Lie Algebra methods for Accelerator Physics in 4D

Chapter 1. Poisson Brackets and Lie Algebra in 4D

In[ ]:=
```
PoissonBracket[a_, b_, q_List, p_List] := Block[{pk, n}, n = Length[q];
  If[n == Length[p],
   pk = Simplify[Sum[D[a, q[[j]]] × D[b, p[[j]]] - D[b, q[[j]]] × D[a, p[[j]]], {j, 1, n}]],
   Print["Incompatible lengths"]]]
```

In[ ]:=
```
PoissonBracket[f[x, px, y, py], g[x, px, y, py], {x, y}, {px, py}]
```

Out[ ]=

$$g^{(0,0,0,1)}[x,px,y,py]\,f^{(0,0,1,0)}[x,px,y,py]-f^{(0,0,0,1)}[x,px,y,py]\,g^{(0,0,1,0)}[x,px,y,py]+g^{(0,1,0,0)}[x,px,y,py]\,f^{(1,0,0,0)}[x,px,y,py]-f^{(0,1,0,0)}[x,px,y,py]\,g^{(1,0,0,0)}[x,px,y,py]$$

In[ ]:=
```
LieOperator[f_] := Function[g, PoissonBracket[f, g, {x, y}, {px, py}]]
```

In[ ]:=
```
LieOperator[f[x, y, px, py]][g[x, y, px, py]]
```

Out[ ]=

$$g^{(0,0,0,1)}[x,y,px,py]\,f^{(0,1,0,0)}[x,y,px,py]-f^{(0,0,0,1)}[x,y,px,py]\,g^{(0,1,0,0)}[x,y,px,py]+g^{(0,0,1,0)}[x,y,px,py]\,f^{(1,0,0,0)}[x,y,px,py]-f^{(0,0,1,0)}[x,y,px,py]\,g^{(1,0,0,0)}[x,y,px,py]$$

In[ ]:=
```
ExpOperator[oprtr_, n_ : 10] :=
 Function[f, Fold[f + oprtr[#1] / #2 &, f, Reverse@Range[n]]]
```

In[ ]:=
```
Simplify[ExpOperator[LieOperator[f[x, y, px, py]], 1][g[x, y, px, py]]]
```

Out[ ]=

$$g[x,y,px,py]+g^{(0,0,0,1)}[x,y,px,py]\,f^{(0,1,0,0)}[x,y,px,py]-f^{(0,0,0,1)}[x,y,px,py]\,g^{(0,1,0,0)}[x,y,px,py]+g^{(0,0,1,0)}[x,y,px,py]\,f^{(1,0,0,0)}[x,y,px,py]-f^{(0,0,1,0)}[x,y,px,py]\,g^{(1,0,0,0)}[x,y,px,py]$$

In[∘]:=
```
Simplify[ExpOperator[LieOperator[f[x, y, px, py]], 2][g[x, y, px, py]]]
```

Out[∘]=

$$
\begin{aligned}
&g[x, y, px, py] + \frac{1}{2} \Big(f^{(0,1,0,0)}[x, y, px, py] \\
&\quad \Big(g^{(0,0,0,2)}[x, y, px, py]\, f^{(0,1,0,0)}[x, y, px, py] - f^{(0,0,0,2)}[x, y, px, py] \\
&\qquad g^{(0,1,0,0)}[x, y, px, py] + g^{(0,0,0,1)}[x, y, px, py] \left(2 + f^{(0,1,0,1)}[x, y, px, py]\right) - \\
&\qquad f^{(0,0,0,1)}[x, y, px, py]\, g^{(0,1,0,1)}[x, y, px, py] + g^{(0,0,1,1)}[x, y, px, py] \\
&\qquad f^{(1,0,0,0)}[x, y, px, py] - f^{(0,0,1,1)}[x, y, px, py]\, g^{(1,0,0,0)}[x, y, px, py] + \\
&\qquad g^{(0,0,1,0)}[x, y, px, py]\, f^{(1,0,0,1)}[x, y, px, py] - \\
&\qquad f^{(0,0,1,0)}[x, y, px, py]\, g^{(1,0,0,1)}[x, y, px, py]\Big) + \\
&\quad f^{(1,0,0,0)}[x, y, px, py] \Big(g^{(0,0,1,1)}[x, y, px, py]\, f^{(0,1,0,0)}[x, y, px, py] - \\
&\qquad f^{(0,0,1,1)}[x, y, px, py]\, g^{(0,1,0,0)}[x, y, px, py] + \\
&\qquad g^{(0,0,0,1)}[x, y, px, py]\, f^{(0,1,1,0)}[x, y, px, py] - f^{(0,0,0,1)}[x, y, px, py] \\
&\qquad g^{(0,1,1,0)}[x, y, px, py] + g^{(0,0,2,0)}[x, y, px, py]\, f^{(1,0,0,0)}[x, y, px, py] - \\
&\qquad f^{(0,0,2,0)}[x, y, px, py]\, g^{(1,0,0,0)}[x, y, px, py] + g^{(0,0,1,0)}[x, y, px, py] \\
&\qquad \left(2 + f^{(1,0,1,0)}[x, y, px, py]\right) - f^{(0,0,1,0)}[x, y, px, py]\, g^{(1,0,1,0)}[x, y, px, py]\Big) + \\
&\quad f^{(0,0,0,1)}[x, y, px, py] \Big(g^{(0,1,0,0)}[x, y, px, py] \left(-2 + f^{(0,1,0,1)}[x, y, px, py]\right) - \\
&\qquad f^{(0,1,0,0)}[x, y, px, py]\, g^{(0,1,0,1)}[x, y, px, py] - \\
&\qquad g^{(0,0,0,1)}[x, y, px, py]\, f^{(0,2,0,0)}[x, y, px, py] + f^{(0,0,0,1)}[x, y, px, py] \\
&\qquad g^{(0,2,0,0)}[x, y, px, py] - g^{(0,1,1,0)}[x, y, px, py]\, f^{(1,0,0,0)}[x, y, px, py] + \\
&\qquad f^{(0,1,1,0)}[x, y, px, py]\, g^{(1,0,0,0)}[x, y, px, py] - g^{(0,0,1,0)}[x, y, px, py] \\
&\qquad f^{(1,1,0,0)}[x, y, px, py] + f^{(0,0,1,0)}[x, y, px, py]\, g^{(1,1,0,0)}[x, y, px, py]\Big) + \\
&\quad f^{(0,0,1,0)}[x, y, px, py] \Big(g^{(0,1,0,0)}[x, y, px, py]\, f^{(1,0,0,1)}[x, y, px, py] - \\
&\qquad f^{(0,1,0,0)}[x, y, px, py]\, g^{(1,0,0,1)}[x, y, px, py] + \\
&\qquad g^{(1,0,0,0)}[x, y, px, py] \left(-2 + f^{(1,0,1,0)}[x, y, px, py]\right) - f^{(1,0,0,0)}[x, y, px, py] \\
&\qquad g^{(1,0,1,0)}[x, y, px, py] - g^{(0,0,0,1)}[x, y, px, py]\, f^{(1,1,0,0)}[x, y, px, py] + \\
&\qquad f^{(0,0,0,1)}[x, y, px, py]\, g^{(1,1,0,0)}[x, y, px, py] - g^{(0,0,1,0)}[x, y, px, py] \\
&\qquad f^{(2,0,0,0)}[x, y, px, py] + f^{(0,0,1,0)}[x, y, px, py]\, g^{(2,0,0,0)}[x, y, px, py]\Big)\Big)
\end{aligned}
$$

In[∘]:=
```
ExpLieOperator[f_, n_ : 10] := Function[g, ExpOperator[LieOperator[f], n][g]]
```

In[∘]:=
```
ExpLieOperator[f[x, y, px, py], 1][g[x, y, px, py]]
```

Out[∘]=

$$
\begin{aligned}
&g[x, y, px, py] + g^{(0,0,0,1)}[x, y, px, py]\, f^{(0,1,0,0)}[x, y, px, py] - \\
&\ f^{(0,0,0,1)}[x, y, px, py]\, g^{(0,1,0,0)}[x, y, px, py] + \\
&\ g^{(0,0,1,0)}[x, y, px, py]\, f^{(1,0,0,0)}[x, y, px, py] - \\
&\ f^{(0,0,1,0)}[x, y, px, py]\, g^{(1,0,0,0)}[x, y, px, py]
\end{aligned}
$$

Chapter 2. Common Accelerator Elements

In[∘]:=
```
Drift[L_, n_ : 10] := Function[g, ExpLieOperator[-0.5 * L * px ^ 2 - 0.5 * L * py ^ 2][g]]
```

In[ ]:=
```
Simplify[Map[Drift[L], {x, y, px, py}]]
```
Out[ ]=

$\{0. + 1.\ L\ px + x,\ 0. + 1.\ L\ py + y,\ 0. + px,\ 0. + py\}$

In[ ]:=
```
ThinQuad[kL_, n_ : 10] := Function[g, ExpLieOperator[-0.5 * kL * x ^ 2 + 0.5 * kL * y ^ 2][g]]
```

In[ ]:=
```
Simplify[Map[ThinQuad[-1 / f], {x, y, px, py}]]
```
Out[ ]=

$$\left\{0. + x,\ 0. + y,\ 0. + px + \frac{1.\ x}{f},\ 0. + py - \frac{1.\ y}{f}\right\}$$

In[ ]:=
```
Simplify[Map[ThinQuad[k], {x, y, px, py}]]
```
Out[ ]=

$\{0. + x,\ 0. + y,\ 0. + px - 1.\ k\ x,\ 0. + py + 1.\ k\ y\}$

In[ ]:=
```
ThinSextupole[s_, n_ : 10] :=
  Function[g, ExpLieOperator[(1 / 3) * s * (x ^ 3 - 3 * x * y ^ 2), n][g]]
```

In[ ]:=
```
Map[ThinSextupole[s], {x, y, px, py}]
```
Out[ ]=

$\left\{x,\ y,\ px + s\left(x^2 - y^2\right),\ py - 2\ s\ x\ y\right\}$

In[ ]:=
```
GetNormalHamiltonian[kn_, n_] := (1 / (1 + n)) * ComplexExpand[Re[kn * (x + I * y) ^ (n + 1)]]
```

In[ ]:=
```
Expand[GetNormalHamiltonian[k, 2]]
```
Out[ ]=

$$\frac{k\ x^3}{3} - k\ x\ y^2$$

In[ ]:=
```
GetSkewHamiltonian[k_, n_] := (1 / (1 + n)) * ComplexExpand[Re[I * k * (x + I * y) ^ (n + 1)]]
```

In[ ]:=
```
Expand[GetSkewHamiltonian[k, 2]]
```
Out[ ]=

$$-k\ x^2\ y + \frac{k\ y^3}{3}$$

In[ ]:=
```
ThinNormalMultipole[ki_, i_, n_ : 10] :=
  Function[g, ExpLieOperator[GetNormalHamiltonian[ki, i], n][g]]
```

In[ ]:=
```
Map[ThinNormalMultipole[k, 2], {x, y, px, py}]
```
Out[ ]=

$\left\{x,\ y,\ px + k\left(x^2 - y^2\right),\ py - 2\ k\ x\ y\right\}$

In[ ]:=
```
ThinSkewMultipole[ki_, i_, n_ : 10] :=
  Function[g, ExpLieOperator[GetSkewHamiltonian[ki, i], n][g]]
```

In[ ]:=
```
Map[ThinSkewMultipole[k2s, 2], {x, y, px, py}]
```
Out[ ]=

$\left\{x,\ y,\ px - 2\ k2s\ x\ y,\ py + k2s\left(-x^2 + y^2\right)\right\}$

In[ ]:=

Hamiltonians of some machine elements (3D)

In general for multipole n:

$$H_n = \frac{1}{1+n}\mathcal{R}e\left[(k_n + \mathrm{i}k_n^{(s)})(x+\mathrm{i}y)^{n+1}\right] + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

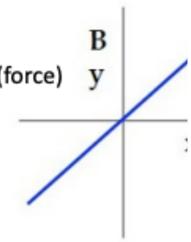
We get for some important types (normal components k_n onl

In[ ]:=

dipole: $$H = -\frac{-x\delta}{\rho} + \frac{x^2}{2\rho^2} + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

quadrupole: $$H = \frac{1}{2}k_1(x^2 - y^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

Such a field (force) we need for focusing



sextupole: $$H = \frac{1}{3}k_2(x^3 - 3xy^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

Out[ ]=

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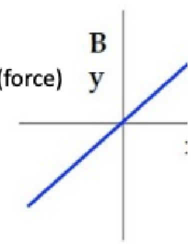

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Chapter 3. Multiple elements and Map Composition

In[ ]:=
```
Map[Composition[Drift[L], Drift[L]], {x, y, px, py}]
```
Out[ ]=
```
{0. + 2. L px + x, 0. + 2. L py + y, 0. + px, 0. + py}
```

In[ ]:=

FODO Lattice

In[ ]:=
```
FODO[L_, K_, n_ : 10] := Function[g, Composition[ThinQuad[-0.5 * K],
    Drift[L], ThinQuad[K], Drift[L], ThinQuad[-0.5 * K]][g]]
```

In[ ]:=
```
Collect[Simplify[FODO[L, -1 / f][x]], {x, y, px, py}]
```
Out[ ]=

$$0. + \left(2.\,L + \frac{1.\,L^2}{f}\right) px + \left(1 - \frac{0.5\,L^2}{f^2}\right) x$$

In[ ]:=
```
Collect[Simplify[FODO[L, -1 / f][y]], {x, y, px, py}]
```

Out[ ]=

$$0. + \left(2.\,L - \frac{1.\,L^2}{f}\right) py + \left(1 - \frac{0.5\,L^2}{f^2}\right) y$$

In[ ]:=
```
Collect[Simplify[FODO[L, -1 / f][px]], {x, y, px, py}]
```

Out[ ]=

$$0. + \left(1 - \frac{0.5\,L^2}{f^2}\right) px + \left(-\frac{0.5\,L}{f^2} + \frac{0.25\,L^2}{f^3}\right) x$$

In[ ]:=
```
Collect[Simplify[FODO[L, -1 / f][py]], {x, y, px, py}]
```

Out[ ]=

$$0. + \left(1 - \frac{0.5\,L^2}{f^2}\right) py + \left(-\frac{0.5\,L}{f^2} - \frac{0.25\,L^2}{f^3}\right) y$$