\section{变分定义} 函数$y(x)$的变分：在同一定义域上的两个函数$y(x), m(x)$,若彼此任意接近，那么$m(x)$与$y(x)$之差，记$\delta y(x) = m(x) - y(x)$称$y(x)$的变分。

变分δy反映了整个函数的变化，而函数增量Δy反应的是同一个函数由于自变量的取值不同引起的变化。

泛函的变分：对任意一个泛函$J[y]$,函数y引起的泛函增量为
$\Delta J = J[y + \delta y] - J[y] = L[y, \delta y] + \dfrac{1}{2!}Q[y, \delta y] + o(||\delta y||^2)$

其中$L(y, \delta y)$是关于δy的线性泛函，$Q[y, \delta y]$是关于δy的两次线性泛函。

则泛函的一阶变分$\delta J = L[y, \delta y]$,二阶变分$\delta^2 J = Q[y, \delta y]$

变分的另一种求法，任给一个齐次函数$\eta(x)$,边界值为零，对任意小的一个实数ε,$y^*(x) = y(x) + \varepsilon\eta(x)$当$\varepsilon = 0$

$$\delta J = \varepsilon \frac{dJ[y + \varepsilon\eta]}{d\varepsilon}$$

$$\delta^2 J = \tfrac{1}{2!}\varepsilon^2 \frac{d^2 J[y + \varepsilon\eta]}{d\varepsilon^2}$$

\section{泛函例子} 例：已知泛函$J[y] = \int_a^b F(x, y, y') \, \mathrm{d}x$

$$\delta J[y] = \int_a^b [\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'] \, \mathrm{d}x$$

$$= \frac{\partial F}{\partial y'}(\delta y) + \int_a^b [\frac{\partial F}{\partial y} + \frac{d}{dx}(\frac{\partial F}{\partial y'})]\delta y \, \mathrm{d}x$$

例：最短线问题，设经过A, B两点距离最短曲线方程为$y^*(x)$,另有任意连续可导函数$\eta = \eta(x)$,且$\eta(x)$满足$\eta(x_0) = \eta(x_1) = 0$,则$y = y^* + \alpha\eta(x)$仍是过$A, B$的连续曲线，其对应长度为
$L(a) = \int_{x_0}^{x_1} \sqrt{1 + (y^*)' + \alpha\eta')^2} \, \mathrm{d}x$

当$a = 0$，$y = y^*(x)$时$L(a)$取极小值，即$\dfrac{dL(a)}{da} = 0$在$a = 0$的地方

$$\frac{dL(a)}{da} = \int_{x_0}^{x_1} \frac{((y^*)' + \alpha\eta')\eta'}{\sqrt{1 + ((y^*)')^2}} \, \mathrm{d}x$$

$$= \int_{x_0}^{x_1} \frac{(y^*)'\eta'}{\sqrt{1 + (y^*)^2}} \, \mathrm{d}x$$

$$= \frac{(y^*)'\eta}{\sqrt{1 + (y^*)^2}} - \int_{x_0}^{x_1} (\frac{(y^*)'}{\sqrt{1 + ((y^*)')^2}})'\eta \, \mathrm{d}x$$

$$= -\int_{x_0}^{x_1} (\frac{(y^*)''}{\sqrt{1 + (y^*)^2}} - \frac{(y^*)'(y^*)'(y^*)''}{\sqrt{1 + (y^*)^3}}))\eta \, \mathrm{d}x$$

$$= -\int_{x_0}^{x_1} \frac{y''}{\sqrt{1 + ((y^*)^2)^3}}\eta \, \mathrm{d}x$$

由$\eta(x)$的任意性，由变分引理可得$\dfrac{y''}{\sqrt{1 + ((y^*)^2)^3}} = 0$

$$\delta J = \varepsilon \frac{dJ[y+\varepsilon\eta]}{d\varepsilon}$$

$$\delta^2 J = \frac{1}{2!}\varepsilon^2 \frac{d^2 J[y+\varepsilon\eta]}{d\varepsilon^2}$$

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