz ineuqality**.

1.4.2 Minkowski's Inequality

Let $1 \leq p \leq \infty$. Let f and g be p-integrable (essentially bounded, if $p = \infty$) and

$$\|f + g\| \leq \|f\|_p + \|g\|_p.$$

1.5 Equivalence Class And Vector Space

1.5.1 Almost everywhere concept

• We say a condition happens in a domain a.e.(almost everywhere) $\Rightarrow$ the codition happens on a set except a subset of measure zero of it.

• A set of measure zero is a sub set of X such that $\mu(A) = 0$ where $A \subset X$.

• We can see cosequently that if two functions f and g are equal a.e. the we write them as $f \sim g$ and we can eventually bring equality in sense of p-integration i.e we definitely observe that $\|f\|_p = \|g\|_p$.

1.5.2 Equivalence class

• Now if we equip the set of all p-integrable functions with a binary relation "$\sim$" and we eventually see that is a equivalence relation and we can now classify the set into some equivalence classes.

1.5.3 Vector space

• We see that if we cosider each equivalence classes as a sigle element and put them in a set $\mathcal{V}$ equipped with natural norm as $\|\cdot\|_p$ then $(\mathcal{V}, \|\cdot\|_p)$ becomes a normed linear space. And we also say that $(\mathcal{V}, \|\cdot\|_p) = \mathcal{L}^p(\mu)$.

• Proposition 1 : Let $(X, \mathcal{S}, \mu)$ be a finite measure space. Then

$$\mathcal{L}^p(\mu) \subset \mathcal{L}^q(\mu)$$

with the inclusion being continuos, whenever $1 \leq q \leq p$.

• Remark: Let $(X, \mathcal{S}, \mu)$ be a finite measure space and let $f \in \mathcal{L}^\infty(\mu)$, $f \neq 0$.

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$

1.5.4 Convergence in $\mathcal{L}^p(\mu)$:

• Convergence in $\mathcal{L}^p(\mu)$: If $f \in \mathcal{L}^p(\mu)$, we say that the sequence $\{f_n\}_{n=1}^\infty \in \mathcal{L}^p(\mu)$ converges to f in $\mathcal{L}^p(\mu)$ if $\|f_n - f\|_p \to 0$ as $n \to \infty$.

• Lemma: Let $1 \leq p < \infty$. Let $(X, \mathcal{S}\mu)$ be a finite measure space. If $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{L}^p(\mu)$, then the sequence is Cauchy in measure.

• Cauchy in Measure: A sequence $\{f_n\}_{n=1}^{\infty}$ is said to be Cauchy in measure $\Rightarrow$ for $\epsilon > 0$ be a fixed real, and for $n, m \in \mathbb{N}$ the set

$$A_{n,m}(\epsilon) = \{x \in X| \, |f_n(x) - f_m(x)| \geq \epsilon\}$$

is of measure zero.

• Let $(X, \mathcal{S}\mu)$ be a measure space. Let $1 \leq p \leq \infty$. Then $\mathcal{L}^p(\mu)$ is a Banach space.

• Let $(X, \mathcal{S}\mu)$ be a measure space and let $f_n \to f$ in $\mathcal{L}^p(\mu)$ for some $1 \leq p \leq \infty$. Then, a subsequence $\{f_{n_k}\}$ such that $f_{n_k}(x) \to f(x)$ a.e.

• Lemma: Let $1 \leq p \leq \infty$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{L}^p(\mu)$ converging pointwise a.e. to a function $f \in \mathcal{L}^p(\mu)$. Then $f_n \to f$ in $\mathcal{L}^p(\mu)$ iff, $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$.

2 Approximation in $\mathcal{L}^p(\mu)$-spaces

2.1 Definition :

• Characteristic function: In a measure space $(X, \mathcal{S})$ $\chi_A : X \to \mathbb{R}$ is called a Characteristic function iff

$$\chi_A(x) = 1, if x \in A$$

$$= 0, o/w$$

• Simple Function : Linear combination of finitely many Characteristic functions. I.e.

$$f(x) = \sum_{k=1}^{N} a_k \chi_{A_k}$$

for some $N \in \mathbb{N}$ and $a_k \in \mathbb{R}$ with each $A_k \subset X$.

2.2 Notations:

• Let $\Omega \subset \mathbb{R}^n$ be a non-empty open set.

• Let $\mathcal{S}$ denote set of all real-valued simple functions defined on Ω which vanishes outside a set of finite Lebesgure measure.

2.3 Lemma :

• Lemma 1: If $1 \leq p < \infty$, a simple function ϕ belongs to $\mathcal{L}^p(\Omega)$ iff, $\phi \in \mathcal{S}$.

• Lemma 2: Let $\Omega \subset \mathbb{R}^N$ be a non-empty open set and let $1 \leq p < \infty$. Then $\mathcal{S}$ is dense in $\mathcal{L}^p(\Omega)$.

• Lemma 3: Let $\Omega \subset \mathbb{R}^N$ be a non-empty open set and let $1 \leq p < \infty$. Let $f \in \mathcal{S}$. Then, f can be approximately by step functions in $\mathcal{L}^p(\Omega)$.

• Lemma 4: Let $\Omega \subset \mathbb{R}^N$ be a non-empty open set and let $1 \leq p < \infty$. Let $\mathcal{C}_c(\Omega)$ denote the space of continuous real-valued functions defined on Ω, having compact support contained in Ω. Then, $\mathcal{C}_c(\Omega)$ is dense in $\mathcal{L}^p(\Omega)$.

Proof: By Lemma 1.2 Lemma 1.3 we have set of step functions are dense in $\mathcal{L}^p(\Omega)$. So, we just have to show that step functions can be approximated by functions from $\mathcal{C}_c(\Omega)$.Which is nothing but mere visualization. Still let's give some mathematical sketch of it.

It can be shown that for a $\varepsilon > 0$ $\exists$ $\varphi \in \mathcal{C}_c(\Omega)$, such that

$$m_N(\{x \in \Omega \mid \varphi(x) \neq f(x)\}) < \left(\frac{\varepsilon}{2\|f\|_\infty}\right)^p$$

and such that

$$\|\varphi\|_\infty \leq \|f\|_\infty.$$

Then

$$\|\varphi - f\|_p^p \leq 2^p \|f\|_\infty^p m_N\left(\{x \in \Omega \mid \varphi \neq f(x)\}\right) < \varepsilon^p$$

so that $\|\varphi - f\|_p < \varepsilon$. This completes the proof.

- The above result is not true for $p = \infty$.
- Lemma 5: For $1 \leq p < \infty$, $\mathcal{L}^p(\Omega)$ is seperable.
- Lemma 6: $\mathcal{L}^\infty(\Omega)$ is not seperable.

3 Applications

3.0.1 Lusin's theorem

Statement : Let $E \subset \mathbb{R}^N$ be a measurable set of finite measure. Let $f : E \to \mathbb{R}$ be a mesurable function. Let $\varepsilon > 0$ be given. Then, $\exists$ $\varphi \in \mathcal{C}_c(\mathbb{R}^{\mathbb{N}})$ such that

$$m_N(\{x \in E \mid \varphi(x) \neq f(x)\}) < \varepsilon$$

.
Further if f is bounded, we can ensure that

$$\|\varphi\|_\infty \leq \|f\|_\infty.$$

Proof: Construct $E_n = \{x \in E | \ |f(x)| \leq n\}$.
Then $E_n \uparrow E$. Since E has finite measure, we can choose $m \in \mathbb{N}$ such that $m_N(E \backslash E_m) < \frac{\varepsilon}{3}$. Now, we define a function $\tilde{f} : \mathbb{R}^N \to \mathbb{R}$ by

$$\tilde{f}(x) = f(x), \ \ if \ x \in E_m$$

$$\tilde{f}(x) = 0, \ \ o/w$$

.
$\tilde{f}$ is integrable on $\mathbb{R}^N$ as $\tilde{f}$ is bounded and E_m has finite measure. Hence, there

exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$ in $\mathcal{C}_c(\mathbb{R}^{\mathbb{N}})$ such that $\varphi_n \to \tilde{f}$ in $\mathcal{L}^1(\mathbb{R}^N)$ by the Lemma 4. Then, there exists a subsequence $\{\varphi_{n_k}$ which converges to $\tilde{f}$ pointwise almost everywhere on $\mathbb{R}^N$.

Now as E_m has finite measure, we can find $F \subset E_m$ such that $m_N(E_m \backslash F) < \frac{\varepsilon}{3}$ such that $\varphi_{n_k} \to \tilde{f}$ uniformly on F, by virtue of Egorof's theorem. Again, since F is of finite measure we can find a compact set K such that $m_N(F \backslash K) < \frac{\varepsilon}{3}$. Hence, $m_N(E \backslash K) < \varepsilon$.

Since $\{\varphi_{n_k}\}$ converges uniformly to $\tilde{f} on K$, it follows that the restriction of $\tilde{f}$ to K is cont. But $K \subset F \subset E_m$ and so $\tilde{f} = f$ for every $X \in K$.

Now, by Tietze extension theorem we can find a continuous function $g : \mathbb{R}^N \to \mathbb{R}$ such that $\|g\|_\infty \leq m$ and such that $g = f$ on K.

Finally, let $\psi \in \mathcal{C}_c(\mathbb{R}^N)$ be such that $0 \leq \psi \leq 1$ and such that $\psi \equiv 1$ on K by Urysohn's lemma. Let $\varphi = \psi g$. Then $\varphi \in \mathcal{C}_c(\mathbb{R}^N)$, and

$$\{x \in E \mid \varphi(x) \neq f(x)\} \subset E \backslash K,$$

measure of which is less that ε. Also $\|\varphi\|_\infty \leq m$ and, if f is bounded, $m \leq \|f\|_\infty \leq M$ where $M \in \mathbb{R}^+$. Hence we get this comparison as $\|\varphi\|_\infty \leq \|f\|_\infty$.

3.0.2 Hardy's inequality

Let $1 < p < \infty$. Let $f \in \mathcal{L}^p(0, \infty)$. For $0 < x < \infty$, define

$$F(x) = \frac{1}{x} \int_{(0,\infty)} f \ dm_1.$$

Then $F \in \mathcal{L}^p(0, \infty)$ and

$$\|F\|_p \leq \frac{p}{p-1} \|f\|_p.$$

Proof:

3.0.3 Examples :

Example 1: The Hardy's inequality is not true when $p = 1$. To show this let's cosider a function $f(x) = e^{-x} \in \mathcal{L}^1(0, \infty)$. Now if we construct

$$F(x) = \frac{1}{x} \int_{(0,x)} f \ dm_1 = \frac{1}{x} \int_0^x e^{-t} \ dt = \frac{1 - e^{-x}}{x}.$$

Before we gonna show Hardy's inequality, we see that $F(x)$ is not integrable over $(0,\infty)$. Why?

As, $F(x) \geq \frac{1-e^{-1}}{x}, x \geq 1$, Then,

$$\int_1^\infty F(x) \geq (1-e^{-1}) \int_1^\infty \frac{1}{x} > \infty.$$

So, basically $F(x) \notin \mathcal{L}^1(0,\infty)$. So, no meaning of Hardy's inequality.

Again, if $p = \infty$ we see that $\|f\|_\infty = 1$. But $\|F\|_\infty > \infty$. So, we see that again $F \notin \mathcal{L}^\infty(0,\infty)$. So, again Hardy's inequality doesn't make sense here.

3.0.4 Hardy's inequality for l_p spaces

Hardy's inequality hods for sequence spaces (l_p) as well, When
$1 < p < \infty$.

To, see this we need a comparision between $\mathcal{L}^p(\Omega)$ space and l_p spaces.
As, any function $f \in \mathcal{L}^p(\Omega)$ has domain of definition Ω like wise in l_p space the domain of definition of any sequence $\{x_n\} \in l_p$ is $\mathbb{N}$.

So, if we look at the p-norm of l_p according as $\mathcal{L}^p(\Omega)$ we see that

$$\int_\Omega |f|^p = \lim_{n\to\infty} \frac{b-a}{n} \sum_{r=1}^{n} |f(a + \frac{b-a}{n} r)|^p$$

(if we set for now $\Omega := (a,b)$).

$$= \lim_{n\to\infty} \frac{n}{n} \sum_{r=1}^{n} |x_r|^p.$$

As our function is $\{x_n\}$ and the domain variable is n. And the partion of the domain $\{1,2,3,......\}$ is $\{(0,1),(1,2),(2,3),.....\}$. where we see that $a = 0$ and we sustitute everything to our equation to get the equivalent form in l_p.

Thus we see that as in Hardy's inequality we construct $F(n) = y_n = \frac{x_1+x_2+....+x_n}{n}$.
And the result thus similarly follow here also as:

$$\|y\|_p \leq \frac{p}{p-1} \|x\|_p.$$

3.0.5 Continuity of $\mathcal{L}^p$-norm:

• Remark: This property is very strong that the contiuity of the function itself. As in $\mathcal{L}^p$ continuity actually defined as convergence in $\mathcal{L}^p$ so, from there it is valid to say the function is continuous when it is in $\mathcal{L}^p$ in the sense of $\mathcal{L}^p$ convergence.

Proposition: Let $1 \leq p < \infty$. Let $f \in \mathcal{L}^p(\mathbb{R}^N)$. For $h \in \mathbb{R}^N$, define

$$\tau_h(f)(X) = f(x-h),\ x \in \mathbb{R}^N.$$

Then

$$\lim_{h \to 0} \|\tau_h(f) - f\|_p = 0.$$

Proof: By the change of variable property of Legesgue integration we see that $\tau_h(f) \in \mathcal{L}^p(\mathbb{R}^N)$,whenever $f \in \mathcal{L}^p(\mathbb{R}^N)$ and aslo that $\|\tau_h(f)\|_p = \|f\|_p$.

Let $\varepsilon > 0$ be given. Choose $\varphi \in \mathcal{C}_c(\mathbb{R}^N)$ such that

$$\|f - \varphi\|_p < \frac{\varepsilon}{3} \tag{1}$$

Then, we also have

$$\|\tau_h(f) - \tau_h(\varphi)\|_p = \|f - \varphi\|_p < \frac{\varepsilon}{3} \tag{2}$$

Let the support of φ be contained in the box $[-a, a]^N$. Since φ is uni-formly continuous, $\exists\ 0 < \delta < 1$ such that, whenever $|h| < \delta$, we have

$$|\varphi(x-h) - \varphi(x)| < \frac{\varepsilon}{3}(2a)^{-\frac{N}{p}},$$

$\forall\ x \in \mathbb{R}^N$. Then for $|h| < \delta$,

$$\int_{\mathbb{R}^N} |\tau_h(\varphi) - \varphi|\ dm_1 = \int_{[-a,a]^n} |\varphi(x-h) - \varphi(x)|^p\ dm_1 < \left(\frac{\varepsilon}{3}\right)^p,$$

so that

$$\|\tau_h(\varphi) - \varphi\|_p < \frac{\varepsilon}{3} \tag{3}$$

Then result now follows on combining the relations (1),(2),(3).

4 Referrence and Useful Tools:

1. Measure and Integration [S.Kesavan].

2. Overleaf software: `https://www.overleaf.com/project/6292570ec3c93ed57bea65cc`