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Question:

Let

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } x^2+y^2 \neq 0 \\ 0, & \text{if } x^2+y^2 = 0 \end{cases}$$

1. Find directonal derivative at $(0,0)$ at the direction $\beta = (l\hat{i} + m\hat{j})$ [where $l^2+m^2=1$].
2. Find $f_x(0,0)$ & $f_y(0,0)$.

Answer:

1. To find the \textbf{directional derivative} of the function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } x^2+y^2 \neq 0 \\ 0, & \text{if } x^2+y^2 = 0 \end{cases}$$

at the point $(0,0)$ in the direction $\beta = (l,m)$ with $l^2+m^2=1$, we use the definition:

$$D_\beta f(0,0) = \lim_{h\to 0} \frac{f((0,0)+h\beta) - f(0,0)}{h}$$

Step 1: Expressing the Increment

Since $h\beta = (hl, hm)$, we evaluate the function at this point:

$$f(hl,hm) = \frac{(hl)(hm)}{(hl)^2+(hm)^2}$$

Since $l^2+m^2=1$, we simplify:

$$f(hl,hm) = \frac{h^2lm}{h^2(l^2+m^2)} = \frac{h^2lm}{h^2} = lm$$

Since $f(0,0)=0$, the directional derivative simplifies to:

$$D_\beta f(0,0) = \lim_{h\to 0} \frac{lm}{h}$$

Step 2: Evaluating the Limit

The expression $\frac{lm}{h}$ does \textbf{not} tend to a finite limit as $h \to 0$; rather, it diverges to \textbf{infinity} unless $lm = 0$.

\section*{Conclusion}

- If $lm \neq 0$, the directional derivative \textbf{does not exist} (since the limit does not converge).
- If $lm = 0$, then $D_\beta f(0,0) = 0$.

Thus, the **directional derivative at (0,0) exists only if either $l = 0$ or $m = 0$, in which case it is 0**. Otherwise, it **does not exist**.

2. To find the partial derivatives $f_x(0,0)$ and $f_y(0,0)$, we use the definitions:

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}, \quad f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h}$$

Finding $f_x(0,0)$:

Substituting $y = 0$ in $f(x,y)$:

$$f(h,0) = \frac{h(0)}{h^2 + 0^2} = 0$$

Since $f(0,0) = 0$, we get:

$$f_x(0,0) = \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

Finding $f_y(0,0)$

Substituting $x = 0$ in $f(x,y)$:

$$f(0,h) = \frac{0(h)}{0^2 + h^2} = 0$$

Since $f(0,0) = 0$, we get:

$$f_y(0,0) = \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

Conclusion

$$f_x(0,0) = 0, \quad f_y(0,0) = 0$$

Notice:

In the part 1 we dicussed that directional derivative exists iff $lm = 0$ that is, either $l = 0$ or $m = 0$. And $l = 0$ means $\beta = (0, m) = m(0, 1)$ and $l^2 + m^2 = 1 \implies m^2 = 1 \implies m = \pm 1$, but $m = 1$ is fine as we are taking $h \to 0+$ and $h \to 0-$, so $\beta = (0, 1)$ or in the direction of y axis that is $f_y(0, 0)$ and $m = 0 \implies \beta = (1, 0)$ similarly. And discussed in part 1 we are going to get the directional derivative zero and yes we get that exactly in part 2. So, we can directly calculate part 2 from part 1. No need to calculate again seperately.