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Exercise: If A is a retract of X, then induced map by inclusion $A \overset{i}{\hookrightarrow} B$ induces $i_* : H_n(A) \longrightarrow H_n(x)$ is injective.

Proof:

Now, there exists a continuous map

$$r : X \longrightarrow A \overset{i}{\hookrightarrow} X$$

Now, we have the following diagram,

$$r : X \to A \overset{i}{\hookrightarrow} X$$
$$C(X) \xrightarrow{r_\#} C(A) \xrightarrow{i_\#} C(X)$$

where $r_\#, i_\#$ are defined as:

$$\begin{aligned} r_\#(a) &= r \circ a \quad && \text{where } a \in C(X) \\ i_\#(b) &= i \circ a && [a : \Delta \to X] \\ & && \text{where } b \in C(A) \\ & && [b : \Delta \to A] \end{aligned}$$

Now, boundary maps and $r_\# \& i_\#$ commutes as follows:

$$\begin{tikzcd} C_{n+1}(X) \arrow[r, "\partial^X_{n+1}"] \arrow[d, shift left, "r_\#"] & C_n(X) \arrow[r, "\partial^X_n"] \arrow[d, shift left, "r_\#"] & C_{n-1}(X) \arrow[r, "\partial^X_{n-1}"] \arrow[d, shift left, "r_\#"] & \cdots \\ C_{n+1}(A) \arrow[r, "\partial^A_{n+1}"] \arrow[u, shift left, "i_\#"] & C_n(A) \arrow[r, "\partial^A_n"] \arrow[u, shift left, "i_\#"] & C_{n-1}(A) \arrow[r, "\partial^A_{n-1}"] \arrow[u, shift left, "i_\#"] & \cdots \end{tikzcd}$$

From the above figure we get,

$$\partial^X_n \circ i_\# = i_\# \circ \partial^A_n ....[i]$$
$$\partial^A_n \circ r_\# = r_\# \circ \partial^X_n ....[ii]$$

Now,

$$H_n(X) = \frac{\ker \partial^X_n}{\operatorname{Im} \partial^X_{n+1}}$$
$$H_n(A) = \frac{\ker \partial^A_n}{\operatorname{Im} \partial^A_{n+1}}$$

Let's define,

$$\begin{aligned} i_* : H_n(A) &\longrightarrow H_n(X) \\ a + \operatorname{Im} \partial_{n+1}^A &\longmapsto i_\#(a) + \operatorname{Im} \partial_{n+1}^X \end{aligned}$$

Now, i_* is well defined:
pf: Let $a + \operatorname{Im} \partial_{n+1}^A = b + \operatorname{Im} \partial_{n+1}^A$

$$\Rightarrow a - b \in \operatorname{Im} \partial_{n+1}^A$$

Now, $i_* \left(a + \operatorname{Im} \partial_{n+1}^A\right) = i_\#(a) + \operatorname{Im} \partial_{n+1}^x$

$$i_* \left(b + \operatorname{Im} \partial_{n+1}^A\right) = i_\#(b) + \operatorname{Im} \partial_{n+1}^x$$

Now,

$$\begin{aligned} & a - b \in \operatorname{Im} \partial_{n+1}^A \\ & \Rightarrow \exists x \in C_{n+1}(A) \text{ st.} \\ & a - b = \partial_{n+1}^A(x) \\ & i_\#(a-b) = i_\# \left(\partial_{n+1}^A(x)\right) \\ & = \partial_{n+1}^X \left(i_\#(x)\right) \text{[by equation [i]]} \end{aligned}$$

$$\begin{aligned} & \Rightarrow i_\#(a-b) \in \operatorname{Im} \partial_{n+1}^X \\ & \Rightarrow \quad i_\#(a) - i_\#(b) \in \operatorname{Im} \partial_{n+1}^X \\ & \Rightarrow i_\#(a) + \operatorname{Im} \partial_{n+1}^X = i_\#(b) + \operatorname{Im} \partial_{n+1}^X \end{aligned}$$

[Because We extended $i_\#$ linearly while defining on generator of $C(A)$]

$$\Rightarrow i \left(a + \operatorname{Im} \partial_{n+1}^A\right) = i \left(b + \operatorname{Im} \partial_{n+1}^A\right)$$

Now, i_* is homomorphism.
It follows easily from the linearity (homomorphism) of $i_\#$.
i_* is injective:
pf:

$$\begin{aligned} & i_* \left(a + \operatorname{Im} \partial_{n+1}^A\right) = 0 \\ & \Rightarrow i_\#(a) + \operatorname{Im} \partial_{n+1}^X = 0 \\ & \Rightarrow i_\#(a) \in \operatorname{Im} \partial_{n+1}^X \\ & \Rightarrow \exists x \in C_{n+1}(X) \text{ st.} \\ & i_\#(a) = \partial_{n+1}^X(x) \\ & \Rightarrow r_\# \left(i_\#(a)\right) = r_\# \left(\partial_{n+1}^X(x)\right) = \partial_{n+1}^A \left(r_\#(x)\right) \text{[By equation[ii]]} \\ & \Rightarrow a = \partial_{n+1}^A \left(r_\#(x)\right) \\ & \Rightarrow a \in \operatorname{Im} \partial_{n+1}^A \\ & \Rightarrow a + \operatorname{Im} \partial_{n+1}^A = 0 \end{aligned}$$