Manifold Theory

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Definition 4.1.1: *Let f be a smooth function on a smooth manifold M. f is said to be a* **Morse function** *if every critical point of f is non-degenerate.*

Example 4.1.1: *The* **height functions** *on the sphere S^2 and the torus $\mathbb{T}^2$ are morse functions.*

Morse Lemma:

Morse Lemma: Lemma 4.2.1:(Morse lemma) Let p be non degenerate critical points of f with index λ. Then there is a local coordinate system $Y : V \subset \mathbb{R}^n \to U_p$ in a neighborhood U_p of p with $0 \in V$ and $Y(0) = p$ such that the identity

$$(f \circ Y)(y_1, y_2, ..., y_n) = f(p) - y_1^2 - .... - y_\lambda^2 + y_{\lambda+1}^2 + ... + y_n^2 ...(4.2.1)$$

holds through out V.

Corollary

Corollary 4.2.1: *Let $f : M \to \mathbb{R}$ be a smooth function on a smooth manifold M. A non-degenerate critical point of a smooth function f is isolated. In particular, f is a Morse function and M is compact, then f has a finite number of critical points.*

proof: By Lemma 4.2.1, we observed that if f has a non-degenerate critical point p, then there is a coordinate chart (U_p, Y^-1) of M about p that satisfies the equation (4.2.1). This chart contains no, other critical points of f other than p since, by equation (4.2.1),

$$d(f \circ Y)_{(y_1, y_2, \ldots.., y_n)} = (-2y_1, \ldots, -2y_\lambda, 2y_{\lambda+1}, \ldots., 2y_n)$$

and $d(f \circ Y)_{(y_1, y_2, \ldots.., y_n)} = 0$ iff $(y_1, y_2, \ldots.., y_n) = 0$. Hence $Y(0) = p$ is the only critical point of f in U_p. Therefore, p is isolated. Now, suppose that M is compact. If the set of all critical points of f is infinite then we can extract a non-constant sequence out of it and it must have a convergent subsequence as compact implies sequentially compact. Then by continuity of

df (as f is C^∞ hence df is contiuous) we get that limit point is also a critical point of f but then critical points of f can't be isolated as per the above scenario, $(\Rightarrow\Leftarrow)$.

Existence Of Morse Function

The goal is to show the existence of Morse functions on any smooth manifold. Since the **Whitney embedding Theorem** tells us that any smooth manifold is embedded in a suitable Euclidean vector space, let M be a smooth $n-$dimensional manifold embedded in $E = \mathbb{R}^{n+k}$ for some $k \in \mathbb{N}$.Hence we will try to prove that any morse function can be viewed as a height function for a compact finite dimensional manifold. And Hence after the existence we can study easily morse functions easily. This actually can be proved that on a compact finite dimensional manifold every morse function can be changed to height function by change of cordinate suitably such that the given manifold can be alligned suitably in upside down position and also height function is a morse function. Hence studying height function is enough on a compact finite dimensional manifold to understand morse theory.

Fundamental Theorem of Morse Theory

In this section, we let $f : M \to \mathbb{R}$ be a real valued function on a smooth manifold M, and let

$$M^a = f^{-1}((-\infty, a]) = \{p \in M : f(p) \leq a\}.$$

1st Fundametal Theorm We first consider the region that f has no critical points as follows:

Theorem 4.4.1: *Let $f : M \to \mathbb{R}$ be a smooth real valued function on a manifold M. Let a and b be regular values of f with $a \leq b$ such that the set*

$$f^{-1}([a, b]) = \{p \in M : a \leq f(p) \leq b\}$$

is compact and contains no critical points of f and M^a is compact subset of M $\forall a \in M$ (or If M is apriori compact). Then M^a is diffeomorphic to M^b. Furthermore, M^a is a deformation retract of M^b, so that the inclusion map $M^a \hookrightarrow M^b$ is a homotopy equivalence.

MFFT

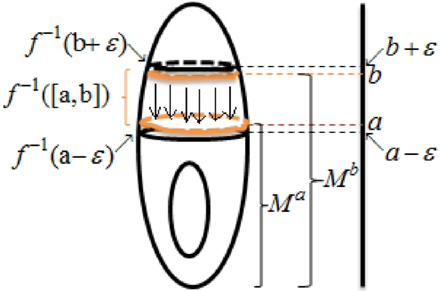


Figure: MFFT

MSFT

Now let us consider a region in which f has one critical point.

Theorem 4.4.2: *Let p be a non degenerate critical point of f with index λ. Let $c = f(p)$ and assume $f^{-1}([c-\epsilon, c+\epsilon])$ is compact and contains no other critical point of f for some $\epsilon > 0$. Then for all sufficiently small ϵ, the set $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with a λ-cell attached.*

Figure

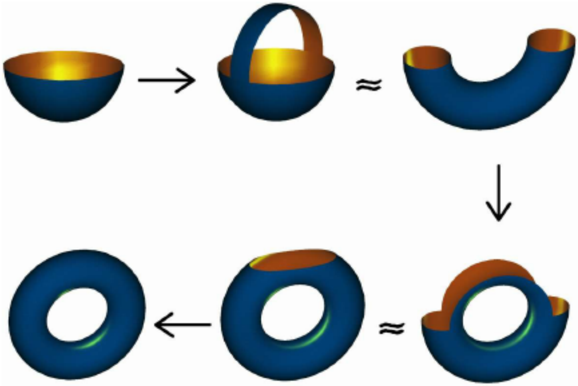


Figure: Attaching cells

Generalisation

Proposition 2.4.1: *(**Generalization of Theorem 2.4.2**) Suppose that $p_1, \cdots, p_k$ are k non-degenerate critical points with indices $\lambda_1, \cdots, \lambda_k$ in $f^{-1}(c)$. Then, $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon} \cup e^{\lambda_1} \cup \cdots \cup e^{\lambda_k}$.*

Example 2.4.1: *In the case $k = 2$, see Figures 2.2 and 2.3.*

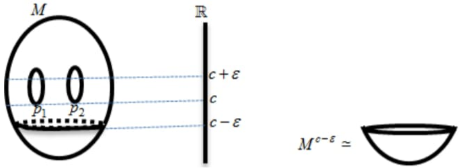


Figure 2.2: p_1 and p_2 are non-degenerate critical points with indices $\lambda_1 = \lambda_2 = 1$ in $f^{-1}(c)$.

Figure 2.3: $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon} \cup e^1 \cup e^1$.

Theorem 4.4.3: If $f : M \to \mathbb{R}$ is a smooth function on a compact smooth manifold M with no degenerate critical points and if each M^a is compact, then M has the homotopy equivalence of a CW-complex, with one cell of dimension λ for each critical point of index λ.
To prove this theorem, we will need the following lemmas.
Lemma 4.4.1: (Whitehead) Let ϕ_0 and ϕ_1 be homotopic maps from the sphere $\partial(e^\lambda)$ to a topological space X. then the identity map of X extends to a homotopy equivalence

$$k : X \sqcup_{\phi_0} e^\lambda \to X \sqcup_{\phi_1} e^\lambda$$

Whitehead fig.

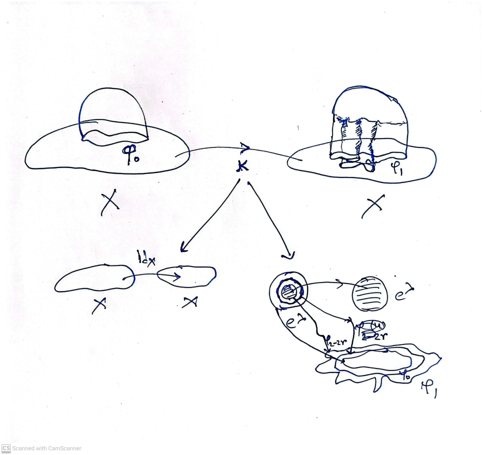


Figure: Whitehead lemma

proof:

Let ϕ_t be a homotopy between ϕ_0 and ϕ_1.
Define $k : X \sqcup_{\phi_0} e^\lambda \to X \sqcup_{\phi_1} e^\lambda$ by

$$k(x) = \begin{cases} x & \text{if } x \in X \\ 2ru & \text{if } x = ru,\ u \in \partial(e^\lambda),\ 0 \leq r \leq \frac{1}{2} \\ \phi_{2-2r}(u) & \text{if } x = ru,\ u \in \partial(e^\lambda)\ , \frac{1}{2} \leq r \leq 1 \end{cases}$$

and $\tilde{k} : X \sqcup_{\phi_1} e^\lambda \to X \sqcup_{\phi_0} e^\lambda$ by

$$\tilde{k}(x) = \begin{cases} x & \text{if } x \in X \\ 2su & \text{if } x = su,\ u \in \partial(e^\lambda)\ , 0 \leq s \leq \frac{1}{2} \\ \phi_{2s-1}(u) & \text{if } x = su,\ u \in \partial(e^\lambda)\ , \frac{1}{2} \leq s \leq 1 \end{cases}$$

Since the functions k and $\tilde{k}$ are continuous, there are the compositions

Proof cont.

$$\tilde{k} \circ k(x) = \begin{cases} x & \text{if } x \in X \\ 4ru & \text{if } x = ru,\ \ u \in \partial(e^{\lambda})\ , 0 \leq r \leq \frac{1}{4} \\ \phi_{4r-1}(u) & \text{if } x = ru,\ \ u \in \partial(e^{\lambda})\ , \frac{1}{4} \leq r \leq \frac{1}{2} \\ \phi_{2-2r}(u) & \text{if } x = ru,\ \ u \in \partial(e^{\lambda})\ , \frac{1}{2} \leq r \leq 1 \end{cases}$$

and

$$k \circ \tilde{k}(x) = \begin{cases} x & \text{if } x \in X \\ 4su & \text{if } x = su,\ \ u \in \partial(e^{\lambda})\ , 0 \leq s \leq \frac{1}{4} \\ \phi_{2-4s}(u) & \text{if } x = su,\ \ u \in \partial(e^{\lambda})\ , \frac{1}{4} \leq s \leq \frac{1}{2} \\ \phi_{2s-1}(u) & \text{if } x = su,\ \ u \in \partial(e^{\lambda})\ , \frac{1}{2} \leq s \leq 1 \end{cases}$$

proof cont.

We want to find a homotopy $h_t : X \sqcup_{\phi_0} e^{\lambda} \to X \sqcup_{\phi_0} e^{\lambda}$, $t \in [0,1]$ such that $h_0 = \tilde{k} \circ k$ and $h_1 = id$. Consider family of maps $h_t : X \sqcup_{\phi_0} e^{\lambda} \to X \sqcup_{\phi_0} e^{\lambda}$ defined by

$$h_t(x) = \begin{cases} x & \text{if } x \in X \\ \frac{4ru}{1+3t} & \text{if } x = ru,\ u \in \partial(e^{\lambda})\ , 0 \leq r \leq \frac{1+3t}{4} \\ \phi_{(\frac{4r}{1+3t}-1)(1-t)}(u) & \text{if } x = ru,\ u \in \partial(e^{\lambda})\ , \frac{1+3t}{4} \leq r \leq \frac{t+1}{2} \\ \phi_{\frac{2-2r}{1+3t}(1-t)}(u) & \text{if } x = ru,\ u \in \partial(e^{\lambda})\ , \frac{t+1}{2} \leq r \leq 1. \end{cases}$$

It is easy to check that h_t is contiuous, $h_0 = \tilde{k} \circ k$ and $h_1 = id$.
We next consider a family of maps $h'_t : X \sqcup_{\phi_1} e^{\lambda} \to X \sqcup_{\phi_1} e^{\lambda}$ defined by

proof cont.

$$h'_t(x) = \begin{cases} x & \text{if } x \in X \\ \frac{4su}{1+3t} & \text{if } x = su,\ u \in \partial(e^\lambda)\ , 0 \le s \le \frac{1+3t}{4} \\ \phi_{1-(\frac{4r}{1+3t}-1)(1-t)}(u) & \text{if } x = su,\ u \in \partial(e^\lambda)\ , \frac{1+3t}{4} \le s \le \frac{t+1}{2} \\ \phi_{1-\frac{2-2r}{1+3t}(1-t)}(u) & \text{if } x = su,\ u \in \partial(e^\lambda)\ , \frac{t+1}{2} \le s \le 1. \end{cases}$$

Again h'_t is continuous and satisfies $h'_0 = k\circ$, $h'_1 = id$.

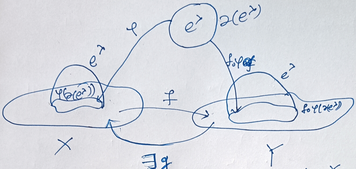
Lemma 4.4.2 : Let $\phi : \partial(e^\lambda) \to X$ be a attaching map. A homotopy equivalence $f : X \to Y$ can be extended to a homotopy equivalence

$$F : X \sqcup_\phi e^\lambda \to Y \sqcup_{f\circ\phi} e^\lambda.$$

Since $f : X \to Y$ is a a homotopy equivalence, there exists a homotopy inverse $g : Y \to X$ to f and $h_t : X \to X$ a homotopy such that $h_0 = g \circ f$ and $h_1 = id_X$. Let $H : [0,1] \times \partial(e^\lambda) \to X$ defined by $H(t,x) = h_t(\phi(x))$. Then we have $H(0,x) = g \circ\ f \circ \phi(x)$ and $H(1,x) = \phi(x)$. Thus $g \circ\ f \circ \phi$ and ϕ are homotopic maps from $\partial(e^\lambda)$ to X. By the Lemma 4.4.1, there exists a homotopy equivalence

$$k : X \sqcup_{g\circ f\circ\phi} e^\lambda \to X \sqcup_\phi e^\lambda.$$

Figure



Given, $X \simeq Y$

$f \circ g \simeq id_Y$

$g \circ f \simeq id_X$

We are showing: $X \sqcup_{\varphi} e^\lambda \simeq Y \sqcup_{f\circ\varphi} e^\lambda$

$\varphi : \partial(e^\lambda) \longrightarrow X$
$x \longmapsto \varphi(x)$

$X \sqcup_{x \sim \varphi(x)} e^\lambda$

identification space (quotient top on $X \sqcup_{\varphi} e^\lambda$)

We show $g \circ f \circ \varphi \simeq \varphi$ [Why? as $g \circ f \simeq id_X$ $\Rightarrow g\circ f\circ\varphi \simeq id_X \circ \varphi$ $\Rightarrow g\circ f\circ\varphi \simeq \varphi$]

by Lemma 4.7.1 $\Rightarrow X \sqcup_{g\circ f\circ\varphi} e^\lambda \simeq X \sqcup_{\varphi} e^\lambda$

proof:

Define the following two maps $F : X \uplus e^{\lambda} \to Y \uplus e^{\lambda}$ and $G : Y \uplus e^{\lambda} \to X \uplus e^{\lambda}$ as follows

$$F(x) = \begin{cases} f(x) & \text{if } x \in X \\ x & \text{if } x \in e^{\lambda} \end{cases}$$

and

$$G(y) = \begin{cases} g(y) & \text{if } y \in Y \\ y & \text{if } y \in e^{\lambda} \end{cases}$$

Then clearly F and G are continuous functions under the disjoint union topology. Now, we will use universal property of qoutient map to get $\tilde{F}$ and $\tilde{G}$ such that the following diagram commutes:

figure

$$
\begin{tikzcd}
X \sqcup_{\varphi} e^{\lambda} \arrow[r, "\tilde{F}"] & Y \sqcup_{f\circ\varphi} e^{\lambda} \\
X \cup e^{\lambda} \arrow[u, "\pi_{\varphi}"] \arrow[r, "F"'] \arrow[ur, "\pi_{f\circ\varphi}\circ F"'] & Y \cup e^{\lambda} \arrow[u, "\pi_{f\circ\varphi}"']
\end{tikzcd}
$$

$$\boxed{\tilde{F}\circ\pi_{\varphi} = \pi_{f\circ\varphi}\circ F}$$

$$
\begin{tikzcd}
X \sqcup_{g\circ f\circ\varphi} e^{\lambda} & Y \sqcup_{f\circ\varphi} e^{\lambda} \arrow[l, "\tilde{G}"'] \\
X \cup e^{\lambda} \arrow[u, "\pi_{g\circ f\circ\varphi}"] & Y \cup e^{\lambda} \arrow[l, "G"] \arrow[u, "\pi_{f\circ\varphi}"'] \arrow[ul, "\pi_{g\circ f\circ\varphi}\circ G"']
\end{tikzcd}
$$

$$\boxed{\tilde{G}\circ\pi_{f\circ\varphi} = \pi_{g\circ f\circ\varphi}\circ G}$$

proof cont.

Now, we will prove that $\tilde{F}$ has a left homotopy inverse viz. $k \circ \tilde{G}$. That is, the composition $k \circ \tilde{G} \circ \tilde{F} : X \sqcup_\phi e^\lambda \to X \sqcup_\phi e^\lambda$ is homotopic to the indentity map. Frorm definiton of k got from lemma of Whitehead we get this.

Similarly $\tilde{G}$ has a left homotopy inverse.

Claim: If a map F has a left and a right homotopy inverse L and R respectively, then F is a homotopy equivalence, and L (or R) is a 2-sided homotopy inverse.

proof

Proof. Since L and R are left and right homotopy inverses to F, we have the relations $LF \simeq \text{id}$ and $FR \simeq \text{id}$. This implies that

$$L \simeq L(FR) = (LF)R \simeq R$$

Hence

$$FL \simeq FR \simeq \text{ id (or } RF \simeq LF \simeq \text{ id)}$$

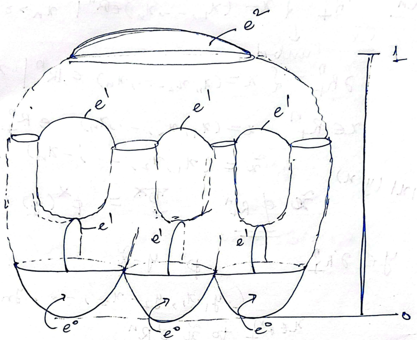
which proves that $L($ or R) is a 2 -sided homotopy inverse. To prove the Lemma 4.4.2, it only remains to prove that F has a right homotopy inverse. By the Claim , we obtain the following: - $k \circ (\tilde{G} \circ \tilde{F}) \simeq \text{id}$ implies that $(\tilde{G} \circ \tilde{F}) \circ k \simeq \text{id}$ since k is known to have a left homotopy inverse (by Lemma 2.4.1). - $G \circ (\tilde{F} \circ k) = (\tilde{G} \circ \tilde{F}) \circ k \simeq \text{id}$ implies that $(\tilde{F} \circ k) \circ \tilde{G} \simeq \text{id}$ since $\tilde{G}$ is known to have a left homotopy inverse. - $\tilde{F} \circ (k \circ \tilde{G}) = (\tilde{F} \circ k) \circ \tilde{G} \simeq \text{id}$ implies that $\tilde{F}$ has $k \circ \tilde{G}$ as a right homotopy inverse. Therefore, F is a homotopy equivalence. This completes the proof of Lemma of Hilton.

Proof of theorem 4.4.3

Proof. (of Theorem 4.4.3) Let $a \in \mathbb{R}$ and p_{ik_i} be critical points belonging to $f^{-1}(c_i)$ with index λ_{ik_i}. If $f^{-1}(a) = \emptyset$, then $M^a = \emptyset$ and so we have nothing to do. If $f^{-1}(a) \neq \emptyset$, then $M^a \neq \emptyset$. Base case: We may assume that $c_1 < a < c_2$. Since M^a is compact, f has a global minimum value $c_1 \in \mathbb{R}$ (i.e, $c_1 \leq f(p)$ for all $p \in M$). $M^{c_1+\epsilon}$ is homotopy equivalent to M^a for some small $\epsilon > 0$. Since the critical points belonging to $f^{-1}(c_1)$ have index 0 , $M^{c_1+\epsilon}$ has the homotopy type of a disjoint union of 0 cells. Therefore, M^a has the homotopy type of a CW-complex.

Induction hypothesis: Suppose that $a \neq c_1, c_2, c_3, \cdots$ such that M^a is homotopy equivalent to a CW-complex K via g. Let $c = c_{j_0}$ be the smallest critical value of f bigger than a. for some small $\epsilon > 0$ we have that $M^{c-\epsilon}$ is homotopy equivalent to M^a via h and that $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon} \cup_{\varphi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{\varphi_{j_0 2}} \cdots \cup_{\varphi_{j_0 k_{j_0}}} e^{\lambda_{j_0 k_{j0}}}$ for some attaching maps $\varphi_{j_0 1}, \cdots, \varphi_{j_0 k_{j_0}}$. Then, by Lemma we see that

Figure



basically $e^0 \simeq e^2$. Hence although it's looking like e^2 it's actually homotopic to e^0.

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proof cont.

$$M^{c-\epsilon} \cup_{\varphi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{\varphi_{j_0 2}} \cdots \cup_{\varphi_{j_0 k_{j0}}} e^{\lambda_{j_0 k_{j_0}}} \simeq M^a \cup_{h0\varphi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{h\circ\varphi_{j_0 2}} \cdots \cup_{h\circ\varphi_{j_0 k_{j0}}} e^{\lambda_{j_0 k_{j_0}}}.$$

Since M^a is homotopy equivalent to K via g, Lemma shows that $M^a \cup_{h\circ\varphi_{j_0 1}} e^{\lambda_{j0 1}} \cup_{h\circ\varphi_{j_0 2}} \cdots \cup_{h\circ\varphi_{j_0 k_{j_0}}} e^{\lambda_{j_0 k_j}} \simeq$
$K \cup_{g\circ h\circ\varphi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{g\circ h\circ\varphi_{j_0 2}} \cdots \cup_{g\circ h\circ\varphi_{jj^k k_0}} e^{\lambda_{j_0 k_j}}$ By cellular approximation, for each $r, 1 \leq r \leq k_{j_0}$, the map $g \circ h \circ \varphi_{j_0 r}$ is homotopic to a cellular map $\psi_{j_0 r} : \partial \left(e^{\lambda_{j_0 r}}\right) \to K^{\left(\lambda_{j_0 r}-1\right)}$, where $K^{\left(\lambda_{j_0 r}-1\right)}$ is the $(\lambda_{j_0 r} - 1)$-skeleton of K. Applying lemma shows that
$K \cup_{g\circ h\circ\varphi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{g\circ h\circ\varphi_{j_0 2}} \cdots \cup_{g\circ h\circ\varphi_{j_0 k_{j_0}}} e^{\lambda_{j_0 k_{j0}}} \simeq$
$K \cup_{\psi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{\psi_{j_0 2}} \cdots \cup_{\psi_{j_0 k_{j_0}}} e^{\lambda_{j_0 k_{j0}}}$.

proof cont.

Hence $K \cup_{\psi_{j_0 1}} e^{\lambda_{j_0 1}} \cup_{\psi_{j_0 2}} \cdots \cup_{\psi_{j_0 k_{j0}}} e^{\lambda_{j_0 k_{j0}}}$ is a CW-complex since the attaching maps are cellular. Therefore, we conclude that $M^{c+\epsilon}$ has the homotopy type of a CW-complex.
By induction, if $\tilde{c}$ is the smallest critical value of c_j 's such that $c_j > c$, then $M^{\tilde{a}}$ has the homotopy type of a CW-complex for every $\tilde{a} \in (c, \widetilde{c})$.
Finally, since M is compact, the Morse function f has a finite number of critical points (see Corollary 2.2.1) and a finite number of critical values. Thus the inductive step above completes the proof for all of M.

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Thanks for your kind attention