Finite Difference Methods for solving ODEs

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Abstract

Four different methods: the Explicit Euler method, the Leapfrog method, the fourth-order Runge-Kutta method (RK4) and the Implicit Euler method were used in order to approximate the solutions of coupled ordinary differential equations. In particular the dynamics of a simple and double pendulum, with damping, were studied in order to determine which method was the best for integrating the system of ODEs. The stability of all methods were taken into account and the RK4 method was determined to be most suitable. This method was then employed in order to study the damped double pendulum. Different parameters were changed in order to analyse the motion and results displayed.

1. Introduction

To be able to find an analytical solution to a differential equation is often a very rare occurrence, especially for complex physical systems of interest where the differential equations that govern the dynamics of the system can be greater than first order. In most cases, numerical methods must be used in order to obtain an approximate solution.

There are four main methods for integrating systems of ODEs: these are the Explicit and Implicit Euler methods, the Leapfrog method and the fourth order Runge-Kutta method (RK4). These work by computing the gradient at an initial point η and projecting it forward using a time step h, by working out a weighted average for the gradient or by using the original, the initial point is updated and the procedure is then repeated.

These algorithms are iterative; hence the efficiency of each method is determined by the computations in one iteration. By modelling the solution at discrete time steps one can observe the motion of a pendulum with respect to its position and velocity.

In the case of a single pendulum the initial conditions were position and velocity. The stability, a measure of how the local error increases or decreases with every step, of each method was determined by looking at the total energy of the system with and without damping. Using the conservation of energy it was trivial to determine when a method became unstable, as the potential and kinetic energy of the pendulum under a lack of damping must remain constant, and the graph of total energy would not.

By studying the stability, accuracy, efficiency and consistency of each method, it was found that RK4 was the most apt method to use when considering damping and step sizes. Subsequently this method was used in studying the dynamics of the double pendulum.

2. Theory

The pair of coupled ODEs which govern the motion of a simple pendulum with a mass *m* and a length *l* making an angle θ to the vertical can be written in matrix notation:

$$\frac{d}{d\tilde{t}}\begin{pmatrix}\theta\\ \omega\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & -\tilde{D}\end{pmatrix}\begin{pmatrix}\theta\\ \omega\end{pmatrix} \rightarrow \frac{d\vec{y}}{d\tilde{t}} = \boldsymbol{L}\vec{y}. \quad (\mathbf{1})$$

This equation has been rescaled in terms of time. In order to make it dimensionless, $\tilde{D}$ is a constant, which has a value of: $\frac{D}{\omega m l}$. *D* is the original damping constant, and $\omega = \sqrt{\frac{g}{l}}$.

The Explicit Euler method uses a tangent to an initial point to step the solution forward by adding a product of the step size *h* and the gradient to the initial point. The method is essentially a first order Taylor expansion for a forward difference scheme [2]:

$$\vec{y}_{n+1} = y_n + h\boldsymbol{L}\vec{y}_n \quad (\mathbf{2})$$

The above equation is written in terms of a matrix operator ***L*** which describes the differential equations. There is only one computation per step in the Explicit Euler method which makes it efficient, it has an accuracy of order *h*.

The Leapfrog method uses a centred difference scheme, which means it requires a previous point in the iteration, alongside the current point in order to step to the final point:

$$\vec{y}_{n+1} = \vec{y}_{n-1} + 2h\boldsymbol{L}\vec{y}_n \quad (\mathbf{3})$$

However, with this method a previous point needs to be stored, also an Euler step must be employed in order to start off the iteration, hence it is not a standalone method. However it is still accurate up to order *h*.

The RK4 method is a single step method using more than one term in the Taylor series expansion. It uses 4 stages of gradient estimation to calculate an average; therefore it has an accuracy of order h^4. Lastly, the Implicit Euler method is much like the Explicit Euler method in that it has a global accuracy of order *h*. However, unlike the explicit method - it is unconditionally stable as the stability analysis shows.

$$\vec{y}_{n+1} = y_n + h\boldsymbol{L}\vec{y}_{n+1} \quad (\mathbf{4})$$

To test the stability of a method, it must be shown that the local error due to truncation of the Taylor series either decreases or stays constant. This can be scrutinised in much more detail when focusing on the total energy of the simple pendulum. Since the total energy is constant (D = 0) or decreasing (D > 0) due to the conservation of energy, any unstable method will be identified by a total energy which either varies or converges to a specific value:

$$E = k(\theta^2 + \omega^2) \quad (\mathbf{5})$$

Equation **(5)** represents the total energy of the system, where k is a constant. This is a good way to test stability as any error in the angular displacement, which is the function being approximated to by these methods, features as a quadratic, therefore the error in E will increase with local error in the method. Eigenvalue analysis was carried out to make sure that the eigenvalues of the update matrix ***T*** were less than one. However, since two of the methods do not use an update matrix, hence, observing the total energy is then the only common way of determining stability.

The double pendulum which modifies Equation **(1)** by a 4x4 matrix replacing ***L*** and a 4-component vector replacing *y*, can be solved by using a finite difference method. RK4 was used in order to approximate a solution to study the system - this was found to be the best scheme, in terms of stability and accuracy.

3. Results

Below are standard results. The first graph in each method represents oscillation without damping (D = 0). The second graph shows damping (D = 0). The second graph shows oscillation with weak damping (D = 0.2). Energy is also plotted as a measure of stability, over a time period of 100 units **(*h = 0.001*)**.

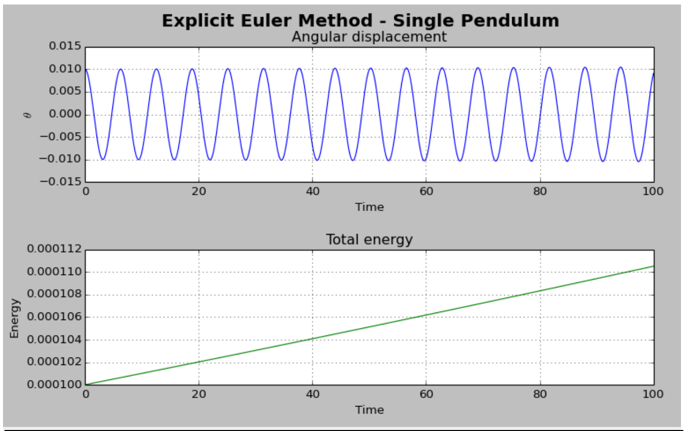


Fig 1. Explicit Euler method at D = 0, instability already present as energy is increasing substantially.

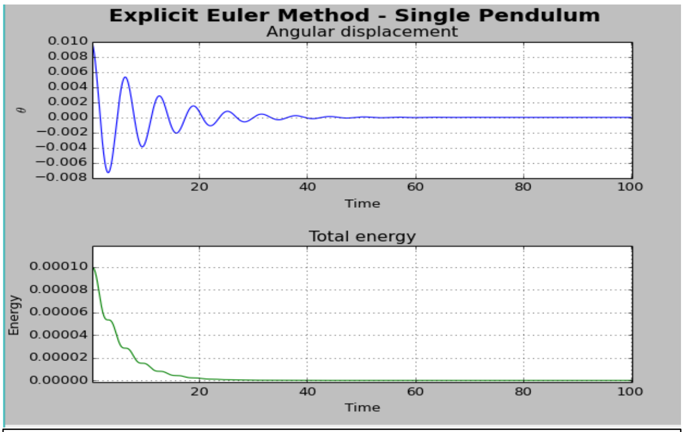


Fig 2. Explicit Euler method at D = 0.2, method looks to be stable when damping is present, physical result.

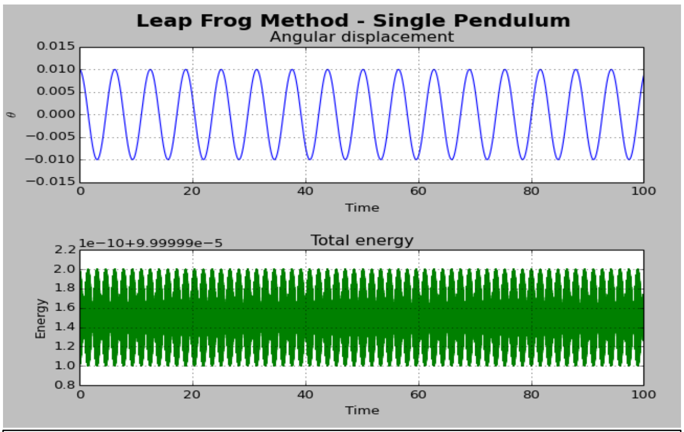


Fig 3. Leapfrog method at D = 0, energy has minute fluctuations, but time-averages out. Method is stable.

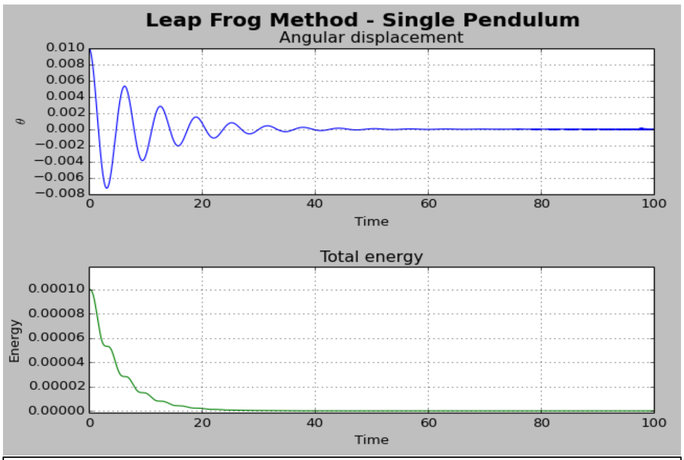


Fig 4. Leapfrog method at D = 0.2, method looks to be stable when damping is present, physical result.

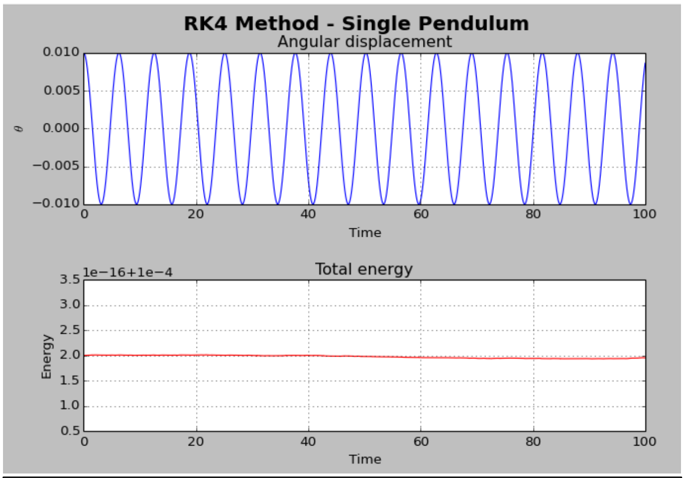


Fig 5. RK4 method at D = 0, method is very stable no minute fluctuations observed, seems to be the best.

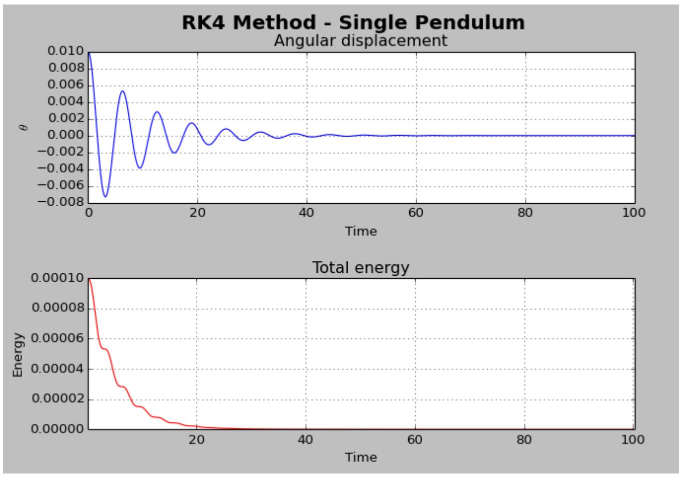


Fig 6. RK4 method at D = 0.2, method looks to be stable when damping is present, physical result.

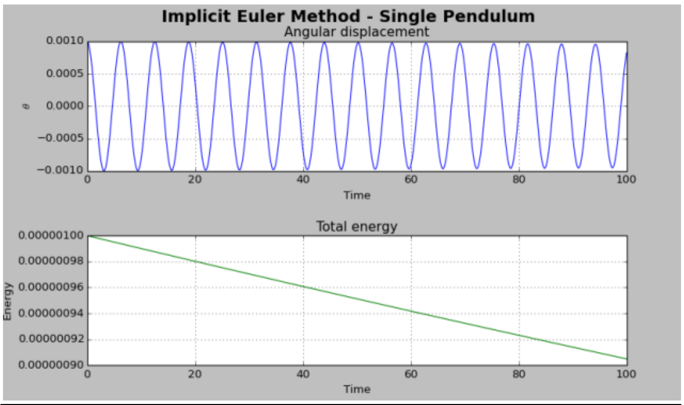


Fig 7. Implicit Euler method at D = 0, instability already present as energy is decreasing substantially.

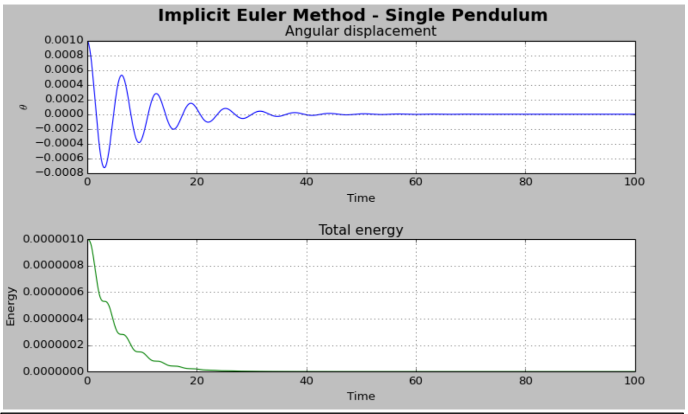


Fig 8. Implicit Euler method at D = 0.2, method looks to be stable when damping is present, physical result.

As seen from **Fig 1 and 7** the Euler methods are not stable when there is no damping. This is due to the fact that it estimates using only one calculation of the gradient; this means when trying to approximate a sinusoid it leads to a build-up of truncation errors with each step. This leads to a skewing in accuracy; the result of which is shown more clearly in the energy plots for each different method.

Although all methods seem to work well for damping, the Leapfrog method will become unstable when D is around 0.5 or above. The leapfrog method is best for when there is a continuous increase in the function or when the function is oscillating in a sinusoidal manner. Therefore the leapfrog method is not fit for use in simulating the damped single pendulum.

Solving the stability condition for the Explicit Euler method states that, when the step, *h*, is greater than the damping, *D*, then instability is encountered.

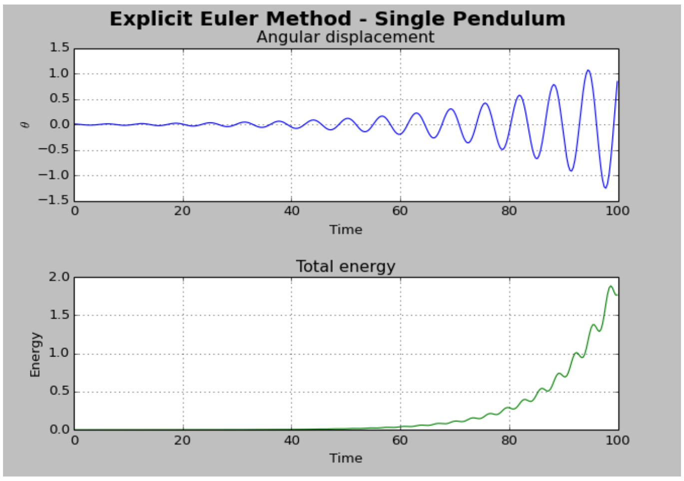


Fig 9. Explicit Euler method at D = 0.2, with $h = 0.3$, highly unstable due to the eigenvalues of the update matrix being greater than 1. Energy seems to be created from nothing thus violating the conservation of energy. Therefore confirming it is unstable; furthermore, the stability condition states $h \leq D$.

Whereas with the same conditions in **Fig 9** RK4 yields:

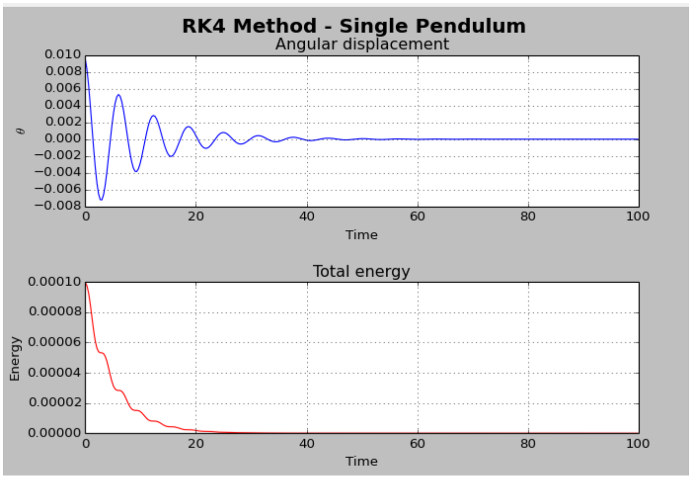


Fig 10. RK4 method at D = 0.2, with $h = 0.3$, method is still very stable.

By examining the above **Fig 5, 6, 10** and looking at the data, it is clear to see that RK4 is the best finite difference method. To confirm that it works with critical damping:

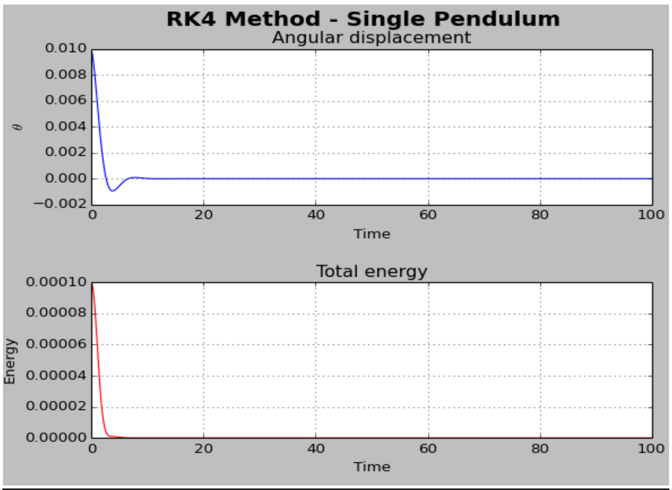


Fig 11. RK4 method when D = 1.2 yields, with *h=0.3*, method is still very stable. Physical result.

Using trial and error is easy to deduce the maximum step length at which RK4 becomes unstable in the un-damped case. This was found to be $2.81 < h < 2.83$.

More exhaustive tests would have to be done on RK4 to confirm that it is, indeed, the best method. However, out of all the methods analysed in this report, this seems to be best suited for the simulation of the damped double pendulum.

3.1 Double Pendulum

The matrix equation which represents the damped double pendulum is [1]:

$$\frac{d}{dt}\begin{bmatrix}\theta\\ \varphi\\ \omega\\ \nu\end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -(R+1) & R & -G & 0\\ (R+1) & -(R+1) & G(1-R^{-1}) & -G/R\end{bmatrix}\begin{bmatrix}\theta\\ \varphi\\ \omega\\ \nu\end{bmatrix}$$

where $R = M/m$ and $G = D/(m\sqrt{gl})$ and time is in units of $\sqrt{l/g}$.

Fig 12.

This can be re-cast as: $\frac{d\vec{y}}{d\tilde{t}} = \boldsymbol{L}\vec{y}$, where $\boldsymbol{L}$ becomes the 4x4 matrix and $\vec{y}$ becomes the vector in **Fig 12**.

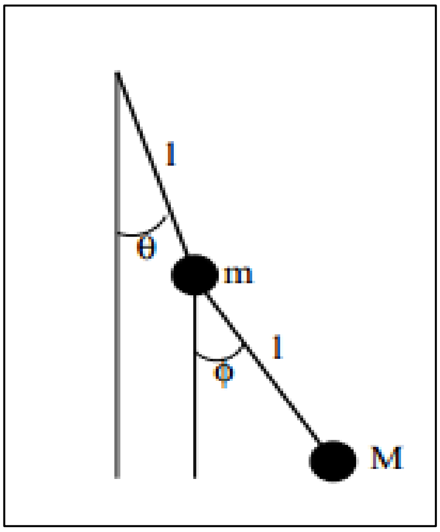


Fig. 13 – Setup of the double pendulum

The matrix $\boldsymbol{L}$ contains important elements such as R and G. Where G is related to the damping of the system and R is the ratio of the two masses.

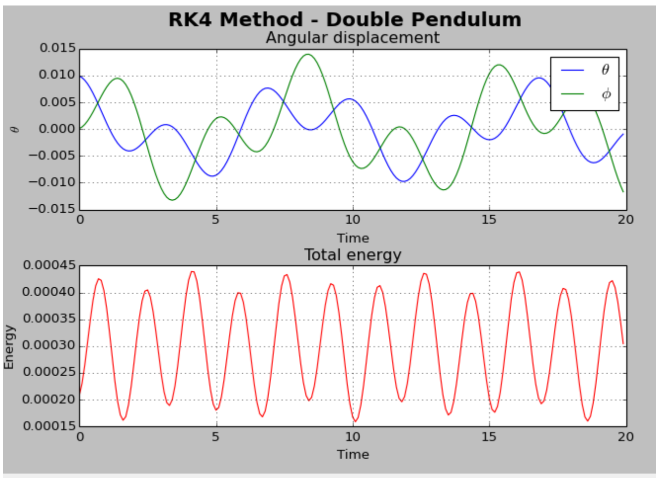
In order to test stability, the formula for the total energy of a double pendulum is used:

$$E = k(\theta^2 + \omega^2 + R(\theta^2 + \varphi^2) + R(\omega + \nu)^2) \quad \textbf{(6)}$$

Here k is the same constant as seen in equation **(5)** for the energy of a single pendulum.

With θ = 0.1, the only non-zero parameter in the vector, and R = 1.0 with G = 0, the result is obtained as seen below.

Fig 13 –RK4 used to simulate pendulum.



In the case where θ = 0.1, R = 10.0 with G = 1.0 the output is:

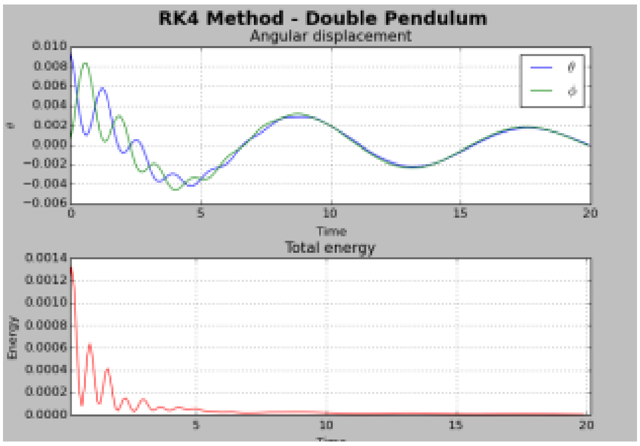
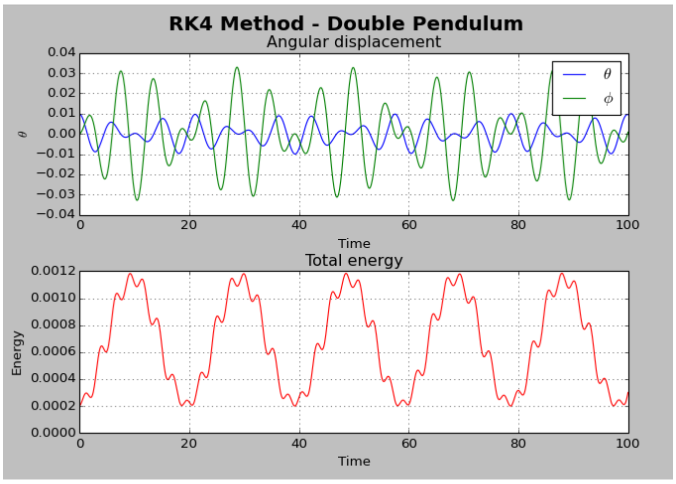


Fig 14 – Double pendulum with damping.

The reason that the total energy decreases and rises rapidly in **Fig 14** is due to the fact that when the pendulum is initially released, the smaller mass is driven roughly in anti-phase with the bigger mass. This means that whenever a small phase difference occurs, the kinetic energy of the bigger mass, which dictates the total energy, will dominate. After a while the masses oscillate in phase, which is the same as considering the double pendulum as a single pendulum. The time step has to be altered as the value of R is changed.

As R is varied, and if it is greater than 1, the heavier mass dominates and a more jagged total energy curve is formed, due to the smaller mass being driven suddenly. When R is less than 1, the smaller mass takes a while to get the bigger mass going; hence one observes a steady increase in the energy, then a decrease.

Fig 15 – Undamped Pendulum R = 0.1



4. Conclusion

Four different methods: the Explicit Euler method, the Leapfrog method, the fourth-order Runge-Kutta method (RK4) and the Implicit Euler method were used in order to approximate the solutions of coupled ordinary differential equations.

The dynamics of a simple and double pendulum, with damping, were studied in order to determine which method was the best for integrating the system of ODEs. The stability of all methods were taken into account and the RK4 method was determined to be most suitable.

This method was then employed in order to study the damped double pendulum. Different parameters were changed in order to observe the motion and the results were analysed.

References

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