Problem

Let Ω be an open subset of $\mathbb{C}$. Suppose f is a meromorphic function on Ω having exactly one zero at z_0 and exactly one pole at z_1. Further, m is the order of zero at z_0 and n is the order of the pole at z_1. If $g : \Omega \to \mathbb{C}$ is analytic and D is an open disc with center a and radius r such that $z_0, z_1 \in D$ and $D \subset \Omega$, then show that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} g(z)\, dz = mg(z_0) - ng(z_1),$$

where $\gamma(t) = a + re^{2\pi i t}$ for $0 \le t \le 1$.

Solution

We apply the residue theorem to the meromorphic function $\dfrac{f'}{f} g$.

Step 1: Behavior of $\dfrac{f'}{f}$ at zeros and poles

Suppose f has a zero of order m at z_0. Then in a neighborhood of z_0, we can write

$$f(z) = (z - z_0)^m h(z),$$

where $h(z)$ is analytic and $h(z_0) \neq 0$. Differentiating,

$$f'(z) = m(z - z_0)^{m-1} h(z) + (z - z_0)^m h'(z),$$

so

$$\frac{f'(z)}{f(z)} = \frac{m}{z - z_0} + \frac{h'(z)}{h(z)}.$$

Hence, $\dfrac{f'}{f}$ has a simple pole at z_0 with residue

$$\operatorname{Res}_{z=z_0} \frac{f'(z)}{f(z)} = m.$$

Step 2: Behavior at poles

Suppose f has a pole of order n at z_1. Then in a neighborhood of z_1, we can write

$$f(z) = \frac{1}{(z - z_1)^n} k(z),$$

where $k(z)$ is analytic and $k(z_1) \neq 0$. Differentiating,

$$f'(z) = -\frac{n}{(z - z_1)^{n+1}} k(z) + \frac{1}{(z - z_1)^n} k'(z),$$

so

$$\frac{f'(z)}{f(z)} = \frac{-n}{z - z_1} + \frac{k'(z)}{k(z)}.$$

Hence, $\dfrac{f'}{f}$ has a simple pole at z_1 with residue

$$\mathrm{Res}_{z=z_1} \frac{f'(z)}{f(z)} = -n.$$

Step 3: Apply residue theorem

The function $\dfrac{f'}{f}g$ is meromorphic on D, and its only singularities inside γ are at z_0 and z_1. By the Cauchy residue theorem,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}g(z)\,dz = \sum_{z\in\{z_0,z_1\}} \operatorname{Res}_z\left[\frac{f'(z)}{f(z)}g(z)\right].$$

At z_0:

$$\operatorname{Res}_{z=z_0}\left[\frac{f'(z)}{f(z)}g(z)\right] = \lim_{z\to z_0}(z-z_0)\frac{f'(z)}{f(z)}g(z) = mg(z_0).$$

At z_1:

$$\operatorname{Res}_{z=z_1}\left[\frac{f'(z)}{f(z)}g(z)\right] = \lim_{z\to z_1}(z-z_1)\frac{f'(z)}{f(z)}g(z) = -ng(z_1).$$

Final Answer

$$\boxed{\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}g(z)\,dz = mg(z_0) - ng(z_1)}$$