

RESONANCE COMPENSATION STUDIES AT THE FNAL RECYCLER RING

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## **ABSTRACT**

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## LIST OF ABBREVIATIONS

<b>MSU</b>	Michigan State University
<b>FNAL</b>	Fermilab National Accelerator Laboratory
<b>RR</b>	Recycler Ring
<b>MI</b>	Main Injector
<b>RDTs</b>	Resonance Driving Terms
<b>NuMI</b>	Neutrinos at the Main Injector
<b>PSB</b>	CERN Proton Synchrotron Booster
<b>ACNET</b>	Accelerator Control Network
<b>TbT</b>	Turn-by-Turn Data
<b>NAFF</b>	Numerical Analysis of Fundamental Frequencies

# **CHAPTER 1**

## **INTRODUCTION**

Particle accelerators are the workhorses for modern scientific discoveries. Experimental nuclear and particle physics research benefit greatly from the progress of accelerator physics and technology. Accelerator physics is a rich field of applied physics living on the intersection of electromagnetism, solid-state and atomic physics, nonlinear mechanics, plasma physics, quantum mechanics, just to name a few [1]. Furthermore, the design and operation of modern accelerator projects require costly enterprises of scientists, engineers, operators, and politicians coming together under one single metaphorical roof. Everyone coming together to perform “megascience” [2].

The scientific principle of particle accelerators involves the acceleration, steering and/or storage of charged particles through electromagnetic manipulations. These manipulations occur through a plethora of devices and components that can control electromagnetic fields, e.g., magnets, electrical cavities. The group of particles that is subject to this electromagnetic handling is referred to as "the beam". The field of beam dynamics studies the interaction between the beam and the steering devices, as well as the Coulomb interactions between the beam itself—this is known as space charge physics. An additional distinction can be made when these steering devices are configured in a circular or linear fashion. This gives rise to the distinction between circular accelerators and linear accelerators (Linacs).

Furthermore, particle accelerators can be categorized by the type of elementary particles that compose the beam and how close to the speed of light they are traveling. The first category refers to the distinction between hadrons and leptons—particles that interact or do not interact through the strong force, respectively [3]. For example, protons and heavy ions are considered hadrons, while electrons and muons are considered leptons. The second category can be summarized if particles in the machine travel at a high or low energy. An example of a low-energy hadron machine is the heavy ion Linac at FRIB (Facility for Rare Isotope Beams) [4]. An example of a high-energy lepton machine was the Stanford Linear Accelerator located at SLAC National Accelerator Laboratory [5]. The two most famous high-energy hadron machines in history are the Tevatron [6], which operated

at Fermilab, and the LHC (Large Hadron Collider), operating at CERN [7]. Furthermore, there are accelerator projects that encompass several categories such as the future EIC (Electron-Ion Collider) being built at Brookhaven National Laboratory [8], which will use an electron ring—a lepton machine—and a heavy-ion circular accelerator—a hadron machine—to probe new physics. This is just to name a few. There’s a plethora of accelerator projects around the world that are either operational, under commissioning or being designed.

The following thesis will explore the beam dynamics of a circular machine. The research results are applied to the Fermilab Recycler Ring, which is used to store high-energy protons.

## 1.1 Circular Accelerators and Storage Rings

As will become more apparent on Ch. 2, a particle accelerator can be thought of as a composition of accelerator-themed LEGO® bricks [9]. Each elemental LEGO® brick can be thought of as an accelerator component performing some mapping on the charged particles entering it. As it turns out, these LEGO® bricks can be assembled together circularly to give rise to circular accelerators. The assembly of these blocks in a particular shape gives rise to what is known as the lattice of the accelerator.

The acceleration part of these structures comes from elements inside the lattice that introduce some sort of electromotive force in the longitudinal direction. The most common example for these blocks are radio-frequency (rf) cavities [1], with super-conducting rf cavities also as an established technology [10]. Particles that go through these elements gain energy on every pass. For the case where there are no acceleration blocks on these structures, a storage ring arises. Nevertheless, storage rings can also have rf cavities just for longitudinal beam manipulation, but no overall acceleration—such is the case of the Recycler Ring.

Circular accelerators are special due to the fact that particles have to pass thousands or even millions of turns through the same LEGO® blocks. This gives birth to very interesting and complex dynamics inside these machines. One of these phenomena are called betatron resonances. The most simple lattice of a high energy machine is composed of focusing and steering blocks, and they are dipole and quadrupole magnets, i.e. the lowest-order multipole magnets. For reasons that

will become apparent in Ch. 2, these elements, in combination with free drift spaces, represent linear blocks. For the simplest circular machine, these linear LEGO® bricks are assembled together to create a linear lattice. A linear lattice is designed to have stable particle orbits all around the accelerator. Nevertheless, accounted or unaccounted elements, described by linear or nonlinear blocks, around the machine can perturb the stable orbits. Ultimately, the effect of these perturbations can add up coherently over many turns to push the beam out of the acceptance of the lattice. This whole process is known as a betatron resonance in a circular accelerator. A mathematical description of this process is described on Ch. 2.

The following thesis describes an effort to mitigate the deleterious effect of these resonances in the Recycler Ring. After dipole and quadrupole, the third order of multipole magnetic fields is the sextupole component. Therefore, sextupole fields around the lattice are the source of third order betatron resonances. Specifically, this thesis explores mitigation techniques to these third order resonances, mainly in the Fermilab Recycler Ring (see Ch. 3, Ch. 5 and Ch. 6), but also with some experiments done at the CERN Proton Synchrotron Booster (see Ch. 4).

## 1.2 Fermilab

The best introduction to Fermilab is to cite an excerpt from Ref. [2]:

[...] A passenger peers through the window of an airplane. As his plane flies into Chicago's O'Hare Field from the west, he notices a large ring on the ground below (see Fig. 1.1). Near it he sees a towering white structure, a group of colorful smaller buildings, an expanse of forest, open fields, and lakes.

"What is that ring?" he asks his neighbor.

"Fermilab," she replies. "It's a physics laboratory. The government supports research there into what the universe is made of."

"Why the ring?"

"It's the four-mile-round main ring of a machine called the Tevatron. It turns protons into tools for looking inside the atomic nucleus. Huge magnets steer the protons

around the ring, while high voltages accelerate them. [...]" (pp. 1)



Figure 1.1 Aerial view of the Fermi National Accelerator Laboratory (FNAL) located in Batavia, IL, USA [11].

The Fermi National Accelerator Laboratory (FNAL), better known as Fermilab, has a long and rich history of designing, building and operating high-energy particle accelerators. Ever since the founding director of Fermilab, Robert R. Wilson, envisioned the 400 GeV Main Ring back in 1967, Fermilab has been at the forefront of accelerator physics [2, 6, 12]. The most famous accelerator project hosted by Fermilab has been the Tevatron, a proton-antiproton circular collider with a circumference of around 6.28 km. This machine was injected protons and antiprotons from smaller machines, that are still in operation or have been repurposed as of 2024, e.g. the Recycler Ring. The Tevatron operated up until 2011, leaving an indelible legacy in the field of high energy and accelerator physics. Nostalgia aside, Fermilab still hosts a deluge of particle physics experiments connected to its main accelerator complex.

The current layout of the Fermilab Accelerator Complex is summarized in Fig. 1.2. As of 2024, the Fermilab Accelerator Complex is composed of an  $H^-$  source that connects to a linear accelerator, accelerating the ions to an energy of 400 MeV. This linear accelerator feeds to the first circular machine—the Booster—where protons are achieved and accelerated to an energy of 8 GeV. After the Booster, the protons are transported to the Recycler Ring (RR), which is the second circular machine. In the RR, protons are stacked and stored in order to increase the beam intensity delivered to the Main Injector (MI). This last circular accelerator is where protons are accelerated from an energy of 8 GeV to 120 GeV. Once at this energy, the protons are transported the Neutrinos at the Main Injector (NuMI) experiment, in order to create the world's most intense neutrino beam [13]. Nevertheless, all throughout the chain of accelerators, beam is also delivered to a plethora of other experiments being conducted at Fermilab. Therefore, the facility has several modes of operation depending on the experiments that are online. A more detailed and technical study of the current Fermilab Accelerator Complex, focusing on the Recycler Ring, is given on Ch. 3.



Figure 1.2 Schematic layout of the Fermilab Accelerator Complex as of 2024. Original plot provided by R. Ainsworth, first published on Ref. [14].

### 1.3 Outline

The following thesis will explore the compensation of third-order resonances in the Fermilab Recycler Ring. Chapter 1 introduces the motivation behind this thesis work. Chapter 2 summarizes single particle dynamics with the help of exponential Lie operators and moves forward to introduce a relevant concept of collective beam dynamics: the space charge tune shift. This theoretical overview gives segue into the Ch. 3 of this thesis, where the Recycler Ring is introduced and described in detail. Motivation for the compensation of third order resonances is given in this chapter under the framework of current and future operation of the RR. With the basic physics concepts and the description of the machine put in place, Ch. 4 describes in full detail the scheme and experiments developed in order to compensate third order resonances at low intensities. Before moving to explore the Recycler Ring at high intensities, Ch. 5 provides an interlude in order to show a series of experiments done at the CERN PS Booster. These experiments explore the use of advanced optimization algorithms to compensate multiple resonance lines simultaneously. Coming back to Fermilab, Ch. 6 showcases the studies and experiments done at high intensities in the RR in order to understand the interplay between the compensation of resonance lines and space charge effects. Finally, Ch. 7 brings down the curtain by providing some general conclusions and future work stemming from this thesis.

## CHAPTER 2

### BEAM DYNAMICS IN RINGS

#### 2.1 Introductory Accelerator Physics

The basic building blocks of a particle accelerator are the elements that steer and focus the beam in the transverse direction. This is done by utilizing and manipulating the Lorentz force  $\vec{F}_L$  by means of the electromagnetic fields  $\vec{E}$  and  $\vec{B}$  to act on some particle with charge  $q$  and velocity  $\vec{v}$ , i.e.,  $\vec{F}_L = q(\vec{E} + \vec{v} \times \vec{B})$ . While there are a handful of electromagnetic devices that can do this, the most prominent ones in high-energy accelerators are magnets which have no electric field,  $\vec{E} = 0$ . In particular, there are pure dipole magnets for steering and pure quadrupole magnets for focusing. Nevertheless, some machines such as the Recycler Ring have combined function magnets which can have both types of magnets—and even higher order magnets—embedded in one, in order to steer and focus the particles. The previous information assumes that every magnet can be expanded and described as a decomposition of magnetic multipoles, where dipole and quadrupole components are the lowest order terms of the expansion. Therefore, using the Beth representation [1], the multipole expansion for an arbitrary multipole magnet reads:

$$\frac{1}{[B\rho]} (B_y(x, y) + iB_x(x, y)) = -\frac{1}{\rho} \sum_{n=0}^{\infty} (b_n + ia_n) (x + iy)^n, \quad (2.1)$$

where  $b_n$  and  $a_n$  are the multipole coefficients defined by

$$b_n = \frac{1}{B_0 n!} \left. \frac{\partial^n B_y}{\partial x^n} \right|_{x=y=0}, \quad a_n = \frac{1}{B_0 n!} \left. \frac{\partial^n B_x}{\partial x^n} \right|_{x=y=0}. \quad (2.2)$$

For Eqs. 2.1 and 2.2,  $x$  and  $y$  represent the Cartesian coordinates, the product  $[B\rho]$  represents the magnetic rigidity of the beam with  $B_0$  being the main dipole field and  $\rho$  is the bending radius, while  $B_x$  and  $B_y$  are the transverse magnetic fields in the magnets. Specifically, the coefficients  $b_n$  and  $a_n$  represent the multipole coefficient of the magnet with the dipole coefficient defined as  $b_0 = 1$ , such that  $B_0 b_0 = -[B\rho]/\rho$ . Following the expansion, the term  $a_0$  is referred to as the dipole roll coefficient,  $b_1$  as the quadrupole coefficient,  $a_1$  as the skew quadrupole coefficient,  $b_2$  as the sextupole coefficient,  $a_2$  for the skew sextupole coefficient,  $b_3$  for the octupole coefficient,  $a_3$  for the skew octupole coefficient, etc. This is all following the U.S. convention.

The most basic circular accelerator of circumference  $C$  is composed of LEGO® blocks chosen from a pile of dipoles, quadrupoles and free drift spaces. Each of these elements are described by Eqs. 2.1 and 2.2, i.e., for the quadrupole case, the only nonzero coefficient is  $b_1$ . The particles inside this ring have a longitudinal relativistic velocity of  $v = \beta_L c$ , and therefore have a revolution frequency of  $f_{rev} = \beta_L c / C$ . The Hamiltonian of a single particle with position coordinates  $x, y$  and momentum coordinates  $p_x, p_y$  traversing through such a system at an independent time coordinate  $s$  is:

$$H = \frac{1}{2} \left( K_x(s)x^2 + K_y(s)y^2 + p_x^2 + p_y^2 \right), \quad (2.3)$$

where  $K_x(s), K_y(s)$  are the effective focusing functions, and are defined as:

$$K_x(s) = \frac{1}{\rho^2} - \frac{b_1(s)}{\rho}, \quad K_y(s) = \frac{b_1(s)}{\rho} \quad (2.4)$$

assuming the definition of  $b_1(s)$  in 2.2 has been extended to describe the distribution of the horizontal quadrupole coefficient around the ring. For the case where  $\rho \gg 1$  (high-energy limit), the function  $K_y(s) = -K_x(s)$ , i.e., horizontally focusing quadrupoles will have a defocusing effect in the vertical direction and vice versa.

In classic accelerator references, such as Refs. [1, 15, 16], the equations of motion derived from the Hamiltonian in Eq. 2.3 are known as Hill's equations. The usual accelerator-physics method to solve this type of equations is to introduce transfer matrices for each type of linear element. This will define linear mappings bringing some initial state vector  $\vec{X}_0 = (x_0, x'_0, y_0, y'_0)$  to a final vector  $\vec{X}_f = (x_f, x'_f, y_f, y'_f)$  using a symplectic matrix  $M$ , i.e.,  $\vec{X}_f = M\vec{X}_0$ . Table 2.1 shows the 4D transfer matrices for common linear elements found in accelerators. It is worth noting that these 4D matrices can be used to include coupling elements that couple the  $x - y$  plane. For this case, the off-block coefficients of the matrices would be non-zero. An example of this last case is shown in Table 2.1 by means of the thin skew quadrupole case. Nevertheless, the starting point of this work is to consider a circular accelerator built from linear non-coupling elements, while other coupling or nonlinear elements are considered perturbative. In Sec. 2.2, Lie operators are used in order to extend these mappings to the nonlinear regime.

Table 2.1 Transfer matrices for common accelerator elements in the high-energy regime and paraxial approximation

Element	4D Transfer Matrix
Drift space of length $L$	$M = \begin{bmatrix} 1 & L & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{bmatrix}$
Dipole of bending radius $\rho$ , length $\ell$ and orbiting angle $\theta = \ell/\rho$	$M = \begin{bmatrix} \cos \theta & \rho \sin \theta & 0 & 0 \\ -\frac{1}{\rho} \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & \ell \\ 0 & 0 & 0 & 1 \end{bmatrix}$
Thin quadrupole of focal length $f = \lim_{\ell \rightarrow 0} \frac{1}{k\ell} > 0$	$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/f & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 1 \end{bmatrix}$
Thick quadrupole of strength $k > 0$ and length $\ell$	$M = \begin{bmatrix} \cos \sqrt{k}\ell & \frac{1}{\sqrt{k}} \sin \sqrt{k}\ell & 0 & 0 \\ -\sqrt{k} \sin \sqrt{k}\ell & \cos \sqrt{k}\ell & 0 & 0 \\ 0 & 0 & \cosh \sqrt{ k }\ell & \frac{1}{\sqrt{ k }} \sinh \sqrt{ k }\ell \\ 0 & 0 & \sqrt{ k } \sinh \sqrt{ k }\ell & \cosh \sqrt{ k }\ell \end{bmatrix}$
Thin skew quadrupole of focal length $f_s = \lim_{\ell \rightarrow 0} \frac{1}{k_s \ell} > 0$	$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1/f_s & 0 \\ 0 & 0 & 1 & 0 \\ 1/f_s & 0 & 0 & 1 \end{bmatrix}$

Just like stacking LEGO® bricks together, these transfer matrices can be stacked up in order to calculate the total mapping through a stack of elements. The total linear mapping of a consecution of accelerator elements is just the matrix multiplication of the corresponding transfer matrices, i.e.,

for a lattice with  $N$  linear blocks the total transfer matrix reads:

$$M_{Total} = M_N M_{N-1} \dots M_1. \quad (2.5)$$

Taking linear lattices one step further, the Courant-Snyder (CS) parametrization can be introduced. This parametrization employs the fact that any linear element or stack of linear elements can be parametrized by the Twiss functions  $\beta_u(s)$ ,  $\alpha_u(s)$  and  $\gamma_u(s)$  and a phase advance defined as  $\phi_u(s) = \int_0^s ds/\beta_u(s)$ , where  $u$  stands for either the  $x$  or  $y$  plane. This results in the following transfer matrix from location  $s = 0$  to an arbitrary location  $s$ :

$$M(s) = \begin{bmatrix} M_x(s) & 0_{2 \times 2} \\ 0_{2 \times 2} & M_y(s) \end{bmatrix}, \quad (2.6)$$

where  $M_u$  is a square matrix defined as:

$$M_u = \begin{bmatrix} \sqrt{\frac{\beta_u(s)}{\beta_u(0)}} (\cos \phi_u(s) + \alpha_u(0) \sin \phi_u(s)) & \sqrt{\beta_u(s)\beta_u(0)} \sin \phi_u(s) \\ \frac{\alpha_u(0)-\alpha_u(s)}{\beta_u(s)\beta_u(0)} \cos \phi_u(s) - \frac{1+\alpha_u(0)\alpha_u(s)}{\beta_u(s)\beta_u(0)} \sin \phi_u(s) & \sqrt{\frac{\beta_u(0)}{\beta_u(s)}} (\cos \phi_u(s) - \alpha_u(s) \sin \phi_u(s)) \end{bmatrix}, \quad (2.7)$$

and  $0_{2 \times 2}$  is a square matrix of size  $2 \times 2$  with zeros in all of its entries. As a consequence of the Twiss parametrization, the position coordinate at an arbitrary  $s$  along a linear lattice may be expressed as:

$$u(s) = \sqrt{2J_u\beta_u(s)} \cos(\phi_u(s) + \phi_{u0}), \quad (2.8)$$

where  $J_u$  is a constant of motion known as the action and  $\phi_{u0}$  is an arbitrary phase constant. Equation 2.8 describes what is known as betatron oscillations. Additionally, the action  $J_u$  can be calculated as

$$2J_u = \gamma_u u^2 + 2\alpha_u uu' + \beta_u u'^2, \quad (2.9)$$

for any  $s$  using the property  $\beta_u \gamma_u = 1 + \alpha_u^2$ . Therefore, the particles traversing through the lattice will exhibit oscillatory motion, with amplitude dictated by the  $\beta_u(s)$  function and frequency dictated by the  $\phi_u(s)$  function.

Up until now, all of this formalism can be applied to either a linear or circular accelerator. No assumption has been made on how the stacking of the LEGO® bricks in the lattice affects

this formalism. Nevertheless, in a circular accelerator of circumference  $C$  there is an additional constraint on the Twiss functions given that there exists periodic boundary conditions, i.e.,  $\beta(s) = \beta(s + C)$ , and similarly for the other Twiss functions. Another important parameter for a circular lattice is the tune  $Q_u$ , defined as the total phase advance over one turn of the machine, i.e.,

$$2\pi Q_u = \phi_u(C) = \int_s^{s+C} \frac{ds}{\beta_u(s)}. \quad (2.10)$$

With the definition of the tune  $Q_u$ , Eq. 2.7 can be rewritten in order to calculate the one-turn matrix  $M(C)$  of the circular accelerator, such that:

$$M(C) = \begin{bmatrix} M_x(C) & 0_{2 \times 2} \\ 0_{2 \times 2} & M_y(C) \end{bmatrix}, \quad (2.11)$$

where each plane will have its own one-turn transfer matrix reading:

$$M_u(C) = \begin{bmatrix} \cos 2\pi Q_u + \alpha_u(C) \sin 2\pi Q_u & \beta_u(C) \sin 2\pi Q_u \\ -\gamma_u(C) \sin 2\pi Q_u & \cos 2\pi Q_u - \alpha_u(C) \sin 2\pi Q_u \end{bmatrix}. \quad (2.12)$$

Ultimately, Eq. 2.12 can be used to calculate the particle's state vector after  $N$  turns, where the total transfer matrix  $M_u(NC)$  will just be  $M_u(NC) = M_u(C)^N$ . In fact, if a particle with some initial conditions is tracked for enough turns and its geometrical coordinates  $(x, x, y, y')$  are recorded through some diagnostic device every turn, the resulting trajectory would lie on an ellipse—the phase space ellipse. The astute reader would have already identified that Eq. 2.9 hinted to this fact, given that this is just the implicit definition for an ellipse. In essence, the Twiss parametrization will give the geometry in order to fully describe the ellipse. These statements are illustrated in Fig. 2.1, where the phase space ellipse is drawn with geometrical quantities associated to the Twiss functions.



Figure 2.1 Phase space ellipse in geometrical coordinates with Twiss parametrization and its counterpart transformation in normalized phase space.

Figure 2.1 also hints at the fact that a linear transformation  $T_u(s)$  can be done in order to transform the phase space ellipse into a circle. This is referred to as a Floquet transformation. Therefore, a change of coordinates from geometrical coordinates  $(u, u')$  to normalized coordinates  $(\hat{u}, \hat{p}_u)$  can be achieved through the following linear transformation:

$$\begin{bmatrix} \hat{u} \\ \hat{p}_u \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{\beta_u}} & 0 \\ \frac{\alpha_u}{\beta_u} & \sqrt{\beta_u} \end{bmatrix} \begin{bmatrix} u \\ u' \end{bmatrix} = \sqrt{2J_u} \begin{bmatrix} \cos \phi_u(s) \\ -\sin \phi_u(s) \end{bmatrix}. \quad (2.13)$$

Ultimately, Floquet and his transformations show that the one-turn Hamiltonian for circular accelerators can be expressed as:

$$H_0 = 2\pi Q_x J_x + 2\pi Q_y J_y, \quad (2.14)$$

which is simpler and more succinct than the Hamiltonian in Eq. 2.3. Conclusively, all the dynamics of a linear circular accelerator with all of its intricacies can be mapped to rotations on a simple circle. The dynamics in normalized phase space is just parametrized by a rotation matrix  $R(s)$ , that will depend on the lattice itself, and is analogous to the transfer matrices  $M(s)$  in geometrical space. Therefore, linear dynamics in a circular accelerator can be summarized with the following

commutative diagram:

$$\begin{array}{ccc}
 \begin{pmatrix} x, x' \\ y, y' \end{pmatrix}_0 & \xrightarrow{M(s)} & \begin{pmatrix} x, x' \\ y, y' \end{pmatrix}_f \\
 T(s) \downarrow & & \downarrow T(s) \\
 \begin{pmatrix} J_x, \phi_x \\ J_y, \phi_y \end{pmatrix}_0 & \xrightarrow{R(s)} & \begin{pmatrix} J_x, \phi_x \\ J_y, \phi_y \end{pmatrix}_f
 \end{array} \tag{2.15}$$

Before proceeding to the next sections, where Lie operators are introduced in order to generalize to non-linear mappings, a short summary is adequate. For a linear circular accelerator, the last section has shown that starting from Hill's equation, linear transformations can be applied to a complex machine such as an accelerator in order to end up with a simple mathematical equation such as the one described in Eq. 2.14. Simply put, nonlinear elements will distort the phase space circle of Fig. 2.1, destroying the linearity of the system. The premise here is that non-linear elements in circular accelerators are inevitable, and they will come from anywhere and everywhere in the lattice, either from accounted or unaccounted sources. Therefore, higher-tier mathematical tools have to be used in order to describe non-linear dynamics in a circular accelerator. Linear matrices will only get you so far.

## 2.2 Lie Maps in Accelerator Physics

The most basic element of a particle accelerator can be thought of as a LEGO® brick acting as a black box transformation for a single particle. This black box takes some single charged particle with initial transverse coordinates  $(x_0, x'_0, y_0, y'_0)$ , as defined in a Frenet-Serret coordinate system, and maps them to some final coordinates  $(x_f, x'_f, y_f, y'_f)$ . For simplicity, any longitudinal effect will not be taken into account for this analysis, but can be easily incorporated. By gathering the initial coordinates into a vector, i.e.  $\vec{X}_0 = (x_0, x'_0, y_0, y'_0)$ , and doing the same for the final coordinates, i.e.,  $\vec{X}_f = (x_f, x'_f, y_f, y'_f)$ , one can define the mapping  $\mathcal{M}$  that relates both vectors,

such that:

$$\vec{X}_f = \mathcal{M} \vec{X}_0. \quad (2.16)$$

Different from the previous section,  $\mathcal{M}$  need not be a linear mapping. For a charged particle inside some accelerator element that can be described using Hamiltonian dynamics, the mapping  $\mathcal{M}$  can be understood in terms of Poisson brackets and exponential Lie operators [9, 15, 17–19].

Let  $\vec{X} = (q_1, p_1, \dots, q_n, p_n)$  be a  $2n$  dimensional vector, made from  $n$  pairs of canonical coordinates  $(q_i, p_i)$  that make up the  $2n$  dimensional phase space. And let two arbitrary functions  $f(\vec{X}; s)$  and  $g(\vec{X}; s)$  be functions of  $\vec{X}$  and  $s$ , where  $s$  plays the role of the independent "time" coordinate. The Poisson brackets  $[\bullet, \bullet]$  can be defined as:

$$[f, g] = \sum_{i=1}^n \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}. \quad (2.17)$$

Using this definition, one can explicitly write out the Poisson bracket definition for a 4 dimensional phase space described by state vector  $\vec{X} = (x, x', y, y')$ . This reads:

$$[f, g] = \frac{\partial f}{\partial x} \frac{\partial g}{\partial x'} - \frac{\partial f}{\partial x'} \frac{\partial g}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial g}{\partial y'} - \frac{\partial f}{\partial y'} \frac{\partial g}{\partial y}. \quad (2.18)$$

The Lie operator :  $f$  : acts on some function  $g$  and is the adjoint operator of the Poisson bracket operator. Its definition reads:

$$: f : g = [f, g]. \quad (2.19)$$

This specific :  $\bullet$  : notation allows for a compact notation in order to define the exponential Lie operator. The exponential Lie operator of an arbitrary function  $f$  is defined as

$$e^{:f:} \bullet = \sum_{k=0}^{\infty} \frac{1}{k!} (: f :)^k \bullet. \quad (2.20)$$

For a Hamiltonian system, the mapping of coordinates from  $\vec{X}_0$  to  $\vec{X}_f$  follows the expression:

$$\vec{X}_f = e^{-\ell:H:} \vec{X} \Big|_{\vec{X}=\vec{X}_0}, \quad (2.21)$$

which is known as a Lie Map [17]. In this case,  $\ell$  corresponds to the integration length of the independent coordinate. For example, for a particle traversing a magnet which has length  $L$ , the

integration length is  $\ell = L$ . When looking at the one-turn map, the integration length corresponds to the circumference  $C$  of the accelerator over an effective Hamiltonian  $H_{eff}$ . Furthermore, if working with action-angle variables, the integration length  $\ell$  would just be the phase advance  $\mu$ . An implementation of exponential Lie operators using *Mathematica* [20] in order to calculate 2D and 4D mappings is presented in Appendix A and in Appendix B, respectively.

### 2.3 One-turn Map and Normal Form

Such as LEGO® bricks can be put together to create complex structures, accelerator elements can be assembled together to create complex ring-shaped structures such as circular accelerators. In such structures, particles will experience the same one-turn mapping over thousands or even millions of turns. The one-turn map  $\mathcal{M}_1$  of a circular accelerator is the composition ( $\circ$ ) of mappings from every LEGO® element in the ring. Choosing an arbitrary initial point at  $s = 0$  and going around the ring, the one-turn map describes the transformation of coordinates after one turn, i.e.,  $\vec{X}_{N=1} = \mathcal{M}_1 \vec{X}_0$ . This map composition reads:

$$\mathcal{M}_1 = M_{N+1} \circ e^{:h_N:} \circ \dots \circ e^{:h_2:} \circ M_2 \circ e^{:h_1:} \circ M_1 = M_{N+1} e^{:h_N:} \dots e^{:h_2:} M_2 e^{:h_1:} M_1, \quad (2.22)$$

where  $M_i$  is the matrix representation of a linear mapping, that does not couple  $x - y$  plane, e.g., drift space mapping, quadrupole mapping. On the other hand, the map  $e^{:h_i:}$  represents any linear or non-linear mapping that can be found around the machine and can be considered a perturbation to the ideal lattice including coupling elements, e.g., skew quadrupoles, higher order multipole elements. Figure 2.2 illustrates the procedure to build the one-turn map for a circular accelerator.



Figure 2.2 Diagram of an arbitrary circular accelerator in order to illustrate the one-turn map.

Through the use of the Baker-Campbell-Hausdorff formula [21], Eq. 2.22 can be collapsed to the expression

$$\mathcal{M}_1 = e^{-C:H_{eff}:}, \quad (2.23)$$

where  $C$  is the circumference of the ring and  $H_{eff}$  is the effective Hamiltonian of the machine over one turn. As mentioned earlier, for most cases, it is of interest to look at the perturbations to the linear uncoupled dynamics of the design lattice. With this in mind, Eq. 2.23 can be rewritten as:

$$\mathcal{M}_1 = e^{h:R}, \quad (2.24)$$

where  $R$  is a rotation matrix encoding the linear uncoupled dynamics of the ideal lattice. On the other hand, the term  $e^{h:}$  encodes the perturbations to this ideal situation. It is worth pointing out that for the case  $h = 0$ , the traditional Courant-Snyder variables are recovered.

The Courant-Snyder variables  $(\hat{x}, \hat{p}_x, \hat{y}, \hat{p}_y)$  or normalized phase space coordinates can be written for a linear uncoupled case as:

$$\hat{u} = \sqrt{2J_u} \cos(\phi_u + \phi_{u_0}); \quad (2.25)$$

$$\hat{p}_u = -\sqrt{2J_u} \sin(\phi_u + \phi_{u_0}), \quad (2.26)$$

where  $u$  can stand either for the  $x$  or  $y$  coordinate,  $J_u$  and  $\phi_u$  correspond to the action-angle variables and  $\phi_{u_0}$  corresponds to the initial phase. For the case where perturbations exist, i.e.,  $h \neq 0$ , the action  $J_u$  is not constant anymore and will be a function of  $\phi_u$ .

The Normal Form formalism is introduced at this point in order to find action-angle coordinates  $I_u$  and  $\psi_u$ , such that the motion just depends on  $\psi_u$  at a constant  $I_u$ , with some initial phase  $\psi_{u_0}$ . These are known as non-linear action-angle variables. The variables  $I_u$  and  $\psi_u$  are calculated from the transformation  $e^{-:F:}$  acting on  $J_u$  and  $\phi_u$ . The whole point is to find these variables that allow for the Hamiltonian to be only amplitude dependent. These Normal Form gymnastics can be summarized by the following commutative diagram:

$$\begin{array}{ccc} \left( \begin{matrix} J_x, \phi_x \\ J_y, \phi_y \end{matrix} \right)_0 & \xrightarrow{e^{:h(J_u, \phi_u):_R}} & \left( \begin{matrix} J_x, \phi_x \\ J_y, \phi_y \end{matrix} \right)_f \\ e^{-:F:} \downarrow & & \downarrow e^{-:F:} \\ \left( \begin{matrix} I_x, \psi_x \\ I_y, \psi_y \end{matrix} \right)_0 & \xrightarrow{e^{:H(I_u):}} & \left( \begin{matrix} I_x, \psi_x \\ I_y, \psi_y \end{matrix} \right)_f. \end{array} \quad (2.27)$$

Without loss of generality, the generating function  $F$  can be written as a Fourier expansion over the objective space  $(I_x, \psi_x, I_y, \psi_y)$  such that:

$$F = \sum_{jklm} f_{jklm} (2I_x)^{\frac{j+k}{2}} (2I_y)^{\frac{l+m}{2}} e^{i[(j-k)(\psi_x + \psi_{x0}) + (l-m)(\psi_y + \psi_{y0})]}. \quad (2.28)$$

Similarly, the argument of the Lie operator  $e^{:h:}$  from Eq. 2.24 can be expanded as:

$$h = \sum_{jklm} h_{jklm} (2J_x)^{\frac{j+k}{2}} (2J_y)^{\frac{l+m}{2}} e^{i[(j-k)(\phi_x + \phi_{x0}) + (l-m)(\phi_y + \phi_{y0})]}. \quad (2.29)$$

For Eqs. 2.28 and 2.29, the integer indices  $j, k, l, m$  run from 0 to infinity, and correspond to the four degrees of freedom for transverse phase space.

The terms  $f_{jklm}$  are known as generating function coefficients. The terms  $h_{jklm}$  are known as Hamiltonian coefficients or resonance driving terms (RDTs). Section 2.5 will take a closer look into how RDTs can be used to characterize the non-linear dynamics of accelerators. The generating function coefficients  $f_{jklm}$  can be related to the Hamiltonian resonance driving terms  $h_{jklm}$  through the following relation [18, 22]:

$$f_{jklm} = \frac{h_{jklm}}{1 - e^{2\pi i[(j-k)Q_x + (l-m)Q_y]}}, \quad (2.30)$$

where  $Q_x$  and  $Q_y$  represent the transverse uncoupled and unperturbed tunes of the accelerator. The transverse tunes of a circular accelerator are defined as the phase advances in each plane over one turn, in units of  $2\pi$ , i.e.,  $Q_u = \phi_u(s = C)/2\pi$ .

In general, the terms  $h_{jklm}$  are defined by the order in which they enter the one-turn normal form Hamiltonian [22]. With the assumption of thin elements around the ring, the general expression to define RDTs reads:

$$h_{jklm} = \Xi_{jklm} \sum_i L_i K_{n-1,i} \beta_{x,i}^{\frac{j+k}{2}} \beta_{y,i}^{\frac{l+m}{2}} e^{i[(j-k)\phi_{x,i} + (l-m)\phi_{y,i}]}, \quad (2.31)$$

where  $\Xi_{jklm}$  is just a constant defined as:

$$\Xi_{jklm} = -\frac{1}{2^n} \frac{1}{n!} \binom{n}{l+m} \binom{j+k}{j} \binom{l+m}{l}. \quad (2.32)$$

For Eqs. 2.31 and 2.32,  $n = j + k + l + m$  represents the order of the RDT. The sum over  $i$  is done over all multipoles of order  $n$  and length  $L_i$  that either have a normal component  $K_{n-1,i} = b_{n-1,i}/\rho$  if  $l + m$  is even, or a skew component  $K_{n-1,i} = K_{n-1,i}^{(s)} = a_{n-1,i}/\rho$  if  $l + m$  is odd, remembering  $\rho$  is the bending radius as it appeared on Eq. 2.1. The notation  $K_{n-1,i}$  is to keep up with the MAD-X convention for naming normal multipole coefficients [23]. The symbols for  $\beta_{x,i}$ ,  $\beta_{y,i}$ ,  $\phi_{x,i}$  and  $\phi_{y,i}$  represent the unperturbed beta functions and phase advances in each plane, respectively.

## 2.4 Resonances in Circular Accelerators

Equation 2.30 diverges for when the denominator goes to zero. Specifically, this happens when the following condition is met:

$$(j - k) Q_x + (l - m) Q_y = p, \quad (2.33)$$

where  $p$  can be any integer. Equation 2.33 defines resonance lines in tune space of order  $n = j + k + l + m$ . If the accelerator is tuned to operate on top of these resonances, the perturbations will add up coherently turn to turn and kick the resonant particles out of their original trajectory. In general, operating close or on top of a resonance line is harmful as particles will be lost. This is specially true for lower order resonances, i.e., for  $n < 4$ . In general, the higher order of a resonance, the weaker it is [16]. This thesis work focuses on third order resonances, i.e.,  $n = 3$ , and how to mitigate their deleterious effect.

Figure 2.3 shows the tune diagram with resonance lines, as defined by Eq. 2.33, drawn up to fifth order. The integer part of both tunes are chosen to include the actual area of operation of the Recycler Ring. Nevertheless, only the fractional part of the tune carries the significant information for resonance diagrams. The operation and tune diagram for the Recycler Ring are described in more detail in Ch. 3. Normally, the operation point of a circular accelerator is chosen to be clear of any resonance line and far away as possible from integer ( $n = 1$ ) and half integer ( $n = 2$ ) resonances. Nevertheless, in reality there are two concepts that complicate things. The first one relates to the fact that resonance lines are not infinitely thin and have some stop bandwidth. The second one, concerns the fact that at high intensities particles will not have localized tunes, but rather a distribution of tunes with some tune spread, i.e., a tune footprint. Section 2.7 takes a closer look at this effect known as space charge tune shift. Ultimately, choosing the operation point on Fig. 2.3 is a matter of localizing a resonance-free region where the intensity-dependent tune footprint can be placed.



Figure 2.3 Tune diagram with resonance lines up to fifth order, enclosing the operation point of the Recycler Ring.

It is worth stopping here and asking what is the driving force behind each of these resonance lines. Classic accelerator references such as Refs. [1, 15, 16] will derive Eq. 2.33 by perturbing Hill's equation with different magnetic multipole orders. A closer look into each perturbation term reveals that half integer resonances are caused by quadrupole terms, third order resonances by sextupole-like terms, fourth order resonances by octupole terms, and so on and so forth. Nevertheless, the story complicates when one takes into account that pure multipole magnets can feed down or up to other order terms if there is installation misalignment, e.g., a misaligned sextupole feeds to skew quadrupole-like terms.

Figure 2.4 zooms into the region of interest for the Recycler Ring operation in the tune diagram, as shown in Fig. 2.3. As mentioned before, the operation point of an accelerator in the tune diagram

is not a singular point but rather a footprint. While the lattice can be tuned to a specific nominal point, particles will interact with other particles through the Coulomb force. Consequently, each particle will feel a different tune shift depending on their position within the bunch of particles. This is called the incoherent space charge tune shift, and it will be the largest for particles in the core of the bunch, i.e., the beam core. At low particle intensities, such as the one used to produce Fig. 2.4, the tune spread of the particles in the bunch is small enough to approximate the physics to single-particle dynamics. For beams with low particle intensities and a small tune spread, such as the one depicted in Fig. 2.4, operating clear from any low order resonance lines is not generally a problem. Nevertheless, at high intensities the situation changes.



Figure 2.4 Approximate operational tune footprint at low intensities calculated with PySCRDT [24], i.e., 1e10 particles per bunch.

Figure 2.4 plots all resonance lines up to fourth order in this region of interest. The half integer line  $Q_x - Q_y = 1$ , also known as a difference coupling resonance, is usually driven by solenoidal

and skew-quadrupole fields in the lattice. The third order lines  $3Q_x = 76$  and  $Q_x + 2Q_y = 74$  are driven by sextupole-like fields. The other third order lines  $3Q_y = 73$  and  $2Q_x + Q_y = 75$  are driven by skew sextupole terms in the lattice. And finally, the fourth order lines  $-Q_x + 3Q_y = 48$  and  $3Q_x - Q_y = 52$  are driven by octupole terms in the lattice and Coulomb (space charge) forces from the bunch itself. This is assuming a rectangular multipole expansion notation of the magnetic field, such as the one presented in Eq. 2.1.

## 2.5 Resonance Driving Terms

The RDTs  $h_{jklm}$  are related to the strength of the resonance  $(j - k)Q_x + (l - m)Q_y$ . Therefore, controlling and measuring these RDTs is of special interest to accelerator physics. The following section explains how to get to a useful expression that can be used in order to measure the  $h_{jklm}$  terms through Fourier expansions.

The whole point of introducing the Normal Form coordinates  $(I_u, \psi_u)$  through the transformation  $e^{-iF}$  as defined in Eq. 2.28 is to transfer complicated non-linear dynamics to simple dynamics that lie on a circle where the action is conserved  $I_u$  and  $\dot{\psi}_u$  is constant. When this happens, a set of canonical coordinates  $\vec{\zeta} = (\zeta_x^+, \zeta_x^-, \zeta_y^+, \zeta_y^-)$  can be defined as:

$$\zeta_u^\pm = \sqrt{2I_u} e^{\mp i(\psi_u + \psi_{u_0})}, \quad (2.34)$$

always keeping in mind that  $I_u$  is a constant of motion and  $\psi_{u_0}$  is a constant initial phase set by the initial conditions. It can be shown that the Poisson brackets for a pair of these quantities are:

$$[\zeta_u^+, \zeta_u^-]_{\psi_u, J_u} = \frac{\partial \zeta_u^+}{\partial \psi_u} \frac{\partial \zeta_u^-}{\partial J_u} - \frac{\partial \zeta_u^+}{\partial J_u} \frac{\partial \zeta_u^-}{\partial \psi_u} = -2i, \quad (2.35)$$

for the same plane  $u$  and using a reduced form of Eq. 2.18. In this notation, the subindices from  $[\bullet, \bullet]_{\psi_u, J_u}$  refer to the variables to be used in order to calculate the Poisson brackets. Using Eq. 2.35, the following useful property can be derived:

$$[\zeta_x^{+j} \zeta_x^{-k} \zeta_y^{+l} \zeta_y^{-m}, \zeta_x^-]_{\psi_x, J_x} = (\zeta_y^{+l} \zeta_y^{-m}) [\zeta_x^{+j} \zeta_x^{-k}, \zeta_x^-]_{\psi_x, J_x} = -2ij \zeta_x^{+j-1} \zeta_x^{-k} \zeta_y^{+l} \zeta_y^{-m}, \quad (2.36)$$

where the last step can be achieved using Leibnitz rule for Poisson brackets, i.e.,  $[fg, h] = [f, h]g + f[g, h]$ .

On the other hand, going back to the Courant-Snyder phase space, a set of coordinates known as a resonance basis  $\vec{h} = (h_x^+, h_x^-, h_y^+, h_y^-)$  can be defined. Similarly to Eq. 2.34, the resonance basis reads:

$$h_u^\pm = \hat{u} \pm \hat{p}_u = \sqrt{2J_u} e^{\mp i(\phi_u + \phi_{u_0})}, \quad (2.37)$$

always keeping in mind that in the Courant-Snyder phase space, the action  $J_u$  is a function of the phase  $\phi_u$ , i.e.,  $J_u = J_u(\phi_u)$  and is not constant. The initial phase  $\phi_{u_0}$  is again a constant set by the initial conditions.

The basis grouped in  $\vec{h}$  and the one grouped in  $\vec{\zeta}$  are related by the transformation:

$$\vec{h} = e^{:F(\vec{\zeta}):} \vec{\zeta}, \quad (2.38)$$

where  $F(\vec{\zeta})$  is the generating function written in terms of the basis  $\vec{\zeta}$ . The inverse transformation to Eq. 2.38 reads:

$$\vec{\zeta} = e^{-:F(\vec{\zeta}):} \vec{h}. \quad (2.39)$$

Writing out the generating function  $F(\vec{\zeta})$  in a general polynomial form, this functions reads:

$$F(\vec{\zeta}) = \sum_{jklm} f_{jklm} \zeta_x^{+j} \zeta_x^{-k} \zeta_y^{+l} \zeta_y^{-m}. \quad (2.40)$$

By inserting the definitions in Eq. 2.34 into Eq. 2.40, the proposed definition in Eq. 2.28 can be recovered.

Expanding Eq. 2.38 by using the exponential Lie operator definition from Eq. 2.20 reads:

$$\vec{h} = \vec{\zeta} + [F(\vec{\zeta}), \vec{\zeta}] + \frac{1}{2} [F[F(\vec{\zeta})], \vec{\zeta}] + \dots, \quad (2.41)$$

where this expression was truncated to second order in the Poisson brackets. By taking only the first two terms of the expansion, and introducing the expression from Eq. 2.40, one can find an approximated expression for  $h_x^-$  which reads:

$$h_x^- \approx \zeta_x^- + [F(\vec{\zeta}), \zeta_x^-] = \zeta_x^- + \sum_{jklm} f_{jklm} [\zeta_x^{+j} \zeta_x^{-k} \zeta_y^{+l} \zeta_y^{-m}, \zeta_x^-], \quad (2.42)$$

At this point is where the usefulness of Eq. 2.36 comes into play. Introducing the explicit result from Eq. 2.36 into Eq. 2.42 yields the following expression:

$$h_x^- \approx \zeta_x^- - 2i \sum_{jklm} j f_{jklm} \zeta_x^{+j-1} \zeta_x^{-k} \zeta_y^{+l} \zeta_y^{-m}. \quad (2.43)$$

Manipulating this expression further, the definition for  $\zeta_u$  as described in Eq. 2.34 can be introduced into Eq. 2.43. This yields:

$$\begin{aligned} h_x^-(N) &= \sqrt{2I_x} e^{i(\psi_x + \psi_{x_0})} \\ &\quad - 2i \sum_{jklm} j f_{jklm} (2I_x)^{\frac{j+k-1}{2}} (2I_y)^{\frac{l+m}{2}} e^{i[(1-j+k)(\psi_x + \psi_{x_0}) + (m-l)(\psi_y + \psi_{y_0})]}. \end{aligned} \quad (2.44)$$

At this point, Eq. 2.44 is starting to look as a useful Fourier expansion. Ultimately, the data that can be extracted from a circular accelerator will come from a diagnostic device triggered every turn, i.e., turn-by-turn data. For that reason, it will be useful to rewrite Eq. 2.44 in terms of the  $N$  number of turns of particles in the accelerator. The expression relating the phase advances to the turn number reads:

$$\psi_u = 2\pi Q_u N, \quad (2.45)$$

where  $2\pi Q_u$  is the respective phase advance over one turn of the accelerator, i.e. the tune of the circular accelerator.

Therefore, the resonance basis can be built by getting the quantity  $h_u^\pm = \hat{u} \pm \hat{p}_u$  in terms of the number of turns  $N$  and using Eq. 2.45. Specifically, for  $h_x^-$  this reads:

$$\begin{aligned} h_x^-(N) &= \sqrt{2I_x} e^{i(2\pi Q_x N + \psi_{x_0})} \\ &\quad - 2i \sum_{jklm} j f_{jklm} (2I_x)^{\frac{j+k-1}{2}} (2I_y)^{\frac{l+m}{2}} e^{i[(1-j+k)(2\pi Q_x N + \psi_{x_0}) + (m-l)(2\pi Q_y N + \psi_{y_0})]}, \end{aligned} \quad (2.46)$$

where  $Q_x$  and  $Q_y$  are the horizontal and vertical uncoupled tune. Note that this analysis can be easily extended to calculate the other elements in  $\vec{h}$ . These calculations are left as an exercise for the reader.

## 2.6 Amplitude-Dependent Tune Shift

The RDT formalism allows to calculate an important quantity in accelerator physics called the amplitude-dependent tune shift. The Hamiltonian for a single particle in a linear circular lattice with perturbation elements reads:

$$H(x, y, s) = H_0(J_x, J_y) + H_1(J_x, \phi_x, J_y, \phi_y), \quad (2.47)$$

where  $H_0$  is the unperturbed linear Hamiltonian with tunes  $Q_x$  and  $Q_y$ , and  $H_1$  is the perturbation Hamiltonian stemming from linear and non-linear unaccounted blocks in the lattice.

From Secs. 2.1 and 2.3, the expressions for  $H_0$  and  $H_1$  have been explicitly written in Eqs. 2.14 and 2.29 in terms of  $J_x, \phi_x, J_y, \phi_y$ , therefore the sum of both expression reads:

$$H_0 + H_1 = 2\pi Q_x J_x + 2\pi Q_y J_y + \sum_{jklm} h_{jklm} (2J_x)^{\frac{j+k}{2}} (2J_y)^{\frac{l+m}{2}} e^{i[(j-k)(\phi_x + \phi_{x0}) + (l-m)(\phi_y + \phi_{y0})]}. \quad (2.48)$$

Nevertheless, it is important to remember that  $H_1$  is the compilation of all perturbations after one turn to the linear Hamiltonian  $H_0$ , and is therefore perturbative.

Consequently, for Eq. 2.48, the independent time variable is the number of turns  $N$ . For this case, the equations of motion taking  $N$  as the number of turns around the circular accelerator are just:

$$\frac{\partial J_u}{\partial N} = -\frac{\partial H}{\partial \phi_u} = -\frac{\partial H_1}{\partial \phi_u}, \quad (2.49)$$

and

$$\frac{\partial \phi_u}{\partial N} = \frac{\partial H}{\partial J_u} = 2\pi Q_u + \frac{\partial H_1}{\partial J_u}. \quad (2.50)$$

This last term  $\partial H_1 / \partial J_u$  in Eq. 2.50 will define the amplitude-dependent tune shift. If the whole lattice were just a linear lattice, the betatron oscillations would just gain a phase change of  $2\pi Q_u$  every turn, i.e.,  $\Delta\phi(N) = 2\pi Q_u N$ . Nevertheless, given that this new term  $H_1$  perturbs the dynamics in the accelerator, the phase change will now depend on the amplitude  $J_u$  of the betatron oscillations of the single particle. Therefore, given that this effect acts on each individual particle, it is an incoherent effect. The effective result from this new term is to detune the circular lattice from its original tune  $Q_u$  for each individual particle.

Explicitly calculating the expression from Eq. 2.50 for  $J_x$  and  $\phi_x$  using the Hamiltonian in Eq. 2.47 yields

$$\frac{\partial \phi_x}{\partial N} = 2\pi Q_x + \sum_{jklm} h_{jklm} (j+k) (2J_x)^{\frac{j+k}{2}-1} (2J_y)^{\frac{l+m}{2}} e^{i[(j-k)(\phi_x+\phi_{x0})+(l-m)(\phi_y+\phi_{y0})]}. \quad (2.51)$$

In particular, it is of interest to look at the average limit where particles have undergone large number of turns around the circular accelerator  $N \rightarrow \infty$ . This is done in order to wash out any oscillatory behavior in  $\partial H_1 / \partial J_u$ . Therefore, the following quantity is of interest:

$$\lim_{N \rightarrow \infty} \left\langle \frac{\partial \phi_x}{\partial N} \right\rangle_N = \lim_{N \rightarrow \infty} \frac{1}{N} \int_0^N dN' \frac{\partial \phi_x}{\partial N'}, \quad (2.52)$$

which is just the definition for the average of  $\partial \phi_x / \partial N$  over  $N$  turns, for many turns. Explicitly calculating this quantity gives:

$$\begin{aligned} \lim_{N \rightarrow \infty} \left\langle \frac{\partial \phi_x}{\partial N} \right\rangle_N &= 2\pi Q_x \\ &+ \sum_{jklm} h_{jklm} (j+k) \lim_{N \rightarrow \infty} \frac{1}{N} \int_0^N dN' (2J_x)^{\frac{j+k}{2}-1} (2J_y)^{\frac{l+m}{2}} e^{i[(j-k)(\phi_x+\phi_{x0})+(l-m)(\phi_y+\phi_{y0})]}. \end{aligned} \quad (2.53)$$

In general, it is known that  $J_u, \phi_u$  depend on the number of turns, i.e.,  $J_u, \phi_u = J_u(N), \phi_u(N)$ . Nevertheless, Eq. 2.53 can be approximated by inserting the unperturbed solution of  $H_0$  which means that  $J_u$  is constant and  $\phi_u = 2\pi Q_u N$ . With this in mind and assuming the constants  $\phi_{u0} = 0$  without loss of generality, Eq. 2.53 reduces to

$$\begin{aligned} \lim_{N \rightarrow \infty} \left\langle \frac{\partial \phi_x}{\partial N} \right\rangle_N &= 2\pi Q_x \\ &+ \sum_{jklm} h_{jklm} (j+k) (2J_x)^{\frac{j+k}{2}-1} (2J_y)^{\frac{l+m}{2}} \lim_{N \rightarrow \infty} \frac{1}{N} \int_0^N dN' e^{2\pi i [(j-k)Q_x + (l-m)Q_y]N'}. \end{aligned} \quad (2.54)$$

A closer look into the integral in Eq. 2.54 reveals that this integral in the limit where  $N \rightarrow \infty$  is just a delta function reading  $\delta((j-k)Q_x + (l-m)Q_y)$ . As a reminder, the RDT approximation breaks down if Eq. 2.33 holds. Therefore, it can be shown that the argument in the delta function can only be zero if  $j = k$  and  $l = m$ . Thus, this delta function effectively becomes two Kronecker deltas— $\delta_{jk}$  and  $\delta_{lm}$ . Inserting this into Eq. 2.54 yields:

$$\lim_{N \rightarrow \infty} \left\langle \frac{\partial \phi_x}{\partial N} \right\rangle_N = 2\pi Q_x + \sum_{jklm} h_{jklm} (j+k) (2J_x)^{\frac{j+k}{2}-1} (2J_y)^{\frac{l+m}{2}} \delta_{jk} \delta_{lm}. \quad (2.55)$$

Reducing this expression further with the properties of Kronecker deltas reads:

$$\lim_{N \rightarrow \infty} \left\langle \frac{\partial \phi_x}{\partial N} \right\rangle_N = 2\pi Q_x + 2 \sum_{jl} h_{jllj} (2J_x)^{j-1} (2J_y)^l. \quad (2.56)$$

Equation 2.56 is stating that the constant detuning terms in the accelerators will be given by the terms where  $j = k$  and  $l = m$ . As a consequence of Eq. 2.31, for this case the RDTs will be real numbers, i.e.,  $h_{jklm} \in \mathbb{R}$ . Therefore, the constant detuning terms will come from even orders of  $n = j + k + l + m$ . It is worth remembering that this is a first approximation given that the assumption is that the dynamics are mainly governed by  $H_0$ . Higher order approximations would involve recursively solving Eqs. 2.49 and 2.50 as a Taylor expansion of the actions  $J_u$  [25].

As an example, Eq. 2.56 can be used to calculate the detuning due to horizontal quadrupole errors  $n = 2$ . For this, the calculation would only involve calculating  $h_{1100}$  from Eq. 2.31, the only surviving term. Therefore, the detuning in  $x$  due to quadrupole errors would read

$$\lim_{N \rightarrow \infty} \left\langle \frac{\partial \phi_x}{\partial N} \right\rangle_N = 2\pi Q_x + 2h_{1100} = 2\pi Q_x - \frac{1}{2} \sum_i L_i K_{1,i} \beta_{x,i} = 2\pi Q_x - \frac{1}{2} \sum_i L_i \frac{b_{1,i}}{\rho} \beta_{x,i}, \quad (2.57)$$

where the sum over  $i$  goes around all the quadrupole errors in the ring. Equation 2.57 gives a well-known result in accelerator physics [1].

Another example, can be used to calculate the detuning terms due to octupole components around the ring, i.e.,  $n = 4$ . For this case, the calculation yields

$$\lim_{N \rightarrow \infty} \left\langle \frac{\partial \phi_x}{\partial N} \right\rangle_N = 2\pi Q_x + 4h_{1111} J_y + 8h_{2200} J_x, \quad (2.58)$$

where if  $h_{1100}$  and  $h_{2200}$  are explicitly calculated from Eq. 2.31, this gives:

$$\lim_{N \rightarrow \infty} \left\langle \frac{\partial \phi_x}{\partial N} \right\rangle_N = 2\pi Q_x + \frac{J_y}{4} \sum_i L_i K_{3,i} \beta_{x,i} \beta_{y,i} + \frac{J_x}{8} \sum_i L_i K_{3,i} \beta_{x,i}^2. \quad (2.59)$$

It is important to clarify that the sum over  $i$  goes around all the octupole perturbations in the ring.

To summarize, the amplitude-dependent tune shift  $\Delta (2\pi Q_u)$  is an important quantity that will change the dynamics of particles inside a circular accelerator. In order to calculate this incoherent effect to first order in perturbation theory, the following quantity has to be calculated:

$$\Delta (2\pi Q_u) = 2\pi Q_u - \lim_{N \rightarrow \infty} \left\langle \frac{\partial \phi_u}{\partial N} \right\rangle_N = \lim_{N \rightarrow \infty} \left\langle \frac{\partial H_1}{\partial J_u} \right\rangle_N. \quad (2.60)$$

## 2.7 Space Charge Tune Shift

Up to this point, the beam dynamics of high energy particle accelerators has been explained in terms of single-particle dynamics. Up until now, a couple of implicit assumptions have been made: (a) particles do not interact with each other; and (b) the basic blocks composing the lattice have been idealized to create the fields but without supplying any electromagnetic boundary conditions. Nevertheless, in order to have a model closer to reality, the interaction between particles through the Coulomb force has to be taken into account. Furthermore, particles also interact with the electromagnetic properties of the basic elements, ultimately, creating unwanted electromagnetic wake fields. This latter phenomenon opens another branch of accelerator physics that studies collective beam instabilities [26]. However, the scope of this thesis is only interested in the first bullet point regarding direct particle-particle interactions through the Coulomb force—widely known as space charge physics. It is worth specifying that this restricts the analysis to a single bunch of particles.

As mentioned in Sec. 2.4, the Coulomb force will act as a detuning force on each individual particle. In order to explain this statement using a Hamiltonian formalism, the starting point needs to be the single-particle Hamiltonian that includes the Coulomb potential from the charge distribution in the bunch [27]. The expression for this system reads:

$$H(x, y) = H_0(x, y) + H_1(x, y) + \Psi(x, y, \tilde{z}), \quad (2.61)$$

keeping in mind that  $x$  and  $y$  are interchangeable for their respective action-phase variables  $J_x, \phi_x, J_y, \phi_y$ . For a bunched beam, the variable  $\tilde{z} = z - z_b$  is introduced in order to represent the longitudinal distance from the center of the bunch, always keeping in mind that the reference system moves with the bunch as described by a Frenet-Serret coordinate system. The center of the bunch with coordinates  $(0, 0, z_b)$  is determined from the centroid of the longitudinal charge distribution. The one-turn space-charge potential  $\Psi$ , is the average potential from all the space charge kicks around the accelerator for one turn. Nevertheless, the functional form should be

similar to Eq. 2.31, in order to apply the same methods of Sec. 2.6. Therefore, this reads::

$$\Psi(J_x, \phi_x, J_y, \phi_y, \tilde{z}) = \sum_{jklm} G_{jklm} (2J_x)^{\frac{j+k}{2}} (2J_y)^{\frac{l+m}{2}} e^{i[(j-k)(\phi_x + \phi_{x0}) + (l-m)(\phi_y + \phi_{y0})]}. \quad (2.62)$$

Analogous to Eq. 2.31, the terms  $G_{jklm}$  are named the global space-charge resonance driving terms (GSCRDTs). These terms are calculated as the one-turn average of the instantaneous driving terms of order  $l, j, k$  and  $m$  which have an explicit  $s$  dependence. Therefore, the definition for the GSCRDTs  $G_{jklm}$  reads:

$$G_{jklm} = \frac{1}{C} \int_{s_0}^{s_0+C} \tilde{V}_{jklm}(s) e^{i[(j-k)(\phi_x(s) + \phi_{x0}) + (l-m)(\phi_y(s) + \phi_{y0})]} ds, \quad (2.63)$$

where the terms  $\tilde{V}_{jklm}(s)$  are the instantaneous driving terms of the expansion for the self-field potential  $\psi$  of the bunch at a point  $s$  of the accelerator, i.e.,

$$\psi(J_u, \phi_u, \tilde{z}, s) = f(\tilde{z}) + \sum_{jklm} \tilde{V}_{jklm}(s) (2J_x)^{\frac{j+k}{2}} (2J_y)^{\frac{l+m}{2}} e^{i[(j-k)(\phi_x + \phi_{x0}) + (l-m)(\phi_y + \phi_{y0})]}. \quad (2.64)$$

The term  $f(\tilde{z})$  will hold any longitudinal dependence of the potential, but in general it is not of relevance to the space charge detuning calculation. This is a similar approach as the one taken by Ref. [28]. All of this assumes the self-field potential has been calculated and a Floquet transformation has been performed using Eq. 2.8. The self-potential  $\psi(x, y, \tilde{z}, s)$  is determined self-consistently from the 3D Poisson equation with a particle number density of the bunch  $n_b(x, y, \tilde{z}, s)$  reading:

$$\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial \tilde{z}} \right) \psi(x, y, \tilde{z}, s) = -\frac{2\pi K_b}{N_b} n_b(x, y, \tilde{z}, s), \quad (2.65)$$

where  $K_b$  is a dimensionless parameter known as the self-field perveance defined as

$$K_b = \frac{2N_b e_b^2}{\gamma_L^3 m_b \beta_L^2 c^2}, \quad (2.66)$$

with  $N_b$  being the total number of particles in the bunch defined as  $N_b = \int dx dy d\tilde{z} n_b(x, y, \tilde{z}, s)$ ,  $e_b$  is the charge of one beam particle,  $m_b$  is the rest mass of one beam particle,  $\gamma_L$  and  $\beta_L$  being the relativistic longitudinal factors of a beam with total energy  $E_T = \gamma_L m_b c^2$ , and  $c$  being the speed of light.

The solution to Eq. 2.65 can be found in any mathematical methods for physics book, see Refs. [29, 30]. The solution to this equation involves using Green's function and finding the convolution with the particle number density  $n_b(x, y, \tilde{z}, s) = n_b(\vec{r}, s)$ . This reads:

$$\psi(x, y, \tilde{z}, s) = -\frac{2\pi K_b}{N_b} \int_{\mathbb{R}^3} d\vec{r}' \frac{n_b(\vec{r}', s)}{|\vec{r} - \vec{r}'|}. \quad (2.67)$$

In section 2.6 it was of interest to look at how particles at different amplitudes undergo different detuning due to  $H_1$  given their  $J_u$  coordinates. The same will be true for the effect of  $\Psi$  on the single-particle dynamics. The quantity of interest will be the detuning due to Coulomb forces—the space-charge tune shift. Nevertheless, it is important to note that space charge will be a continuous force all around the accelerator. Therefore, the terms  $G_{jklm}$  in order to enter the one-turn Hamiltonian in Eq. 2.61 will have to be an average of the space-charge force around the ring for a single particle, just like it was defined in Eq. 2.63. Ultimately, the space-charge tune shift will also be an incoherent quantity.

For now, let  $H_1 = 0$ , but let space-charge dictate the functional form of  $\Psi$ . For this case, and in analogy to Eq. 2.50, the space charge tune shift will be dictated by:

$$\frac{\partial \phi_u}{\partial N} = \frac{\partial H}{\partial J_u} = 2\pi Q_u + \frac{\partial \Psi}{\partial J_u}. \quad (2.68)$$

The non-trivial part about this calculation is figuring  $\partial\Psi/\partial J_u$  for a specific bunch distribution after calculating the integral in Eq. 2.67.

The first case to analyze is when the bunch has a uniform charge density enclosed in a 3D ellipsoid. The charge bunch distribution reads:

$$n_b(x, y, \tilde{z}, s) = \begin{cases} \hat{n}_b(s), & \text{if } 0 \leq \frac{x^2}{a^2(s)} + \frac{y^2}{b^2(s)} + \frac{\tilde{z}^2}{c^2(s)} \leq 1 \\ 0, & \text{if else.} \end{cases}, \quad (2.69)$$

where  $a(s)$ ,  $b(s)$  and  $c(s)$  are the length of the semi-axes of the ellipsoid around the accelerator, and can be calculated from the beam-envelope equations [27]. Additionally, the constant particle number density over an ellipsoid is defined as  $\hat{n}_b = N_b/V_e$ , where  $V_e$  is the volume of an ellipsoid reading  $V_e = (4/3)\pi abc$ . Inputting this distribution into Eq. 2.67 gives the expression for the

potential. This involves solving a 3D integral inside the ellipsoid. This reads:

$$\psi(x, y, \tilde{z}, s) = -\frac{2\pi K_b \hat{n}_b}{N_b} \int_{\mathbb{S}} d\vec{r}' \frac{1}{|\vec{r} - \vec{r}'|}, \quad (2.70)$$

where  $\mathbb{S}$  is the region enclosed by the ellipsoid. The integral has been studied with ellipsoidal coordinates and the solution can be found in Refs. [27, 31]. In these references, it is shown that the solution can be expressed as a quadratic function of  $x, y$  and  $\tilde{z}$  such that:

$$\psi(x, y, \tilde{z}, s) = -\frac{\pi K_b \hat{n}_b abc}{N_b} [-Ax^2 - By^2 - \tilde{C}\tilde{z}^2 + D], \quad (2.71)$$

such that  $A, B, \tilde{C}$  and  $D$  are calculated as:

$$A(s) = \int_0^\infty \frac{d\varsigma}{(a^2 + \varsigma) \sqrt{(a^2 + \varsigma)(b^2 + \varsigma)(c^2 + \varsigma)}}, \quad (2.72)$$

$$B(s) = \int_0^\infty \frac{d\varsigma}{(b^2 + \varsigma) \sqrt{(a^2 + \varsigma)(b^2 + \varsigma)(c^2 + \varsigma)}}, \quad (2.73)$$

$$\tilde{C}(s) = \int_0^\infty \frac{d\varsigma}{(c^2 + \varsigma) \sqrt{(a^2 + \varsigma)(b^2 + \varsigma)(c^2 + \varsigma)}}, \quad (2.74)$$

$$D(s) = \int_0^\infty \frac{d\varsigma}{\sqrt{(a^2 + \varsigma)(b^2 + \varsigma)(c^2 + \varsigma)}}. \quad (2.75)$$

Furthermore, Eq. 2.71 can be rewritten using Eq. 2.8, in order to explicitly highlight the dependence on  $J_u, \phi_u$ . This reads:

$$\begin{aligned} \psi(J_u, \phi_u, \tilde{z}, s) = & -\frac{\pi K_b \hat{n}_b abc}{N_b} \left[ -A(2\beta_x J_x) \left( \frac{e^{2i(\phi_x + \phi_{x0})} + e^{-2i(\phi_x + \phi_{x0})} + 2}{4} \right) \right. \\ & \left. - B(2\beta_y J_y) \left( \frac{e^{2i(\phi_y + \phi_{y0})} + e^{-2i(\phi_y + \phi_{y0})} + 2}{4} \right) - \tilde{C}\tilde{z}^2 + D \right]. \end{aligned} \quad (2.76)$$

Equation 2.76 can be cross-examined using Eq. 2.64 in order to retrieve the driving terms  $\tilde{V}_{jklm}$  and  $f(\tilde{z})$ . Therefore, for a uniform distribution, the following expansion for the self-potential holds:

$$\begin{aligned} \psi(J_u, \phi_u, \tilde{z}, s) = & f_0(\tilde{z}) + \tilde{V}_{2000} J_x e^{2i(\phi_x + \phi_{x0})} + \tilde{V}_{0200} J_x e^{-2i(\phi_x + \phi_{x0})} + \tilde{V}_{1100} J_x \\ & + \tilde{V}_{0020} J_y e^{2i(\phi_y + \phi_{y0})} + \tilde{V}_{0002} J_y e^{-2i(\phi_y + \phi_{y0})} + \tilde{V}_{0011} J_y. \end{aligned} \quad (2.77)$$

The instantaneous driving terms  $\tilde{V}_{jklm}$  from Eq. 2.77 are summarized in Table 2.2. Table 2.2 also shows how to calculate the SCRTDs  $G_{jklm}$  from averaging out the  $\tilde{V}_{jklm}(s)$  all around the circular accelerator as an example of using Eq. 2.63. The integrals for the  $G_{jklm}$  should be carried out with previous knowledge of the unperturbed beta functions and phase advances all around the ring. The envelope equations, as described in Ref. [27], should also be solved previously in order to get  $a(s)$ ,  $b(s)$  and  $c(s)$  around the lattice, ultimately, aiding in the evaluation of  $A(s)$ ,  $B(s)$  and  $C(s)$ .

Table 2.2 Instantaneous driving terms and GSCRTDs for a uniform beam

$\tilde{V}_{jklm}$	$\tilde{V}_{jklm}(s)$	$G_{jklm}$	$G_{jklm}$
$\tilde{V}_{2000}$	$\frac{3}{8}K_b A(s)\beta_x(s)$	$G_{2000}$	$\frac{3K_b}{8C} \int_{s_0}^{s_0+C} A(s)\beta_x(s)e^{2i(\phi_x(s)+\phi_{x0})} ds$
$\tilde{V}_{0200}$	$\frac{3}{8}K_b A(s)\beta_x(s)$	$G_{0200}$	$\frac{3K_b}{8C} \int_{s_0}^{s_0+C} A(s)\beta_x(s)e^{-2i(\phi_x(s)+\phi_{x0})} ds$
$\tilde{V}_{1100}$	$\frac{3}{4}K_b A(s)\beta_x(s)$	$G_{1100}$	$\frac{3K_b}{4C} \int_{s_0}^{s_0+C} A(s)\beta_x(s) ds$
$\tilde{V}_{0020}$	$\frac{3}{8}K_b B(s)\beta_y(s)$	$G_{0020}$	$\frac{3K_b}{8C} \int_{s_0}^{s_0+C} B(s)\beta_y(s)e^{2i(\phi_y(s)+\phi_{y0})} ds$
$\tilde{V}_{0002}$	$\frac{3}{8}K_b B(s)\beta_y(s)$	$G_{0002}$	$\frac{3K_b}{8C} \int_{s_0}^{s_0+C} B(s)\beta_y(s)e^{-2i(\phi_y(s)+\phi_{y0})} ds$
$\tilde{V}_{0011}$	$\frac{3}{4}K_b B(s)\beta_y(s)$	$G_{0011}$	$\frac{3K_b}{4C} \int_{s_0}^{s_0+C} B(s)\beta_y(s) ds$

Just like in Sec. 2.6, it is of interest to look at the terms where  $j = k$  and  $l = m$ . This will give nonlinear detuning terms due to having a uniform charge distribution. Therefore, similar to Eq.

2.56, the nonlinear detuning terms due to space charge for the horizontal plane are:

$$\lim_{N \rightarrow \infty} \left\langle \frac{\partial \phi_x}{\partial N} \right\rangle_N = 2\pi Q_x + 2 \sum_{jl} G_{jlli} j (2J_x)^{j-1} (2J_y)^l. \quad (2.78)$$

Therefore, it would be of interest to look at the terms  $G_{1100}$  and  $G_{0011}$  which will cause constant detuning of the unperturbed tunes.

The previous analysis has been done for a uniform beam, i.e., the particle number density is constant inside the bunch. Nevertheless, this is an idealized case for a circular accelerator. A more realistic approach would be to consider a Gaussian distribution. This multivariate distribution assuming no correlation reads:

$$n_b(x, y, \tilde{z}, s) = \frac{1}{(2\pi)^{3/2} \sigma_x \sigma_y \sigma_{\tilde{z}}} e^{-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{\tilde{z}^2}{2\sigma_{\tilde{z}}^2}} \quad (2.79)$$

This is the approach taken in Refs. [24, 28]. Specifically, the Python library developed at CERN called PySCRDT [24] allows for calculation of GSCRDTs, as well as calculation of the tune footprint, for a Gaussian beam distribution. As mentioned before in the discussion of Fig. 2.4, the tune footprint depends on the intensity of the beam interchangeable with the beam space charge perveance parameter  $K_b$ . Figure 2.5 shows the space charge tune footprint for the Recycler Ring for typical beam parameters used in studies. The footprint has been calculated using PySCRDT. Nevertheless, the approach behind PySCRDT is similar to the example provided with a uniform beam distribution. One special feature of the Gaussian beam is that the space charge potential is its largest at the core of the beam. Hence, the detuning is largest for particles close to the middle of the bunch. In contrast, the particles away from the core of the beam will not be subject to the full space charge potential, and therefore, will oscillate close to the unperturbed nominal tune of  $Q_u$ . Figure 2.5 illustrates this concept by drawing a color map at different actions, close to the beam core and  $3\sigma_u$  away from the center of the bunch—location where particles are oscillating close to nominal tunes.



Figure 2.5 Tune footprint for a Gaussian beam in the Recycler Ring at an intensity of 5e10 particles per bunch created with PySCRDT.

Up until now, for this section the assumption  $H_1 = 0$  holds. Nevertheless, in high-intensity particle accelerators there will be an interplay between the nonlinear Hamiltonian and the space charge potential. Ultimately, the terms  $h_{jklm}$  and  $G_{jklm}$  will dictate the perturbation to the linear Hamiltonian by their effective detuning. Therefore, the following Hamiltonian will describe the perturbation due to nonlinear elements in the circular lattice and due to space charge forces:

$$H(J_x, \phi_x, J_y, \phi_y, \tilde{z}) = 2\pi Q_x J_x + 2\pi Q_y J_y + \sum_{jklm} (h_{jklm} + G_{jklm}) (2J_x)^{\frac{j+k}{2}} (2J_y)^{\frac{l+m}{2}} e^{i[(j-k)(\phi_x + \phi_{x0}) + (l-m)(\phi_y + \phi_{y0})]}. \quad (2.80)$$

## CHAPTER 3

### THE FNAL RECYCLER RING

#### 3.1 Introduction

The Fermilab Recycler Ring (RR) is one of the circular accelerators located in the Fermilab Accelerator Complex. It was originally designed to store and accumulate antiprotons that remained from a Tevatron event [32]. The recycling of antiprotons was deemed ineffective and was never operationally implemented [33]. Since 2011, the RR has been repurposed to act as a pre-injector to the Main Injector (MI) by storing and accumulating protons [14]. It is worth pointing out, that the MI and the RR share the same tunnel, which has a circumference of 3.319 km (2.062 mi). The work done for this thesis focuses on the Recycler Ring. The following chapter starts by giving a general description for the operation and physics of the Recycler Ring. The next sections introduce and motivate the compensation of third order resonances for high intensity operation.

The MI/RR complex is fed protons by the Proton Source, which by itself consists of the Pre-Accelerator, the Linear Accelerator (Linac), and the Booster. The Pre-Accelerator systems provide  $H^-$  ions to the Linac, where they are accelerated to an energy of 400 MeV. After this, the beam is injected into the Booster Ring. The Booster is a rapid-cycling synchrotron operating at a 15 Hz repetition rate. During this injection process, the  $H^-$  beam passes through a carbon stripping foil, and it incorporates to the circulating proton beam. The Booster ramps the energy up from 400 MeV to 8 GeV. This 8 GeV proton beam can either go to the Booster Neutrino Experiments or get injected into the Recycler Ring. Once in RR the beam has two possible destinations: 1) high energy neutrino experiments through MI or 2) Muon Campus. For the latter, proton beam gets rebunched from 53 MHz to 2.5 MHz and transported to Muon Campus. For high energy neutrino experiments, the proton beam gets slip-stacked, hence doubling the intensity that gets injected into Main Injector. Once in MI, the beam is accelerated to 120 GeV and sent to the NuMI (Neutrinos at the Main Injector) beam facility [13, 14, 33]. A description of the current accelerator complex is shown in figure 3.1, including the experimental beamlines which feed neutrino, muon and fixed target experiments.



Figure 3.1 Current operational layout of the Fermilab Accelerator Complex as of 2024. Original plot provided by R. Ainsworth, first published on Ref. [14], but modified for this document.

The Proton Improvement Plan II (PIP-II) is the first step in establishing the Fermilab Accelerator Complex as a multi-MW proton facility [34]. The near-future objective is to deliver a 1.2 MW proton beam to the Deep Underground Neutrino Experiment (DUNE) through the Long-Baseline Neutrino Facility (LBNF) [35], still in construction. In order to meet this goal, several upgrades are being planned in the accelerator complex, including a new 800 MeV superconducting linear accelerator. The future plan for the layout of the Fermilab Accelerator Complex is shown in Fig. 3.2. With minimal upgrades to the Main Injector and Recycler Ring, but with a substantial overhaul of the Booster Ring, this will allow for a 50% increase in particles per pulse intensity. Table 3.1 also specifies some upgrades that will happen for the PIP-II era. Some examples include an increase of the particle per bunch intensity, a shortening of the Main Injector acceleration ramp and an increase in the Booster ramping rate. As the Recycler Ring starts to deal with higher intensities from the PIP-II upgrade, it is important to mitigate the effects of space charge as discussed in Secs. 2.4 and 2.7. Particles along the bunch will experience space charge forces leading to detuning in their betatron frequencies. Given the incoherent nature of this process, this leads to the beam having a larger tune spread in the tune diagram and having particles operate on top of resonances.



Figure 3.2 Future layout of the Fermilab Accelerator Complex for the Proton Improvement Plan II (PIP-II). Original plot provided by R. Ainsworth, first published on Ref. [14], but modified for this document.

### 3.2 General Description

The RR is a permanent magnet storage ring operating at a fixed momentum of  $8.835 \text{ GeV}/c$  equivalent to an energy of 8 GeV. The basic cell structures of this machine are FODO (Focusing Quadrupole - Drift - Defocusing Quadrupole - Drift) cells. During its conception, the need for a quick and non-expensive design spurred the idea of combining quadrupole and dipole magnets into one combined function magnet. These combined function magnets can be seen in Fig. 3.3 as the green covered magnets on the top ring—the Recycler Ring. In order to further reduce costs during its construction, these magnets were chosen to be permanent magnets made out of a strontium ferrite [32]. Some advantages of having permanent magnets is that there is no need for power supplies, cooling systems or power distribution cables. Consequently, these type of magnets are very stable against time and temperature. Nevertheless, the magnetic field of such magnets does degrade over time. Reference [14] shows how the fields in RR-type magnets can degrade around 1% after 20 years. Ultimately, this slightly changes the nominal energy in the machine.



Figure 3.3 Picture of the Main Injector (blue and red magnets in the bottom) and the Recycler Ring (green magnets up top) tunnel.

Figure 3.3 reminds the reader that the Main Injector and the Recycler Ring share the same tunnel. This tunnel is divided into six sections with labels: 100, 200, 300, 400, 500 and 600. Injection from the Booster Ring into the Recycler takes place just after the beginning of section 100. Specifically, this happens at a Lambertson magnet labelled as LAM102. This is just an example of how every element in the RR or MI will be labelled according to their position in one of these sections. Figure 3.4 shows a schematic of the Recycler Ring section with some labels for important subsystems inside them. In particular, it shows the location close to 232 where beam is transferred from the Recycler Ring to the Main Injector. Figure 3.4 also presents the location in section 500 where beam is transferred from RR to the Muon Campus, as well as the location after 401 where beam is dumped towards the abort line [14, 36].

An important subsystem as highlighted in Fig. 3.4 is the tune trombone. The RR has two of them, one located in sections 601-609 and another one in 301-309. A tune trombone or phase

trombone is a linear insert composed of quadrupoles used to introduce local phase advances. These quadrupoles are powered in such a way that they add a controlled phase advance, while leaving the Twiss parameters unchanged at the end of the insert and matched to the start of the trombone [32, 37]. Ultimately, these subsystems introduce local tune changes that allow to control both tunes in a range of  $\pm 0.5$ , i.e.,  $\Delta Q_u = \pm 0.5$ . The Recycler Ring has accelerator applications through ACNET that allow to control these tune trombones and set the tunes to some desired values. Furthermore, this application allows introducing tune ramps, continuous linear changes in the tunes. This feature is crucial for building dynamic loss maps—an important tool to probe resonance lines as will be described in Sec. 4.5.1.

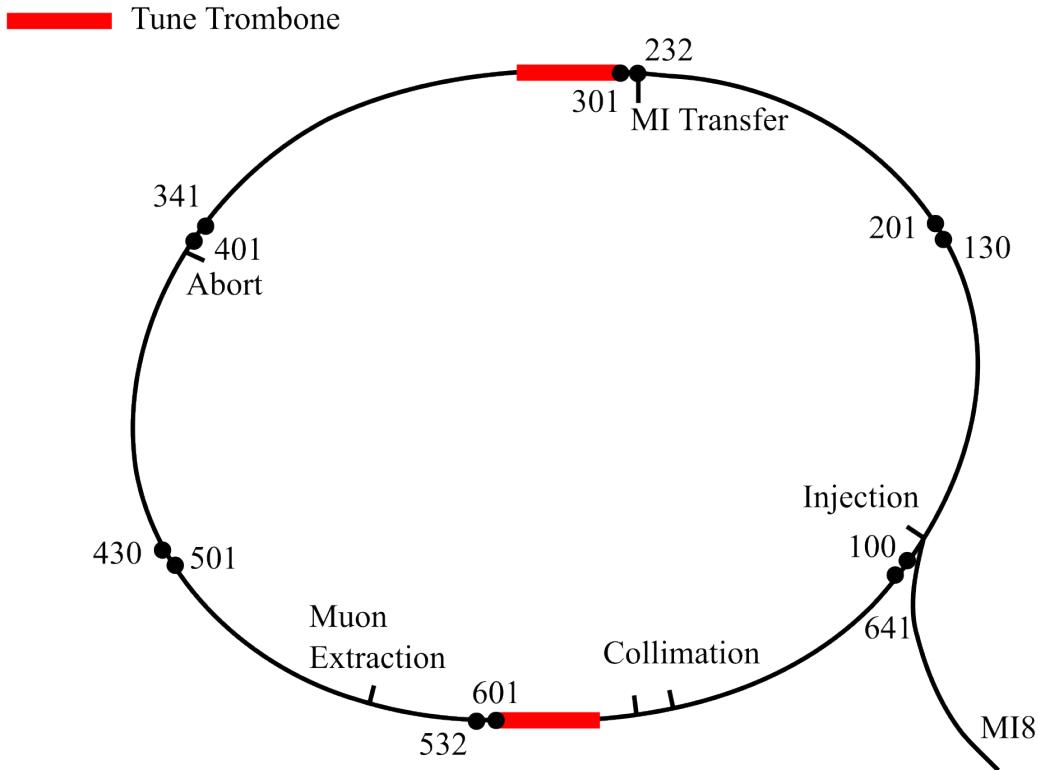


Figure 3.4 Schematic layout of the Recycler Ring and its corresponding sections. Original plot provided by R. Ainsworth, first published on Ref. [14].

The Recycler Ring has 104 FODO cells distributed into 3 main groups around its 3319.4-meter circumference [32]. The first group of cells are composed of two permanent magnet quadrupoles arranged in such a way that each cell has a phase advance of  $90^\circ$ . There are 18 cells of this first

group throughout the Recycler. The second group is defined by a cell with two combined function magnets in order to bend and focus the beam at the same time. This type of cell also has a phase advance of  $90^\circ$ , with 54 cells of this type in the RR. Ultimately, the configuration of these cells will dictate the Twiss parameters around the ring. Figure 3.5 shows the beta functions  $\beta_u(s)$  around the Recycler Ring tuned to some particular set of tunes. Some horizontal BPMs are marked in the upper horizontal axis as reference to note the corresponding section in the tunnel.

Before describing the third group, it is worth clarifying to the reader that the transverse motion inside any circular accelerator is dictated by:

$$u(s) = u_\beta(s) + D_u(s)\delta, \quad (3.1)$$

where  $u$  is either the  $x$  or  $y$  plane,  $u_\beta(s)$  is the betatron motion as described by Eq. 2.8,  $D_u(s)$  is known as the dispersion function and  $\delta = \Delta p/p_0$  is the fractional momentum deviation of an individual particle with respect to reference longitudinal momentum  $p_0$ . In Ch. 2, specifically in Sec. 2.2, an assumption was made that the transverse coordinates were not going to be influenced by any longitudinal property, focusing on on-momentum particles, i.e.,  $\delta = 0$ . Nevertheless, for this section it is relevant to introduce the dispersion function  $D_u(s)$  in order to describe the Recycler Ring in all its glory and detail. The dispersion function is dictated by the distribution of quadrupole and dipole component throughout the ring [1]. Even so, the scope of this work regarding resonance compensation will not take into account any longitudinal-transverse coupling.

With all this said, the third group of cells is composed of special dispersion suppressor cells,  $D_x(s) = 0$  for these regions. Each one having two combined function magnets and globally having a betatron phase advance of again  $90^\circ$ . This type of cells will allow having dispersion-free regions which are necessary for injection, extraction and other special subsystems in the Recycler. There are 32 cells of this type. Figure 3.6 shows the dispersion functions for the Recycler Ring. The horizontal dispersion function  $D_x(s)$  has regions of close-to-zero values thanks to this third type of FODO cells. The vertical dispersion is effectively negligible, i.e.,  $D_y(s) \approx 0$ .



Figure 3.5 Beta functions for the Recycler Ring lattice tuned to  $Q_x = 25.44$  and  $Q_y = 24.39$ . Lattice functions calculated from lattice file using SYNERGIA.



Figure 3.6 Dispersion functions for the Recycler Ring lattice tuned to  $Q_x = 25.44$  and  $Q_y = 24.39$ . Lattice functions calculated from lattice file using SYNERGIA.

The ultimate role of the Recycler Ring is to smooth the way for high-intensity proton beam injection into the Main Injector. In particular, for the high-intensity beam that gets delivered to the neutrino experiments, the Recycler performs a beam manipulation known as slip-stacking. Although this longitudinal manipulation is out of the main scope of this thesis, it is relevant to briefly explain this procedure since this is the Recycler Ring's main feature. Slip-stacking is the process through which a pair of bunches are manipulated in longitudinal phase space in order to merge them into one higher-intensity bunch [36]. In particular, the RR utilizes two RF stations to

decelerate and accelerate trains of bunches. The Recycler can fit 7 Booster Rings, hence 6+6 beam batches are injected and slip-stacked, while 1 batch space is left as a gap for the kickers to fire. Each batch is composed of 81 bunches in accordance to the harmonic number of the Recycler. A more detailed explanation of the Recycler Ring's slip-stacking procedure can be found in Refs. [14, 36, 38]. Ultimately, this slip-stacking process is relevant because it doubles the particles per bunch intensity (ppb) which has important consequences for the transverse dynamics, as explained in Sec. 2.7. Furthermore, Table 3.1 summarizes some important parameters for the RR slip-stacking and for the general operation of neutrino-bound beam.

Table 3.1 Typical Recycler Ring properties for beam sent to NuMI, with some PIP-II nominal parameters.

Parameter	Value	Unit
Circumference	3319.4	m
Momentum	8.835	GeV/c
Revolution Period	11.1	$\mu$ s
Revolution Frequency	90.1	kHz
RF Frequency	52.8	MHz
RF Voltage	80	kV
Harmonic Number	588	
Synchrotron Tune	0.0028	
Slip Factor	$-8.6 \times 10^{-3}$	
Superperiodicity	2	
Horizontal Tune	25.43	
Vertical Tune	24.445	
Horizontal Chromaticity	-6	
Vertical Chromaticity	-7	
95% Normalized Emittance	15	$\pi$ mm mrad
95% Longitudinal Emittance	0.08	eV s
Intensity	$5 \times 10^{10}$	ppb
	$8 \times 10^{10}$ (PIP-II)	ppb
MI Ramp Time	1.2	s
	1.133	s
	1.067	s
Booster Frequency	15	Hz
	20 (PIP-II)	Hz

### 3.3 Tune Diagram and Resonances

Table 3.1 specifies nominal horizontal and vertical tunes for the operation of the Recycler Ring. While this pair of tunes can be switched around during operation in order to reduce losses, they will always stay clear of the surrounding transverse betatron resonances, as introduced in Sec. 2.4. The resonance lines that are relevant to the Recycler Ring operation are plotted in Fig. 3.7, with their corresponding order in the multipole expansion. Specifically, this study looks at normal sextupole lines  $3Q_x = 76$  and  $Q_x + 2Q_y = 74$ , plus skew sextupole lines  $3Q_y = 73$  and  $2Q_x + Q_y = 75$ .



Figure 3.7 Portion of the tune diagram enclosing the operational tunes of the Recycler Ring.

Starting from the lowest order resonance line, Fig. 3.7 shows the second-order linear coupling line  $Q_x - Q_y = 1$ . This resonance line comes from skew-quadrupole component accumulated all around the ring. While operating on top of this resonance line won't lead to chaotic motion, the dynamics are characterized by action-exchange between the betatron modes [1]. This exchange can cause emittance growth, eventually leading up to beam loss. Due to this fact, the Recycler Ring has two families of skew quadrupoles in order to correct this resonance line. Consequently, operating close to this resonance line is feasible with this correction.

The third order resonance lines are the main focus of this thesis. They are driven by normal and skew sextupole component distributed all around the ring. In particular, it is known that the combined function magnets are the main drivers of these resonance lines. Even though these magnets were designed to be quadrupole and dipole only, they can still have sextupole component from construction errors. Nevertheless, since they are permanent magnets little can be done about the magnets themselves. As a result of this, the following work explores a compensation technique aimed towards bringing down the lattice RDTs that drive these resonance lines. This involves the usage of additional compensation sextupoles, elements which will be described in Sec. 3.6. Chapter 4 goes into a deep dive explaining this approach from a theoretical and experimental point of view.

Finally, there are also fourth order resonance lines close to the nominal operation of the Recycler. These lines are driven by octupole components around the ring. In practice,  $-Q_x + 3Q_y = 48$  and  $3Q_x - Q_y = 52$  are not particularly concerning to the operation of the RR. Their corresponding lattice RDTs are not large enough to consider them dangerous during the RR cycle, i.e., beam is stored for approximately 1 second. Nevertheless, these lines are interesting because fourth order resonance lines are susceptible to be amplified by the space charge potential through the SCRDTs, as described in Sec. 2.7.

### 3.4 High Intensity and Tune Footprint

In order to achieve the PIP-II beam power objective, the Recycler will be required to store and accumulate 50% more beam than current operations [34]. As it was explained in Sec. 2.7, larger

beam intensities lead to larger space charge tune spread. Figure 3.8 shows how for typical operation to the neutrino experiments under future PIP-II specifications, there is a space charge tune spread of around 0.1 in both planes. At nominal tunes, it is clear that for high intensity operation the particles in the core of the beam will start to operate on top of third order resonances. One thing complicates this picture further, which is that particles will undergo synchrotron oscillations within the diamond shape region. This means that particles will go from the core of the beam to its outskirts, and its incoherent space charge tune shift will fluctuate accordingly. Therefore, particles will be crossing in and out of these third order resonances at a synchrotron period. Ultimately, Fig. 3.8 just tries to take a snapshot of the region of these excursions.



Figure 3.8 Approximate operational tune footprint at high intensities, i.e.,  $1e11$  particles per bunch.

Figure 3.8 and the latter discussion motivate the need for mitigating the deleterious effects of  $3Q_x = 76$  and  $Q_x + 2Q_y = 74$ , and weaker effects from  $3Q_y = 73$  and  $2Q_x + Q_y = 75$ . If the RDT

of each resonance line is reduced, then their strength becomes less, losses in those regions become less, and ultimately there will be a larger area in the tune diagram for operation. This is desirable for PIP-II. Nevertheless, as it will be explored in Ch. 4, compensating one resonance line might make another one worse and vice versa. Here lies the difficulty of this approach. Additionally, before going to PIP-II intensities, these third order resonance lines need to be characterized properly at low intensities. Experimentally, at high intensities, everything starts to blend together in the tune diagram due to the space charge tune shift. This motivates the fact to have a distinction between Ch. 4 and 6.

### 3.5 Diagnostic Devices

The Recycler Ring has two main diagnostic devices that fall under the scope of this work: the Beam Position Monitors (BPMs) and the Ion Profile Monitors (IPMs). Although both systems are used to quantify properties of the beam, each one gives different information about the beam distribution. In particular, BPMs are used to probe the first moment of the transverse distribution, i.e., the mean position or beam centroid of the bunch. This is done either in one plane or in both planes simultaneously, depending on the BPM system. On the other hand, IPMs go one order higher and give information about the second order moment of the transverse beam distribution—information about the variance and spread of the bunch. Specifically for the Recycler case, there is one IPM for the horizontal direction and another one for the vertical case.

There are 208 BPMs in the Recycler Ring in total. Specifically, there are 104 horizontal BPMs and 104 vertical BPMs. Each class is oriented in order to measure the corresponding plane. The BPMs are labelled according to their position around the ring as it was described in Sec. 3.2 and in Fig. 3.4. One BPM consists of two parallel pick-up electrodes that produce electric signal once the beam passes through. The beam position is determined from the relative amplitude between the signals of the opposing channels [39]—also known as the  $(A - B)/(A + B)$  signal. This signal is digitized and calibrated to include the scaling factors and offsets, ultimately, in order to represent the transverse beam position. The digitized signal from all 208 BPMs is interfaced to ACNET, in order to be used for accelerator applications. The resulting data is digitized every turn, hence

known as turn-by-turn (TbT) data. Figure 3.9 shows an example of this TbT BPM data for a beam that was pinged in the horizontal direction and recorded for 2048 turns.

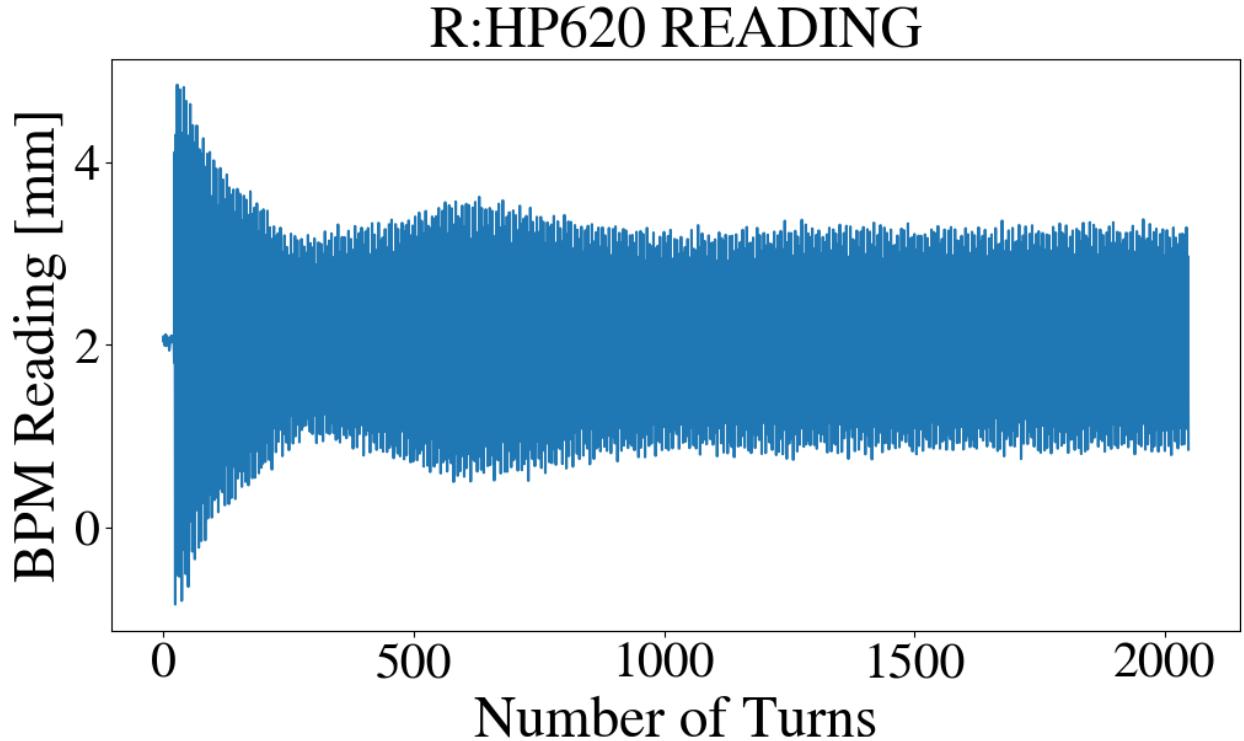


Figure 3.9 BPM turn-by-turn data for an arbitrary kick at horizontal BPM R:HP620.

One important example of using TbT data is using it in order to perform tune measurements. As mentioned in Sec. 2.1, specifically in Eq. 2.8, the motion inside the Recycler Ring will exhibit betatron oscillations. In particular, the main harmonic of these oscillations will be dictated by the tune frequency. Say a particular set of BPM TbT data has been recorded for  $N$  number of turns. Therefore, a Fast-Fourier Transform (FFT) will help uncover and measure to FFT accuracy— $\sigma_Q \approx 1/N$ —the main frequency of these oscillations. Figure 3.10 shows how the main peak of the FFT can be identified in order to measure the horizontal tune  $Q_x$  of the circular accelerator. Furthermore, Ch. 4 will explore how a more involved Fourier Transform algorithm, such as NAFF (Numerical Analysis of Fundamental Frequencies) [40], will help analyze the spectrum of TbT data and measure higher order harmonics, ultimately, leading to the measurement of RDTs in the RR.



Figure 3.10 Fast Fourier Transform amplitude for the turn-by-turn data presented in Fig. 3.9.

The second type of diagnostics relevant to this work are the Ion Profile Monitors (IPMs). As mentioned before, this device is used to measure the beam size in the Recycler Ring. The RR has two IPM systems: one for the horizontal plane and another one for the vertical. One special characteristic of IPMs relies on their non-destructive nature. The working principle of this system is to collect ions created in the interaction between the proton beam and residual molecules in the vacuum. The secondary ions are detected with micro-channel plates (MCP), which are biased at an arbitrary high voltage [41]. The counts at each MCP are logged in and digitized in order to be compatible with ACNET. Similar to the BPM systems, an application was developed to control and use the IPM system.



Figure 3.11 Reconstructed beam profile for the horizontal plane assuming Gaussian distributions along 1024 turns for an arbitrary beam in the Recycler Ring.

Figure 3.11 shows the reconstruction of the horizontal beam profile from the raw data of the IPM micro-channel plates. At every turn, the raw count data from each of the 96 MCP is fit to Gaussian beam distribution resulting in some variance  $\sigma$ . The color map shows the evolution of this beam profile along 1024 turns. The IPM system can record up to 64000 turns. Nevertheless, it batches the data in such a way that it can fit into a  $1024 \times 96$  array. In particular, Fig. 3.11 shows how for this arbitrary experimental conditions, the size of the beam grows with the number of turns—this is pointing at emittance growth. Other distributions different from a Gaussian can be used to fit the raw data in order to extract more information about the skewness or tail population of the beam, e.g., a q-Gaussian or a Skew-Gaussian distribution.

### 3.6 Sextupoles for Resonance Compensation

The ideal linear lattice of a circular accelerator would be composed only of dipoles and quadrupoles—assuming the beam is composed of mono-energetic particles. Nevertheless, this is a strong assumption given that a bunch of particles will always have an energy spread. Particles orbiting at different energies will feel different quadrupole fields resulting in betatron detuning. This is the definition of chromaticity, and therefore its mathematical definition reads:

$$C_u = \frac{d(\Delta Q_u)}{d\delta}. \quad (3.2)$$

Every real ring will have a non-zero chromaticity. Sextupoles are crucial elements to control and correct chromaticity [1], therefore, they are indispensable in any lattice. Nevertheless, as it was explained in Ch. 2, introducing non-linear elements inevitably leads to driving betatron resonances, e.g., sextupoles will drive third order resonances. In this sense, and dramatically speaking, sextupoles are double-edged swords. They are essential for chromaticity correction but deleterious by inducing resonances in the accelerator.

The Recycler Ring has 76 sextupoles, including normal and skew sextupoles. Additionally, each gradient magnet, as described in Sec. 3.2, has end shims on both ends and are used to correct the field shape [42]. These end shims can also induce sextupole field on the beam. It is also worth reminding the reader that the gradient magnets, themselves, also have sextupole components in their permanent fields. All of these builds up to the fact that the chromaticity of the Recycler Ring can be controlled in a range of -2 to -20, in both planes, i.e.,  $Q_u \in (-2, -20)$ .

Table 3.2 List of sextupoles used for the purpose of compensating third-order resonances in the Recycler Ring.

Name	Type
SC220(a/b)	Normal Sextupole
SC222(a/b)	Normal Sextupole
SC319(a/b)	Normal Sextupole
SC321(a/b)	Normal Sextupole
SS222(a/b)	Skew Sextupole
SS319(a/b)	Skew Sextupole
SS321(a/b)	Skew Sextupole
SS323(a/b)	Skew Sextupole

In particular, out of all the sextupoles in the Recycler Ring, this thesis work is interested in the ones listed in Table 3.2. These elements have been placed in the Recycler Ring with the sole objective of mitigating the harmful effect of the third order resonances [38]. The elements listed in Table 3.2 are in regions of close-to-zero dispersion,  $D_x \approx 0$ , in order to cancel out any additional chromaticity due to their inclusion. In principle, normal sextupoles are used to correct  $3Q_x = 76$  and  $Q_x + 2Q_y = 74$ , while skew sextupoles are used to correct  $3Q_y = 73$  and  $2Q_x + Q_y = 75$ . Nevertheless, normal sextupoles can have small tilts around the magnetic axis, which introduces skew sextupole field, and vice versa.

Figure 3.12 shows an example of these sextupoles installed in the Recycler Ring. As it can be seen, there are two yellow-coated elements installed in the top ring. While there are two elements close together, they are powered by the same power supply. Hence, the notation in Table 3.2 alluding to a and b elements, e.g., normal sextupoles SC220(a/b). The numbers in the name allude to their location in the RR/MI tunnel, as illustrated in Fig. 3.4. Given that each pair of elements, a and b, are connected to the same power supply, these two elements effectively become one knob. The currents for the power supply are controlled through ACNET. This is important for the reader to remember, given that in Ch. 4, the currents of the sextupoles listed in Table 3.2 will be the main knobs turned in order to mitigate the strength of these resonances.



Figure 3.12 Picture of compensation sextupoles (yellow magnets on top) installed in the Recycler Ring. Picture provided by Dr. Robert Ainsworth.

## CHAPTER 4

### COMPENSATION OF THIRD-ORDER RESONANCES AT LOW INTENSITIES

#### 4.1 Global RDTs and Lattice Model

The following chapter explores how to mitigate the effect of third order resonances from the Recycler Ring by means of minimizing the Resonance Driving Terms (RDTs) that drive each resonance. The resonances in question are introduced in Figs. 3.7 and 3.8, and are summarized in Table 4.1. The RDTs for each of these third order resonance lines can be calculated from Eqs. 2.31 and 2.32. Table 4.1 shows the explicit expression for each third-order RDT of relevance to this work. The sum over  $i$ , goes through each element of the lattice beam line and asks if it has some sort of sextupole component in its definition—it can be normal  $K_{2,i}$  or skew  $K_{2,i}^{(s)}$  sextupole component. If it has this multipole, it will add it to the RDT sum by weighting it with the beta functions  $\beta_{u,i}$  and phase advances  $\phi_{u,i}$  from the linear approximation at those particular locations. Ultimately, the  $h_{jklm}$  RDT will be a complex number whose amplitude  $|h_{jklm}|$  should be minimized.

Table 4.1 Corresponding RDTs and spectral lines for each resonance line.

Resonance Line	RDT Expression
$3Q_x = 76$	$h_{3000} = -\frac{1}{48} \sum_i K_{2,i} L_i \beta_{x,i}^{\frac{3}{2}} e^{3i\phi_{x,i}}$
$Q_x + 2Q_y = 74$	$h_{1020} = -\frac{1}{16} \sum_i K_{2,i} L_i \beta_{x,i}^{\frac{1}{2}} \beta_{y,i} e^{i[\phi_{x,i} + 2\phi_{y,i}]}$
$3Q_y = 73$	$h_{0030} = -\frac{1}{48} \sum_i K_{2,i}^{(s)} L_i \beta_{y,i}^{\frac{3}{2}} e^{3i\phi_{y,i}}$
$2Q_x + Q_y = 75$	$h_{2010} = -\frac{1}{16} \sum_i K_{2,i}^{(s)} L_i \beta_{x,i} \beta_{y,i}^{\frac{1}{2}} e^{i[2\phi_{x,i} + \phi_{y,i}]}$

Figure 4.1 shows a visual representation for the calculation of the  $h_{3000}$  RDT. This plot shows the amplitude of the complex cumulative sum as it goes around the ring (thick solid orange line). Additionally, this plot also shows the amplitude of each individual contribution for every  $i$ -th element in the lattice with sextupole component (thin purple line). This particular quantity can be used to visualize where on how the sextupole component is distributed around the ring. Ultimately,

after doing this sum around the ring, the final result is a complex number with some amplitude and phase which corresponds to the  $h_{3000}$  RDT, as calculated from some arbitrary location in the lattice. The amplitude of the  $h_{3000}$  term is plotted in Fig. 4.1 with a red dashed line. Figure 4.2 shows a similar exercise for the  $h_{1020}$  term. All of these calculations are done with a lattice model that has a list of components and magnet coefficients, that, in principle, should be very close to what's inside the tunnel. The particular RR model used was the RR2020V0922FLAT lattice model, provided by R. Ainsworth and M. Xiao.

A question that promptly arises is: does the arbitrary starting position for the sum of Eq. 2.31 change the RDT result? The short answer is yes, the RDT will change depending on the initial position for the sum. Reference [19], specifically in its Ch. 5, goes into depth as to how to correlate the RDT calculated from a starting point  $s_1$  to one measured at starting point  $s_2$ . The difference in this case relates to the amount of multipole component between both calculation points, e.g., the amount of elements that have sextupole component between an  $s_1$  and  $s_2$  observation point. Nevertheless, given that there is an infinite amount of  $s_1$  and  $s_2$  observation points, and only so much real state in this thesis, I am plotting for an arbitrary observation point in the lattice. Additionally, given that the sextupole components are evenly distributed around the ring, the RDT values will not oscillate much.

As mentioned in Ch. 3, the Recycler Ring is made up of permanent gradient magnets. From looking at Figs. 4.1 and 4.2, one can see that the sextupole component is evenly distributed around some sections of the ring. If one were to plot the distribution of permanent gradient on these plots, the location of them would coincide with the peaks of the individual contributions to the RDT of Figs. 4.1 and 4.2. Therefore, the sources that drive the Recycler Ring's normal sextupole resonances come from the permanent magnets themselves—this is known as a systematic-driven resonance as opposed to a random-error-driven resonance. Highly periodic machines and highly linear machines use the fluctuations between RDT measurements from BPMs to locate any sextupole errors in the lattice and try to fix them [19]. Nevertheless, this is not the case for the Recycler given its low superperiodicity of 2 and its uniform sextupole component distribution.

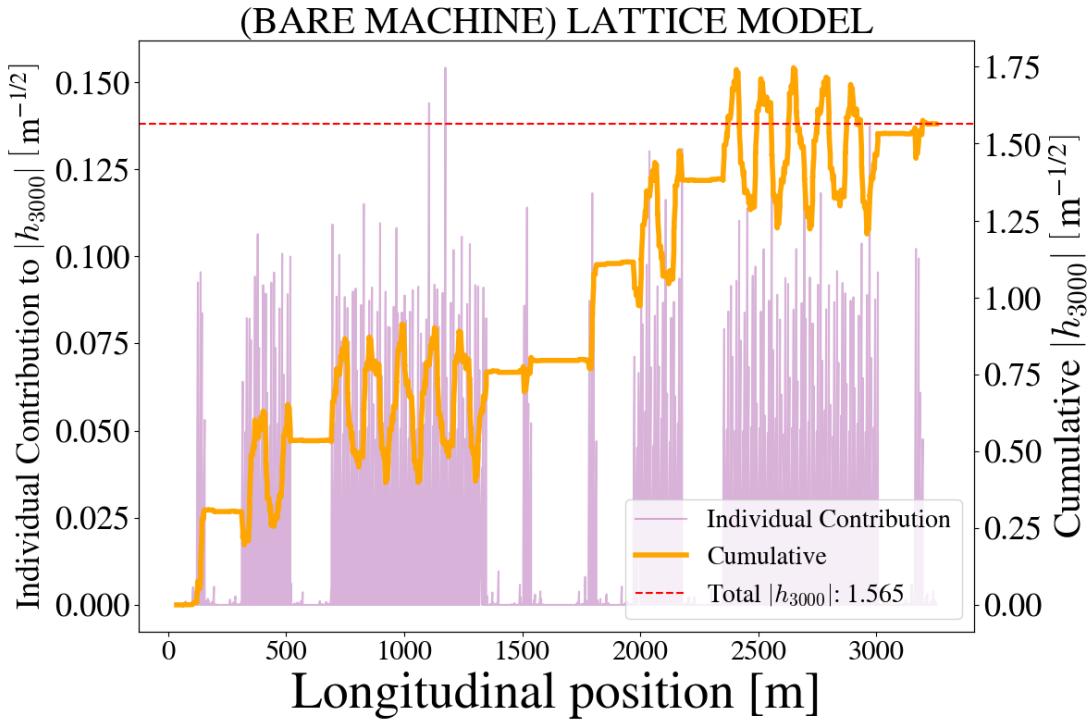


Figure 4.1 Distribution of the  $h_{3000}$  term around the ring with individual contributions from each relevant element and the cumulative sum from an arbitrary starting point.

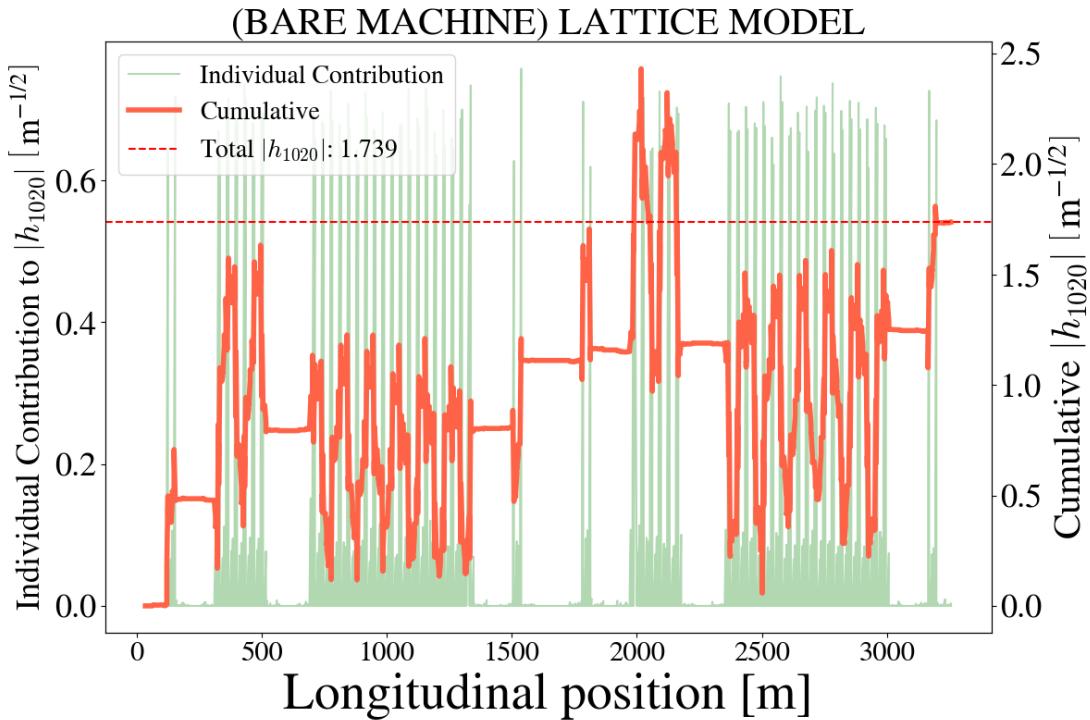


Figure 4.2 Distribution of the  $h_{1020}$  term around the ring with individual contributions from each relevant element and the cumulative sum from an arbitrary starting point.

## 4.2 Measurement of Third Order RDTs

Calculating the theoretical RDTs from the lattice model is a matter of calculating the sums outlined in Table 4.1. Nevertheless, the measurement of the third order RDTs requires following a long and involved recipe. This recipe is based on previous work from Refs. [19, 22], but with a lot of original steps specifically developed for the Recycler Ring. In summary, these are the steps used in order to measure RDTs at the Recycler:

1. Kick the beam with dipole kickers and save turn-by-turn BPM data for offline analysis.
2. Go through data and estimate momentum coordinate with previously calculated transfer matrices from lattice model.
3. Estimate Twiss parameters and normalized coordinates  $(\hat{u}, \hat{p}_u)$  at every BPM location.
4. Create resonance basis  $h_u^\pm$  (see Eq. 2.37).
5. Get spectrum of resonance basis using NAFF (Numerical Analysis of Fundamental Frequencies) through the SUSSIX software [43].
6. Identify resonance lines from the spectrum that correspond to the RDT of interest.
7. Calculate the RDT at each BPM location from the equivalence relation of spectral lines and RDT expansion (see Eq. 2.46).

The following subsections will explore in more detail the steps outlined in the previous list.

#### 4.2.1 Dipole Kick and BPM data

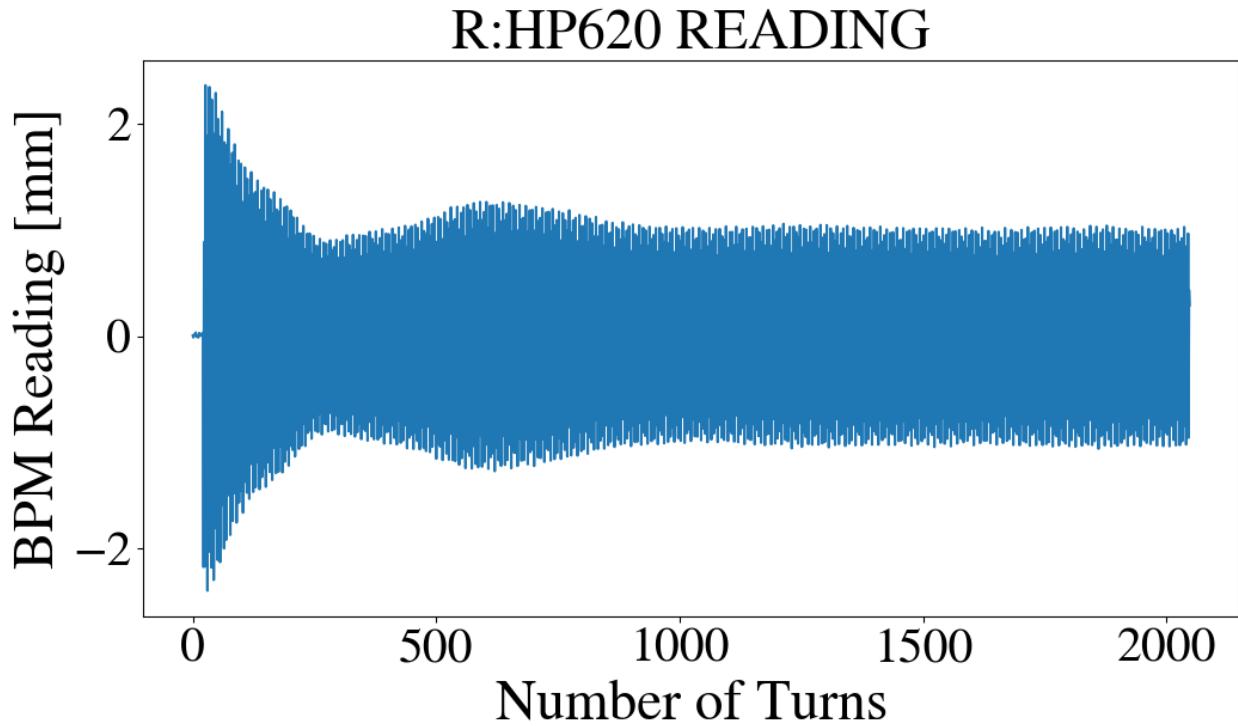


Figure 4.3

#### 4.2.2 Estimation of Momentum Coordinates

#### 4.2.3 Twiss Parameters and Normalized Data

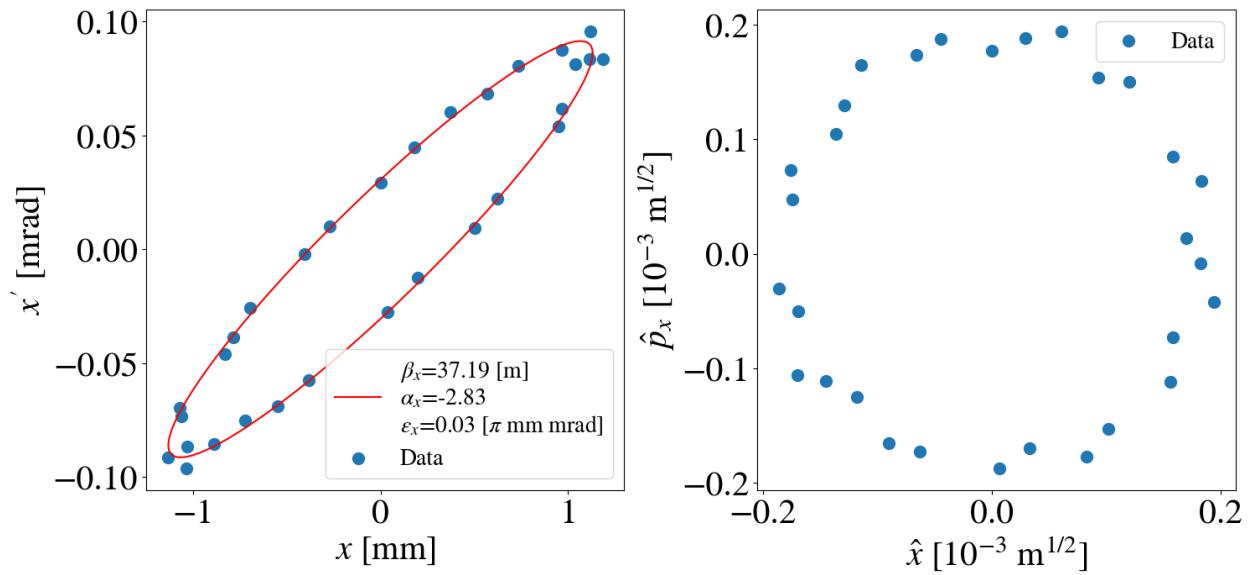


Figure 4.4

#### 4.2.4 Resonance basis and Spectral Decomposition

$$h_x^-(N) = \hat{x} \pm \hat{p}_x = \sum_{jklm} HSL_{jklm} e^{2\pi i N [(1-j+k)Q_x + (m-l)Q_y]} \quad (4.1)$$

$$h_y^-(N) = \hat{y} \pm \hat{p}_y = \sum_{jklm} VSL_{jklm} e^{2\pi i N [(k-j)Q_x + (1-l+m)Q_y]} \quad (4.2)$$

$$\begin{aligned} h_x^-(N) &= \sqrt{2I_x} e^{i(\psi_x + \psi_{x_0})} \\ &\quad - 2i \sum_{jklm} j f_{jklm} (2I_x)^{\frac{j+k-1}{2}} (2I_y)^{\frac{l+m}{2}} e^{i[(1-j+k)(\psi_x + \psi_{x_0}) + (m-l)(\psi_y + \psi_{y_0})]}. \end{aligned} \quad (4.3)$$

$$\begin{aligned} h_y^-(N) &= \sqrt{2I_y} e^{i(\psi_y + \psi_{y_0})} \\ &\quad - 2i \sum_{jklm} l f_{jklm} (2I_x)^{\frac{j+k}{2}} (2I_y)^{\frac{l+m-1}{2}} e^{i[(k-j)(\psi_x + \psi_{x_0}) + (1-l+m)(\psi_y + \psi_{y_0})]}. \end{aligned} \quad (4.4)$$

#### 4.2.5 Resonance Basis Spectrum

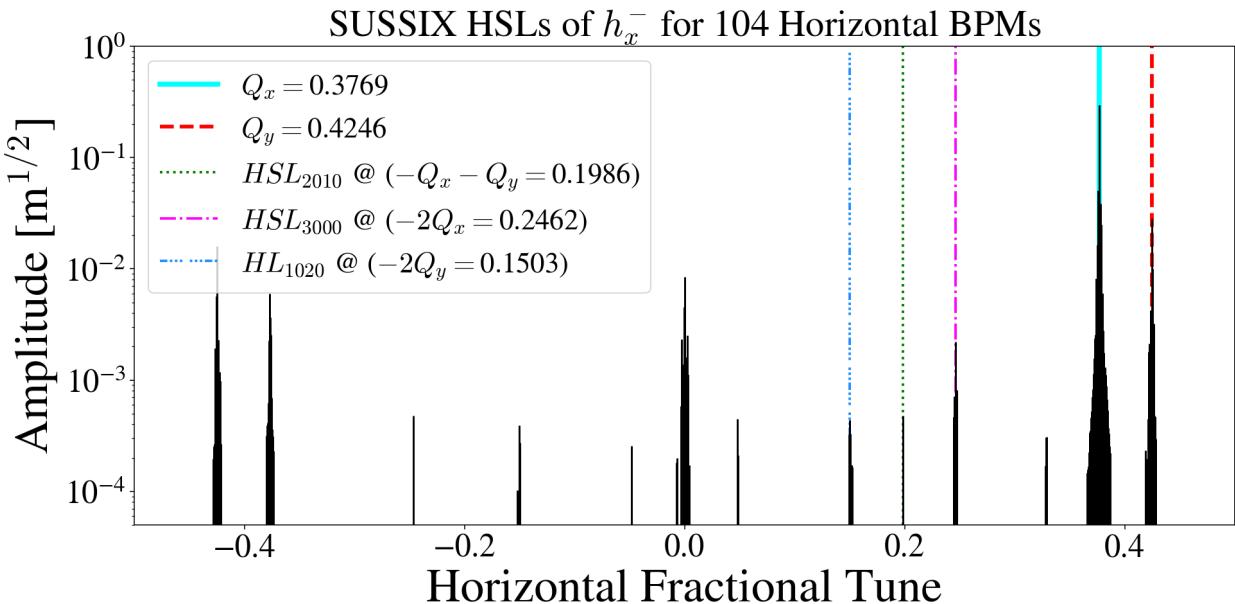


Figure 4.5 Spectral lines of  $h_x^-$  calculated with SUSSIX [43]. The  $h_x^-$  signal was reconstructed for the 104 Horizontal BPMs. The spectrum for all BPMs is superimposed in this plot.

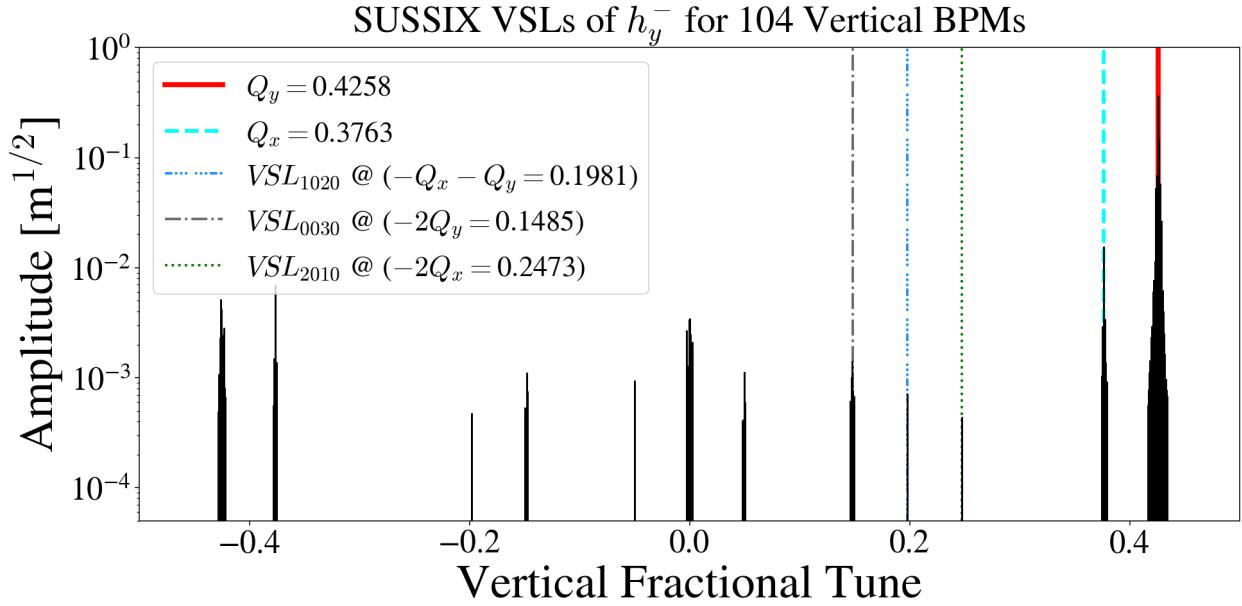


Figure 4.6 Spectral lines of  $h_y^-$  calculated with SUSSIX [43]. The  $h_y^-$  signal was reconstructed for the 104 Vertical BPMs. The spectrum for all BPMs is superimposed in this plot.

#### 4.2.6 Spectral Lines and RDT calculation

Table 4.2 Corresponding RDTs and location of spectral lines for each resonance line.

Resonance Line	Source	RDT	Hor. Spect.	Vert. Spect.
$3Q_x = 76$	Normal Sextupole	$h_{3000}$	(-2,0)	-
$Q_x + 2Q_y = 74$	Normal Sextupole	$h_{1020}$	(0,-2)	(-1,-1)
$3Q_y = 73$	Skew Sextupole	$h_{0030}$	-	(0,-2)
$2Q_x + Q_y = 75$	Skew Sextupole	$h_{2010}$	(-1,-1)	(-2,0)

#### 4.2.7 Third Order RDTs at every BPM location

### 4.3 Compensation of RDTs

For resonance compensation we have four dedicated normal sextupoles with currents that can be set to  $(I_{sc220}, I_{sc222}, I_{sc319}, I_{sc321})$  and four dedicated skew sextupoles with currents that can be set to  $(I_{ss323}, I_{ss323}, I_{ss319}, I_{ss321})$ . As shown in the previous section one RDT can be cancelled out with the right kick from the correction elements, which means the resonances are corrected to first order.

Nevertheless, by compensating one resonance line, other resonances might become worse. This is why for simultaneous compensation, compensation currents will vary depending on the subsets of resonances to compensate. In principle, the currents  $I_x$  needed in each correction element in order to cancel out the four bare machine RDTs, are given by the solution to this linear system of equations:

$$\begin{bmatrix} -|h_{3000}| \cos(\psi_{3000}) \\ -|h_{3000}| \sin(\psi_{3000}) \\ -|h_{1020}| \cos(\psi_{1020}) \\ -|h_{1020}| \sin(\psi_{1020}) \\ -|h_{0030}| \cos(\psi_{3000}) \\ -|h_{0030}| \sin(\psi_{3000}) \\ -|h_{2010}| \cos(\psi_{1020}) \\ -|h_{2010}| \sin(\psi_{1020}) \end{bmatrix}_{(Bare)} = \mathbf{M} \begin{bmatrix} I_{sc220} \\ I_{sc222} \\ I_{sc319} \\ I_{sc321} \\ I_{ss223} \\ I_{ss323} \\ I_{ss319} \\ I_{ss321} \end{bmatrix} \quad (4.5)$$

where  $M_{ij}$  is the response matrix for the RDTs with respect to the currents, and includes any roll that can happen for the correction sextupoles. This response matrix  $M_{ij}$  can be calculated by scanning the currents in each correction element and looking at the response from the real and imaginary part of the RDTs, i.e.,  $h_{jklm} = |h_{jklm}|e^{i\psi_{jklm}}$ .

In reality, there are limitations to solving equation 4.5. The first one is that all the RDTs ( $h_{jklm}$ ) may not be accessible for measurement, given that they may not show up as a spectral line. Another limitation is that the solution for the currents may be outside the maximum limits for the correction elements.

One can also try to cancel out a subset of RDTs from equation 4.5, including only one RDT. For example, in order to compensate  $3Q_x = 76$ , the system of equations to be solved is:

$$\begin{bmatrix} -|h_{3000}| \cos(\psi_{3000}) \\ -|h_{3000}| \sin(\psi_{3000}) \\ 0 \\ 0 \end{bmatrix}_{(Bare)} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} I_{sc220} \\ I_{sc222} \\ I_{sc319} \\ I_{sc321} \end{bmatrix} \quad (4.6)$$

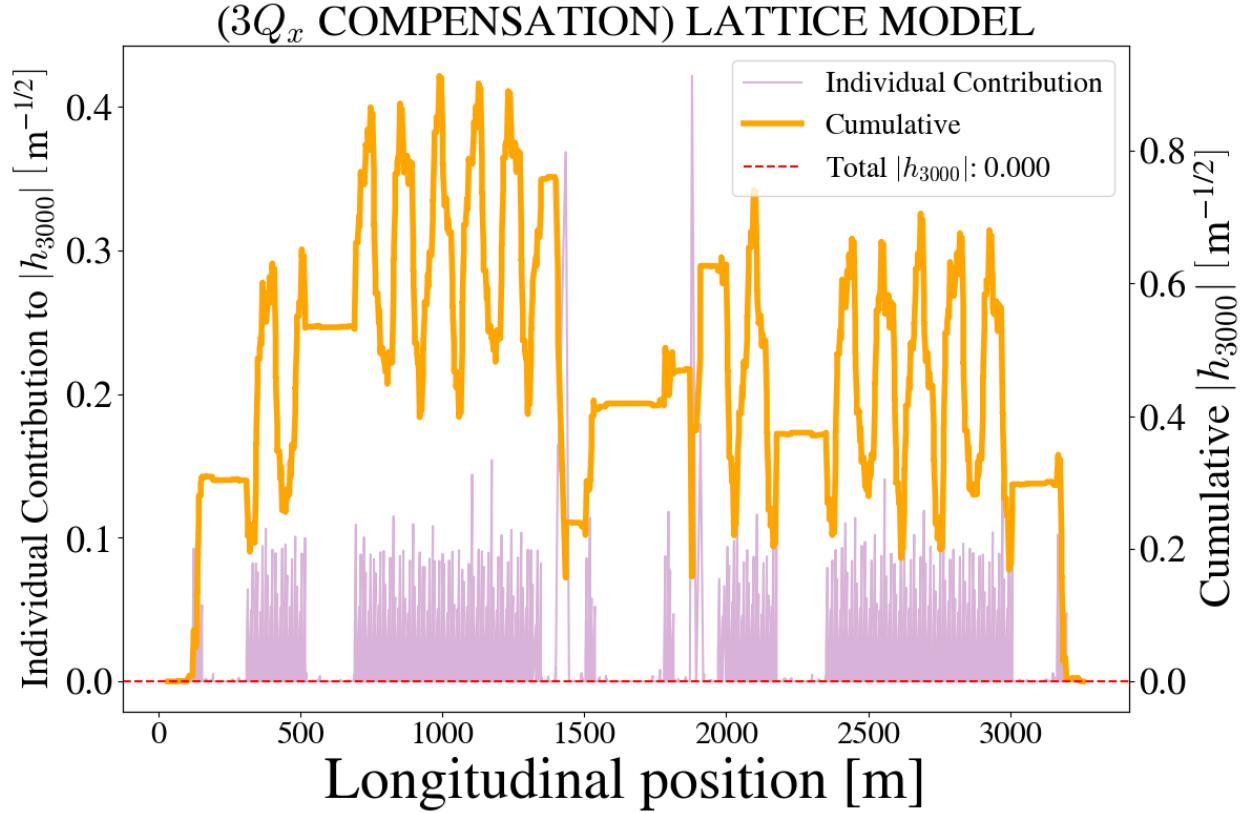


Figure 4.7 Distribution of the  $h_{3000}$  term around the ring with individual contributions from each relevant element and the cumulative sum from an arbitrary starting point when correction elements are set to compensate  $3Q_x = 76$ , i.e.,  $h_{3000} = 0$ .

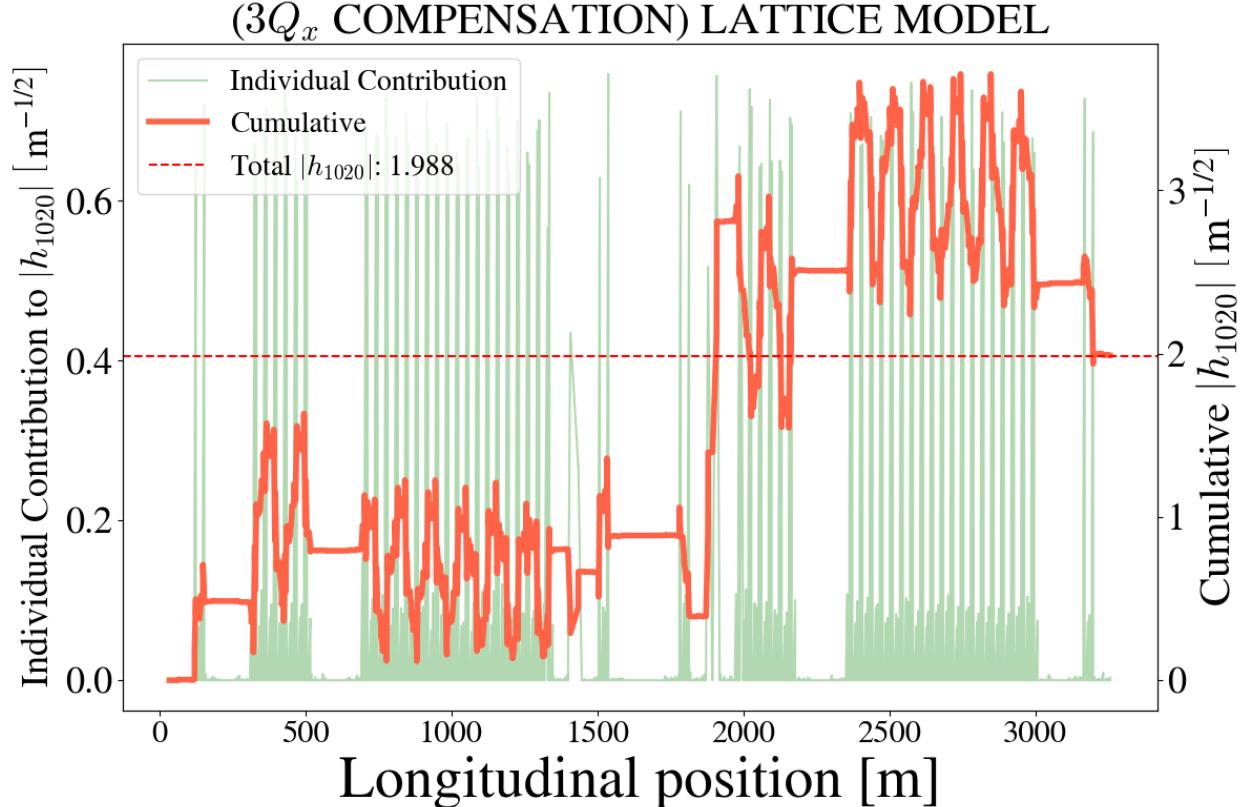


Figure 4.8 Distribution of the  $h_{1020}$  term around the ring with individual contributions from each relevant element and the cumulative sum from an arbitrary starting point when correction elements are set to compensate  $3Q_x = 76$ , i.e.,  $h_{3000} = 0$ .

#### 4.4 Optimization of Compensation Currents

#### 4.5 Experimental Verification of Compensation

##### 4.5.1 Dynamic Loss Maps

One effective method for visualizing resonance compensation involves constructing dynamic loss maps. To generate these representations, specialized quadrupoles responsible for controlling the tune of the Recycler are gradually adjusted to map out the desired tune area. Throughout this process, the beam loss rate is meticulously measured and interpolated across the specified region. This is done in the horizontal and vertical direction. The initial horizontal scan is generated by maintaining a constant vertical tune while implementing a horizontal tune ramp ranging from  $Q_x = 25.47$  to  $Q_x = 25.31$ . Subsequently, the vertical tune, initially set as *constant* at  $Q_y = 24.47$ ,

is adjusted incrementally to  $Q_y = 24.31$  in steps of 0.005, with intensity data recorded at each step. Conversely, for the vertical scan, the roles are reversed: the horizontal tune remains constant while a vertical tune ramp progresses from  $Q_y = 24.47$  to  $Q_y = 24.31$ . Then, the *constant* horizontal tune is varied from  $Q_x = 25.47$  to  $Q_y = 25.31$  in steps of 0.005. The resulting intensity data from both scans can be differentiated, normalized by the instantaneous intensity, and interpolated within a two-dimensional grid to construct plots akin to those depicted in Figs. 4.9 and 4.10. Figure 4.9 demonstrates the initial machine scan without any compensation. If plotted alongside the theoretical positions of the lines as in Fig. 4.10, the beam loss bands align with the resonance lines.

Figure 4.10 illustrates the correspondence between the loss patterns and the theoretical positions of resonance lines. A slight deviation exists between the set tune and the actual tune due to calibration adjustments from the tune trombone program. However, despite this variance, the resonance line configuration within the loss pattern facilitates the visualization of each resonance's strength. Specifically, a higher normalized loss at a particular tune location indicates a stronger Resonance Driving Term (RDT) for the corresponding resonance line. Within Figs. 4.9 and 4.10, third, fourth, and even traces of fifth-order resonance lines are discernible, with third-order resonance lines exhibiting the greatest prominence.

Figure 4.11 depicts dynamic loss maps representing various configurations of the compensation sextupoles. Specifically, Fig. 4.11a illustrates the loss map for the bare machine, where no compensation sextupoles are activated, while Fig. 4.11b and Fig. 4.11c demonstrate compensation for a single resonance line each. In the case of  $3Q_x$  compensation, the four normal sextupoles are adjusted to the calculated compensation currents using the RDT response matrix method. Moreover, a comparison between Fig. 4.11b and Fig. 4.11a clearly indicates a reduction of normalized losses by two orders of magnitude at the  $3Q_x$  line with compensation. This observation holds true for the  $3Q_y$  compensation as well, as shown in Fig. 4.11c.

Figures 4.11d, 4.11e, and 4.11f showcase the optimal configurations of compensation sextupoles designed to address multiple resonance lines simultaneously. It's important to note that while

attempting to compensate for one or multiple resonance lines, there's a possibility that other resonance lines may strengthen. This is evident in the explicit case depicted in Fig. 4.11f, where compensating for  $3Q_y$  and  $Q_x + 2Q_y$  leads to the amplification of the  $2Q_x + Q_y$  resonance. Such occurrences pose a limitation when aiming to compensate for more than two resonance lines, as the compensation currents tend to increase. There exists a constraint on the currents supplied to the compensation sextupoles. For instance, in compensating both normal sextupole lines,  $3Q_x$  and  $Q_x + 2Q_y$ , the required currents exceed the current limit. Ongoing efforts are focused on reducing the compensation currents in this specific scenario. Section 4.6 summarizes some of these efforts, where additional sextupoles have been installed in order to decrease the currents that cancel out both the  $h_{3000}$  and  $h_{1020}$  RDTs.

Another notable detail evident in Figs. 4.11a-4.11f is the presence of white areas within the loss maps, indicating regions where there was insufficient beam to accurately map out the losses. In certain configurations of the compensation sextupoles, the combined weakening of the third-order resonance lines occurs in a manner that leaves some beam remaining beyond these lines. It could be argued that conducting two additional scans, injecting from the left and bottom, could effectively map out these inaccessible regions. Such an enhancement could be considered as a future upgrade to these dynamic loss maps. Ultimately, all the plots presented in Fig. 4.11 demonstrate various potential configurations that open up regions of tune space for utilization during operations, enabling the accommodation of high-intensity beams.

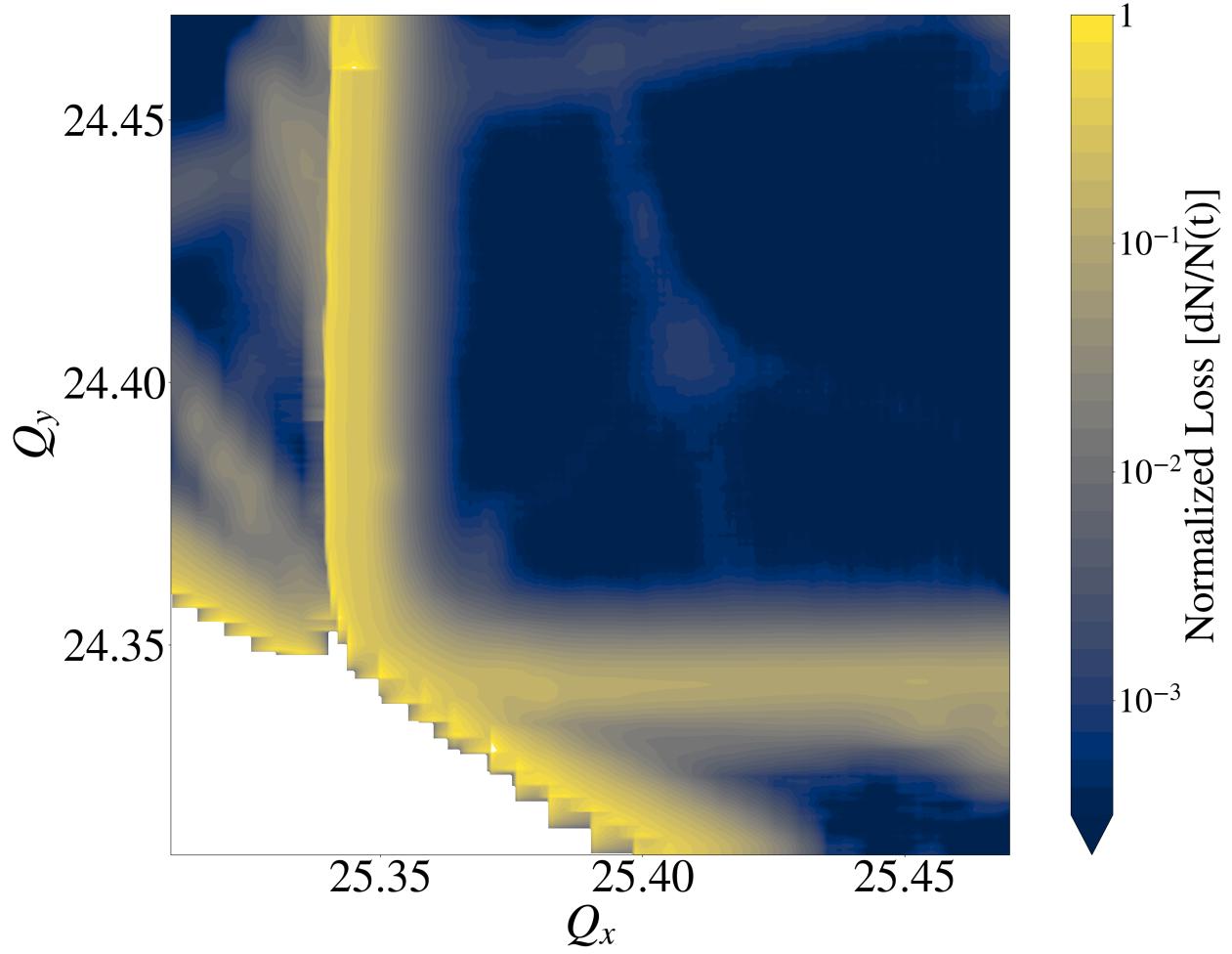


Figure 4.9 Dynamic loss map from ramping the tunes with an interval of  $\Delta Q_u = 0.005$  in both directions. The directions of scan are from left to right and top to bottom. The results are superimposed in this plot.

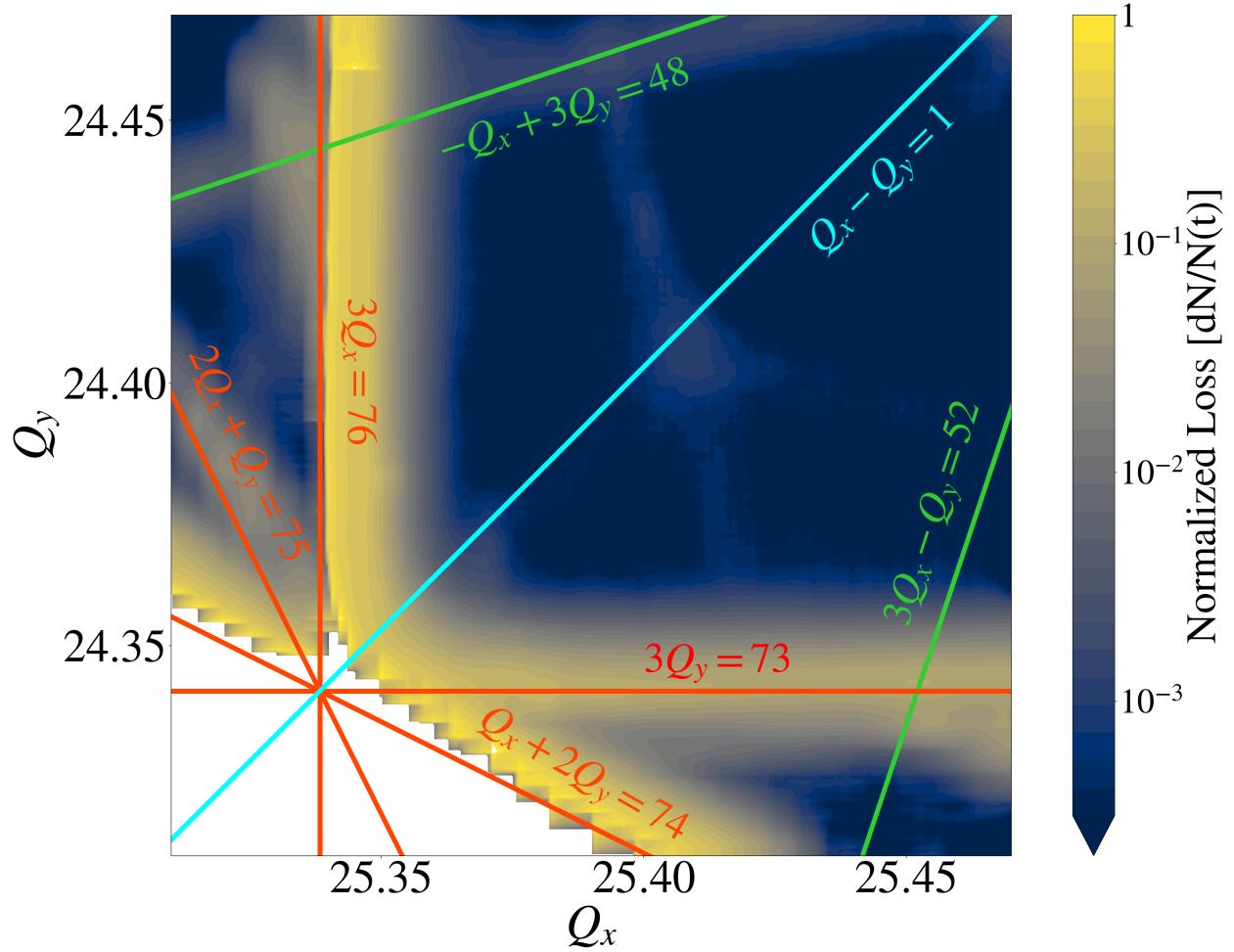


Figure 4.10 Dynamic loss map with the corresponding lines from Fig. 3.7 drawn on top.

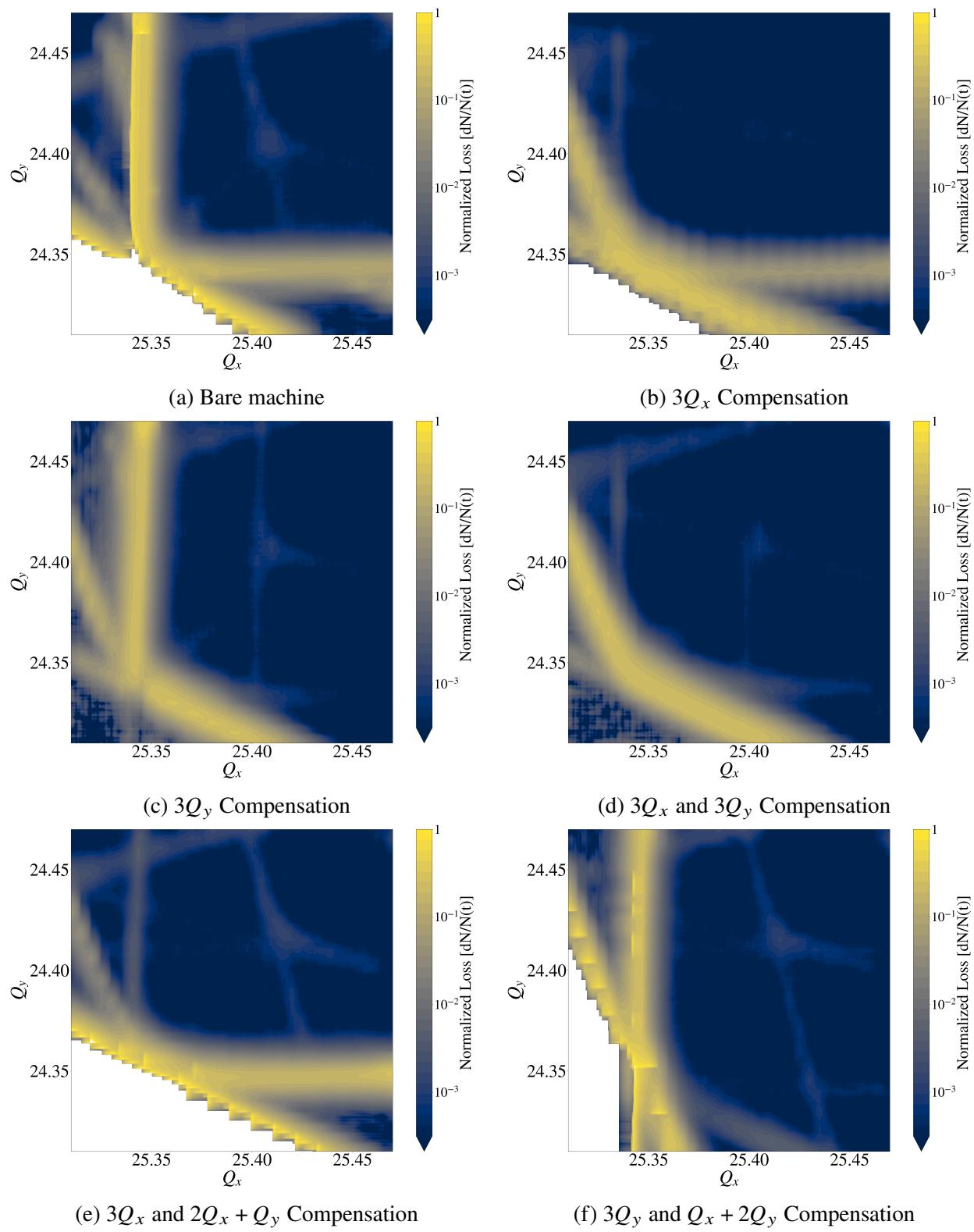


Figure 4.11 Dynamic loss maps for several configurations of compensation sextupoles

#### 4.5.2 Static Tune Scans

Another method for visualizing resonance compensation involves static tune scans. While the loss maps detailed earlier illustrate the dynamic crossing of resonance, an alternative method entails setting the tune to a specific value and assessing the beam survival ratio alongside the beam size over a defined time interval. The beam size is quantified using the Ion Profile Monitor System (IPM) within the ring, and the results are expressed in arbitrary units to indicate a relative effect.

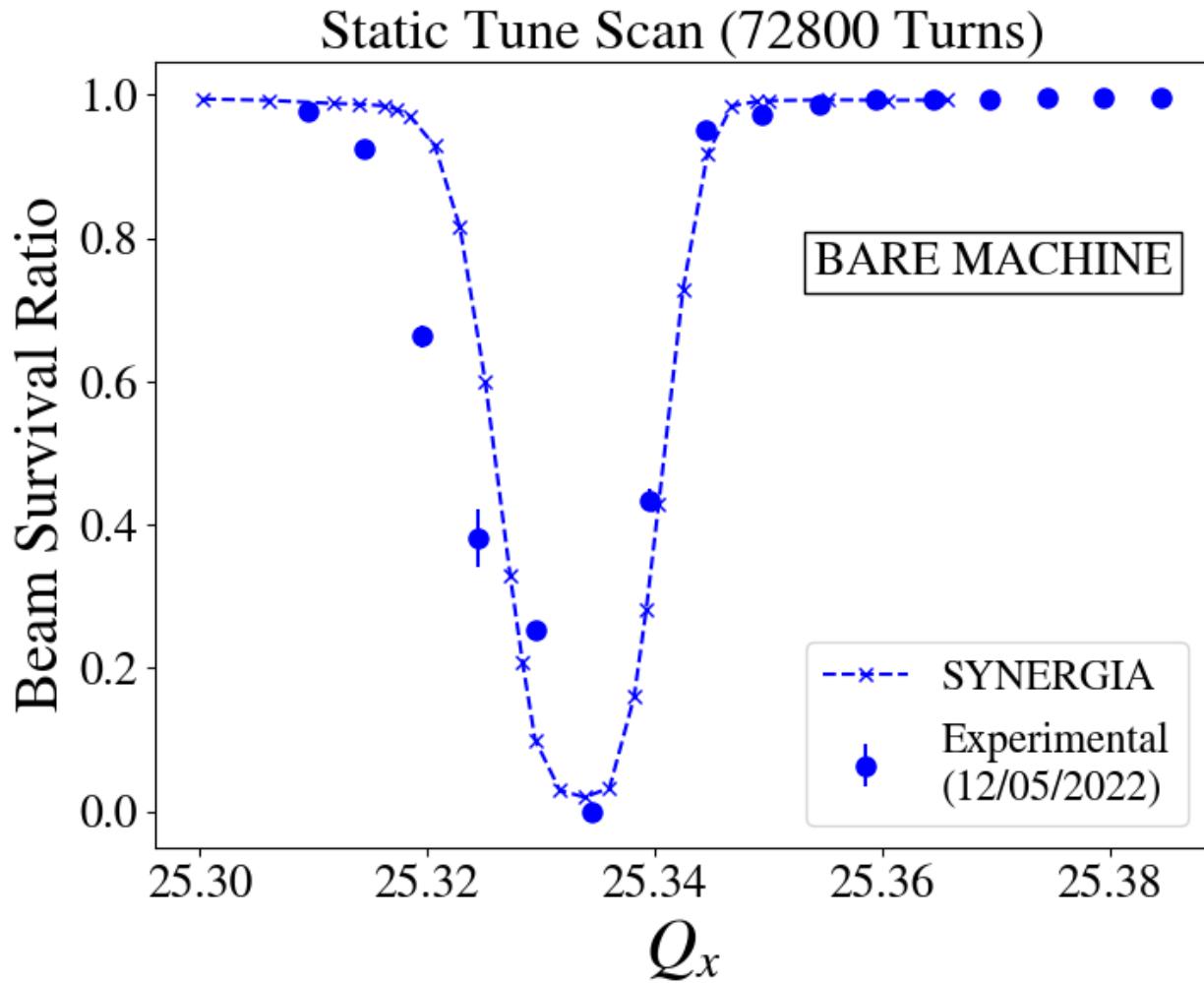


Figure 4.12 Static tune scan for bare machine with comparisons between experimental data and SYNERGIA simulations.

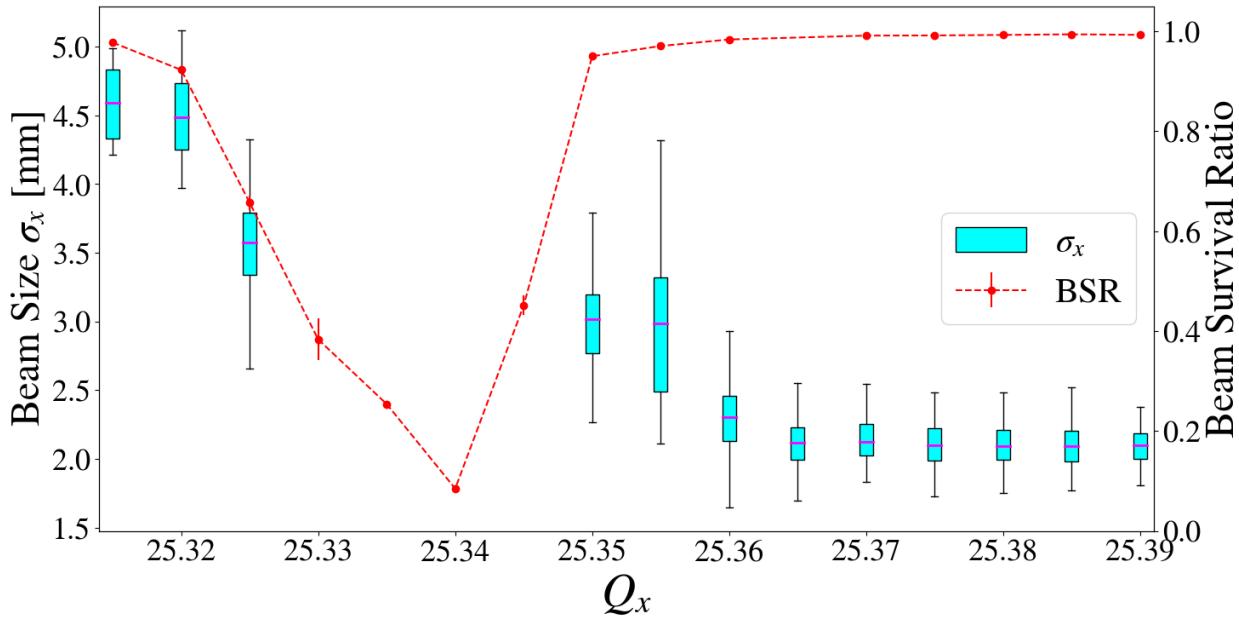


Figure 4.13 Static tune scan with beam survival ratio and IPM data box plots for bare machine at 2 Booster Turns of equivalent intensity.

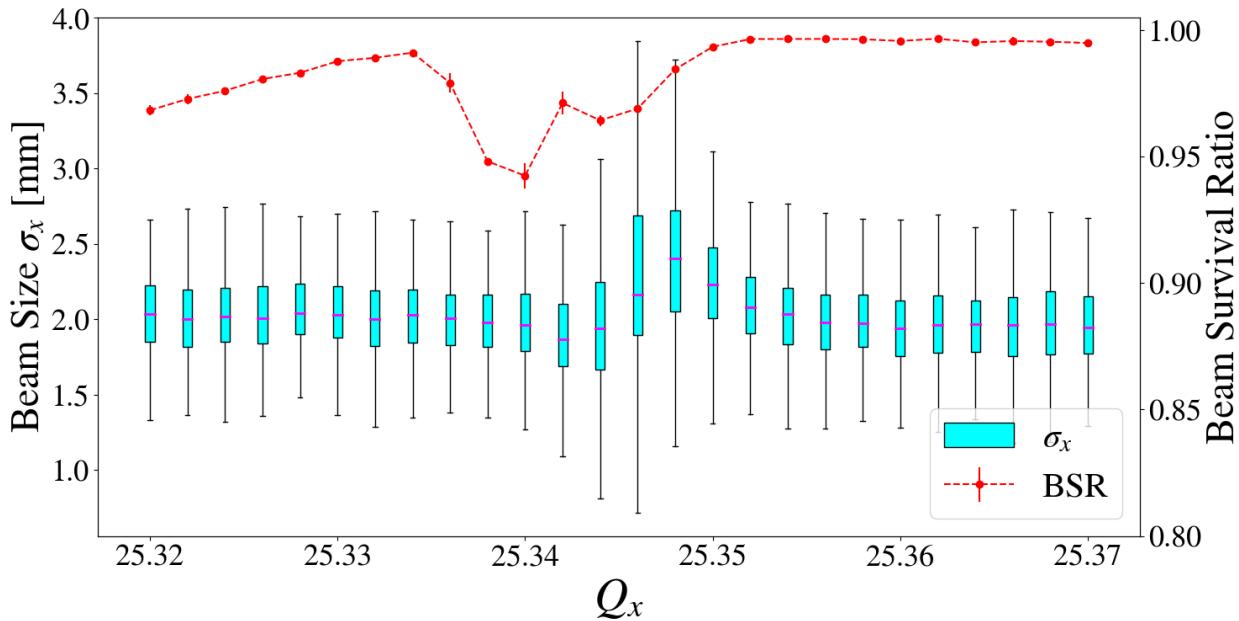


Figure 4.14 Static tune scan with beam survival ratio and IPM data box plots for machine with  $3Q_x$  compensation at 2 Booster Turns of equivalent intensity.

## 4.6 Additional Sextupoles for Resonance Compensation

$$\begin{bmatrix} -|h_{3000}| \cos \psi_{3000} \\ -|h_{3000}| \sin \psi_{3000} \\ -|h_{1020}| \cos \psi_{1020} \\ -|h_{1020}| \sin \psi_{1020} \end{bmatrix}_{(Bare)} = \mathbf{M} \begin{bmatrix} k_2^{(sc220)} \\ k_2^{(sc222)} \\ k_2^{(sc319)} \\ k_2^{(sc321)} \end{bmatrix} \quad (4.7)$$

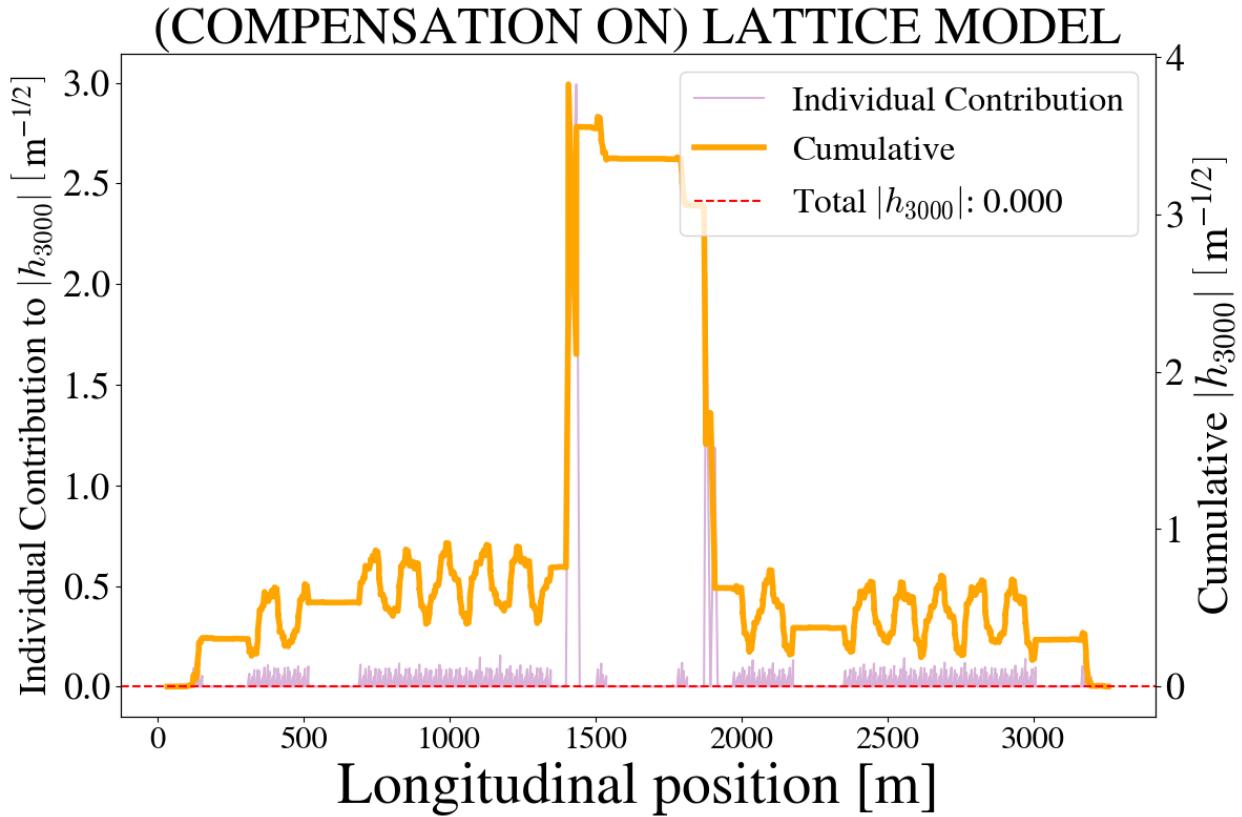


Figure 4.15 Distribution of the  $h_{3000}$  term around the ring with individual contributions from each relevant element and the cumulative sum from an arbitrary starting point. This is with existing sextupoles powered at the correct currents to cancel out  $h_{3000}$  and  $h_{1020}$  simultaneously.

## (COMPENSATION ON) LATTICE MODEL

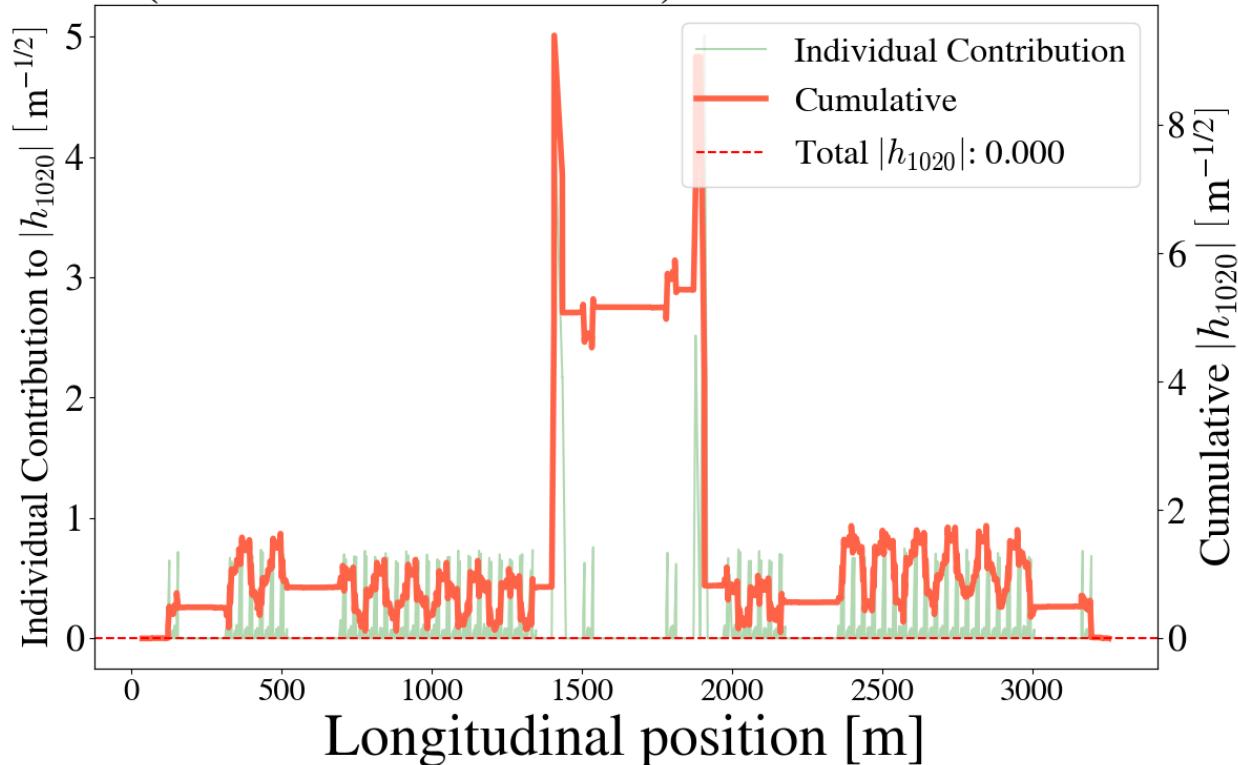


Figure 4.16 Distribution of the  $h_{1020}$  term around the ring with individual contributions from each relevant element and the cumulative sum from an arbitrary starting point. This is with existing sextupoles powered at the correct currents to cancel out  $h_{3000}$  and  $h_{1020}$  simultaneously.

$$\begin{bmatrix} -|h_{3000}| \cos \psi_{3000} \\ -|h_{3000}| \sin \psi_{3000} \\ -|h_{1020}| \cos \psi_{1020} \\ -|h_{1020}| \sin \psi_{1020} \end{bmatrix}_{(Bare)} = \mathbf{M} \begin{bmatrix} k_2^{(sc220)} \\ k_2^{(sc222)} \\ k_2^{(sc319)} \\ k_2^{(sc321)} \\ k_2^{(1)} \\ k_2^{(2)} \end{bmatrix} \quad (4.8)$$

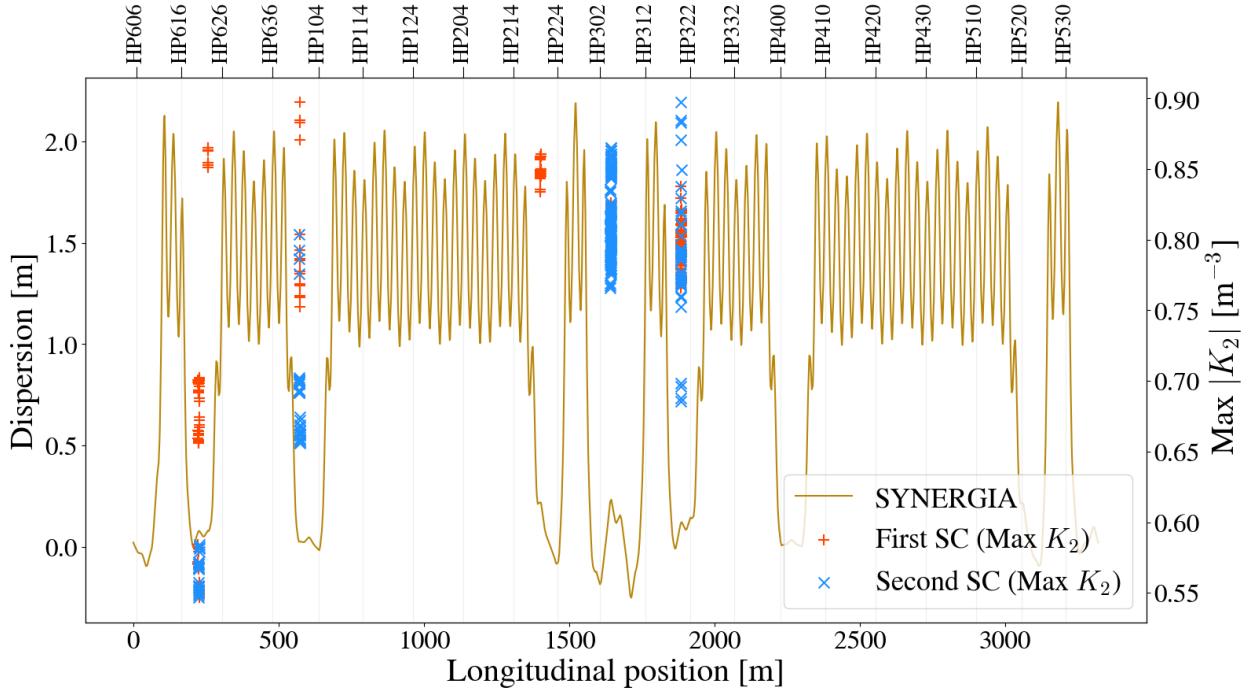


Figure 4.17 Dispersion function of the Recycler Ring with possible new locations to introduce a pair of sextupole magnets that cancel out  $h_{3000}$  and  $h_{1020}$  simultaneously. The right y-axis shows the maximum compensation current needed with these new candidates.

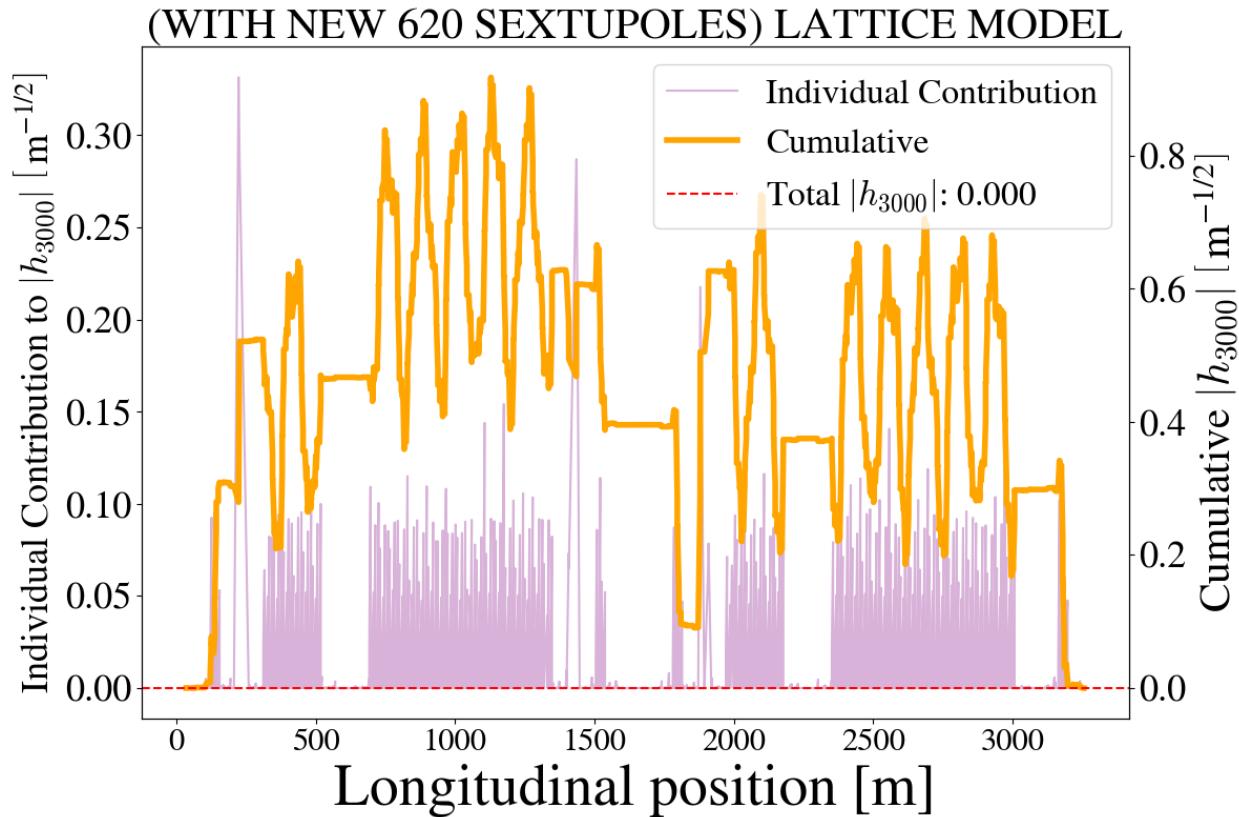


Figure 4.18 Distribution of the  $h_{3000}$  term around the ring with individual contributions from each relevant element and the cumulative sum from an arbitrary starting point. This is with the new 620 sextupoles powered at the correct currents to cancel out  $h_{3000}$  and  $h_{1020}$  simultaneously.

(WITH NEW 620 SEXTUPOLES) LATTICE MODEL

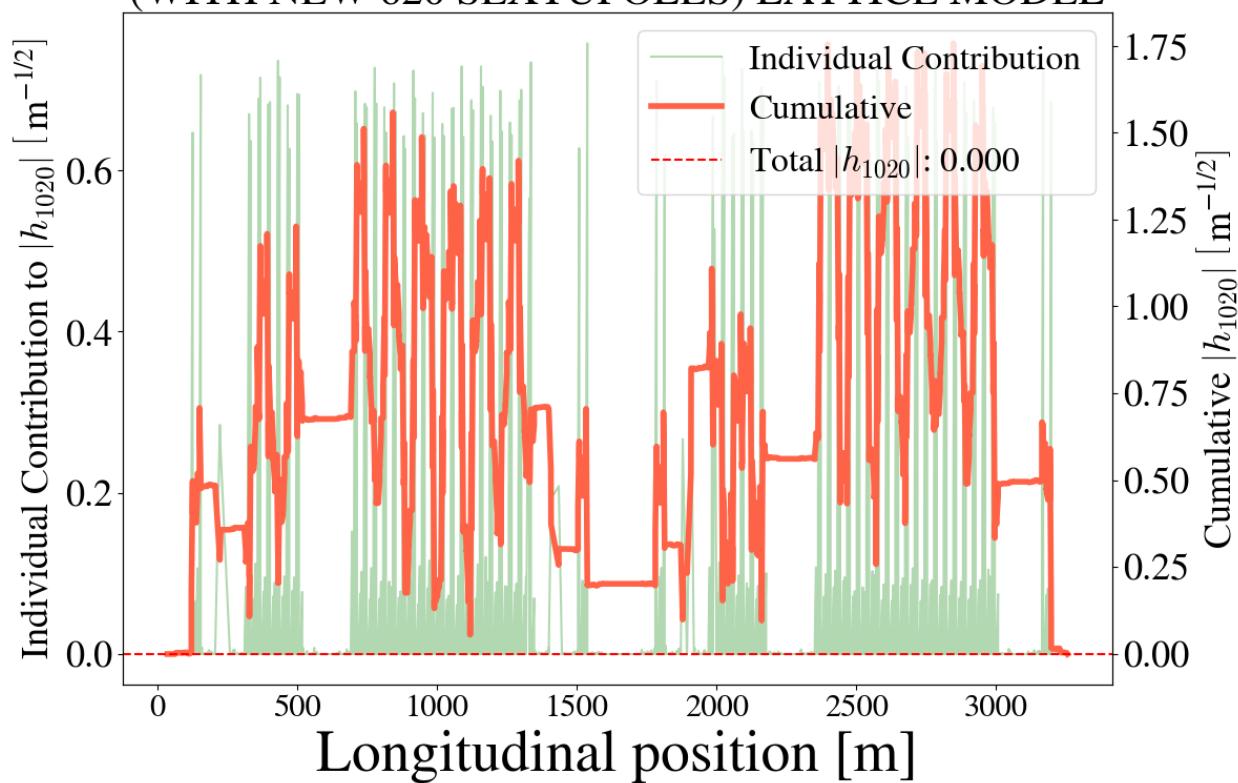


Figure 4.19 Distribution of the  $h_{1020}$  term around the ring with individual contributions from each relevant element and the cumulative sum from an arbitrary starting point. This is with the new 620 sextupoles powered at the correct currents to cancel out  $h_{3000}$  and  $h_{1020}$  simultaneously.

## CHAPTER 5

### RESONANCE COMPENSATION STUDIES AT THE CERN PROTON SYNCHROTRON BOOSTER

#### 5.1 General Description

The Proton Synchrotron Booster (PSB) is the first circular accelerator in the CERN accelerator complex that ultimately leads to the LHC. Figure 5.1 shows the entire chain of accelerators at CERN, feeding a variety of physics experiments [44]. Following the successful implementation of the LHC Injectors Upgrade (LIU) [45], the PSB receives  $H^-$  ion beam coming from the Linac4 at an energy of 160 MeV. Interestingly enough, the PSB is not just one ring, but rather four identical synchrotron rings stacked on top of each other. This is to counteract space charge effects which are largest at low energy machines. Once the ion beam enters the PSB rings, the electrons are stripped off through a charge-exchange process with a carbon foil and proton beam is achieved [46]. The proton beam is then accelerated from an energy of 160 MeV to 2 GeV. The beam from the four rings gets merged together and then gets injected to the Proton Synchrotron (PS). This is true for LHC-type beams, nevertheless, the PSB can also accelerate lead ions and deliver to other customers such as the heavy ion experiments, e.g., ISOLDE [47].

One ring of the PS Booster has a total circumference of 157.08 meters. Multiplying this quantity by the four rings you get a length of 628.32 meters, which is the exact circumference of the next accelerator, the Proton Synchrotron (PS). The PSB has a superperiodicity of 16, meaning it has 16 identical fundamental cells. Each cell has a length of 9.82 meters housing a sequence of bending magnet, focusing quadrupole, defocusing quadrupole, focusing quadrupole and bending magnet [48]. In between these main components there are drift spaces used for RF insertions, diagnostic devices, injection and extraction devices and additional multipole corrector magnets. At injection energy, 160 MeV, protons have a revolution period of  $1.01 \mu s$ , while at the extraction energy of 2.0 GeV they have a revolution period of  $0.553 \mu s$ . The overall cycle in the PS Booster takes around 1.2 seconds.

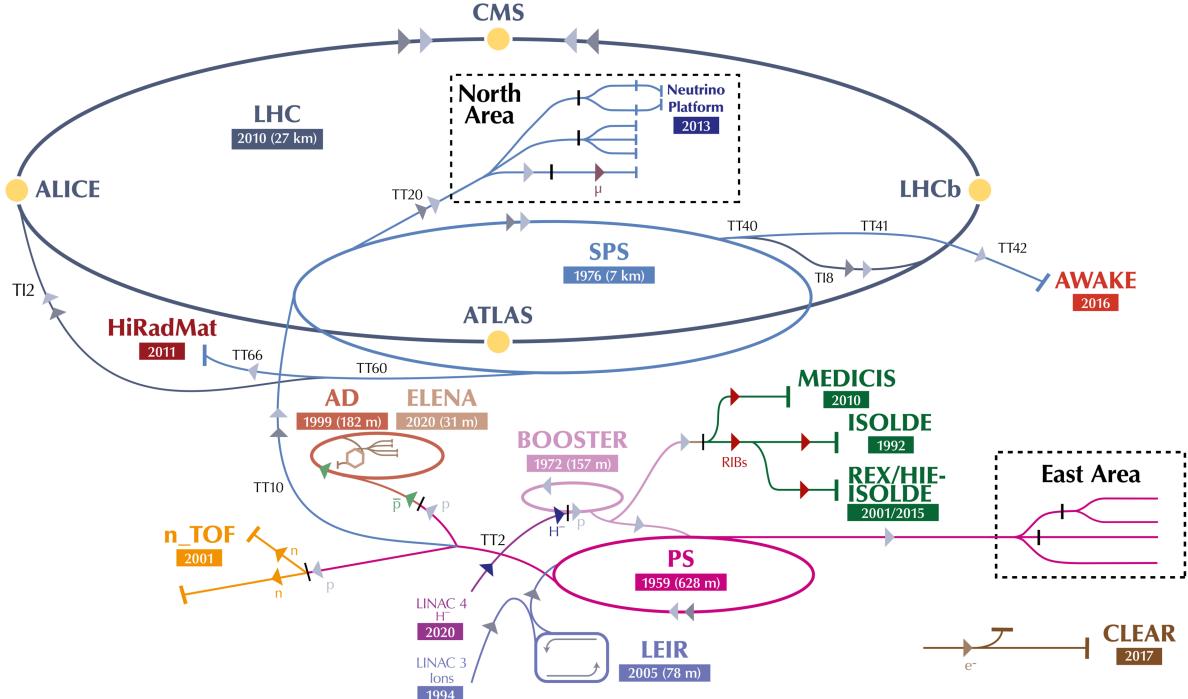


Figure 5.1 A graphic overview of all accelerators in operation at CERN as of 2022. Original image taken from Ref. [44]. This file is licensed under the Creative Commons Attribution 4.0 International license.

## 5.2 Tune Diagram and Operation

Figure 5.2 illustrates the tune diagram dynamics that LHC-type beam undergoes at the PS Booster [47, 49, 50]. As mentioned before, beam gets injected at an energy of 160 MeV. At this low energy, the tune footprint is large enough that the spread can reach values of up to 0.5, i.e.,  $\Delta Q_u \approx -0.5$ . The nominal injection tunes are around  $Q_x = 4.40$  and  $Q_y = 4.45$ , in order to accommodate the footprint between the integer resonance lines  $Q_u = 4.0$  and the half-integer line  $2Q_y = 9$ . As the beam is accelerated, the quadrupoles are ramped up to match the increasing beam rigidity, but, additionally, a tune ramp is introduced in order to move the shrinking footprint to a less resonance-populated area in the tune diagram. The nominal extraction tunes are around  $Q_x = 4.17$  and  $Q_y = 4.23$ . At extraction, the beam tune footprint has shrunk by a factor of  $(\gamma_L^3 \beta_L^2)$ , as explained by the beam perveance definition in Eq. 2.66. At extraction, the footprint is smaller than 0.05, i.e.,  $|\Delta Q_u| \lesssim 0.05$ .

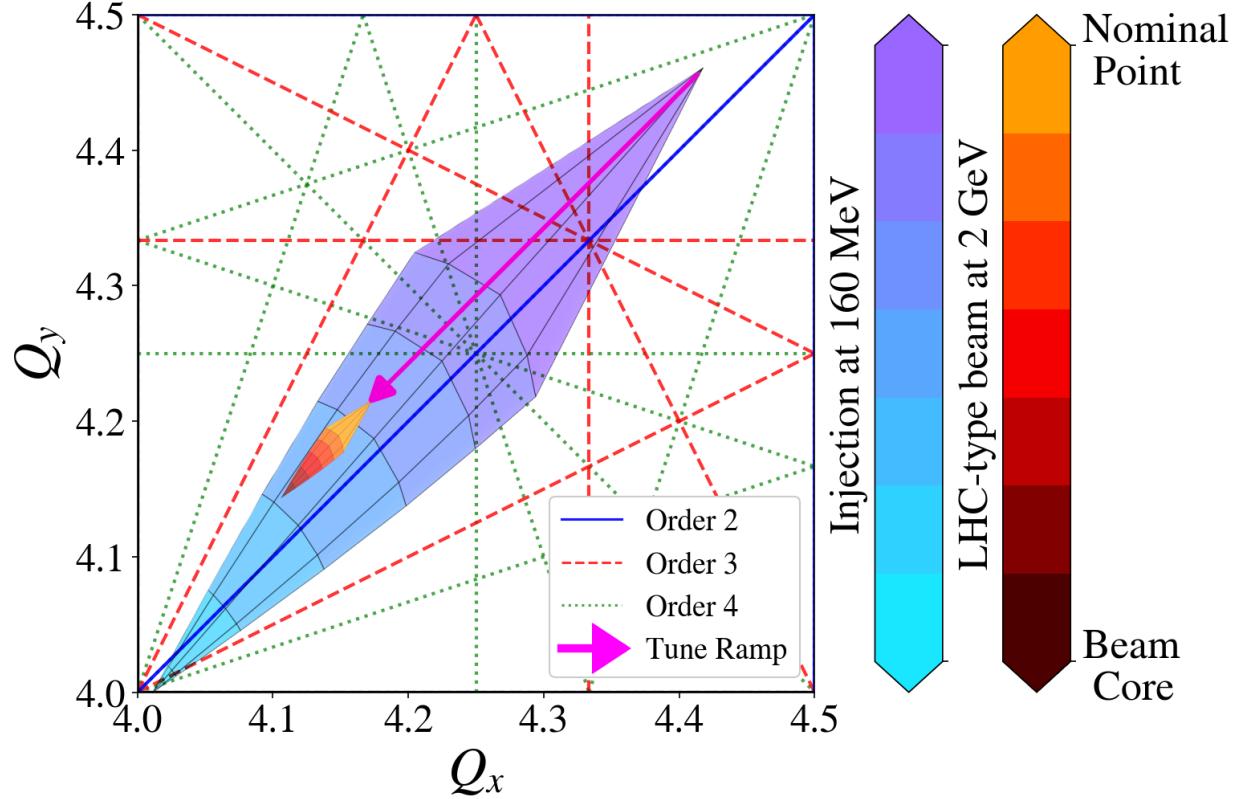


Figure 5.2 Operational tune footprint for PSB beam at injection (cool color map) and footprint after beam has been accelerated to 2 GeV (warm color map). During acceleration, there is a tune ramp illustrated with the fuchsia arrow.

The nominal tune ramp, as illustrated in Fig. 5.2 with the fuchsia arrow, crosses several resonance lines. Uncorrected for, these resonance lines will lead to beam loss during the tune ramping. These include 4 third-order resonance lines and 4 fourth-order lines. It is worth reminding the reader that the third order lines are excited by sextupole component in the ring, while fourth-order lines are excited by octupole fields in the ring. The third order resonances include two normal sextupole lines,  $3Q_x = 13$  and  $Q_x + 2Q_y = 13$ , and two skew sextupole lines,  $3Q_y = 13$  and  $2Q_x + Q_y = 13$ . For the octupole case, these include two normal octupole lines,  $4Q_x = 17$  and  $Q_x + 3Q_y = 17$ , two skew octupole lines,  $4Q_y = 17$  and  $3Q_x + Q_y = 17$ , and the octupole coupling sum resonance,  $2Q_x + 2Q_y = 17$ . Figure 5.3 shows all of these resonance lines summarized in one tune diagram. All of these resonance lines have different strengths in each Booster ring. It is worth pointing out that the coupling resonance  $Q_x - Q_y = 0$  is already being corrected for with skew

quadrupoles, similar to the Recycler Ring case.

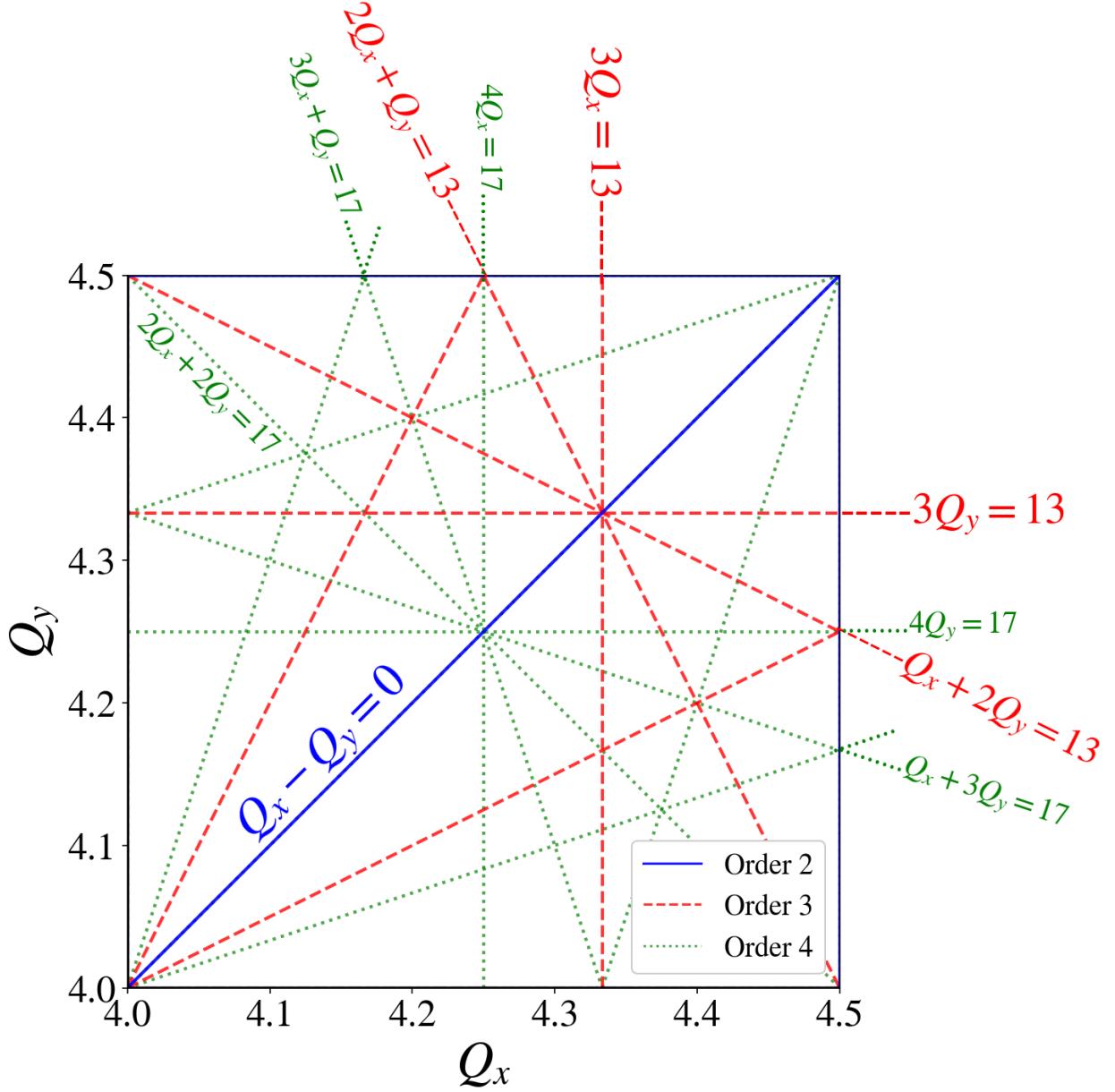


Figure 5.3 Portion of the tune diagram enclosing the operational tunes of the PS Booster with relevant resonance lines labelled.

Similar to the plots shown in Fig. 4.9 and Fig. 4.10, loss maps can be used to visualize the strength of the resonance lines in the CERN PS Booster. Figure 5.4 shows loss maps for each of the four rings in the PSB. These plots are created slightly different from the ones in the Recycler Ring. The plots shown in Fig. 5.4 are an average from four different loss maps. One where losses are

mapped from (a) left to right, i.e., fixing  $Q_y$  and ramping from  $Q_x \approx 4.49$  to  $Q_x \approx 4.15$ , (b) another one from right to left, (c) one from top to bottom, i.e., fixing  $Q_x$  and ramping from  $Q_y \approx 4.49$  to  $Q_y \approx 4.15$ , (d) and one from bottom to top. Therefore, these plots show an average from mapping the losses in four different directions.

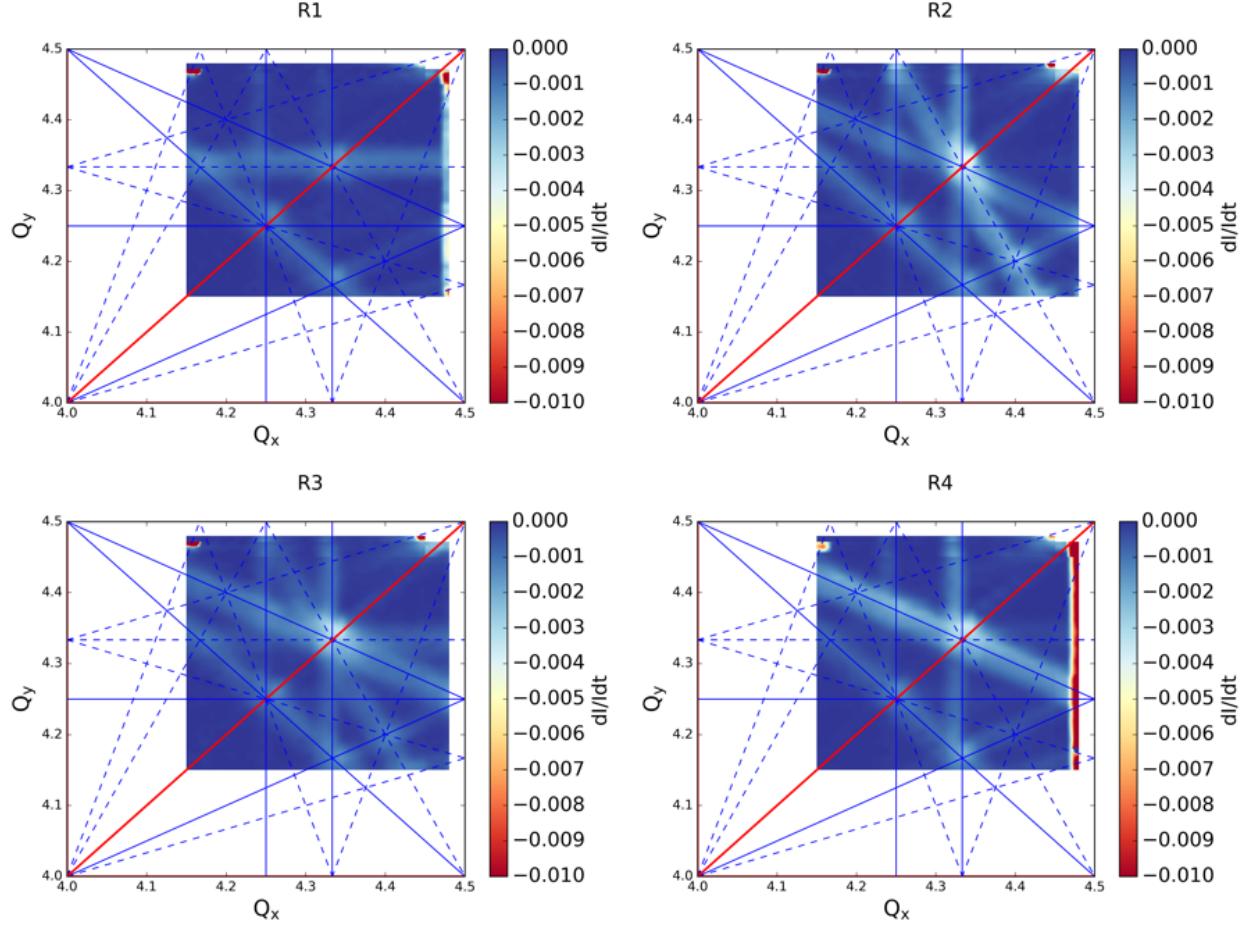


Figure 5.4 Dynamic loss maps for the bare machine of the 4 rings (R1, R2, R3 and R4) in the PS Booster. The plots are an average of scanning in 4 directions. Plot provided by F. Asvesta.

Figure 5.4 shows how resonances are excited differently for each ring at the Proton-Synchrotron Booster. For example, third-order lines are more excited for Ring 2, then Ring 1. This hints at the fact that every ring will have different values for the corrector magnets used for resonance compensation. The work in the following sections looks to calculate the currents of the corrector magnets used for resonance compensation. This involves using two optimization procedures to find the values that clear out the losses from the plots in Fig. 5.4.

### 5.3 Optimization Algorithms for Resonance Compensation

The whole objective of the following work is to minimize the losses that happen during the operational cycle of the PS Booster, as explained in Fig. 5.2. The losses during the cycle come from particles falling on top of third-order and fourth-order resonance lines, as identified in Fig. 5.3. In order to decrease the strength of these resonances, sextupoles and octupoles can be used in order to control their Resonance Driving Terms (RDTs). Nevertheless, measuring and minimizing the RDTs may not always be ideal. For this particular work, the main observable to optimize was the beam loss coming from crossing the resonance lines, which is related to the amplitude of the RDTs. Figure 5.5 shows an illustration of the experimental setup used in order to measure this beam loss.

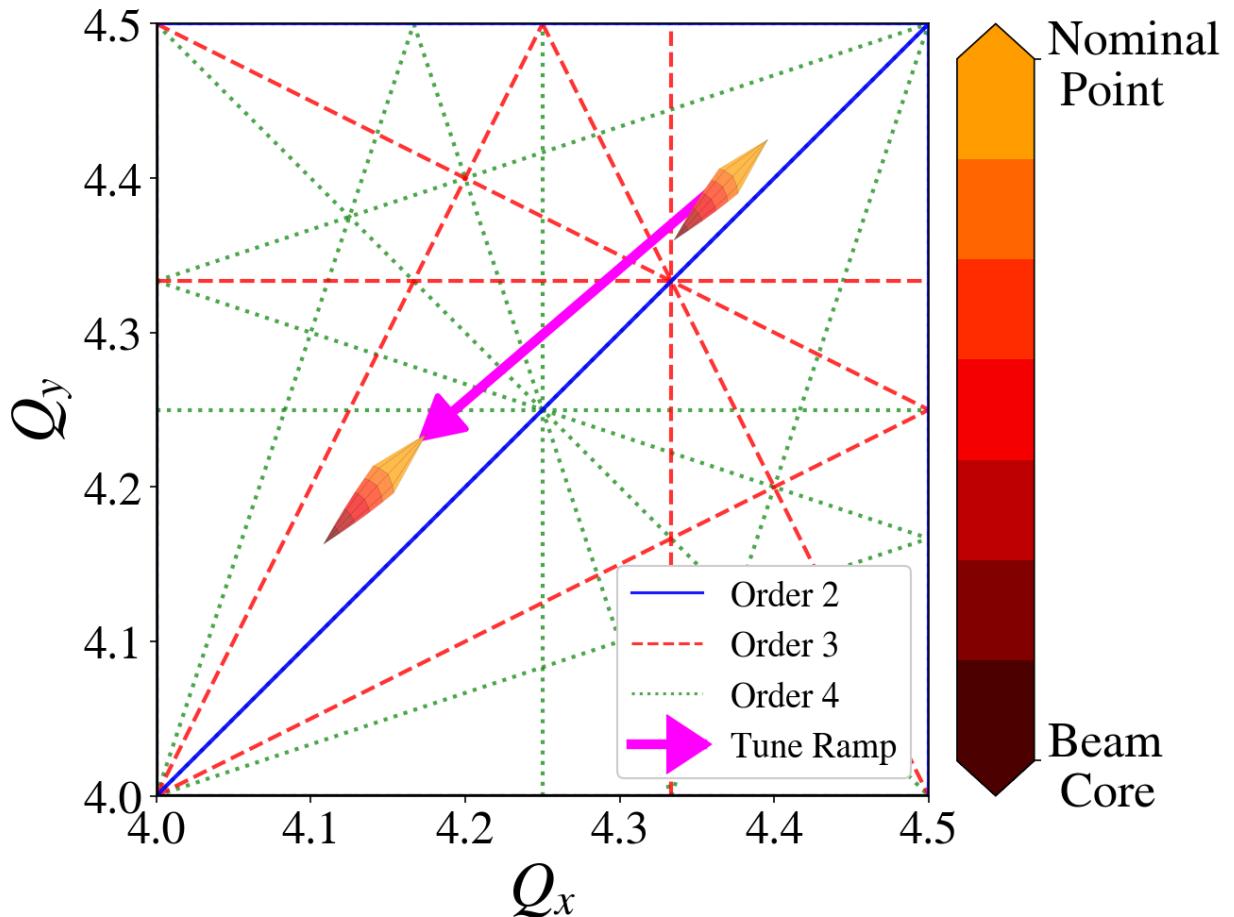


Figure 5.5 Experimental setup of the tune diagram dynamics for the optimization of resonance compensation used in the PS Booster.

The experimental setup introduced in the PS Booster for finding the optimal compensation currents involved several steps. First, low-brightness beam was injected into every ring at an energy of 160 MeV. Second, a tune ramp was programmed into the quadrupoles in the ring, in order to go from initial tunes of  $Q_x = 4.40$  and  $Q_y = 4.45$  to a final setting of  $Q_x = 4.17$  and  $Q_y = 4.23$ . This particular setting was introduced in order to mimic the operational tune ramp of LHC-type beam. The start of this tune ramp occurs within  $t_0 = 300$  ms of the start of the cycle, and ends at  $t_f = 600$  ms, i.e., all of this occurs within a 300 ms time window. During this time window, the beam loss is measured by comparing the beam current at the end of the window with respect to the initial value of the beam current. The currents fed to the corrector magnets remain constant during this measurement. Figures 5.5 and 5.6 summarize the experimental setup just explained.

While monitoring the beam loss, the corrector magnets used for compensation are varied every cycle according to the optimization algorithm. Table 5.1 summarizes the 11 elements used for resonance compensation for this work. Out of these elements, there are 6 normal sextupoles, 3 skew sextupoles, and 2 normal octupoles. Figure 5.6 shows an example of the power cycle in these magnets for one optimization step. Before the tune ramp, most actors show currents at or near zero. As preparation for the tune ramp, the magnets are powered to the set values—per the optimizer calculation. During the tune ramp, they are set to a constant value and then powered off once the cycle is finished. These magnets were varied for each ring, and each ring had its own independent optimization run. Theoretically, in order to fully correct 8 resonance lines one needs at least 16 correctors. Nevertheless, the objective of this work was to find a solution to this over-constrained problem through advanced optimization algorithms.

The two optimization algorithms used were Bayesian Optimization and BOBYQA (Bound Optimization BY Quadratic Approximation). In order to implement these algorithms, the special application GeOFF (Generic Optimization Frontend and Framework) was used [51]. This graphical application is designed to facilitate numerical optimization through various algorithms and reinforcement learning on CERN accelerators. It incorporates programmable interfaces that can be used to specify the hyperparameters of the optimization algorithms.

Table 5.1 List of elements (optimization actors) in the PS Booster at CERN used for resonance compensation optimization as present in all four PS Booster rings.

Actor	Name	Type
1	XN04L1	Normal Sextupole
2	XN06L1	Normal Sextupole
3	XN09L1	Normal Sextupole
4	XN012L1	Normal Sextupole
5	XN0311L1	Normal Sextupole
6	XN0816L1	Normal Sextupole
7	ON0311L1	Normal Octupole
8	ON0816L1	Normal Octupole
9	XSK2L4	Skew Sextupole
10	XSK4L1	Skew Sextupole
11	XSK6L4	Skew Sextupole

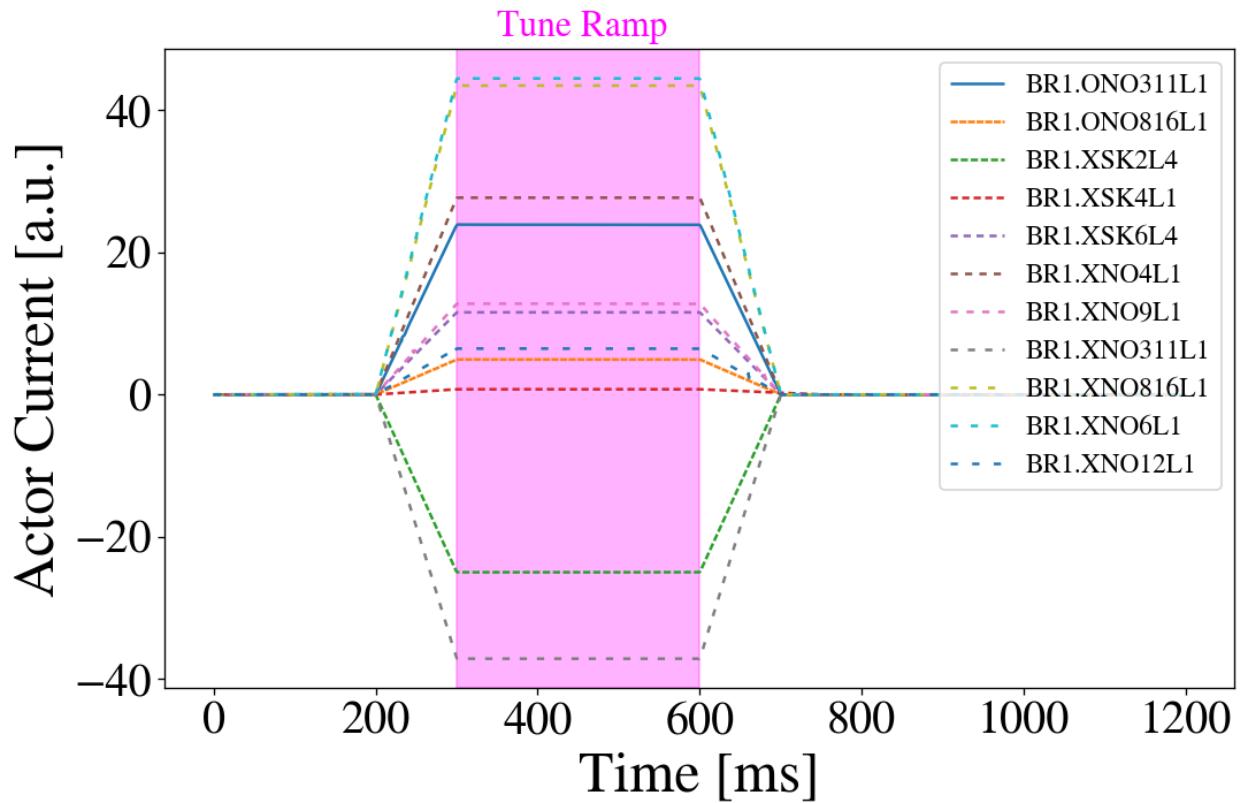


Figure 5.6 Waveform for the currents fed to the correctors (actors) as a function of time in the study cycle, for one particular iteration of an arbitrary optimization instance.

### 5.3.1 Bayesian Optimization

Bayesian optimization is a well-established technique used in machine learning and optimization tasks to efficiently search for the optimal solution within a parameter space, particularly when the objective function is expensive or time-consuming to evaluate—this is the case for an accelerator. It combines probabilistic modeling with the principles of Bayesian inference to iteratively update a model of the objective function based on observed data, gradually refining its understanding of the parameter space [52]. By balancing exploration (searching for promising regions) and exploitation (leveraging known information to identify the best areas), Bayesian optimization aims to find the global optimum while minimizing the number of evaluations required. This method is particularly useful in hyperparameter tuning for complex models, where traditional grid search or random search approaches may be impractical due to computational costs.

At the heart of Bayesian optimization, are Gaussian processes. A Gaussian process (GP) is a statistical model, that represents a collection of random variables, any finite subset of which has a joint Gaussian distribution [52]. GPs are defined by a mean function and a covariance function, which capture the prior beliefs about the underlying function being modeled and the correlations between different points in the input space, respectively. These special type of statistical models are key to estimate the underlying function that wants to be optimized. Through Bayesian inference, GPs can be updated with observed data, yielding posterior distributions that can inform predictions and lead to an optimized sampling for the Bayesian optimization algorithm.

Figure 5.7 shows an example for the evolution of the objective function during a Bayesian optimization procedure. In this case, the objective function is the normalized beam loss after the tune ramp illustrated by Fig. 5.5. It can be seen how the Bayesian optimizer finds solutions that effectively cancel out the beam loss from crossing the resonances. Nevertheless, given that this optimizer is built to find a global minimum, it will keep sampling other regions to make sure the best solution is not a local minimum. The color map of Fig. 5.7 shows how for some of these cases, the optimizer prioritizes exploration and drifts to some unknown region where the losses are high. For these cases, the underlying Gaussian process will learn that there is no worth in exploring

these regions. Ultimately, the configuration with the least amount of relative beam loss—the best configuration—is saved and kept as the optimum solution.

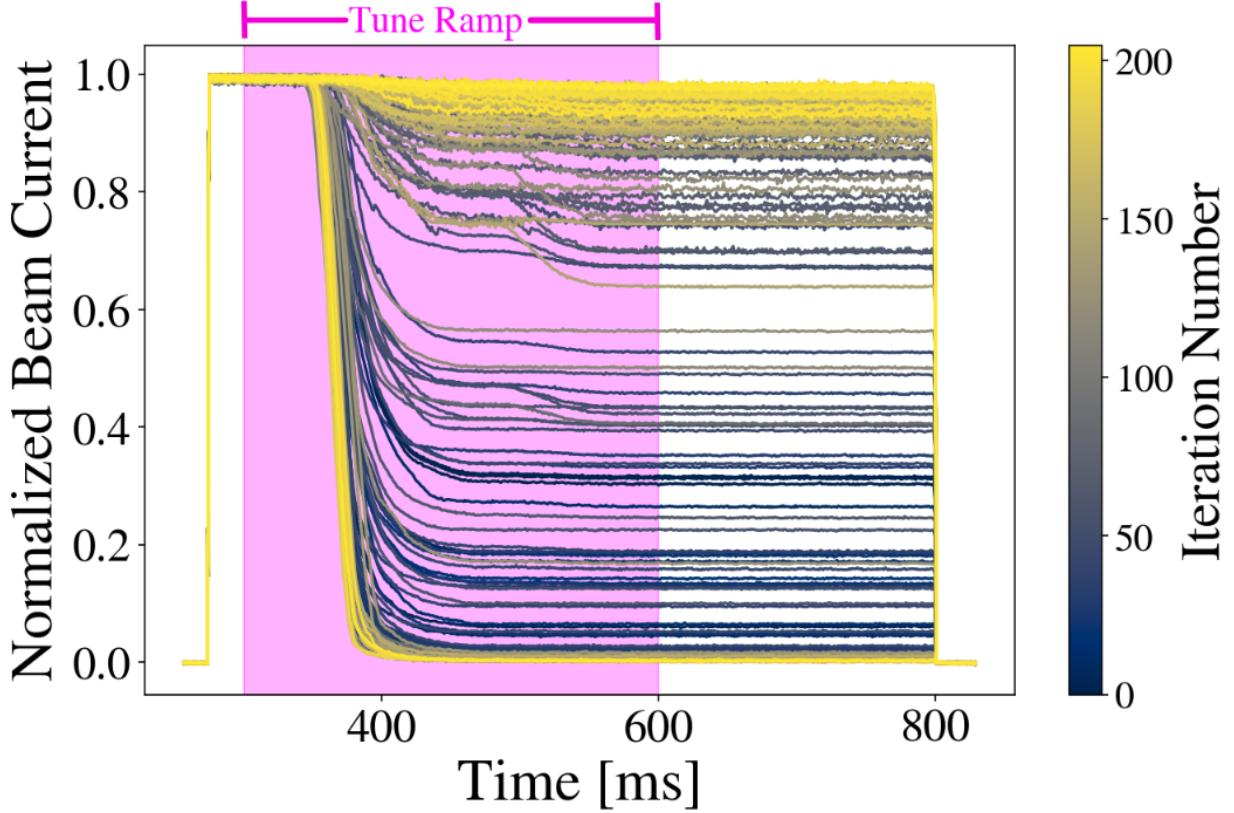


Figure 5.7 Normalized beam current plots for the Bayesian Optimization method done at Ring 1.

The bottom plot of Fig. 5.8 shows the explicit steps of each actor versus the number of iterations. Additionally, the top plot shows the trend of the objective function (relative beam loss) as the number of iterations increases. There are significant oscillations in the relative beam loss values, especially at the beginning, but a general trend towards minimization as the algorithm progresses through iterations. The early iterations reflect the exploration phase. BO is sampling points that give a broad understanding of the objective function’s landscape. As iterations progress, there is a trend toward certain regions in the parameter space. This indicates a shift from exploration to exploitation, where the algorithm starts to sample more from areas it believes to be near the optimum. The narrowing of actor current variability suggests a reduction in uncertainty about the location of the minimum beam loss, as the GP model becomes more informed. At the end of the optimization instance, such

as the one shown in Fig. 5.8, the configuration that gave the smallest relative beam loss is saved. GeOFF sets the default configuration of the correctors to these best values.

One important thing to discuss from Fig. 5.8 are the corrector values for the best configurations. It can be seen from Fig. 5.8 some correctors land on the limit values, e.g., the limits for the normal sextupoles (XNO magnets) were [-50,50]. This is specially apparent for the octupole correctors (ONO magnets) which have limits from -60 to 60, e.g., the magnet ONO816L1 is maxed out. Nevertheless, this was expected given that only 2 octupole correctors were used to correct 4 fourth order lines, where to fully cancel out all the fourth order RDTs one would need 8 correctors. It is important to remind the reader that in order to efficiently implement these type of algorithms with several actors, the currents need to be normalized between [0,1], in order to ensure that all the data is on the same scale and improve the model performance. This is done by GeOFF internally.

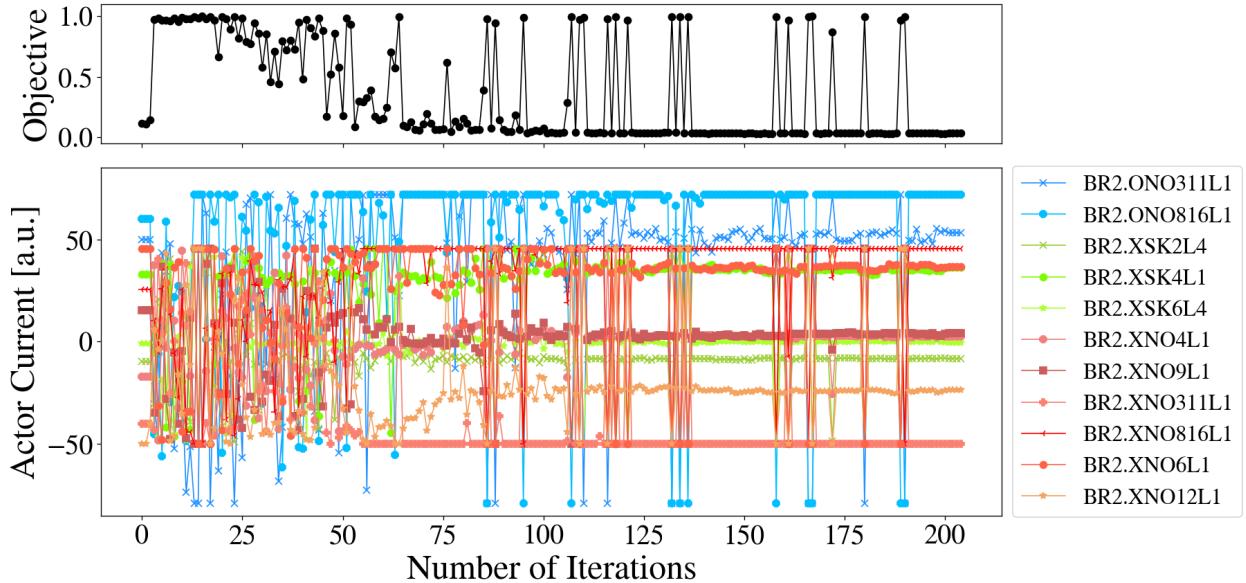


Figure 5.8 Summary for Bayesian optimization of resonance compensation applied to Ring 2 in the CERN PSB.

While Fig. 5.8 shows a relatively smooth Bayesian optimization instance for Ring 2, Figs. 5.9 and 5.10 show a different story for Ring 3. This particular optimization instance failed to find a configuration yielding a relative beam loss less than 5% after 200 iterations—unacceptable for operation. This can be seen on the top plot of Fig. 5.9 which follows the objective function across

iterations. In order to find an acceptable configuration, the BO instance was performed again, but in this case, the initial Gaussian process was trained with the data from the previous incomplete run. The additional 350 iterations from this second instance are shown in Fig. 5.10. This second instance yielded configurations that led to acceptable values for the relative beam loss. Possible explanations of this hurdle include inadequate exploration of the parameter space, poor choice of kernel for the Gaussian Process, poor choice of hyperparameters for this BO, or the presence of noise in the measurements.

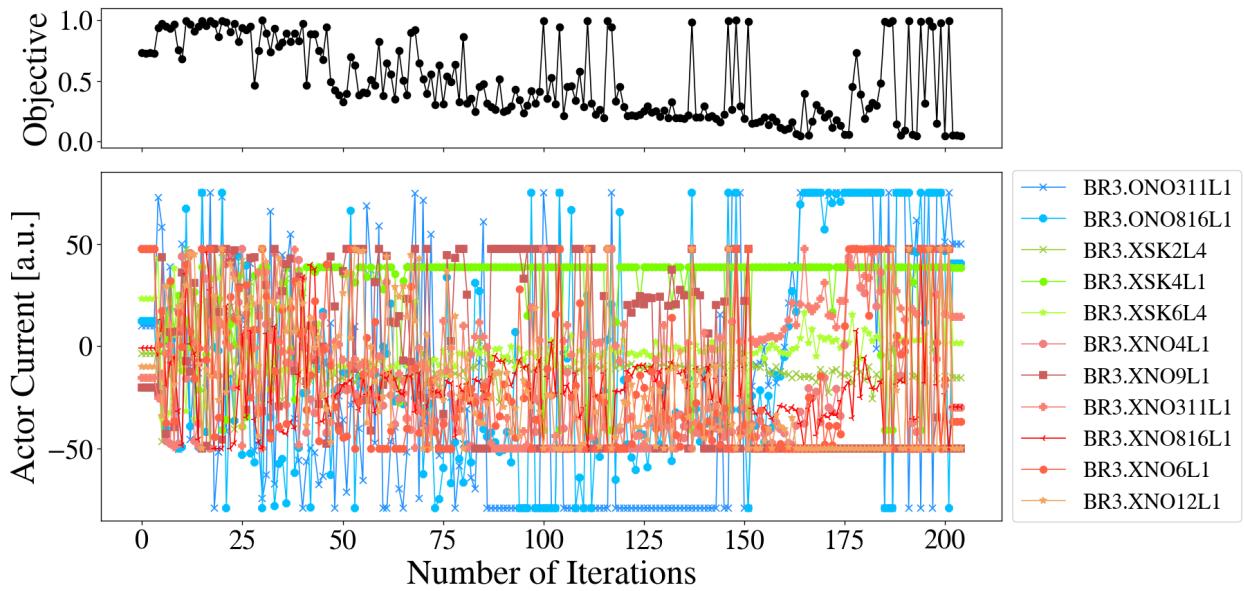


Figure 5.9 Summary for the first instance of Bayesian optimization towards resonance compensation applied to Ring 3 in the CERN PSB.

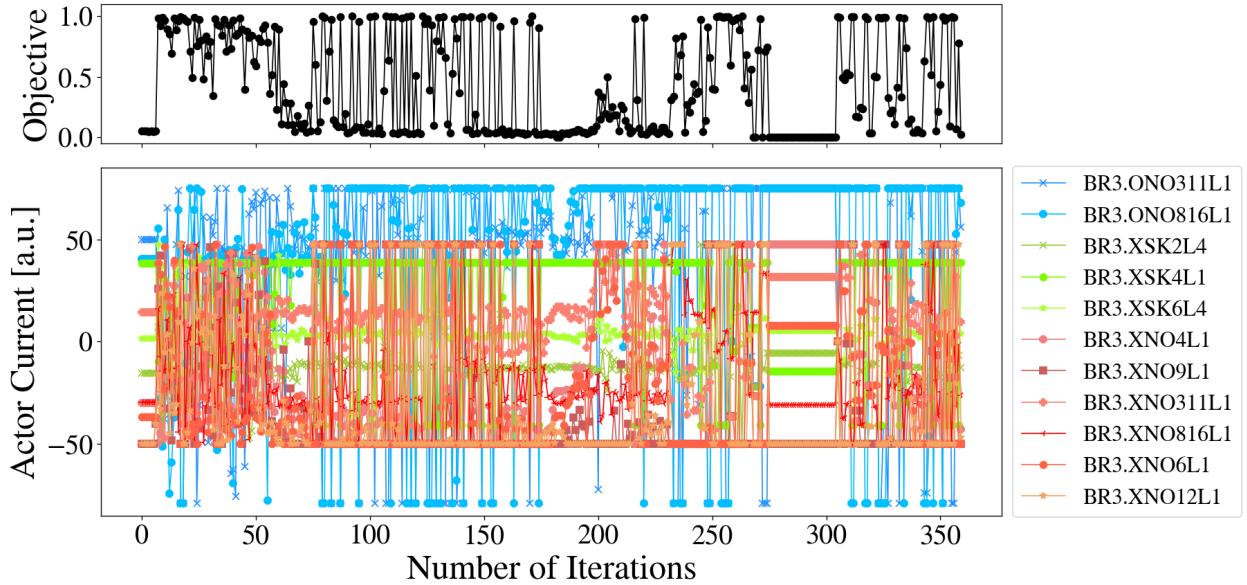


Figure 5.10 Summary for the second instance of Bayesian optimization of resonance compensation applied to Ring 3 in the CERN PSB. This second instance of optimization was trained with the data from the first instance.

### 5.3.2 BOBYQA (Bound Optimization BY Quadratic Approximation)

BOBYQA, which stands for Bound Optimization BY Quadratic Approximation, is an optimization algorithm commonly used for solving constrained optimization problems [53]. Unlike gradient-based methods, BOBYQA belongs to the class of derivative-free optimization algorithms, making it suitable for scenarios where the objective function is not differentiable or computationally expensive to evaluate. The algorithm iteratively builds a quadratic approximation of the objective function within a trust region, which is a bounded area around the current solution. By iteratively updating the quadratic model and moving towards the predicted optimum within the trust region, BOBYQA efficiently explores the search space while minimizing the number of function evaluations. Additionally, it employs a bound constraint strategy to ensure that the search remains within specified bounds. In this case, BOBYQA was used to solve the same optimization task from the last section but using a special type of beam at the PSB which is the BCMS (Bunch Compression Merging and Splitting) beam.

Similar to the exercise done in Fig. 5.7, Fig. 5.11 shows the relative beam loss profiles for an

instance of the BOBYQA optimization done on Ring 1. As the iterations progress (transitioning from blue to yellow), the spread in beam current values starts to decrease. This suggests that the algorithm is refining its search based on the feedback from the objective function and moving towards regions of the parameter space that yield better optimization results. The later iterations, indicated by yellow lines, show beam current trajectories that are closer together. This pattern of convergence signifies that the algorithm has likely identified a promising region in the parameter space and is now fine-tuning the parameters. It is minimizing the variance between iterations, suggesting an approach toward a stable solution.

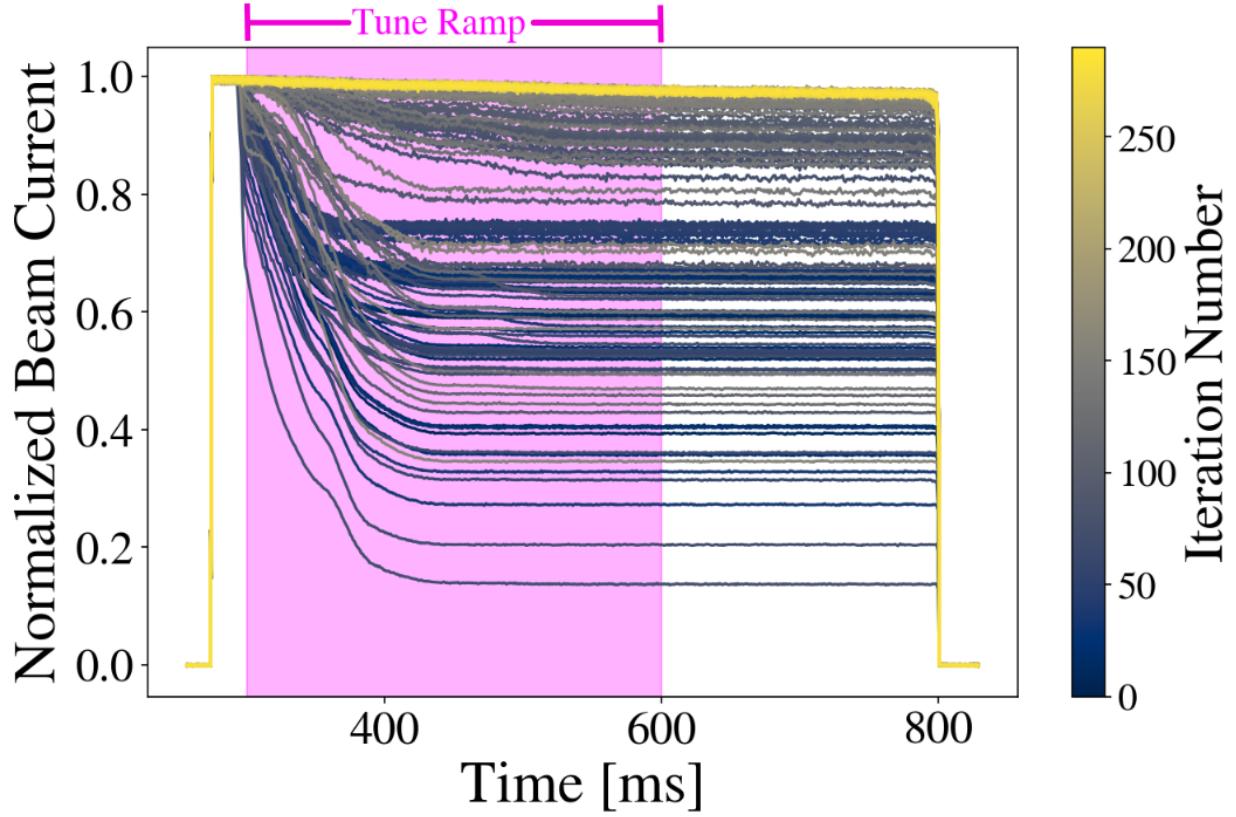


Figure 5.11 Normalized beam current plots for the BOBYQA method done at Ring 1.

Similar to the exercise done for the Bayesian optimization, Fig. 5.12 shows the evolution of the objective function and actor currents for a BOBYQA optimization instance on BCMS beam. The initial values were set from the Bayesian optimization results for each ring, as discussed in the last section. The trend of the objective function (top plot) shows a general decrease in relative

beam loss as the number of iterations increases, which indicates that the optimization is successful in finding parameters that result in less beam loss. The bottom graph illustrates the variation of the actor currents in arbitrary units across iterations. Each actor is distinguished by a unique line pattern and color. Some correctors are kept at 0 given that they were not available for Ring 3. The implementation of BOBYQA through GeOFF ensures the algorithm doesn't fall to a local minimum by bumping the settings to new configuration once it falls to some stable configuration. That can be seen from the sudden oscillations found in the objective function plot from Fig. 5.12.

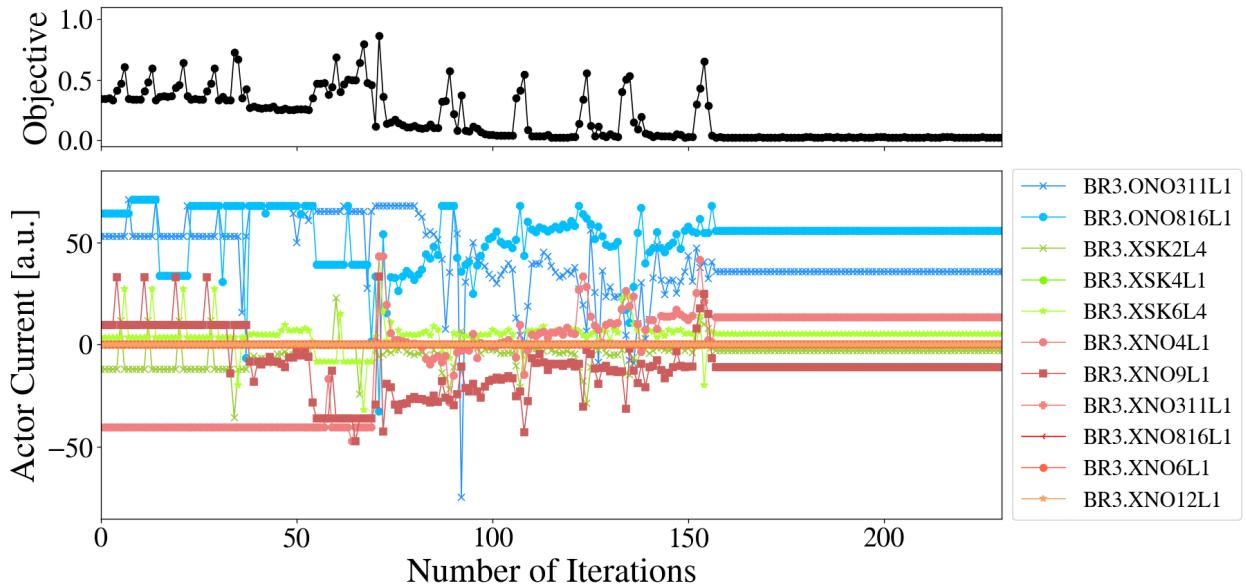


Figure 5.12 Summary for BOBYQA optimization of resonance compensation applied to Ring 3 in the CERN PSB for BCMS (Bunch Compression Merging and Splitting) beam.

Comparing this BOBYQA instance with the Bayesian optimization results, one can see that the exploration phase is much shorter and less broad for the BOBYQA instance. Similarly, the variation steps from BOBYQA in the actor currents are smaller than BO. This is true because of the underlying mechanism used to estimate the objective function and its uncertainty. BOBYQA uses deterministic local quadratic models, while Bayesian Optimization uses probabilistic models which provide a measure of uncertainty. BOBYQA is generally considered a local search technique, meaning it may be more prone to finding a local minimum rather than the global minimum. Hence, the initial point for BOBYQA holds more importance. Bayesian optimization is generally used for global

optimization and is particularly strong in high-dimensional spaces or when function evaluations are very expensive. In summary, while both algorithms are powerful tools for optimization in an accelerator physics problems, the choice between BOBYQA and Bayesian optimization will depend on the specific characteristics of the problem at hand, including the nature of the objective function, the presence of constraints, the dimensionality of the problem, practical considerations like the availability of computational and experimental resources, and, ultimately, availability of study/machine development time.

#### 5.4 Experimental Verification of Compensation

The whole point of the optimization algorithms explained in the last sections were to reduce the losses that show up in the loss map from Fig. 5.4. Figure 5.13 shows a new loss map with the best configurations found for each ring using the Bayesian optimization procedure on LHC-type beam. When comparing both loss maps, it is clear that with these new configurations, the loss maps have been cleared out of losses in the region of interest. The immediate losses are decreased by nearly one order of magnitude in the region of occupied by the tune ramp. In particular, the third order resonances that dominated the losses in Fig 5.4 are largely suppressed with these new configurations. Nevertheless, there are still some resonance lines visible in the loss maps on Fig. 5.13, e.g.,  $Q_x - 2Q_y = -4$ . Given that these lines are not in the region of PSB operation, they are not of particular concern.

This chapter has shown a different approach to resonance compensation from the one presented on Ch. 4. Chapter 4 showed a physics-informed approach by minimizing the Recycler's RDTs, and, ultimately, leading to the reduction of beam loss. On the other hand, the previous sections of this chapter, showed an optimization-based approach, which can be considered a brute-force line of action. In this approach, all the actors are thrown to an optimization algorithm, which finds a numerical solution that minimizes the objective function. No physics involved just numerical optimization. In particular, BOBYQA's deterministic approach can quickly home in on a solution when the accelerator is already operating near a resonance compensation optimum. Bayesian Optimization, with its probabilistic nature, is better suited for exploring unknown or poorly understood

operational regimes. All the previous approaches have been proven to work and yield satisfactory results in the context of resonance compensation. The choice between optimization-based and physics-informed approaches—or a combination thereof—depends on the specific context of the particle accelerator’s operational goals, the available data, and the accuracy of the model.

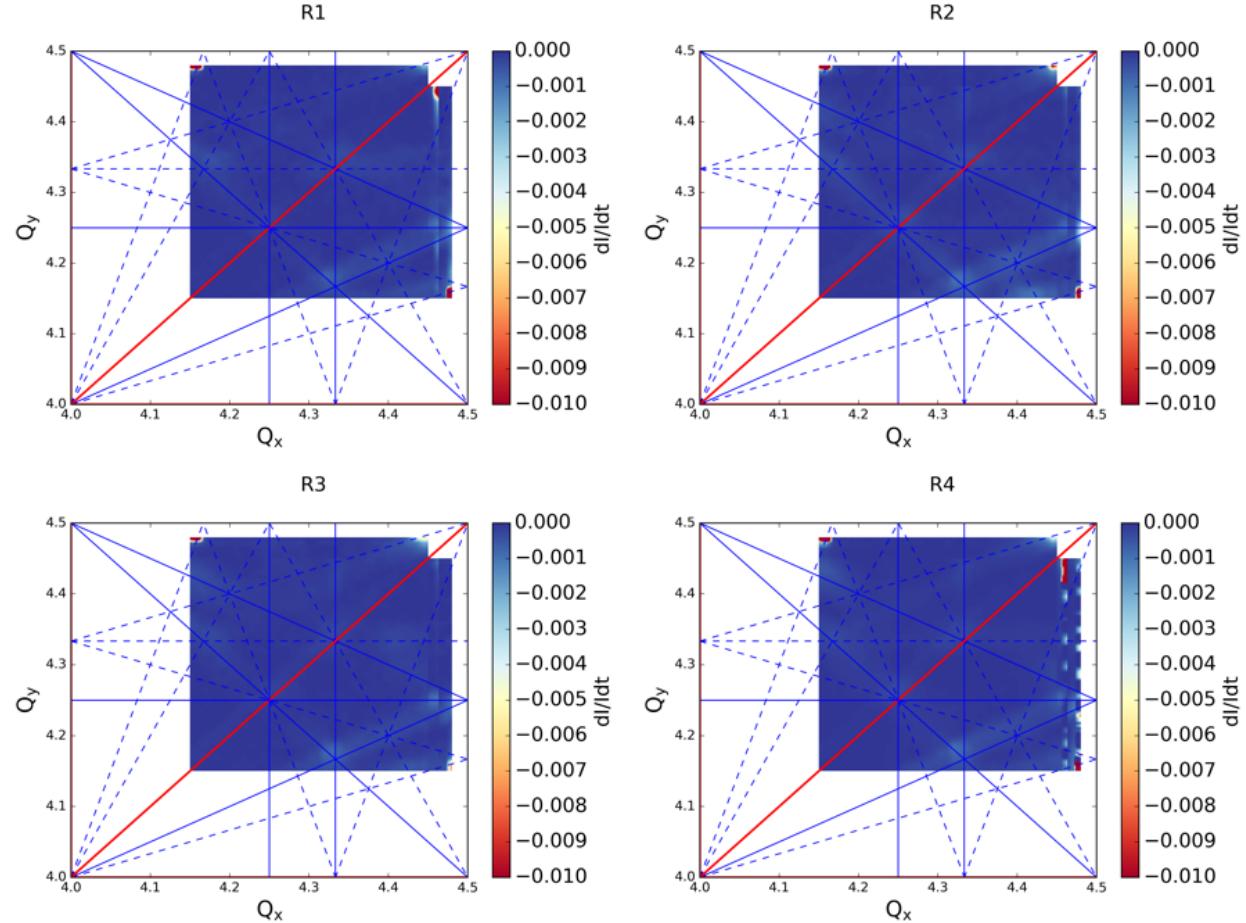


Figure 5.13 Dynamic loss maps for the 4 rings in the PS Booster with the best configuration from the Bayesian optimization of the resonance compensation. The plots are an average of scanning in 4 directions. Plot provided by F. Asvesta.

# CHAPTER 6

## HIGH INTENSITY STUDIES

### 6.1 Global RDTs and Intensity-Dependent Effects

[54]

### 6.2 Space Charge Tune Shift

$$\Delta\nu_{sc} = \frac{-3Nr_0RS}{4\sigma_z M\beta\gamma^2\varepsilon_{N,95\%}} \quad (6.1)$$

### 6.3 Measurement of Tune Spread

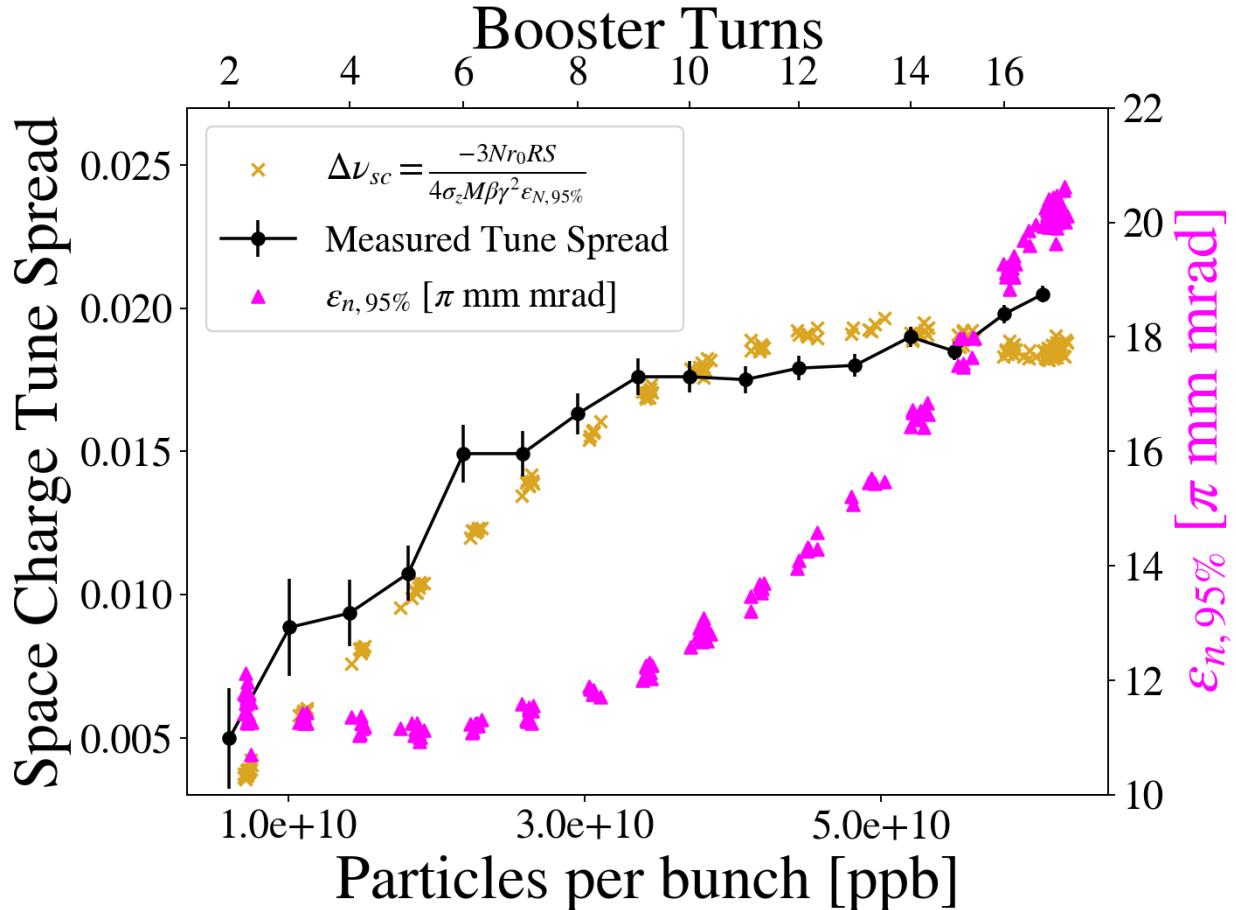


Figure 6.1 Measurement of tune spread.

## 6.4 Static Tune Scans at Different Intensities

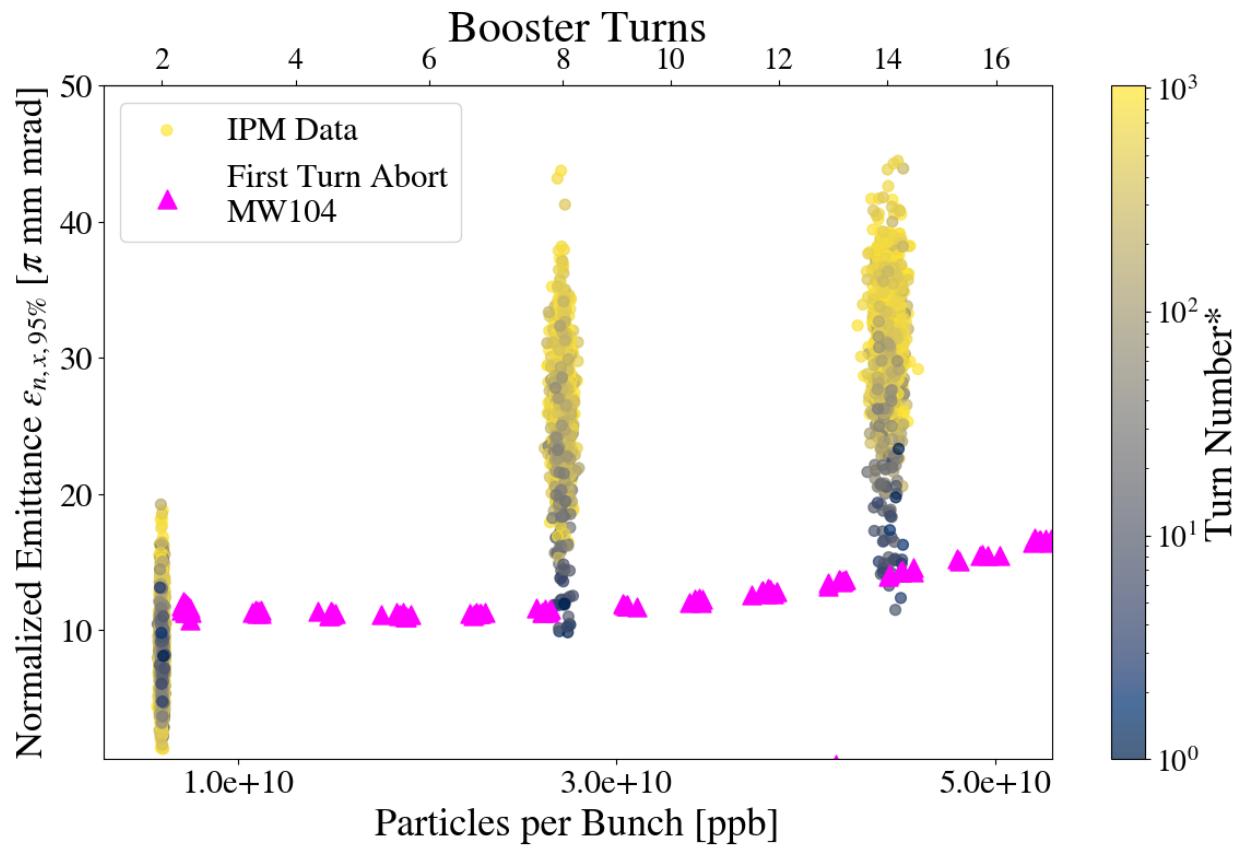


Figure 6.2 IPM data and multi-wire first turn abort data for  $Q_x = 25.370$  at different intensities.

## 6.5 Effect of Transverse Dampers

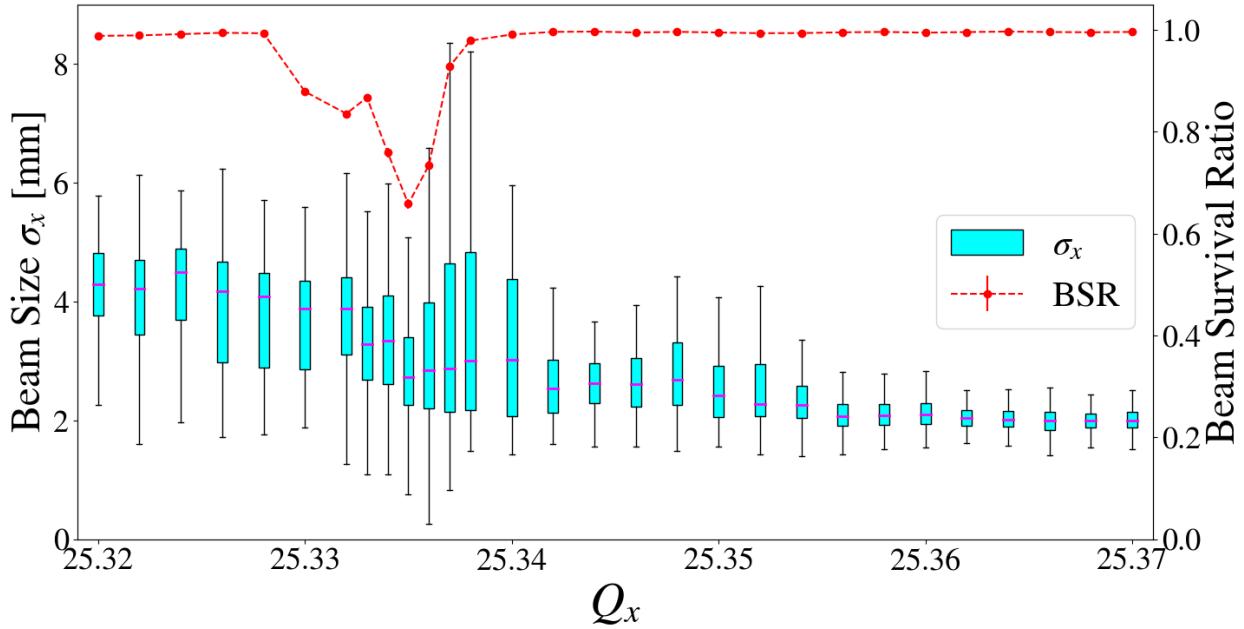


Figure 6.3 Static tune scan with beam survival ratio and IPM data box plots with  $3Q_x$  compensation, transverse dampers ON and 2 Booster Turns of equivalent intensity.

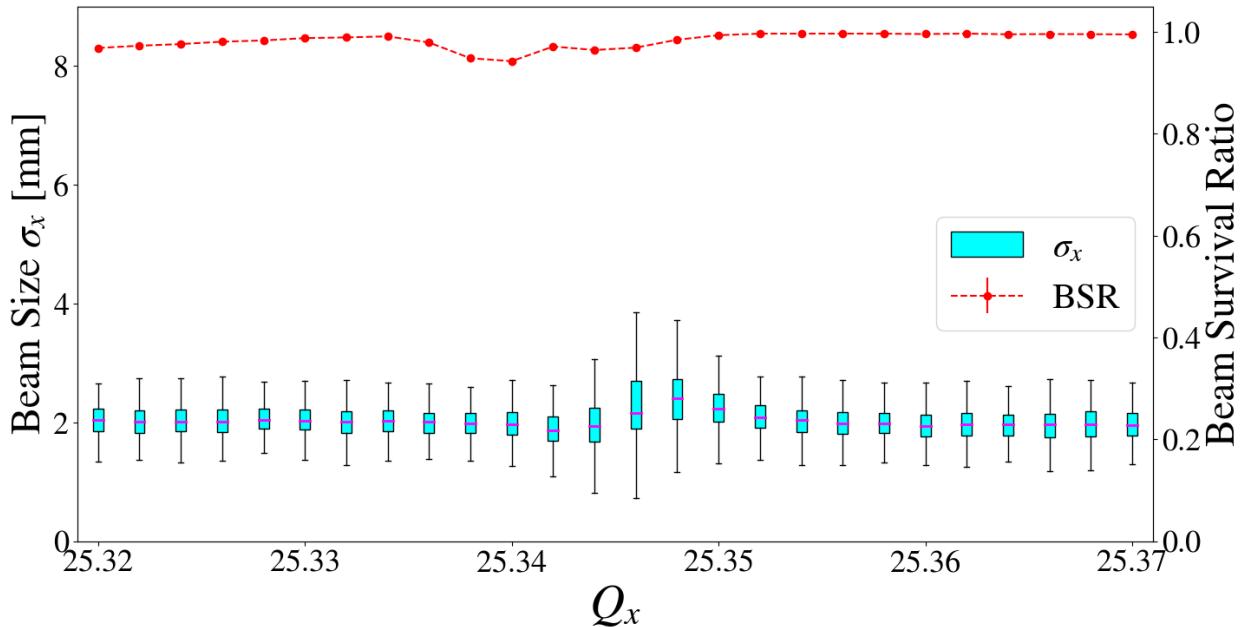


Figure 6.4 Static tune scan with beam survival ratio and IPM data box plots with  $3Q_x$  compensation, transverse dampers OFF and 2 Booster Turns of equivalent intensity.

## CHAPTER 7

### CONCLUSIONS AND FUTURE WORK

#### **7.1 Conclusions**

##### **7.1.1 RDTs and Resonance Compensation**

##### **7.1.2 Physics-Informed vs. Optimization-Based Compensation**

##### **7.1.3 High-Intensity Resonance Compensation**

##### **7.1.4 Dampers and High-Intensity Compensation**

#### **7.2 Future Work**

##### **7.2.1 Validation of Newly-Installed Sextupoles**

##### **7.2.2 Resonances and Transverse Dampers at High Intensities**

##### **7.2.3 Effect of MI Ramp on RDTs**

##### **7.2.4 Limits of Resonance Compensation**

##### **7.2.5 Space Charge RDTs**

[55] [56]

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## **APPENDIX A**

### **LIE ALGEBRA METHODS FOR ACCELERATOR PHYSICS IN 2D USING MATHEMATICA**

# Lie Algebra methods for Accelerator Physics in 2D

## 1. Poisson Brackets, Lie Operator and Exponential Lie Operators

```
In[1]:= PoissonBracket[f_, g_, q_Symbol, p_Symbol] :=
  Simplify[D[f, q] * D[g, p] - D[f, p] * D[g, q]]

In[2]:= PoissonBracket[f[x, p], g[x, p], x, p]
Out[2]=
  g^{(0,1)} [x, p] f^{(1,0)} [x, p] - f^{(0,1)} [x, p] g^{(1,0)} [x, p]

In[3]:= LieOperator[f_] := Function[g, PoissonBracket[f, g, x, p]]
LieOperator[f[x, p]][g[x, p]]
Out[3]=
  g^{(0,1)} [x, p] f^{(1,0)} [x, p] - f^{(0,1)} [x, p] g^{(1,0)} [x, p]

In[4]:= ExpOperator[oprtr_, n_ : 10] :=
  Function[f, Fold[f + oprtr[#1] / #2 &, f, Reverse@Range[n]]]

In[5]:= DiffOperator := Function[f, D[f, x]]
Simplify[ExpOperator[DiffOperator, 4][f[x]]]
Out[5]=
  f[x] + f'[x] + \frac{f''[x]}{2} + \frac{1}{6} f^{(3)} [x] + \frac{1}{24} f^{(4)} [x]

In[6]:= ExpOperator[LieOperator[f[x, p]], 1][g[x, p]]
Out[6]=
  g[x, p] + g^{(0,1)} [x, p] f^{(1,0)} [x, p] - f^{(0,1)} [x, p] g^{(1,0)} [x, p]
```

## 2. Lie Methods for common accelerator elements

```
In[1]:= Drift[L_, n_ : 10] := Function[g, ExpOperator[LieOperator[-0.5 * L * p^2], n][g]]
```

```

In[1]:= Map[Drift[L], {x, p}]
Out[1]= {0. + 1. L p + x, 0. + p}

In[2]:= ThinQuad[kL_, n_: 10] := Function[g, ExpOperator[LieOperator[-0.5 * kL * x^2], n][g]]
In[3]:= Map[ThinQuad[kL], {x, p}]
Out[3]= {0. + x, 0. + p - 1. kL x}

In[4]:= ThinKick[k_, m_, n_: 10] := Function[g, ExpOperator[LieOperator[k * x^m], n][g]]
In[5]:= Map[ThinKick[\[Lambda], n], {x, p}]
Out[5]= {x, p + n x^{-1+n} \[Lambda]}

In[6]:= ThickQuad[k_, L_, n_: 10] := Function[g,
ExpOperator[LieOperator[-Sign[k] * 0.5 * L * (k * x^2 + Sign[k] * p^2)], n][g]]
In[7]:= Collect[Assuming[k < 0, ThickQuad[k, L, 3][x]], {x, p}]
Out[7]= L (1. + 0.166667 k L^2) p + (1 + 0.5 k L^2) x

In[8]:= Collect[Assuming[k < 0, ThickQuad[k, L, 3][p]], {x, p}]
Out[8]= (1 + 0.5 k L^2) p + (1. k L + 0.166667 k^2 L^3) x

In[9]:= ThickFQuad[k_, L_, n_: 10] :=
Function[g, ExpOperator[LieOperator[-0.5 * L * (k * x^2 + p^2)], n][g]]
In[10]:= Collect[ThickFQuad[k, L, 10][p], {x, p}]
Out[10]= (1 - 0.5 k L^2 - 2.75573 \times 10^{-7} k^2 L^3 (-151200. L + 5040. k L^3 - 90. k^2 L^5 + 1. k^3 L^7)) p +
(-1. k L - 2.75573 \times 10^{-6} k^2 L^3 (-60480. + 3024. k L^2 - 72. k^2 L^4 + 1. k^3 L^6)) x

In[11]:= ThickDQuad[k_, L_, n_: 10] :=
Function[g, ExpOperator[LieOperator[0.5 * L * (k * x^2 - p^2)], n][g]]
In[12]:= Collect[ThickDQuad[k, L, 6][p], {x, p}]
Out[12]= (1 + 0.5 k L^2 + 0.0416667 k^2 L^4 + 0.00138889 k^3 L^6) p +
(1. k L + 0.166667 k^2 L^3 + 0.00833333 k^3 L^5) x

In[13]:= Collect[ExpOperator[LieOperator[(0.5 * p^2 + k * x^2) * s]][x], {x, p}]
Out[13]= 
$$\frac{1}{3} p s (-3. + 1. k s^2 - 0.1 k^2 s^4 + 0.0047619 k^3 s^6 - 0.000132275 k^4 s^8) +$$


$$\left(1 + \frac{1}{3} k s^2 (-3. + 0.5 k s^2 - 0.0333333 k^2 s^4 + 0.00119048 k^3 s^6 - 0.000026455 k^4 s^8)\right) x$$


```

```
In[6]:= Collect[ExpOperator[LieOperator[-(0.5*p^2 + k*x^3/6)*s], 3][x], {x, p}]
Out[6]= 1. p s + (1 - 0.166667 k p s^3) x - 0.25 k s^2 x^2
```

### 3. Concatenating elements by composition of maps

```
In[7]:= Map[Composition[Drift[L], Drift[L]], {x, p}]
Out[7]= {0. + 2. L p + x, 0. + p}
```

#### FODO Lattice

```
In[8]:= Collect[Simplify[Composition[ThinQuad[-1/(2 f)],
Drift[L], ThinQuad[1/f], Drift[L], ThinQuad[-1/(2 f)]][x]], {x, p}]
Out[8]= 0. + (2. L - 1. L^2/f) p + (1 - 0.5 L^2/f^2) x

In[9]:= Collect[Simplify[Composition[ThinQuad[-1/(2 f)],
Drift[L], ThinQuad[1/f], Drift[L], ThinQuad[-1/(2 f)]][p]], {x, p}]
Out[9]= 0. + (1 - 0.5 L^2/f^2) p + (-0.5 L/f^2 - 0.25 L^2/f^3) x
```

## **APPENDIX B**

### **LIE ALGEBRA METHODS FOR ACCELERATOR PHYSICS IN 4D USING MATHEMATICA**

# Lie Algebra methods for Accelerator Physics in 4D

## Chapter 1. Poisson Brackets and Lie Algebra in 4D

```
In[1]:= PoissonBracket[a_, b_, q_List, p_List] := Block[{pk, n}, n = Length[q];
If[n == Length[p],
pk = Simplify[Sum[D[a, q[[j]]] × D[b, p[[j]]] - D[b, q[[j]]] × D[a, p[[j]]], {j, 1, n}]],
Print["Incompatible lengths"]]]

In[2]:= PoissonBracket[f[x, px, y, py], g[x, px, y, py], {x, y}, {px, py}]
Out[2]=
g(0,0,0,1) [x, px, y, py] f(0,0,1,0) [x, px, y, py] -
f(0,0,0,1) [x, px, y, py] g(0,0,1,0) [x, px, y, py] +
g(0,1,0,0) [x, px, y, py] f(1,0,0,0) [x, px, y, py] -
f(0,1,0,0) [x, px, y, py] g(1,0,0,0) [x, px, y, py]

In[3]:= LieOperator[f_] := Function[g, PoissonBracket[f, g, {x, y}, {px, py}]]
In[4]:= LieOperator[f[x, y, px, py]][g[x, y, px, py]]
Out[4]=
g(0,0,0,1) [x, y, px, py] f(0,1,0,0) [x, y, px, py] -
f(0,0,0,1) [x, y, px, py] g(0,1,0,0) [x, y, px, py] +
g(0,0,1,0) [x, y, px, py] f(1,0,0,0) [x, y, px, py] -
f(0,0,1,0) [x, y, px, py] g(1,0,0,0) [x, y, px, py]

In[5]:= ExpOperator[oprtr_, n_ : 10] :=
Function[f, Fold[f + oprtr[#1] / #2 &, f, Reverse@Range[n]]]
In[6]:= Simplify[ExpOperator[LieOperator[f[x, y, px, py]], 1][g[x, y, px, py]]]
Out[6]=
g[x, y, px, py] + g(0,0,0,1) [x, y, px, py] f(0,1,0,0) [x, y, px, py] -
f(0,0,0,1) [x, y, px, py] g(0,1,0,0) [x, y, px, py] +
g(0,0,1,0) [x, y, px, py] f(1,0,0,0) [x, y, px, py] -
f(0,0,1,0) [x, y, px, py] g(1,0,0,0) [x, y, px, py]
```

```
In[1]:= Simplify[ExpOperator[LieOperator[f[x, y, px, py]], 2][g[x, y, px, py]]]
Out[1]=
g[x, y, px, py] +  $\frac{1}{2} \left( f^{(0,1,0,0)}[x, y, px, py] - f^{(0,0,0,2)}[x, y, px, py] \right.$ 
 $\left. (g^{(0,0,0,2)}[x, y, px, py] f^{(0,1,0,0)}[x, y, px, py] - f^{(0,0,0,2)}[x, y, px, py] \right.$ 
 $\left. g^{(0,1,0,0)}[x, y, px, py] + g^{(0,0,0,1)}[x, y, px, py] (2 + f^{(0,1,0,1)}[x, y, px, py]) - \right.$ 
 $\left. f^{(0,0,0,1)}[x, y, px, py] g^{(0,1,0,1)}[x, y, px, py] + g^{(0,0,1,1)}[x, y, px, py] \right.$ 
 $\left. f^{(1,0,0,0)}[x, y, px, py] - f^{(0,0,1,1)}[x, y, px, py] g^{(1,0,0,0)}[x, y, px, py] + \right.$ 
 $\left. g^{(0,0,1,0)}[x, y, px, py] f^{(1,0,0,1)}[x, y, px, py] - \right.$ 
 $\left. f^{(0,0,1,0)}[x, y, px, py] g^{(1,0,0,1)}[x, y, px, py] \right) +$ 
 $f^{(1,0,0,0)}[x, y, px, py] (g^{(0,0,1,1)}[x, y, px, py] f^{(0,1,0,0)}[x, y, px, py] -$ 
 $f^{(0,0,1,1)}[x, y, px, py] g^{(0,1,0,0)}[x, y, px, py] +$ 
 $g^{(0,0,0,1)}[x, y, px, py] f^{(0,1,1,0)}[x, y, px, py] - f^{(0,0,0,1)}[x, y, px, py]$ 
 $g^{(0,1,1,0)}[x, y, px, py] + g^{(0,0,2,0)}[x, y, px, py] f^{(1,0,0,0)}[x, y, px, py] -$ 
 $f^{(0,0,2,0)}[x, y, px, py] g^{(1,0,0,0)}[x, y, px, py] + g^{(0,0,1,0)}[x, y, px, py]$ 
 $(2 + f^{(1,0,1,0)}[x, y, px, py]) - f^{(0,0,1,0)}[x, y, px, py] g^{(1,0,1,0)}[x, y, px, py] \right) +$ 
 $f^{(0,0,0,1)}[x, y, px, py] (g^{(0,1,0,0)}[x, y, px, py] (-2 + f^{(0,1,0,1)}[x, y, px, py]) -$ 
 $f^{(0,1,0,0)}[x, y, px, py] g^{(0,1,0,1)}[x, y, px, py] -$ 
 $g^{(0,0,0,1)}[x, y, px, py] f^{(0,2,0,0)}[x, y, px, py] + f^{(0,0,0,1)}[x, y, px, py]$ 
 $g^{(0,2,0,0)}[x, y, px, py] - g^{(0,1,1,0)}[x, y, px, py] f^{(1,0,0,0)}[x, y, px, py] +$ 
 $f^{(0,1,1,0)}[x, y, px, py] g^{(1,0,0,0)}[x, y, px, py] - g^{(0,0,1,0)}[x, y, px, py]$ 
 $f^{(1,1,0,0)}[x, y, px, py] + f^{(0,0,1,0)}[x, y, px, py] g^{(1,1,0,0)}[x, y, px, py] \right) +$ 
 $f^{(0,0,1,0)}[x, y, px, py] (g^{(0,1,0,0)}[x, y, px, py] f^{(1,0,0,1)}[x, y, px, py] -$ 
 $f^{(0,1,0,0)}[x, y, px, py] g^{(1,0,0,1)}[x, y, px, py] +$ 
 $g^{(1,0,0,0)}[x, y, px, py] (-2 + f^{(1,0,1,0)}[x, y, px, py]) - f^{(1,0,0,0)}[x, y, px, py]$ 
 $g^{(1,0,1,0)}[x, y, px, py] - g^{(0,0,0,1)}[x, y, px, py] f^{(1,1,0,0)}[x, y, px, py] +$ 
 $f^{(0,0,0,1)}[x, y, px, py] g^{(1,1,0,0)}[x, y, px, py] - g^{(0,0,1,0)}[x, y, px, py]$ 
 $f^{(2,0,0,0)}[x, y, px, py] + f^{(0,0,1,0)}[x, y, px, py] g^{(2,0,0,0)}[x, y, px, py] \right)$ 
```

```
In[2]:= ExpLieOperator[f_, n_ : 10] := Function[g, ExpOperator[LieOperator[f], n][g]]
```

```
In[3]:= ExpLieOperator[f[x, y, px, py], 1][g[x, y, px, py]]
```

```
Out[3]=
g[x, y, px, py] + g^{(0,0,0,1)}[x, y, px, py] f^{(0,1,0,0)}[x, y, px, py] -
```

 $f^{(0,0,0,1)}[x, y, px, py] g^{(0,1,0,0)}[x, y, px, py] +$ 
 $g^{(0,0,1,0)}[x, y, px, py] f^{(1,0,0,0)}[x, y, px, py] -$ 
 $f^{(0,0,1,0)}[x, y, px, py] g^{(1,0,0,0)}[x, y, px, py]$ 

## Chapter 2. Common Accelerator Elements

```
In[4]:= Drift[L_, n_ : 10] := Function[g, ExpLieOperator[-0.5 * L * px^2 - 0.5 * L * py^2][g]]
```

```

In[1]:= Simplify[Map[Drift[L], {x, y, px, py}]]
Out[1]= {0. + 1. L px + x, 0. + 1. L py + y, 0. + px, 0. + py}

In[2]:= ThinQuad[kL_, n_:10]:= Function[g, ExpLieOperator[-0.5*kL*x^2+0.5*kL*y^2][g]]
In[3]:= Simplify[Map[ThinQuad[-1/f], {x, y, px, py}]]
Out[3]= {0. + x, 0. + y, 0. + px +  $\frac{1. x}{f}$ , 0. + py -  $\frac{1. y}{f}$ }

In[4]:= Simplify[Map[ThinQuad[k], {x, y, px, py}]]
Out[4]= {0. + x, 0. + y, 0. + px - 1. k x, 0. + py + 1. k y}

In[5]:= ThinSextupole[s_, n_:10]:= Function[g, ExpLieOperator[(1/3)*s*(x^3-3*x*y^2), n][g]]
In[6]:= Map[ThinSextupole[s], {x, y, px, py}]
Out[6]= {x, y, px + s (x^2 - y^2), py - 2 s x y}

In[7]:= GetNormalHamiltonian[kn_, n_] := (1/(1+n))*ComplexExpand[Re[kn*(x + I y)^(n+1)]]
In[8]:= Expand[GetNormalHamiltonian[k, 2]]
Out[8]=  $\frac{k x^3}{3} - k x y^2$ 

In[9]:= GetSkewHamiltonian[k_, n_] := (1/(1+n))*ComplexExpand[Re[I*k*(x + I y)^(n+1)]]
In[10]:= Expand[GetSkewHamiltonian[k, 2]]
Out[10]= -k x^2 y +  $\frac{k y^3}{3}$ 

In[11]:= ThinNormalMultipole[ki_, i_, n_:10]:= Function[g, ExpLieOperator[GetNormalHamiltonian[ki, i], n][g]]
In[12]:= Map[ThinNormalMultipole[k, 2], {x, y, px, py}]
Out[12]= {x, y, px + k (x^2 - y^2), py - 2 k x y}

In[13]:= ThinSkewMultipole[ki_, i_, n_:10]:= Function[g, ExpLieOperator[GetSkewHamiltonian[ki, i], n][g]]
In[14]:= Map[ThinSkewMultipole[k2s, 2], {x, y, px, py}]
Out[14]= {x, y, px - 2 k2s x y, py + k2s (-x^2 + y^2)}

```

In[15]:=

## Hamiltonians of some machine elements (3D)

**In general for multipole  $n$ :**

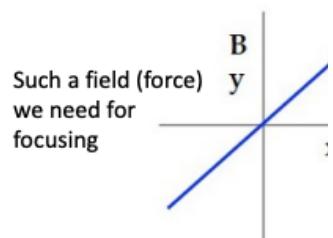
$$H_n = \frac{1}{1+n} \mathcal{R}e \left[ (k_n + ik_n^{(s)}) (x + iy)^{n+1} \right] + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

**We get for some important types (normal components  $k_n$  only)**

**dipole:**  $H = -\frac{-x\delta}{\rho} + \frac{x^2}{2\rho^2} + \frac{p_x^2 + p_y^2}{2(1+\delta)}$

**quadrupole:**  $H = \frac{1}{2} k_1 (x^2 - y^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$

**sextupole:**  $H = \frac{1}{3} k_2 (x^3 - 3xy^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$



Out[=]

## Hamiltonians of some machine elements (3D)

**In general for multipole  $n$ :**

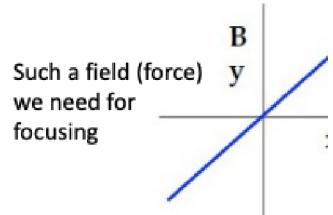
$$H_n = \frac{1}{1+n} \operatorname{Re} [(k_n + ik_n^{(s)})(x + iy)^{n+1}] + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

**We get for some important types (normal components  $k_n$  only)**

**dipole:**  $H = -\frac{x\delta}{\rho} + \frac{x^2}{2\rho^2} + \frac{p_x^2 + p_y^2}{2(1+\delta)}$

**quadrupole:**  $H = \frac{1}{2}k_1(x^2 - y^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$

**sextupole:**  $H = \frac{1}{3}k_2(x^3 - 3xy^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$



# Chapter 3. Multiple elements and Map Composition

In[=]:= Map[Composition[Drift[L], Drift[L]], {x, y, px, py}]

Out[=]=

$$\{0. + 2. L \text{px} + x, 0. + 2. L \text{py} + y, 0. + \text{px}, 0. + \text{py}\}$$

In[=]:=

### FODO Lattice

In[=]:= FODO[L\_, K\_, n\_ : 10] := Function[g, Composition[ThinQuad[-0.5 \* K], Drift[L], ThinQuad[K], Drift[L], ThinQuad[-0.5 \* K]][g]]

In[=]:= Collect[Simplify[FODO[L, -1/f][x]], {x, y, px, py}]

Out[=]=

$$0. + \left(2. L + \frac{1. L^2}{f}\right) \text{px} + \left(1 - \frac{0.5 L^2}{f^2}\right) x$$

In[6]:= Collect[Simplify[FODO[L, -1/f][y]], {x, y, px, py}]

Out[6]=

$$0. + \left( 2. L - \frac{1. L^2}{f} \right) p y + \left( 1 - \frac{0.5 L^2}{f^2} \right) y$$

In[7]:= Collect[Simplify[FODO[L, -1/f][px]], {x, y, px, py}]

Out[7]=

$$0. + \left( 1 - \frac{0.5 L^2}{f^2} \right) p x + \left( -\frac{0.5 L}{f^2} + \frac{0.25 L^2}{f^3} \right) x$$

In[8]:= Collect[Simplify[FODO[L, -1/f][py]], {x, y, px, py}]

Out[8]=

$$0. + \left( 1 - \frac{0.5 L^2}{f^2} \right) p y + \left( -\frac{0.5 L}{f^2} - \frac{0.25 L^2}{f^3} \right) y$$