Information Fusion in Localization

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May 2023

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Motivation

Suppose we have two independent estimates of x as

$$x_1 = x + n_1 \text{ and } x_2 = x + n_2$$

- $n_1 \sim \mathcal{N}(0, \sigma_1^2)$ and $n_2 \sim \mathcal{N}(0, \sigma_2^2)$ are independent measurement noise
- $\sigma_1^2 > \sigma_2^2$, x_1 is a *better* estimate of x
- Can we get a better estimate of x by combining x_1 or x_2?
- Suppose $\sigma_1 = 0.1\,\mathrm{m}$, $\sigma_2 = 10\,\mathrm{m}$, how much better is the accuracy by combining x_1 and x_2? Does information fusion even make sense?

Example: Mobile Anchor-Assisted Underwater Localization

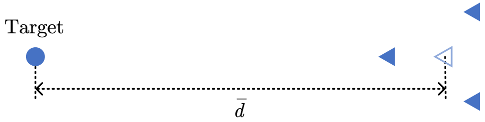


Fig. 1: Geometrical distribution of target and anchors.

How it works?

(a) the target is constantly broadcasting a beacon signal

(b) an AUV is receiving the beacon signal signal in real-time

(c) the target can be localized with enough measurements

Why mobile anchor?

(a) Better coverage (b) Improved Accuracy (c) Easy deployment

Doppler-Based Localization

Protocol

- Target located at $\mathbf{x}$, broadcasts at f_0 Hz
- The AUV is located at $\mathbf{x}_a$
- AUV moves at $\mathbf{v} = (v_x, v_y)$
- The radial velocity is

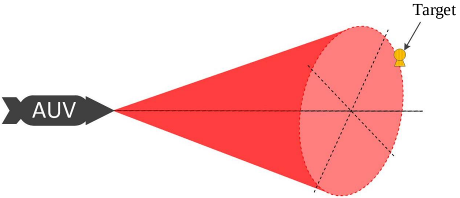
$$v_r = \frac{(\mathbf{x}_a - \mathbf{x})^T \mathbf{v}}{\|\mathbf{x}_a - \mathbf{x}\|}. \tag{1}$$

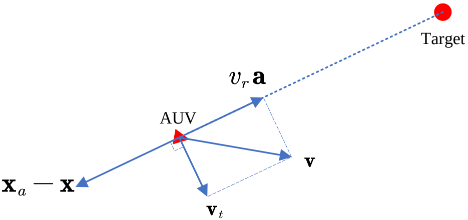
- The tangential velocity

$$\mathbf{v}_t = \mathbf{v} - v_r \mathbf{a} = (\mathbf{I} - \mathbf{a}\mathbf{a}^T)\mathbf{v}, \tag{2}$$

- The Doppler shift is measured as

$$f_D = -f_0 \frac{v_r}{c}. \tag{3}$$





Doppler-Based Localization-CRLB

- The gradient of f_D with respect to $\mathbf{x}$

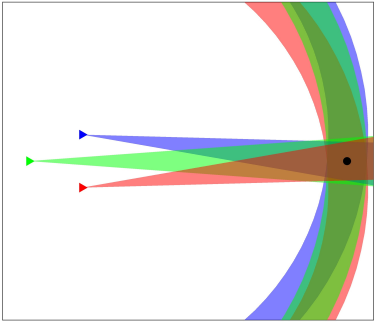
$$\nabla_{\mathbf{x}} f_D = \frac{f_0}{c}\frac{\mathbf{v}_t}{d} \tag{4}$$

- The AUV takes M measurements
- Consider i.i.d. Doppler estimation error
- The CRLB:

$$\mathbf{R}_{\mathbf{x}}^{Dop} = \sigma_f^2 \bar{d}^2 \lambda_0^2 \left(\sum_m \mathbf{v}_{t,m} \mathbf{v}_{t,m}^T \right)^{-1} \tag{5}$$

- $\mathbf{v}_{t,m}$ is the tangential velocity related to the m-th measurement

- Positioning error is proportional to distance
- In the far-field, the GDOP (Geometrical Dilution of Precision) will be poor



ToA-Based Localization

PDF of Propagation Delay

$$f(\hat{\boldsymbol{\tau}};\mathbf{x}) \propto \exp\left\{-\frac{1}{2\sigma_t^2}\sum_m (\hat{\tau}_m - \tau_m)^2\right\}$$

$\hat{\tau}_m$: measured propagation delay of the m-th period;

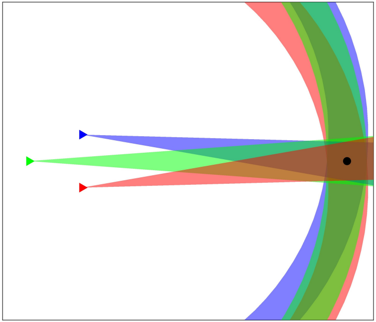
τ_m: true propagation delay

σ_t^2: timing error

The CRLB is given as:

$$\mathbf{R}_{\mathbf{x}}^{ToA} = c^2\sigma_t^2\left(\sum_m \mathbf{a}_m\mathbf{a}_m^T\right)^{-1} \quad (6)$$

- Positioning error is not dependent on distance
- Poor GDOP in far-field because $\mathbf{a}_m$'s are almost parallel



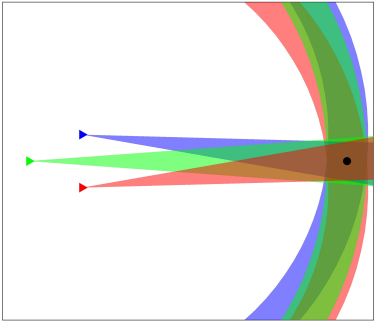
TDoA-Based Localization

- Why TDoA: Tx-Rx synchronization might impractical
- There will be extra unknown: Δt
- CRLB is given as

$$\mathbf{R}_{\mathbf{x}} = \frac{1}{c^2\sigma_t^2}\left(\sum_m (\mathbf{a}_m - \bar{\mathbf{a}})(\mathbf{a}_m - \bar{\mathbf{a}})^T\right)^{-1}$$

- $\mathbf{a}_m - \mathbf{a} \approx (\mathbf{x}_m - \bar{\mathbf{x}})/\bar{d}$

- $\mathbf{R}_{\mathbf{x}}^{TDoA} \succeq \mathbf{R}_{\mathbf{x}}^{ToA}$
- Poor GDOP in far-field
- Positioning error is dependent on distance $\{\mathbf{a}_m - \mathbf{a}\}$'s are almost parallel



How do we get Doppler shift & ToA/TDoA simultaneously?

- The target broadcasts a LFM signal of the following form

$$s(t) = Ae^{j(2\pi f_0 t + k\pi t^2 + \phi)}, \quad (t \in [0, T]).$$

- The received signal is another LFM signal:

$$r(t) = s(\rho(t - \tau)) + w(t), \tag{7}$$

ρ: timing scaling factor: $\rho = 1 - v_r/c$
τ: propagation delay
$w(t)$: additive white noise

- Received signal is another LFM signal:

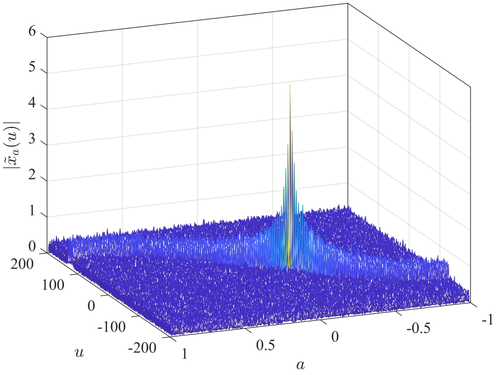
$$r(t) = \tilde{A} \exp\left[j\left(2\pi \tilde{f}_0 t + \tilde{k} t^2 + \tilde{\phi}\right)\right] + w(t), \tag{8}$$

Joint Doppler Shift & ToA Estimation (FrFT)

The parameters of the received LFM signal are functions of τ and ρ:

$$\tilde{f}_0 = f_0\rho - k\rho^2\tau,\ \tilde{k} = k\rho^2,\ \tilde{\phi} = k\rho^2\tau^2 - 2f_0\rho\tau + \phi,$$

Fractional Fourier Transform can be used for joint estimation:



Doppler+ToA/TDoA-CRLB

Doppler+ToA

- Define $\phi = [\tilde{f}_0, \tilde{k}_0]^T$
- CRLB of ϕ is

$$\mathbf{R}_{\phi} \propto \frac{\sigma^2}{\tilde{A}^2 T^2} \begin{bmatrix} 16 & -30/T \\ -30/T & 60/T^2 \end{bmatrix}.$$

- CRLB is given as

$$\mathbf{R}_{\mathbf{x}} = \left(\sum_m \mathbf{P}_m^T \mathbf{F}_{\boldsymbol{\eta}_m} \mathbf{P}_m \right)^{-1}.$$

Doppler+TDoA

- The CRLB in this case will be

$$\tilde{\mathbf{R}}_{\mathbf{x}} = \left(\mathbf{R}_{\mathbf{x}}^{-1} - \mathbf{f}\mathbf{f}^T / p \right)^{-1}, \tag{9}$$

- $\mathbf{f} = \sum_m \mathbf{P}_m^T \mathbf{F}_{\boldsymbol{\eta}_m} \mathbf{p}$
- $p = \sum_m \mathbf{p}^T \mathbf{F}_{\boldsymbol{\eta}_m} \mathbf{p}$.
- By using TDoA instead of ToA, positioning error will increase:

$$\tilde{\mathbf{R}}_{\mathbf{x}} \succeq \mathbf{R}_{\mathbf{x}} \tag{10}$$

Simulation Parameters

f_0	10 kHz
B	400 Hz
k	200 Hz/s
T	2 s
c	1500 m/s

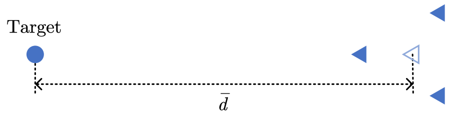


Fig. 2: Geometrical distribution of target and anchors.

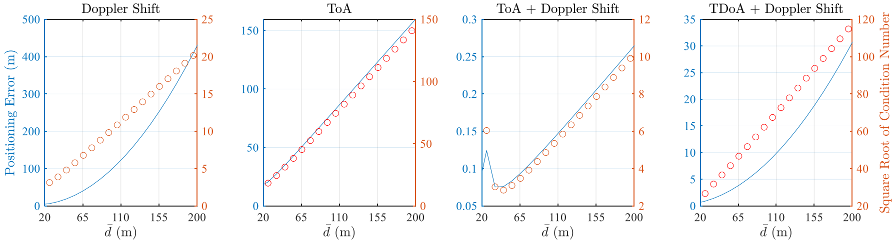


Fig. 3: Comparison of ToA-, TDoA, and Doppler-based localization.

Normalized Positioning Error

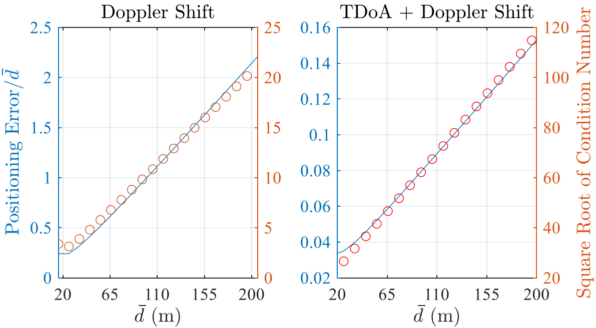


Fig. 4: TDoA- and Doppler-based localization error with distance.

Geometrical Explanation

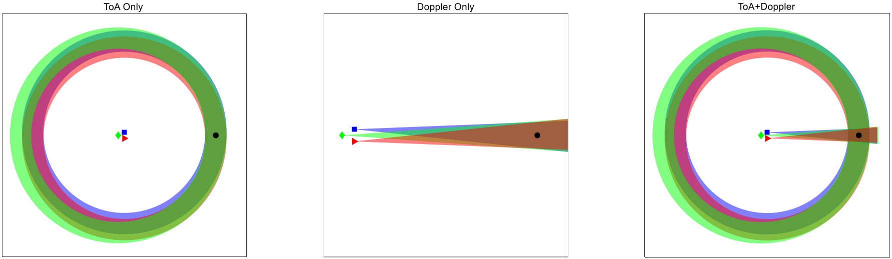


Fig. 5: Geometrical explanation of the huge performance improvement.

Mathematical Explanation

- The FIM of Doppler shift-based positioning system

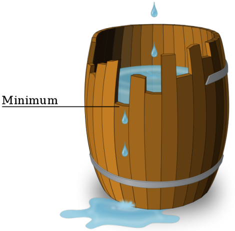
$$\mathbf{F}_d = [\mathbf{f}_0, \mathbf{f}_1] \begin{bmatrix} \lambda_{d,0} & 0 \\ 0 & \lambda_{d,1} \end{bmatrix} \begin{bmatrix} \mathbf{f}_0^T \\ \mathbf{f}_1^T \end{bmatrix}.$$

- $\lambda_{d,0} \gg \lambda_{d,1}$
- The condition number: $c_d = \lambda_{d,0}/\lambda_{d,1}$
- Variance of positioning error:

$$\sigma_d^2 = 1/\lambda_{d,0} + 1/\lambda_{d,1} \approx \frac{c_d}{\lambda_{d,0} + \lambda_{d,1}}. \qquad (11)$$

- $$\sigma_d^2 = \frac{\lambda_{d,0} + \lambda_{d,1}}{\lambda_{d,0}\lambda_{d,1}} \approx \frac{1}{\lambda_{d,1}}.$$
- So the positioning accuracy is highly dependent on the *smallest* eigenvalue.

The *Wooden Barrel Theory*: the capacity of a barrel is determined not by the longest wooden bars, but by the shortest. In our case, the smallest eigenvalue of the FIM is the bottleneck of system performance.



Mathematical Explanation

- The FIM of ToA-based system is

$$\mathbf{F}_t \approx [\mathbf{f}_0, \mathbf{f}_1] \begin{bmatrix} \lambda_{t,0} & 0 \\ 0 & \lambda_{t,1} \end{bmatrix} \begin{bmatrix} \mathbf{f}_0^T \\ \mathbf{f}_1^T \end{bmatrix}.$$

- $\lambda_{t,0} \ll \lambda_{t,1}$. That is to say, most of the energy of $\mathbf{F}_t$ lies on the sub-space of $\mathbf{f}_1$, while most energy of $\mathbf{F}_d$ lies on $\mathbf{f}_0$.
- The condition number is approximately

$$c_t = \lambda_{t,1}/\lambda_{t,0}.$$

- The variance of positioning error will be

$$\sigma_t^2 = 1/\lambda_{t,0} + 1/\lambda_{t,1} = \frac{\lambda_{t,0} + \lambda_{t,1}}{\lambda_{t,0}\lambda_{t,1}} = \frac{c_t + 1}{\lambda_{t,1}} \approx \frac{c_t}{\lambda_{t,0}}. \tag{12}$$

- $$\sigma_t^2 = \frac{\lambda_{t,0} + \lambda_{t,1}}{\lambda_{t,0}\lambda_{t,1}} \approx \frac{1}{\lambda_{t,0}}.$$

- Again the positioning accuracy is highly dependent on the smallest eigenvalue.

Mathematical Explanation

- The FIM of the ToA+Doppler shift-based positioning system will be

$$\mathbf{F}_{t+d} = \mathbf{F}_t + \mathbf{F}_{\mathbf{d}} = [\mathbf{f}_0, \mathbf{f}_1] \begin{bmatrix} \lambda_{t,0} + \lambda_{d,0} & 0 \\ 0 & \lambda_{t,1} + \lambda_{d,0} \end{bmatrix} \begin{bmatrix} \mathbf{f}_0^T \\ \mathbf{f}_1^T \end{bmatrix}.$$

- none of these two eigenvalues are small any more
- We have the upper and lower bounds of the positioning error as

$$\min\left\{\frac{1}{\lambda_{t,0}}, \frac{1}{\lambda_{d,1}}\right\} < \frac{2}{\lambda_{t,0} + \lambda_{d,1}} \leq \sigma_{t+d}^2 \leq \frac{1}{\lambda_{t,0}} + \frac{1}{\lambda_{d,1}}$$

- In this inequality, we can clearly see that the positioning error is no longer related to the condition numbers c_d and c_t.
- This is the power of diversity of measurements! (diversity gain in wireless communications).

Take Home Messages

- For Doppler Shift/TDoA-based localization, the positioning error in the far-field is poor for two reasons:
  - Huge condition number of the FIM
  - Large distance
- For ToA-based localization, the poor positioning accuracy mainly comes from the huge condition number
- By combing Doppler shift with ToA/TDoA measurements, positioning accuracy can be significantly improved