Derivatives of Common Functions

The Power Function

The power function has the form

$$f(x) = x^n$$

According to the definition of derivatives, we have

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

Use the Binomial Theorem, we have:

$$(x+h)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}h^k$$

Because the first term in the above summzation is x^n, so we have:

$$\frac{(x+h)^n - x^n}{h} = \frac{\sum_{k=1}^{n}\binom{n}{k}x^{n-k}h^k}{h}$$

Please be aware of the change of the range of k from 0 to 1 in the summation.

Then, let us cancel k, the denominator, and get:

$$\frac{\sum_{k=1}^{n}\binom{n}{k}x^{n-k}h^k}{h} = \sum_{k=1}^{n}\binom{n}{k}x^{n-k}h^{k-1}$$

Please be aware of the change of the superscription of h from k to $k-1$.

This change inspires us to substitue $k-1$ by k', which ranges from 0 to $n-1$:

$$\sum_{k=1}^{n}\binom{n}{k}x^{n-k}h^{k-1} = \sum_{k'=0}^{n-1}\binom{n}{k'}x^{n-k}h^{k'}$$

When $h \to 0$, most terms in above summation becomes 0, except for the first one

$$\binom{n}{1}x^{n-1}h^0 = nx^{n-1}$$

So we have

$$\frac{d}{dx}x^n = nx^{n-1}$$

The Exponential Function

The exponential function has the form

$$f(x) = a^x$$

I didn't derive its derivative from the definition; instead, I used a property that the derivative of the *natural exponential function* is itself, or

$$\frac{d}{dx}e^x = e^x$$

We will derive this property later. For now, we just use it to derive the derivative of the general exponential function

$$\frac{d}{dx}a^x = \frac{d}{dx}\left(e^{\ln(a)}\right)^x = \frac{d}{dx}e^{x\ln(a)}$$

Here we need to use the chain rule by assuming that $g(x) = x\ln(a)$, so we have

$$\frac{d}{dx}e^{x\ln(a)} = \frac{d}{dg}e^g\frac{d}{dx}x\ln(a) = e^{x\ln(a)}\ln(a) = a^x\ln(a)$$

The Natural Exponential Function

By the definition of derivatives, we have

$$\frac{d}{dx}e^x = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}e^x\frac{(e^h-1)}{h}$$

By the definition of e, or the Eular's number, we have

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Please be aware that $n\to\infty$ implies that $1/n\to 0$. By introducing $h = 1/n$, we get another form of the definition of e:

$$e = \lim_{h\to 0}(1+h)^{\frac{1}{h}}$$

Substitue this definition into the above derivative

$$\frac{d}{dx}e^x = \lim_{h\to 0} e^x \frac{((1+h)^{\frac{1}{h}})^h - 1}{h} = \lim_{h\to 0} e^x$$

The Logarithm Function

The natural logarithm function has the form

$$f(x) = \ln(x)$$

Given that

$$e^{\ln(x)} = x$$

The derivatives of both sides should be the same

$$\frac{d}{dx}e^{\ln(x)} = \frac{d}{dx}x = 1$$

Using the chain rule for the left side

$$e^{\ln(x)}\frac{d}{dx}\ln(x) = 1$$

So

$$\frac{d}{dx}\ln(x) = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

We can extends to logarithm with arbitrary base

$$f(x) = \log_a(x)$$

1. Using the quotient rule:

$$\frac{d}{dx}\log_a(x) = \frac{d}{dx}\frac{\ln(x)}{\ln(a)} = \frac{\ln'(x)\ln(a) - \ln(x)\ln'(a)}{\ln^2(a)}$$

Because $\ln'(a) = 0$, so

$$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$$

1. Using the equality trick

$$\frac{d}{dx}a^{\log_a(x)} = \frac{d}{dx}x = 1$$

and by the chain rule

$$\frac{d}{dx}a^{\log_a(x)} = a^{\log_a(x)}\ln(a)\frac{d}{dx}\log_a(x) = x\ln(a)\frac{d}{dx}\log_a(x) = 1$$

so we have

$$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$$