The shallow water equations are

$$\partial_t \boldsymbol{u} + g\boldsymbol{\nabla} h + 2\Omega \boldsymbol{e}_z \times \boldsymbol{u} = \boldsymbol{F} - \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}, \tag{1}$$
$$\partial_t h + H\boldsymbol{\nabla}\cdot\boldsymbol{u} = F_h - \boldsymbol{\nabla}\cdot(h\boldsymbol{u}), \tag{2}$$
$$\partial_t c = f(h) - \boldsymbol{u}\cdot\boldsymbol{\nabla} c. \tag{3}$$

Instead of writing this in terms of u_θ and u_ϕ, we want to write the equations in terms of

$$v_\pm = \frac{1}{\sqrt{2}}\left(u_\theta \mp i u_\phi\right). \tag{4}$$

Then we get

$$\partial_t u_+ + g\nabla_+ h + 2\Omega \left[\boldsymbol{e}_z \times \boldsymbol{u}\right]_+ = F_+ - \left[\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}\right]_+, \tag{5}$$
$$\partial_t u_- + g\nabla_- h + 2\Omega \left[\boldsymbol{e}_z \times \boldsymbol{u}\right]_- = F_- - \left[\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}\right]_-, \tag{6}$$
$$\partial_t h + H\left(\nabla_+ u_- + \nabla_- u_+\right) = F_h - \nabla_+(h\boldsymbol{u})_- - \nabla_-(h\boldsymbol{u})_+, \tag{7}$$
$$\partial_t c = f(h) - \boldsymbol{u}\cdot\boldsymbol{\nabla} c. \tag{8}$$

We need to evaluate $\left[\boldsymbol{e}_z \times \boldsymbol{u}\right]_+$ and $\left[\boldsymbol{e}_z \times \boldsymbol{u}\right]_-$. We have

$$\boldsymbol{e}_z \times \boldsymbol{u} = \cos(\theta)\boldsymbol{e}_r \times \boldsymbol{u}, \tag{9}$$

where we have neglected terms in the radial direction. Then

$$\begin{aligned} \boldsymbol{e}_r \times u_+\boldsymbol{e}_+ &= u_+\boldsymbol{e}_r \times \frac{1}{\sqrt{2}}\left(\boldsymbol{e}_\theta - i\boldsymbol{e}_\phi\right) = u_+\frac{1}{\sqrt{2}}\left(\boldsymbol{e}_\phi + i\boldsymbol{e}_\theta\right) \\ &= iu_+\frac{1}{\sqrt{2}}\left(\boldsymbol{e}_\theta - i\boldsymbol{e}_\phi\right) = iu_+\boldsymbol{e}_+. \end{aligned} \tag{10}$$

Similarly,

$$\begin{aligned} \boldsymbol{e}_r \times u_-\boldsymbol{e}_- &= u_-\boldsymbol{e}_r \times \frac{1}{\sqrt{2}}\left(\boldsymbol{e}_\theta + i\boldsymbol{e}_\phi\right) = u_-\frac{1}{\sqrt{2}}\left(\boldsymbol{e}_\phi - i\boldsymbol{e}_\theta\right) \\ &= -iu_-\frac{1}{\sqrt{2}}\left(\boldsymbol{e}_\theta + i\boldsymbol{e}_\phi\right) = -iu_-\boldsymbol{e}_-. \end{aligned} \tag{11}$$

Thus, if we use the cosine multiplication operator, $\mathcal{C}$, we have

$$\partial_t u_+ + g\nabla_+ h + i2\Omega\mathcal{C}u_+ = F_+ - \left[\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}\right]_+, \tag{12}$$
$$\partial_t u_- + g\nabla_- h - i2\Omega\mathcal{C}u_- = F_- - \left[\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}\right]_-, \tag{13}$$
$$\partial_t h + H\left(\nabla_+ u_- + \nabla_- u_+\right) = F_h - \nabla_+(h\boldsymbol{u})_- - \nabla_-(h\boldsymbol{u})_+, \tag{14}$$
$$\partial_t c = f(h) - \boldsymbol{u}\cdot\boldsymbol{\nabla} c. \tag{15}$$