Manifold Theory

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Respected, sir,
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Contents

1 Preliminaries

1.1 Definition of Topological space

• Let X be a any non-empty set and let $\tau \subseteq \mathcal{P}$(X). Now if τ has the following properties viz. :

1. $\phi, X \in \tau$.
2. Let $\{U_\alpha\}_{\alpha \in J}$ be any arbitrary collection of elements of τ, then $\cup_{\alpha \in J} U_\alpha \in \tau$.
3. Let $\{U_i\}_{i=1}^n \subseteq \tau$ the $\cap_{i=1}^n U_i \in \tau$.

if the above three properties are satisfied the (X, τ) is called a topological space.

1.2 Definition of Manifold

• Let (X, τ) be any topological space which is hausdorff and second countable and locally euclidean i.e. locally homeomorphic to a open subset of $\mathbb{R}^n$ for a fixed n. That is there is a homeomorhism $\phi_\alpha : U_\alpha^{'}(\subseteq X) \to U_\alpha(\subseteq \mathbb{R}^n)$.

1.2.1 Chart:

For a point $p \in X$ if there is a homomorphism as in the above definition then we say $(\phi_\alpha, U_\alpha^{'})$ is a chart around p.

Two charts $(U, \phi), (V, \psi)$ are called compatible iff $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is diffeomorphic. This map is called **transition map!**

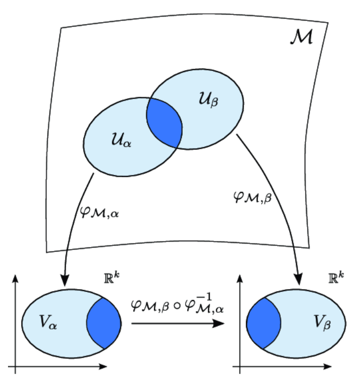


Figure 1:

1.2.2 Dimension of Manifold:

We define local dimension of a manifold at a point $p \in X$ to be the $n \in \mathbb{N}$ such that at that point we get a chart say $\phi : U \to U^{'}(\subseteq \mathbb{R}^n)$.
Is the above definition well defined? I.e. can't we get some other chart viz. $\psi : V \to V^{'} \subseteq \mathbb{R}^k$, where $k \neq n$.
If so, then $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is homeomorphism but as u see $\psi(U \cap V) \subseteq V^{'} \subseteq \mathbb{R}^k$ is open subset of $\mathbb{R}^k$ but $\phi(U \cap V) \subseteq U^{'} \subseteq \mathbb{R}^n$ is a open subset of $\mathbb{R}^n$. But two open subsets of different dimensional euclidean spaces can't be homeomorphic as if they be homeorphic then the whole this boils down to the homeomophism between $\mathbb{R}^n$ and $\mathbb{R}^k$ when $k \neq n$. And this is not possible can be proved in various ways. One of the way is using higher order fundamental groups of higher dimensional spheres.

We now proved that local dimension is well defined! Now, we moved forward to defining global dimension. For this

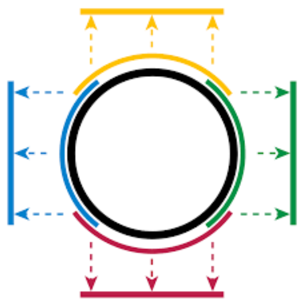
•**Lemma0 :** we restrict ourselves to connected spaces in particular connected manifolds, then then local dimension becomes global dimension.
proof: If the manifold is connected then there exits no chart in an atlas not intersecting with other charts otherwise that chart and union of the rest charts will form a seperation! But as we have seen in the above discussion that intersection of two charts will make same local dimension through out the both charts making the local dimension transmitted globally.

1.2.3 Atlas:

Collection of pairwise compatible charts whose union is whole X. An atlas is denoted by $\mathcal{A} = \{U_\alpha : \cup_\alpha U_\alpha = X \text{ and } U_\alpha, U_\beta \text{ is compatible}\}$.

1.2.4 Differentiable Structure:

There may be many atlases on a particular manifold for example,



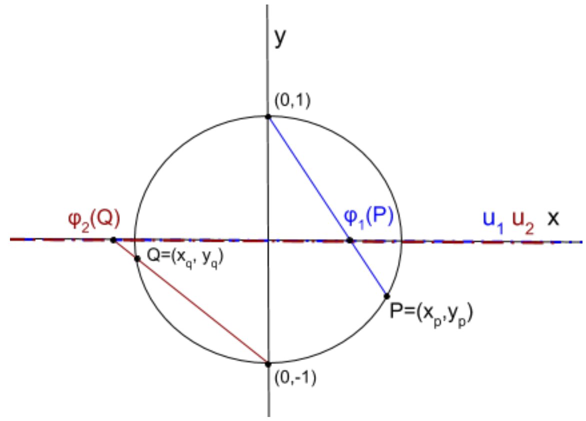


Figure 2: Vertical Projection Charts(left) and Stereographic projection chart(right)

Hence, we get two atlases viz. $\mathcal{A}_1 = \{yellow, red, green, blue\}$,
$\mathcal{A}_2 = \{circle - \{southpole\}, circle - \{northpole\}\}$. But if you calculate, you will eventually find that $\mathcal{A}_1 \cup \mathcal{A}_2$ is again an atlas. Hence we can take only one larger collection viz. the union! But not always all atlas union end upto again an atlas!
Hence, we introduce concept of equivalence classes on collection of all possible atlases on a given manifold X.
• Two atlases $\mathcal{A}_1, \mathcal{A}_2$ is called related iff $\mathcal{A}_1 \cup \mathcal{A}_2$ is again an atlas!
Now, it is easy to verify that this is an equivalence relation of collection of atlases on a given manifold!

•**Lemma1:** The above relation is an equivalence relation!
proof: reflexivity and symmetricity is trivial. Only non-trivial to check is transitivity!
Let $\mathcal{A}_1 \sim \mathcal{A}_2$ and $\mathcal{A}_2 \sim \mathcal{A}_3$, then we take any $U \in \mathcal{A}_1, W \in \mathcal{A}_3$. Now let $\mathcal{A}_2 = \{V_\alpha\}$. We need to prove U, W is compatible and that completes the proof! Then let $\phi : U \to U^{'}$ and $\psi : V \to V^{'}$. Then, $\phi \circ \psi^{-1} : \psi(U \cap V_\alpha \cap W) \to \phi(U \cap V_\alpha \cap W)$ can be written as $\phi \circ \zeta_\alpha^{-1} \circ \zeta_\alpha \circ \psi^{-1} = \phi \circ \psi^{-1}$ $\forall \alpha$. But $\forall \alpha$, $\phi \circ \zeta_\alpha^{-1}, \zeta_\alpha \circ \psi^{-1}$ are diffeomorphic due to compatibility of $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_2, \mathcal{A}_3$. Hence, $\phi \circ \psi^{-1}$ is diffeomorphic on $U \cap V_\alpha \cap W$ and we have $U \cap W = \cup_\alpha (U \cap V_\alpha \cap W)$. Then by very definition of differentiability we get $\phi \circ \psi^{-1}$ is diffeomorphic on $U \cap W$.
Provind the lemma!
Hence, we get a equivalence class of atlases on a manifold! And we call one equivalence class a $\underline{Differentiable}$ $\underline{structure}$!

Now, we can have a multipe differentiable structures on a manifold of various dimesions! Below is a chart of differentible stuctures on various dimensional spheres!

Dimension	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Smooth types	1	1	1	≥1	1	1	28	2	8	6	992	1	3	2	16256	2	16	16	523264	24

Figure 3: Differentiable structures on spheres

We see clearly that dim≤ 3, we have only one differentiable structure. This is in general true for any manifold with dim less than or equal to 3.

A manifold with a differentiable structure is called a differentiable maifold and if the transition maps are smooth maps(i.e. all order partial derivatives exits). Now, we move to define our most basic and important setup viz. differentiability of a function from one manifold to other!

Now we can draw some remark from the above discussions viz.

1. If $p \in U \subset X$, then $\phi(p) = (\phi_1(p), \phi_2(p), ..., \phi_n(p)) \in \mathbb{R}^n$.
2. Since ϕ is cont., so $\phi_i : U \to \mathbb{R}$ is a cont. for each $i = 1(1)n$.
3. The pair (U, ϕ) is called a coordinate neighborhood!(or a coordinate chart or a chart) of X.
4. $(\phi_1, \phi_2, ..., \phi_n)$ is called local coordinate system on (U, ϕ).

Ex: The n-sphere $\mathbb{S}^n = \{x = (x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} : ||x|| = 1\}$ $\forall n \in \mathbb{N}$ can be endowded with a stereographic atlas on it viz. $\mathcal{A} = \{U_1, U_2\}$ where $U_1 = \mathbb{S}^n - \{(0, 0, 0..., 1)\}$ and $U_2 = \mathbb{S}^n - \{(0, 0, 0..., -1)\}$. And $\phi_1(x_1, x_2, ..., x_{n+1}) = (\frac{x_1}{1-x_{n+1}}, \frac{x_2}{1-x_{n+1}}, ..., \frac{x_n}{1-x_{n+1}})$ and $\phi_2(x_1, x_2, ..., x_{n+1}) = (\frac{x_1}{1+x_{n+1}}, \frac{x_2}{1+x_{n+1}}, ..., \frac{x_n}{1+x_{n+1}})$.
Now, $\phi_1^{-1}(x_1, x_2, ..., x_n) = (\frac{2x_1}{x_1^2+x_2^2+...+x_n^2+1}, ..., \frac{x_1^2+x_2^2+...+x_n^2-1}{x_1^2+x_2^2+...+x_n^2+1})$ and $\phi_2^{-1}(x_1, x_2, ..., x_n) = (\frac{2x_1}{x_1^2+x_2^2+...+x_n^2+1}, ..., \frac{1-x_1^2-x_2^2-...-x_n^2}{x_1^2+x_2^2+...+x_n^2+1})$.
Now, $\phi_1 \circ \phi_2^{-1}(x_1, x_2, ..., x_n) = (\frac{x_1}{x_1^2+x_2^2+...+x_n^2}, ..., \frac{x_n}{x_1^2+x_2^2+...+x_n^2})$. Now, this one is differentiable function and all partial derivatives exists except for origin. And our domain of transition function does not include origin. Hence we get the transition map to be smooth. Similarly, we can prove $\phi_2 \circ \phi_1^{-1}$ is also smooth in its domain of definition. And hence we get a smooth structure on $\mathbb{S}^n$ as $[\mathcal{A}]$. Hence $(\mathbb{S}^n, [\mathcal{A}])$ is a smooth manfold.

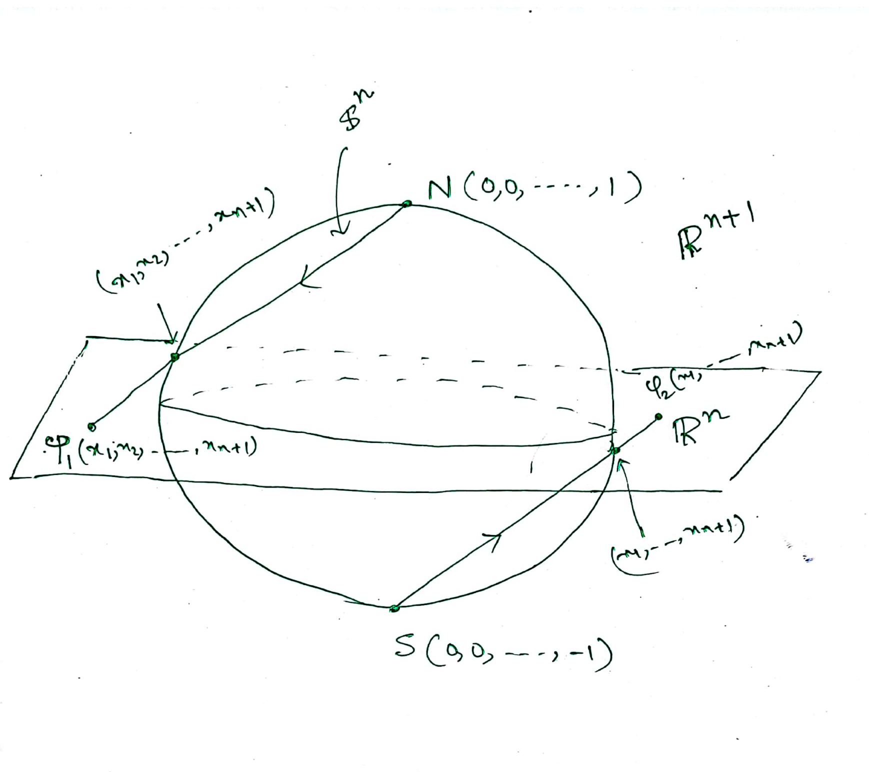


Figure 4: Stereogrphic Projection to Hyperplane

1.3 Deferrentiability:

Let X_1, X_2 be two smooth manifolds of dim m and n resp. A map $f : X_1 \to X_2$ is called smooth at $p \in X_1$ if given a chart (V, ψ) at $f(p) \in X_2$ there exist a chart (U, ϕ) at $p \in X_1$ such that $f(U) \subseteq V$ and the mapping $\psi \circ \phi^{-1} : \phi(U)(\subseteq \mathbb{R}^m) \to \psi(V)(\subseteq \mathbb{R}^n)$ is smooth at $\phi(p)$. A map f is smooth if it's smootg at every p^t of X_1. The set of all smooth funtios from X_1 to X_2 is denoted by $C^\infty(X_1, X_2)$.
In particular, a map $f : X \to \mathbb{R}$ on a smooth manifold is called smooth if fo all $p \in X$, there exits a chart (U, ϕ) such that the map $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth. We denote by $C^\infty(X, \mathbb{R}) = C^\infty(X)$, the set of all real valued smooth functions on X.

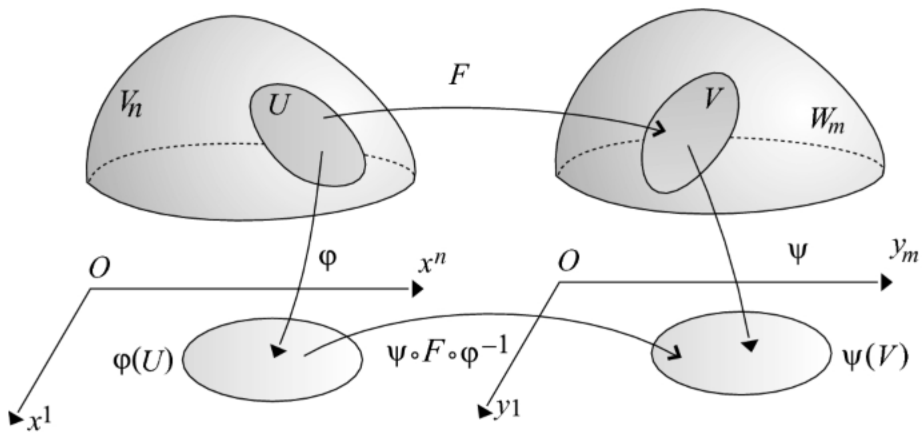


Figure 5: Defferentiability

•**Definition:** Let X_1, X_2 be two smooth manifolds, We say that a mapping $\phi : M \to Nis$

1) A diffeomorphism if its bijection and the maps ϕ and ϕ^{-1} are smooth.

2) is a local diffeomorphism at $p \in X$ if there exist neighbourhood U of p and V of $\phi(p)$ such that the map $\phi|_U : U \to V$ is a diffeomorphism .

•**Remark:**While defining differentiability on a smooth manifold we actually use the charts but if we change charts then does it lead to non differentiabilty of the function?
Answer: No as long as we are resctricted ourselves in one differentiable structure! That result leads to the fact that you don't need to worry about analysing differentiabilty usin only one representative atlas from the differentiable structure!
Proof: Let we have two charts (U, ϕ) and (V, ψ) around a point $p \in X$ such that they belong to $\mathcal{A}_1, \mathcal{A}_2$ respectively, and $\mathcal{A}_1, \mathcal{A}_2 \in [\mathcal{A}]$. Then we see that $(f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1}) = f \circ \psi^{-1}$ on the domain of definition $U \cap V$. Hence we get smoothness and if range set charts change then say we have two compatible charts as far same reasoning viz. $(W_1, \eta_1), (W_2, \eta_2)$ containg $f(p)$, then $(\eta_2 \circ \eta_1^{-1}) \circ (\eta_1 \circ f) = \eta_2 \circ f$. Then this is smooth from the above logic. Whole together we get, $f(U), f(V) \subseteq W_1, W_2$, and hence $f(U \cap V) \subseteq W_1 \cap W_2$. and moreover, we have, $(\eta_2 \circ \eta_1^{-1}) \circ (\eta_1 \circ f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1}) = \eta_2 \circ f \circ \psi^{-1}$ telling RHS to be smooth. And Hence as long as e are in fixedd differential stuctres on both domain and range manifolds differetiability can be checked using only one reprsentative atlas taken from each differentiable structures respectively!

1.3.1 Hessian , Regular Point , Critical Point of a realvalued function:

Let $f : X \to \mathbb{R}$ be any smooth function, where X be a smooth manifold.Then we define the following:

•Definition:

1. Differential of f: Differential of f t any point $p \in X$ is defined w.r.t. a chart (U, ϕ) about p as differential of $f \circ \phi^{-1}|_{\phi(p)}$. As, we know $f \circ \phi^{-1} : \phi(U) \subseteq (\mathbb{R}^n) \to \mathbb{R}$. Then, it's differential is calculated as normally, $d(f \circ \phi^{-1})|_{\phi(p)} = (\frac{\partial(f \circ \phi^{-1})}{\partial x_1}, ..., \frac{\partial(f \circ \phi^{-1})}{\partial x_n})|_{\phi(p)}$. And we define defferential of f wrt (U, ϕ) at point p as differential of $f \circ \psi^{-1}$ at point $\phi(p)$ and denoted by $df|_p = d(f \circ \phi^{-1})|_{\phi(p)}$.

2. Hessian of f: The hessian of f w.r.t. (U, ϕ) at point p is defined as hessian of $f \circ \phi^{-1}$ at point $\phi(p)$, and denoted by $H(f)|_p$. Then,

$$H(f)|_p = H(f \circ \phi^{-1})|_{\phi(p)} = J(d(f \circ \phi))|_{\phi(p)}$$

   i.e.

$$H(f)|_p = \begin{bmatrix} \frac{\partial^2 F}{\partial^2 x_1^2}|_{\phi(p)} & \frac{\partial^2 F}{\partial x_1 \partial x_2}|_{\phi(p)} & \cdots & \frac{\partial^2 F}{\partial x_1 \partial x_n}|_{\phi(p)} \\ \frac{\partial^2 F}{\partial x_2 \partial x_1}|_{\phi(p)} & \frac{\partial^2 F}{\partial^2 x_2^2}|_{\phi(p)} & \cdots & \frac{\partial^2 F}{\partial x_2 \partial x_n}|_{\phi(p)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 F}{\partial x_n \partial x_1}|_{\phi(p)} & \frac{\partial^2 F}{\partial x_n \partial x_2}|_{\phi(p)} & \cdots & \frac{\partial^2 F}{\partial^2 x_n^2}|_{\phi(p)} \end{bmatrix}_{n \times n}$$

   Here, $F = f \circ \phi^{-1}$.

3. Critical points of f: If p is a critical point or singular point of f if $d(F)|_{\phi(p)}$ is not surjective, this means that the partial derivtives $\frac{\partial F}{\partial x_i}(\phi(p)) = 0$ for all $i = 1, ..., n$.
   And the real value $f(p)$ is called the critical point of f.
   •Remark: Here, all we assume that $\phi(U)$ has dim n and is subset of $\mathbb{R}^n$. We we introduce local coordinates as $\phi(p) = (x_1(p), ..., (x_n(p))$. Now, as ϕ is bijective on U, for on U we can think of $\phi(p)$'s local neighbourhood points as $(x_1, ..., x_n)$ where x_i's are reals such that $||x - \phi(p)|| \le \varepsilon$ for which $B(\phi(p), \varepsilon) \subseteq \phi(U)$ as there is always a pre-image in U for each such point as ϕ being surjective.

4. Regular Point: Any point which is nit a critical point is called regular point of f and any real value which is not a critical value of f is called regular value of f.

5. Non-degenerate Critical Points: $p \in X$ is called non-degenerate critical point of f if the Hessian of f at p is non singular matrix. I.e.

$$det(H(F)|_{\phi(p)}) \neq 0$$

6. Degenerate Critical point: Any critical point whose Hessian is singluar is called degenerate critical point.

7. Index of a non-degenerate critical point: The index of a non-degenerate critical point is the number of negative eigen values of the Hessian matrix $H(F)|_{\phi(p)}$.

2 Morse Function

2.1 Definition:

A smooth map on a smooth manifold X, $f : X \to \mathbb{R}$ is a morse function if its all critical points are non-degenerate.

2.1.1 Example of Morse Function:

A very basic example of morse function is height function on a sphere and torus embedded vertically in $\mathbb{R}^3$.

(a) Let $X = \mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. The function $f : X \to \mathbb{R}$ is defined by $(x, y, z) \mapsto z$ is a morse function.
proof: Let $\phi_1(x_1, x_2, x_3) = (\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3})$ and $\phi_2(x_1, x_2, x_3) = (\frac{x_1}{1+x_3}, \frac{x_2}{1+x_3})$ be two charts of $\mathbb{S}^2$. The inverses of ϕ_1, ϕ_2 are given by $\phi_1^{-1}(x_1, x_2) = (\frac{2x_1}{x_1^2+x_2^2+1}, \frac{2x_2}{x_1^2+x_2^2+1}, \frac{x_1^2+x_2^2-1}{x_1^2+x_2^2+1})$ and $\phi_2^{-1}(x_1, x_2) = (\frac{2x_1}{x_1^2+x_2^2+1}, \frac{2x_2}{x_1^2+x_2^2+1}, \frac{1-x_1^2-x_2^2}{x_1^2+x_2^2+1})$ respectively.In order to determine the critical points of f, consider the map $f \circ \phi_i^{-1} : \mathbb{R}^2 \to \mathbb{R}$ for each $i = 1, 2$. Note that $(\mathbb{S}^2 - \{S\}, \phi_2)$ is the coordinate chart around $(0, 0, 1)$ and define a map $g = f \circ \phi_2^{-1}$ by $g(x_1, x_2) = \frac{1-x_1^2-x_2^2}{1+x_1^2+x_2^2}$.
Now, as we see that $\phi_2(\mathbb{S}^2 - \{S\}) = \mathbb{R}^2$ and on whole $\mathbb{R}^2$ g is differentiable and smooth. This neighbourhood works for all point except for south pole i.e.$S = (0, 0, -1)$.
Similarly we can prove that at point $N = (0, 0, 1)$, f smooth.

Since, $dg(x_1, x_2) = (\frac{-4x_1}{(1+x_1^2+x_2^2)^2}, \frac{-4x_2}{(1+x_1^2+x_2^2)^2})$,

We have $dg(x_1, x_2) = 0$ iff $x_1, x_2 = 0$.

Hence, $\phi_2^{-1}(0, 0) = (0, 0, 1)$ is the only critical point of f in $\mathbb{S}^2 - \{S\}$.
We will find the Hessian of f at point $(0, 0, 1)$.
$H(g)|_{\phi((0,0,1))} = H(g)|_{(0,0)} = (\frac{\partial^2 g}{\partial x_i \partial x_j}(0, 0))_{1 \leq i,j \leq 2} =$

$$\begin{bmatrix} \frac{-4(1-3x_1^2+x_2^2)}{(1+x_1^2+x_2^2)^3}|_{(0,0)} & \frac{16x_1x_2}{(1+x_1^2+x_2^2)^3}|_{(0,0)} \\ \frac{16x_1x_2}{(1+x_1^2+x_2^2)^3}|_{(0,0)} & \frac{-4(1+x_1^2-3x_2^2)}{(1+x_1^2+x_2^2)^3}|_{(0,0)} \end{bmatrix} =$$

$$\begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$$

This shows that $(0,0,1)$ is a non-degenerate critical point of f with index 2. For, the point $(0,0,-1)$ we use the chart $(\mathbb{S}^2 - \{N\}, \phi_1)$ and similarly shows that $(0,0,-1)$ is the only critical point of f which is non-degenerate with index 0.

•**Remark:** We get on $\mathbb{S}^2$, $(x,y,z) \mapsto z$ is a morse function with only two critical points.

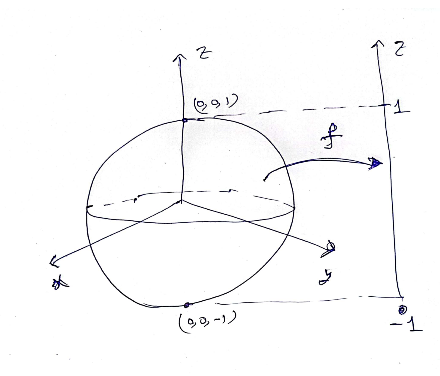


Figure 6: Morse function on Sphere

•**Question:** Is the converse true?
i.e. If X be a compact, connected, smooth manifold. Suppose there exists a Morse fucntion on X with exactly two critical points. Then M is homeomorphic to a sphere of the corresponding dimension of the manifold.
Answer: Yes! This is proved by mathematician Reeb's.
Finally,

(b) Let r, R be real numebrs satisfying $0 \leq r \leq R$, and let

$$X = \mathbb{T}^2 = \{(x,y,z) : x^2 + (\sqrt{y^2+z^2} - R)^2 = r^2\}$$

be a two dimensional torus. Then function $f : \mathbb{T}^2 \to \mathbb{R}$ defined by $f(x,y,z) = z$ is a morse function which has four non-degenerate critical ponits, $(0,0,-(R+r)), (0,0,-(R-r)), (0,0,R-r), (0,0,R+r)$.

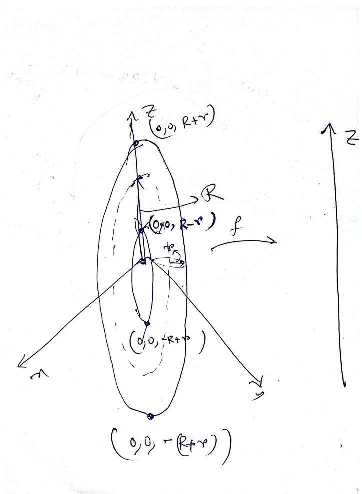


Figure 7: Morse Function on torus

3 Appendix:

3.1 My Conclusion:

After reading and making this project I am led to the following conclusions:

(a) Manifold theory is basically a generalisation of euclidean space but rather more realistic than Euclidean Spac.Basically, manifold study comes from the need of studying real life objects and calculus on them to find maximum and minimum of some height function!
For example, studying contour and landcapes and studying what happens if we passes through some points where every directional derivative vanishes, what changes happen in the topology of the submanifold defined by $X_p = f^{-1}(-\infty, f(p))$.

(b) In this purpose we define calculus on manifold or arbitrary structures of real life which we can visualize as a euclidean space when zoomed in locally. Then differentiation being local property we can relate it to more theoritical aspect the euclidean space and study it easily as mathematics is easy in euclidean spaces. But euclidean space is not realistic. It just came to visualize manifolds or real life objets in the most symmetric and easy way so, that we can develop mathematics on it symmetrically then can extend it on Manifolds via approximating them locally to euclidean spaces.

3.2 My future Plan after Mid-term:

We move to analyze the part (a) of my above conclusion more clearly. And will see that how topology changes when we crosses a citical point using Handle bodies and handle decomposition theorem.

In this regards, we use homology theory, homotopy from algebraic topology to prove morse lemma's. Then finally, we end proving h-cobordism theoreorem and using it proving smooth poincare conjecture for dimension greater than or equal to 5.

After Mid Sem

4 Morse Theory:

4.1 Morse Function

Definition 4.1.1: *Let f be a smooth function on a smooth manifold M. f is said to be a **Morse function** if every critical point of f is non-degenerate.*

Example 4.1.1: *The **height functions** on the sphere S^2 and the torus $\mathbb{T}^2$ are morse functions.*

Example 4.1.2: *Let $[z_0, z_1, ..., z_n]$ be an equivalence class of (n+1)-tuples $(z_0, z_1, ..., z_n)$ of complex numbers, with $\sum_{j=0}^{n}|z_j|^2 = 1$, and let $M = \mathbb{CP}^n = [z_0, z_1, ..., z_n]$ be the complex projective n-space. Define $f : M \to \mathbb{R}$ by*

$$[z_0, z_1, ..., z_n] \mapsto \sum_{j=0}^{n} c_j|z_j|^2,$$

where $c_0, c_1, ..., c_n$ are distinct real numbers. Such a function f is a Morse Function.
proof: In order to determine the critical points of f and their indices, we consider the following local coordinate system. For each $j \in \{0, 1, ..., n\}$, let U_j be the set of equivalence classes of (n+1)-tuples $(z_0, z_1, ..., z_n)$ of complex numbers with $z_j \neq 0$. That is,

$$U_j = \{[z_0, z_1, ..., z_n] : z_j \neq 0\} =$$

4.2 Morse Lemma

Lemma 4.2.1:(Morse lemma) *Let p be non degenerate critical points of f with index λ. Then there is a local coordinate system $Y : V \subset \mathbb{R}^n \to U_p$ in a neighborhood U_p of p with $0 \in V$ and $Y(0) = p$ such that the identity*

$$(f \circ Y)(y_1, y_2, ..., y_n) = f(p) - y_1^2 - .... - y_\lambda^2 + y_{\lambda+1}^2 + ... + y_n^2 ...(4.2.1)$$

holds through out V.

Corollary 4.2.1: *Let $f : M \to \mathbb{R}$ be a smooth function on a smooth manifold M. A non-degenerate criical point of a smooth function f is isolated. In particular, f is a Morse function and M is compact, then f has a finite number of critical points.*

proof: By Lemma 4.2.1, we observed that if f has a non-degenerate critical point p, then there is a coordinate chart (U_p, Y^-1) of M about p that satisfies the equation (4.2.1). This chart contains no, other critical points of f other than p since, by equation (4.2.1),

$$d(f \circ Y)_{(y_1, y_2, \ldots.., y_n)} = (-2y_1, \ldots, -2y_\lambda, 2y_{\lambda+1}, \ldots., 2y_n)$$

and $d(f \circ Y)_{(y_1, y_2, \ldots.., y_n)} = 0$ iff $(y_1, y_2, \ldots.., y_n) = 0$. Hence $Y(0) = p$ is the only critical point of f in U_p. Therefore, p is isolated. Now, suppose that M is compact. If the set of all critical points of f is infinite then we can extract a non-constant sequence out of it and it must have a convergent subsequence as compact implies sequentially compact. Then by continuity of df (as f is C^∞ hence df is contiuous) we get that limit point is also a critical point of f but then critical points of f can't be isolated as per the above scenario, ($\Rightarrow\Leftarrow$).

4.3 Existence of Morse Functions

The goal of this section is to show the existence of Morse functions on any smoot manifold. Since the **Whitney embedding Theorem** tells us that any smooth manifold is embedded in a suitable Euclidean vector space, let M be a smooth $n-$dimensional manifold embedded in $E = \mathbb{R}^{n+k}$ for some $k \in \mathbb{N}$.Hence we will try to prove that any morse function can be viewed as a height function for a compact finite dimensional manifold. And Hence after the existence we can study easily morse functions easily.
To do so, let us recall some useful results:

Recall: $E^* = \{\alpha | \alpha : E \to \mathbb{R}$ is a linear map$\}$ is the dual space of real vector space E, and the following useful definitions:

Definition 4.3.1: *The dual of the tangent space T_xM of a smooth manifold M is called the cotangent space at x denoted by*

$$T_x^*M = (T_xM)^*.$$

An element of T_x^*M is called cotangent vector or covector.

Definition 4.3.2: *Let $f : M$ be a smooth finite dimensional manifolds. The differetial map of f at x is the linear map $df_x : T_xM \to T_{f(x)}N$ and is given by $df_x(v) = \beta^{'}(0)$
Where $\beta = g \circ \alpha$ where $\alpha(0) = x$ and $\alpha : (-\epsilon, \epsilon) \to M$ is a curve smooth at 0.*

*(1) f is called **immersion** if df_x is injective for every $x \in M$.*

(2) f is called ***submersion*** *if df_x is surjective for every $x \in M$.*

4.4 Fundamental Theorems of Morse Theory

In this section, we let $f : M \to \mathbb{R}$ be a real valued function on a smooth manifold M, and let

$$M^a = f^{-1}((-\infty, a]) = \{p \in M : f(p) \leq a\}.$$

4.4.1 Morse's First Fundamental Theorem(MFFT)

We first consider the region that f has no critical points as follows:

Theorem 4.4.1: *Let $f : M \to \mathbb{R}$ be a smooth real valued function on a manifold M. Let a and b be regular values of f with $a \leq b$ such that the set*

$$f^{-1}([a, b]) = \{p \in M : a \leq f(p) \leq b\}$$

is compact and contains no critical points of f. Then M^a is diffeomorphic to M^b. Furthermore, M^a is a deformation retract of M^b, so that the inclusion map $M^a \hookrightarrow M^b$ is a homotopy equivalence.

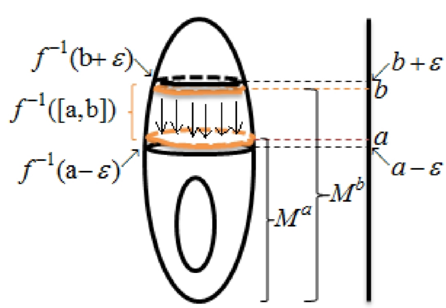


Figure 8: MFFT

proof: Since $f^{-1}([a, b])$ is compact and contains no critical points, then there exits $\epsilon > 0$ small enough such that the set $f^{-1}((a-\epsilon, b+\epsilon))$ contains no critical points of f. Let $\rho : M \to \mathbb{R}$ be a smooth function defined by

$$\rho(x) = \begin{cases} \frac{1}{||gradf(x)||^2} & if \text{ x} \in f^{-1}((a-\epsilon, b+\epsilon)) \\ 0 & otherwise \end{cases}$$

Now we can define a smooth vector field X on M by,

$$X_x = \rho(x) gradf(x) \quad \forall x \in M.$$

That is,

$$X_x = \begin{cases} \frac{1}{||gradf(x)||^2} gradf(x) & if \text{ x} \in f^{-1}((a-\epsilon, b+\epsilon)) \\ 0 & otherwise \end{cases} \quad ....(4.4.1)$$

By a lemma X generates a 1-parameter group of diffeomorphism $\phi : \mathbb{R} \times M \to M$. Then for each fixed $p \in M$ the map $c := \phi_p : \mathbb{R} \to M$ is a smooth curve in M defined by $c(t) = \phi_t(p)$ and $c(0) = \phi_0(p) = p$, because $\phi_0 = id_M$.

Lemma 4.4.1:

- The vector field $gradf(p) = 0$ if p is a critical point of f.
- If we have a curve $c : \mathbb{R} \to M$ with velocity vector $\frac{dc}{dt}$, then

$$\frac{d(f \circ c)}{dt} = df_{c(t)}(\frac{c(t)}{dt}) = \left\langle \frac{dc}{dt}, gradf(c(t)) \right\rangle.$$

Therefore, by lemma 4.4.1,

$$\begin{aligned} \frac{d(f \circ \phi_t(p))}{dt} &= \frac{d(f \circ \ c)}{dt} \\ &= \left\langle \frac{dc}{dt}, gradf(c(t)) \right\rangle \\ &= \left\langle \frac{d\phi_t(p)}{dt}, gradf(\phi_t(p)) \right\rangle \\ &= \left\langle X_{\phi_t(p)}, gradf(\phi_t(p)) \right\rangle \end{aligned}$$

since $\frac{d\phi_t(p)}{dt} = X_{\phi_t(p)}$. Hence, the last equality together with equation (4.4.1) give us that

$$\frac{df(\phi_t(p))}{dt} = \begin{cases} 1 & \text{if } \phi_t(p) \in f^{-1}((a-\epsilon, b+\epsilon)) \\ 0 & \text{o/w} \end{cases}$$

We then have

$$f(\phi_t(p) = \begin{cases} t + f(p) & \text{if } \phi_t(p) \in f^{-1}((a-\epsilon, b+\epsilon)) \\ f(p) & \text{o/w} \end{cases} \quad ....(4.4.2)$$

since $\phi_0(p) = p$. In addition, $f(\phi_t(p))$ is increasing

since $\frac{df(\phi_t(p))}{dt} \geq 0 \; \forall \; t \in \mathbb{R}$ and $p \in M$.

Consider the diffeomorphism

$\phi_{b-a} : M \to M$. We claim that $\phi_{b-a}\Big|_{M^a} : M^a \to M^b$ is a diffeomorphism.

First, we prove that ϕ_{b-a} maps M^a into M^b. We wish to prove that for every $x \in M^a$, then $f(\phi_{b-a}(x)) \leq b$(i.e. $\phi_{b-a}(x) \in M^b$). Let $x \in M^a$. Since $f(\phi_t(p))$ is increasing,

$$f(\phi_0(x)) = f(x) < f(\phi_{b-a}(x)).$$

Therefore are two cases:

- if $f(\phi_{b-a}(x) \leq b)$, then $\phi_{b-a}(x) \in M^b$.
- if $f(\phi_{b-a}(x)) > b$, then by (4.4.2), $f(x) - a > 0$ and $f(x) > b$ which is a contradiction.

Therefore, ϕ_{b-a} maps M^a into M^b. Since $\phi_t : M \to M$ is a diffemorphism for each t, then the restriction of ϕ_{b-a} to M^a is also one to one. So, we only remain to prove tat ϕ_{b-a} maps M^a onto M^b. Let $y \in M^b$. There exit $x = \phi_{a-b}(y) \in M^a$ because, by (4.4.2), we have

$$f(x) = f(\phi_{a-b}(y)) \leq b$$

and if $f(\phi_{a-b}(y)) > a$, since $f(\phi_t(p))$ is increasing, we obtain

$$a < f(\phi_{a-b}(y)) < f(\phi_{f(x)-b}(y)) \leq b,$$

and this implies that

$$f(\phi_{a-b}(y)) = a - b + f(y) \leq a - b + b = a$$

which is a contradiction. Therefore, the map ϕ_{b-a} is onto since

$$\phi_{b-a}(x) = \phi_{b-a}(\phi_{a-b}(y)) = \phi_0(y) = y.$$

Now we proceed to prove the second part: M^a is deformation retract to M^b. Consider the family of maps $r_t : M^a \to M^b$ defined by

$$r_t(x) = \begin{cases} x & \text{if } x \in M^b \\ \phi_{(a-f(x))t}(x) & \text{if } a \leq f(x) \leq b \end{cases}, t \in [0,1]$$

If $x \in M^a$, then $r_t(x) = x \in M^a \subset M^b$. If $a \leq f(x) \leq b$, then $(a-f(x))t \leq 0$ and by the monotonicity of $f(\phi_t(p))$, this implies that $f(\phi_{(a-f(x))t}(x)) \leq f(\phi_0(x)) = f(x) \leq b$. Thus $r_t(x) = \phi_{(a-f(x))t}(x) \in M^b$. This family also satisfies the following condtions:

- $r_t(x)$ is continuous on the product topology $M^b \times [0,1]$.
- $r_0(x) = x$ for all $x \in M^b$
- $r_1(x) = \phi_{(a-f(x))(x)} \in M^a$. Indeed, if $x \in M^a$, then $r_1(x) = x \in M^a$ and by the monotonicity of $f(\phi_t(p))$, if $a \leq f(x) \leq b$, then

  $$f(r_1(x)) = f(\phi_{a-f(x)}(x) \leq f(\phi_0(x)) = f(x) \leq b$$

  .

  Case 1: if $f(r_(x)) \leq a$, then $r_1(x) \in M^a$. Case 2: if $a \leq f(r_1(x)) \leq b$, then $f(r_1(x)) = a - f(x) + f(x) = a$. Hence $r_1(x) \in M^a$.
- It is clear that $r_1(x) = x$ for all $x \in M^a$.

  Therefore , M^a is a deformation retraction of M^b, so that the inclusion map $M^a \hookrightarrow M^b$ is a homotopy equivalence.

4.4.2 Morse's Second Fundamental Theorem(MSFT)

Now let us consider a region in which f has one critical point.

Theorem 4.4.2:*Let p be a non degenerate critical point of f with index λ. Let $c = f(p)$ and assume $f^{-1}([c-\epsilon, c+\epsilon])$ is compact and contains no other critical point of f for some $\epsilon > 0$. Then for all sufficiently small ϵ, the set $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with a λ-cell attached.*

4.4.3 Consequence of the Fundamental Theorems

Theorem 4.4.3: If $f : M \to \mathbb{R}$ is a smooth function on a compact smooth manifold M with no degenerate critical points and if each M^a is compact, then M has the homotopy equivalence of a CW-complex, with one cell of dimension λ for each critical point of index λ.

To prove this theorem, we will need the following lemmas.

Lemma 4.4.1: (Whitehead) Let ϕ_0 and ϕ_1 be homotopic maps from the sphere $\partial(e^\lambda)$ to a topological space X. then the identity map of X extends to a homotopy equivalence

$$k : X \sqcup_{\phi_0} e^\lambda \to X \sqcup_{\phi_1} e^\lambda$$

Proof. Let ϕ_t be a homotopy between ϕ_0 and ϕ_1.
Define $k : X \sqcup_{\phi_0} e^\lambda \to X \sqcup_{\phi_1} e^\lambda$ by

$$k(x) = \begin{cases} x & \text{if } x \in X \\ 2ru & \text{if } x = ru,\ u \in \partial(e^\lambda),\ 0 \le r \le \frac{1}{2} \\ \phi_{2-2r}(u) & \text{if } x = ru,\ u \in \partial(e^\lambda)\ , \frac{1}{2} \le r \le 1 \end{cases}$$

and $\tilde{k} : X \sqcup_{\phi_1} e^\lambda \to X \sqcup_{\phi_0} e^\lambda$ by

$$\tilde{k}(x) = \begin{cases} x & \text{if } x \in X \\ 2su & \text{if } x = su,\ u \in \partial(e^\lambda)\ , 0 \le s \le \frac{1}{2} \\ \phi_{2s-1}(u) & \text{if } x = su,\ u \in \partial(e^\lambda)\ , \frac{1}{2} \le s \le 1 \end{cases}$$

Since the functions k and $\tilde{k}$ are continuous, there are the compositions

$$\tilde{k} \circ k(x) = \begin{cases} x & \text{if } x \in X \\ 4ru & \text{if } x = ru,\ u \in \partial(e^\lambda)\ , 0 \le r \le \frac{1}{4} \\ \phi_{4r-1}(u) & \text{if } x = ru,\ u \in \partial(e^\lambda)\ , \frac{1}{4} \le r \le \frac{1}{2} \\ \phi_{2-2r}(u) & \text{if } x = ru,\ u \in \partial(e^\lambda)\ , \frac{1}{2} \le r \le 1 \end{cases}$$

and

$$k \circ \tilde{k}(x) = \begin{cases} x & \text{if } x \in X \\ 4su & \text{if } x = su,\ u \in \partial(e^\lambda)\ , 0 \le s \le \frac{1}{4} \\ \phi_{2-4s}(u) & \text{if } x = su,\ u \in \partial(e^\lambda)\ , \frac{1}{4} \le s \le \frac{1}{2} \\ \phi_{2s-1}(u) & \text{if } x = su,\ u \in \partial(e^\lambda)\ , \frac{1}{2} \le s \le 1 \end{cases}$$

We want to find a homotopy $h_t : X \sqcup_{\phi_0} e^\lambda \to X \sqcup_{\phi_0} e^\lambda$, $t \in [0,1]$ such that $h_0 = \tilde{k} \circ k$ and $h_1 = id$. Consider family of maps $h_t : X \sqcup_{\phi_0} e^\lambda \to X \sqcup_{\phi_0} e^\lambda$ defined by

$$h_t(x) = \begin{cases} x & \text{if } x \in X \\ \frac{4ru}{1+3t} & \text{if } x = ru,\ u \in \partial(e^\lambda)\ , 0 \le r \le \frac{1+3t}{4} \\ \phi_{(\frac{4r}{1+3t}-1)(1-t)}(u) & \text{if } x = ru,\ u \in \partial(e^\lambda)\ , \frac{1+3t}{4} \le r \le \frac{t+1}{2} \\ \phi_{\frac{2-2r}{1+3t}(1-t)}(u) & \text{if } x = ru,\ u \in \partial(e^\lambda)\ , \frac{t+1}{2} \le r \le 1. \end{cases}$$

It is easy to check that h_t is contiuous, $h_0 = \tilde{k} \circ k$ and $h_1 = id$.

We next consider a family of maps $h'_t : X \sqcup_{\phi_1} e^\lambda \to X \sqcup_{\phi_1} e^\lambda$ defined by

$$h'_t(x) = \begin{cases} x & \text{if } x \in X \\ \frac{4su}{1+3t} & \text{if } x = su,\ u \in \partial(e^\lambda)\ , 0 \le s \le \frac{1+3t}{4} \\ \phi_{1-(\frac{4r}{1+3t}-1)(1-t)}(u) & \text{if } x = su,\ u \in \partial(e^\lambda)\ , \frac{1+3t}{4} \le s \le \frac{t+1}{2} \\ \phi_{1-\frac{2-2r}{1+3t}(1-t)}(u) & \text{if } x = su,\ u \in \partial(e^\lambda)\ , \frac{t+1}{2} \le s \le 1. \end{cases}$$

Again h'_t is continuous and satisfies $h'_0 = k \circ k$, $h'_1 = id$. □

Lemma 4.4.2: (Hilton) Let $\phi : \partial(e^\lambda) \to X$ be a attaching map. A homotopy equivalence $f : X \to Y$ can be extended to a homotopy equivalence

$$F : X \sqcup_\phi e^\lambda \to Y \sqcup_{f \circ \phi} e^\lambda.$$

Proof. Since $f : X \to Y$ is a a homotopy equivalence, there exists a homotopy inverse $g : Y \to X$ to f and $h_t : X \to X$ a homotopy such that $h_0 = g \circ f$ and $h_1 = id_X$. Let $H : [0,1] \times \partial(e^\lambda) \to X$ defined by $H(t,x) = h_t(\phi(x))$. Then we have $H(0,x) = g \circ \circ \phi(x)$ and $H(1,x) = \phi(x)$. Thus $g \circ \circ \phi$ and ϕ are homotopic maps from $\partial(e^\lambda)$ to X. By the Lemma 4.4.1, there exists a homotopy equivalence

$$k : X \sqcup_{g \circ f \circ \phi} e^\lambda \to X \sqcup_\phi e^\lambda.$$

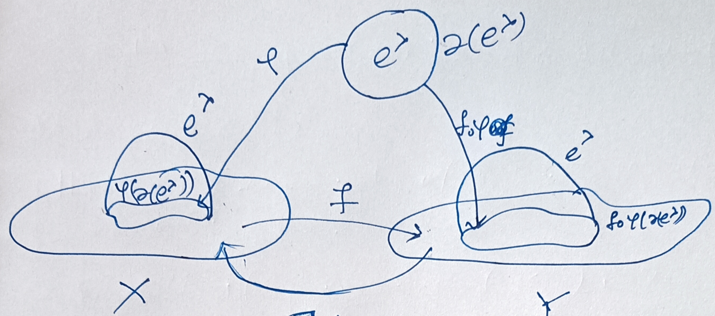
Define the following two maps $F : X \uplus e^\lambda \to Y \uplus e^\lambda$ and $G : Y \uplus e^\lambda \to X \uplus e^\lambda$ as follows

$$F(x) = \begin{cases} f(x) & \text{if } x \in X \\ x & \text{if } x \in e^\lambda \end{cases}$$

and

$$G(y) = \begin{cases} g(y) & \text{if } y \in Y \\ y & \text{if } y \in e^\lambda \end{cases}$$

Then clearly F and G are continuous functions under the disjoint



$\exists g.$

Given, $X \simeq Y$

$f \circ g \simeq id_Y$

We are showing :

$X \underset{\varphi}{\sqcup} e^{\lambda} \simeq Y \underset{f\circ\varphi}{\sqcup} e^{\lambda}$

$g \circ f \simeq id_X$

$$\varphi : \partial(e^{\lambda}) \longrightarrow X$$
$$x \longmapsto \varphi(x)$$

$$X \underset{x \sim \varphi(x)}{\sqcup} e^{\lambda}$$

identification space (quotient top on $X \underset{\varphi}{\sqcup} e^{\lambda}$)

We show, $g \circ f \circ \varphi \simeq \varphi$ [Why? as $g \circ f \simeq id_X$

$\Rightarrow g \circ f \circ \varphi \simeq id_X \circ \varphi$

$\Rightarrow g \circ f \circ \varphi \simeq \varphi$]

by Lemma 4.4.1

$\Rightarrow X \underset{g\circ f\circ\varphi}{\sqcup} e^{\lambda} \simeq X \underset{\varphi}{\sqcup} e^{\lambda}$

union topology. Now, we will use universal property of qoutient map to get $\tilde{F}$ and $\tilde{G}$ such that the following diagram commutes: Now,

$$\begin{tikzcd}
X \sqcup_{\varphi} e^{\lambda} \arrow[r, "\tilde{F}"] & Y \sqcup_{f\circ\varphi} e^{\lambda} \\
X \cup e^{\lambda} \arrow[u, "\pi_{\varphi}"] \arrow[ur, "\pi_{f\circ\varphi}\circ F"] \arrow[r] & Y \cup e^{\lambda} \arrow[u, "\pi_{f\circ\varphi}"']
\end{tikzcd}$$

$$\boxed{\tilde{F}\circ\pi_{\varphi} = \pi_{f\circ\varphi}\circ F}$$

$$\begin{tikzcd}
X \sqcup_{g\circ f\circ\varphi} e^{\lambda} & Y \sqcup_{f\circ\varphi} e^{\lambda} \arrow[l, "\tilde{G}"'] \\
X \cup e^{\lambda} \arrow[u, "\pi_{g\circ f\circ\varphi}"] & Y \cup e^{\lambda} \arrow[l, "G"] \arrow[u, "\pi_{f\circ\varphi}"'] \arrow[ul, "\pi_{g\circ f\circ\varphi}\circ G"']
\end{tikzcd}$$

$$\boxed{\tilde{G}\circ\pi_{f\circ\varphi} = \pi_{g\circ f\circ\varphi}\circ G}$$

we will prove that $\tilde{F}$ has a left homotopy inverse viz. $k \circ \tilde{G}$. That is, the composition $k \circ \tilde{G} \circ \tilde{F} : X \sqcup_{\phi} e^{\lambda} \rightarrow X \sqcup_{\phi} e^{\lambda}$ is homotopic to the indentity map. From the definiton of $k, \tilde{F}$ and $\tilde{G}$, we note that

$$k \circ \tilde{G} \circ \tilde{F} = \Big\{$$

Claim 2.4.4: If a map F has a left and a right homotopy inverse L and R respectively, then F is a homotopy equivalence, and L (or R) is a 2-sided homotopy inverse.

Proof. Since L and R are left and right homotopy inverses to F, we have the relations $LF \simeq \mathrm{id}$ and $FR \simeq \mathrm{id}$. This implies that

$$L \simeq L(FR) = (LF)R \simeq R$$

Hence

$$FL \simeq FR \simeq \text{ id (or } RF \simeq LF \simeq \text{ id)}$$

which proves that $L($ or R) is a 2 -sided homotopy inverse. To prove the Lemma 2.4.2, it only remains to prove that F has a right homotopy inverse. By the Claim 2.4.4, we obtain the following: - $k \circ (G \circ F) \simeq \mathrm{id}$ implies that $(G \circ F) \circ k \simeq \mathrm{id}$ since k is known to have a left homotopy inverse (by Lemma 2.4.1). - $G \circ (F \circ k) = (G \circ F) \circ k \simeq \mathrm{id}$ implies that $(F \circ k) \circ G \simeq \mathrm{id}$ since G is known to have a left homotopy inverse. - $F \circ (k \circ G) = (F \circ k) \circ G \simeq \mathrm{id}$ implies that F has $k \circ G$ as a right homotopy inverse. Therefore, F is a homotopy equivalence. This completes the proof of Lemma 2.4.2.

Proof. (of Theorem 4.4.3) Let $a \in \mathbb{R}$ and p_{ik_i} be critical points belonging to $f^{-1}(c_i)$ with index λ_{ik_i}. If $f^{-1}(a) = \emptyset$, then $M^a = \emptyset$ and so we have nothing to do. If $f^{-1}(a) \neq \emptyset$, then $M^a \neq \emptyset$. Base case: We may assume that $c_1 < a < c_2$. Since M^a is compact, f has a global minimum value $c_1 \in \mathbb{R}$ (i.e, $c_1 \leq f(p)$ for all $p \in M$). According to the Theorem 2.4.1, 44 CRV CHAPTER 2. MORSE THEORY $M^{c_1+\epsilon}$ is homotopy equivalent to M^a for some small $\epsilon > 0$. Since the critical points belonging to $f^{-1}(c_1)$ have index 0 , by Proposition 2.4.1, $M^{c_1+\epsilon}$ has the homotopy type of a disjoint union of 0 cells. Therefore, M^a has the homotopy type of a CW-complex.

Induction hypothesis: Suppose that $a \neq c_1, c_2, c_3, \cdots$ such that M^a is homotopy equivalent to a CW-complex K via g. Let $c = c_{j_0}$ be the smallest critical value of f bigger than a. According to the Theorem 2.4.1 and Proposition 2.4.1, for some small $\epsilon > 0$ we have that $M^{c-\epsilon}$ is homotopy equivalent to M^a via h and that $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon} \cup_{\varphi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{\varphi_{j_0 2}} \cdots \cup_{\varphi_{j_0 k_{j_0}}} e^{\lambda_{j_0 k_{j_0}}}$ for some attaching maps $\varphi_{j_0 1}, \cdots, \varphi_{j_0 k_{j_0}}$. Then, by Lemma 2.4 .2 we see that

$$M^{c-\epsilon} \cup_{\varphi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{\varphi_{j_0 2}} \cdots \cup_{\varphi_{j_0 k_{j_0}}} e^{\lambda_{j_0 k_{j_0}}} \simeq M^a \cup_{h 0 \varphi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{h \circ \varphi_{j_0 2}} \cdots \cup_{h \circ \varphi_{j_0 k_{j_0}}} e^{\lambda_{j_0 k_{j_0}}} .$$

Since M^a is homotopy equivalent to K via g, Lemma 2.4 .2 shows that

$$M^a \cup_{h \circ \varphi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{h \circ \varphi_{j_0 2}} \cdots \cup_{h \circ \varphi_{j_0 k_{j_0}}} e^{\lambda_{j_0 k_j}} \simeq K \cup_{g \circ h \circ \varphi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{g \circ h \circ \varphi_{j_0 2}} \cdots \cup_{g \circ h \circ \varphi_{j j k_{k_0}}} e^{\lambda_{j_0 k_j}}$$

By cellular approximation, for each $r, 1 \leq r \leq k_{j_0}$, the map $g \circ h \circ \varphi_{j_0 r}$ is homotopic to a cellular map $\psi_{j_0 r} : \partial \left(e^{\lambda_{j_0 r}}\right) \rightarrow K^{\left(\lambda_{j_0 r}-1\right)}$, where $K^{\left(\lambda_{j_0 r}-1\right)}$ is the $(\lambda_{j_0 r} - 1)$-skeleton of K. Applying lemma 2.4.1 shows that

$$K \cup_{g \circ h \circ \varphi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{g \circ h \circ \varphi_{j_0 2}} \cdots \cup_{g \circ h \circ \varphi_{j_0 k_{j_0}}} e^{\lambda_{j_0 k_{j_0}}} \simeq K \cup_{\psi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{\psi_{j_0 2}} \cdots \cup_{\psi_{j_0 k_{j_0}}} e^{\lambda_{j_0 k_{j_0}}} .$$

Hence $K \cup_{\psi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{\psi_{j_0 2}} \cdots \cup_{\psi_{j_0 k_{j_0}}} e^{\lambda_{j_0 k_{j_0}}}$ is a CW-complex since the attaching maps are cellular. Therefore, we conclude that $M^{c+\epsilon}$ has the homotopy type of a CW-complex.

By induction, if $\tilde{c}$ is the smallest critical value of c_j 's such that $c_j > c$, then $M^{\tilde{a}}$ has the homotopy type of a CW-complex for every $\tilde{a} \in (c, \widetilde{c})$.

Finally, since M is compact, the Morse function f has a finite number of critical points (see Corollary 2.2.1) and a finite number of critical values. Thus the inductive step above completes the proof for all of M. $\square$

4.5 Referrences:

I used Internet for figures and I read the following books:

i. Kirby - The topology of 4-manifolds

ii. Alexandru Scorpan - The Wild World of 4-Manifolds -American Mathematical Society (2005)

iii. Morse Theory Wiki-Pedia.