Modeling of mechatronics systems

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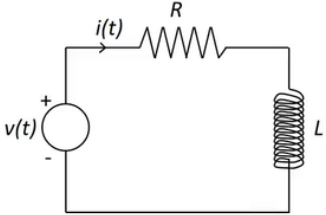
Summary

Example – Electrical system modeling

- Derive a mathematical model for the system:



Three phase method

- Structuring
  - Divide into subsystems
  - Inputs, outputs, internal variables?
- Basic equations
  - Conservation laws
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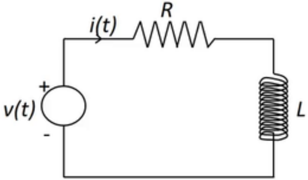

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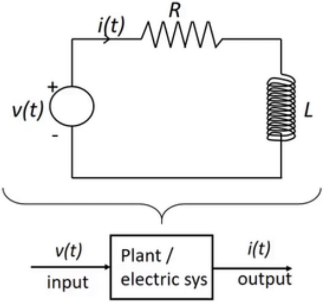

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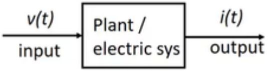

Conservation laws (balance equation):

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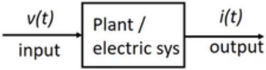

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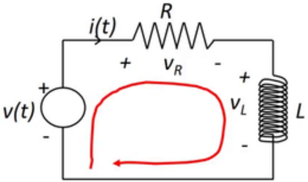
Conservation laws (balance equation):

What is it that changes? What is causing this change?



- Voltage balance [V] (Kirchhoff's voltage law)

$$\sum_i v_i = 0$$

$$v - v_R - v_L = 0$$

Electrical System Modeling

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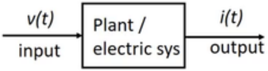

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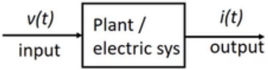
Constitutive relations:

Three phase method



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Conservation laws (balance equation):

- Voltage balance [V] (Kirchhoff's voltage law)

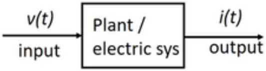
$$v - v_R - v_L = 0$$

Constitutive relations:

$$v_R = Ri$$ (resistor: voltage and current)

$$v_L = L\frac{di}{dt}$$ (inductor: voltage and current)

Three phase method



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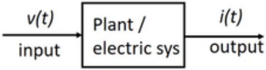
$$v - v_R - v_L = 0$$

$$v_R = Ri \qquad v_L = L\frac{di}{dt}$$

$$\dot{x} = \frac{dx}{dt} = f(x, u, d)$$

$$y = h(x, u, d)$$

Three phase method



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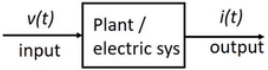
Choose state variables:

Three phase method



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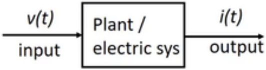

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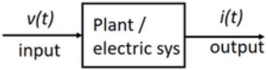
Choose state variables:

- What is changing?
  - current: i

Electrical System Modeling

Three phase method



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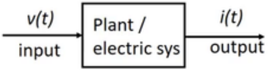
Choose state variables: i

$$\frac{di}{dt} =$$

Three phase method



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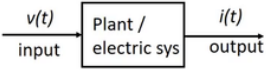
Choose state variables: i

$$\frac{di}{dt} = \frac{1}{L}v_L$$

Three phase method



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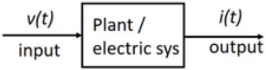

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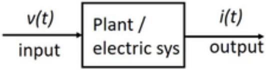

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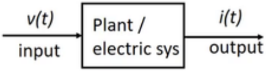
$$v - v_R - v_L = 0$$

$$v_R = Ri$$

Choose state variables: i

$$\frac{di}{dt} = \frac{1}{L}(v - v_R)$$

Three phase method



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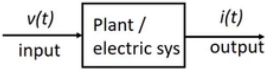

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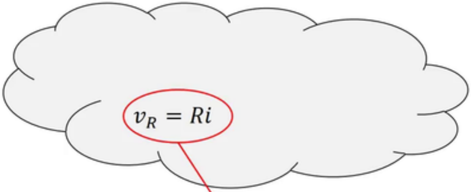

Choose state variables: i

$$\frac{di}{dt} = \frac{1}{L}(v - v_R)$$

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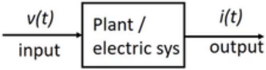

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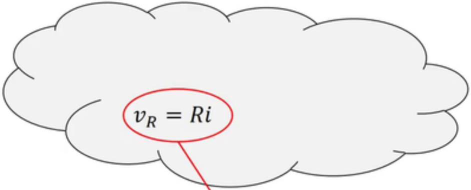

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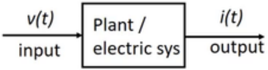

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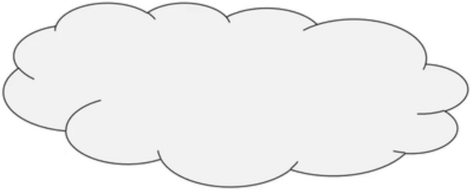

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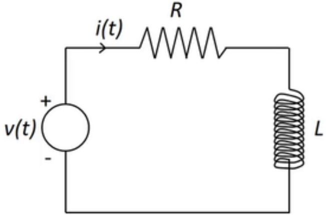
$$\left\{ \frac{di}{dt} = \frac{1}{L}(v - Ri) \right.$$

Variable change: $x = i, u = v \; and \; y = x$

$$\left\{ \begin{aligned} \frac{dx}{dt} &= -\frac{R}{L}x + \frac{1}{L}u \\ y &= x \end{aligned} \right.$$

Example – Electrical system modeling

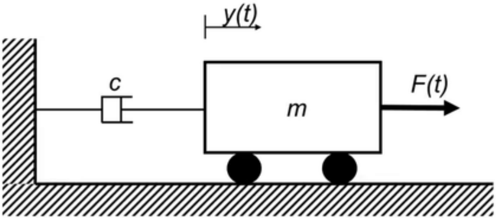
- Derive a mathematical model for the system:



$$\left.\right\} \quad \begin{aligned} \frac{dx}{dt} &= -\frac{R}{L}x + \frac{1}{L}u \\ y &= x \end{aligned}$$

Example – Mechanical system modeling

- Derive a mathematical model for the system:



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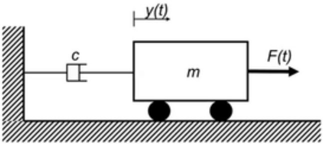

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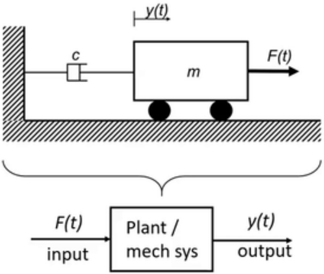

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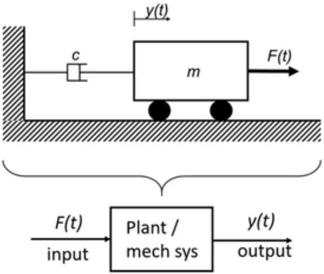
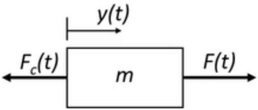


Free body diagram:



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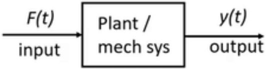

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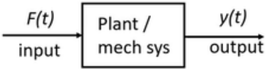

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What is it that changes? What is causing this change?

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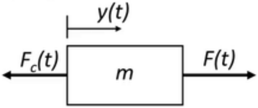

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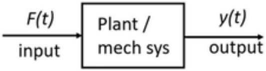

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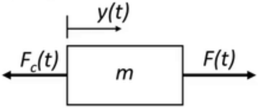
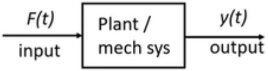


- Force balances [$kgm/s^2 = N$] (Newton's law)

$$ma = \sum_i F_i$$

$$ma = F - F_c$$

Three phase method



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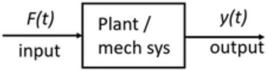

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Constitutive relations:

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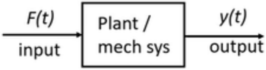

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Constitutive relations:

$$a = \frac{dv}{dt} = \dot{v}$$ (acceleration and velocity)

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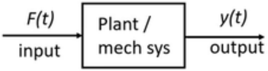

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Conservation laws (balance equation):

- Force balances [kgm/s^2 = N] (Newton's law)

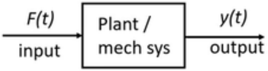
$$ma = F - F_c$$

Constitutive relations:

$$a = \frac{dv}{dt} = \dot{v}$$ (acceleration and velocity)

$$v = \frac{dy}{dt} = \dot{y}$$ (velocity and position)

Three phase method



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Conservation laws (balance equation):

- Force balances [kgm/s^2 = N] (Newton's law)

$$ma = F - F_c$$

Constitutive relations:

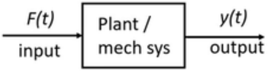
$$a = \frac{dv}{dt} = \dot{v}$$ (acceleration and velocity)

$$v = \frac{dy}{dt} = \dot{y}$$ (velocity and position)

$$F_c = cv$$ (force and velocity)

Mechanical System Modeling

Three phase method



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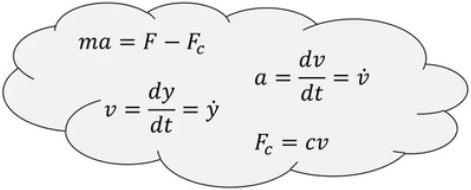
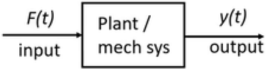


Three phase method



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$$ma = F - F_c$$

$$a = \frac{dv}{dt} = \dot{v}$$

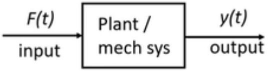
$$v = \frac{dy}{dt} = \dot{y}$$

$$F_c = cv$$

$$\dot{x} = \frac{dx}{dt} = f(x, u, d)$$

$$y = h(x, u, d)$$

Three phase method



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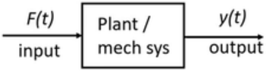
$$F_c = cv$$

Choose state variables:

- What is changing?

Mechanical System Modeling

Three phase method



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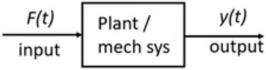
Choose state variables:

- What is changing?
  - position: y
  - velocity: v

Three phase method



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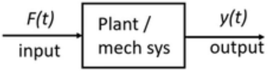
Choose state variables: $y \; and \; v$

Three phase method



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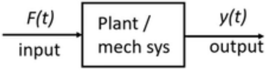
Choose state variables: $y \; and \; v$

$$\frac{dy}{dt} = v$$

Three phase method



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Choose state variables: y *and* v

$$\frac{dy}{dt} = v$$

Mechanical System Modeling

Three phase method

F(t) input → Plant / mech sys → y(t) output

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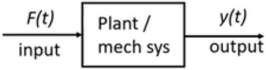
$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} =$$

Three phase method



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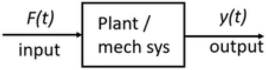
Three phase method



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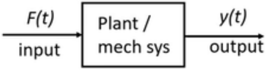
Mechanical System Modeling

Three phase method



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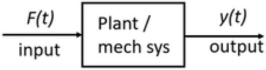
Three phase method



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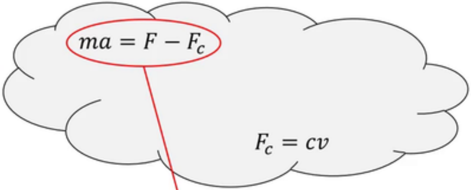
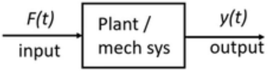


Choose state variables: y and v

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = \frac{1}{m}(F - F_c)$$

Three phase method



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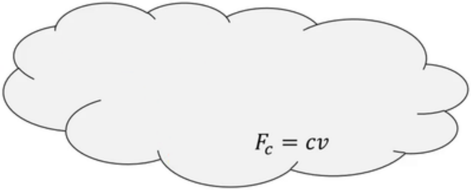

$$F_c = cv$$

Choose state variables: $y \text{ and } v$

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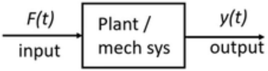
Mechanical System Modeling

Three phase method



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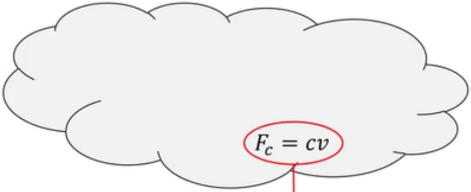

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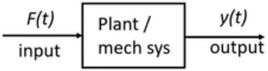

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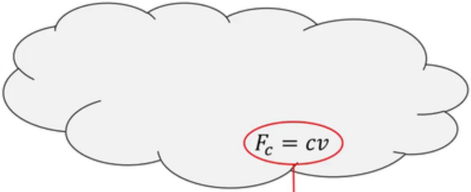

Choose state variables: y and v

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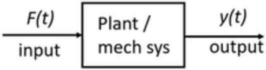
Mechanical System Modeling

Three phase method



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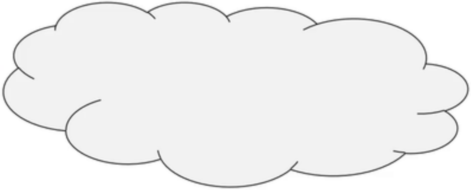

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Variable change:

$$x_1 = y, x_2 = v, u = F \; and \; y = x_1$$

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$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = \dfrac{1}{m}(u - cx_2) \end{cases}$$

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Rewrite in matrix form:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

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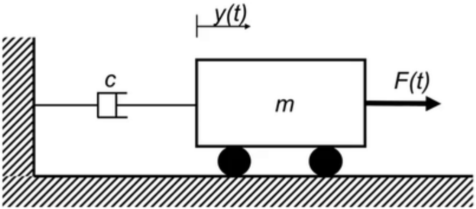
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u$$

Example – Mechanical system modeling

- Derive a mathematical model for the system:



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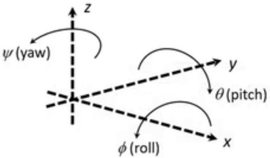
Example – Single track vehicle modeling

Derive a mathematical model for studying vehicle handling dynamics:

Planar motion:

- longitudinal and lateral velocities and the rotational velocity (yaw rate)



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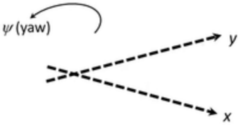

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Types of models:

- Two track models
- Single track models (bicycle models)
  - Linear single track model

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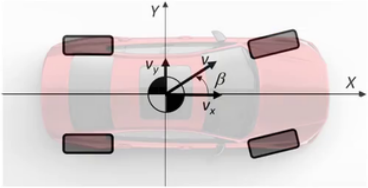

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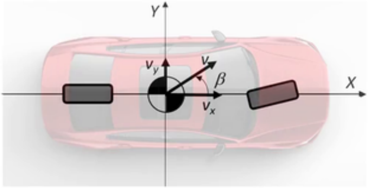

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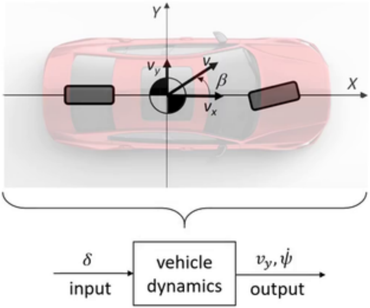

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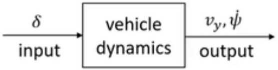

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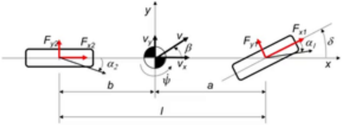

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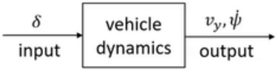

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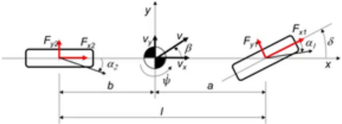

- Force balances [kgm/s² = N] (Newton's law)

$$ma = \sum_i F_i$$

- Torque balance [kgm²/s² = Nm] (Newton's law)

$$J\ddot{\psi} = \sum_i T_i$$

Three phase method

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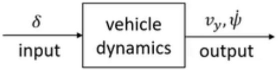

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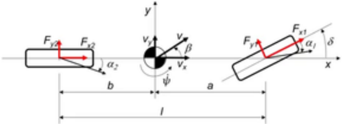

- Force balances [kgm/s² = N] (Newton's law)

$$ma = \sum_i F_i \begin{cases} ma_x = F_{x1}\cos\delta + F_{x2} - F_{y1}\sin\delta \\ ma_y = F_{x1}\sin\delta + F_{y1}\cos\delta + F_{y2} \end{cases}$$

- Torque balance [kgm²/s² = Nm] (Newton's law)

$$J\ddot{\psi} = \sum_i T_i$$

Single Track Vehicle Modeling

Three phase method

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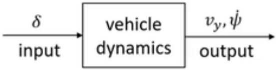
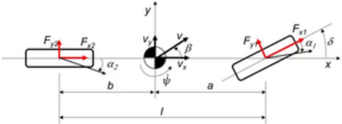


Conservation laws (balance equation):



- Force balances [kgm/s² = N] (Newton's law)

$$ma = \sum_i F_i \begin{cases} ma_x = F_{x1}\cos\delta + F_{x2} - F_{y1}\sin\delta \\ ma_y = F_{x1}\sin\delta + F_{y1}\cos\delta + F_{y2} \end{cases}$$

- Torque balance [kgm²/s² = Nm] (Newton's law)

$$J\ddot{\psi} = \sum_i T_i \left\{ J\ddot{\psi} = aF_{x1}\sin\delta + aF_{y1}\cos\delta - bF_{y2} \right.$$

Three phase method

- Structuring
  - Divide into subsystems
  - Inputs, outputs, internal variables?
- Basic equations
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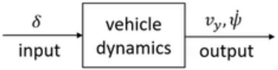
  - Constitutive relations
- Form state-space model
  - Choose state variables
  - Form $\dot{x} = \cdots$

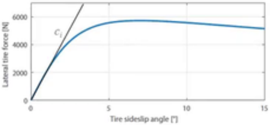


Conservation laws (balance equation):

$$ma_x = F_{x1}\cos\delta + F_{x2} - F_{y1}\sin\delta$$
$$ma_y = F_{x1}\sin\delta + F_{y1}\cos\delta + F_{y2}$$
$$J\ddot{\psi} = aF_{x1}\sin\delta + aF_{y1}\cos\delta - bF_{y2}$$

Constitutive relations:

Tire dynamics:



Three phase method

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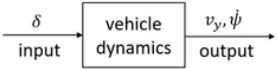

Conservation laws (balance equation):

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Constitutive relations:

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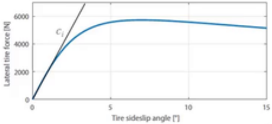
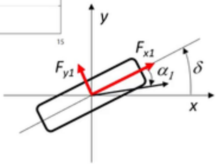


$$\alpha_i = -\tan^{-1}\left(\frac{v_{yi}}{v_{xi}}\right)$$



Three phase method

- Structuring
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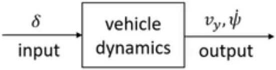


Conservation laws (balance equation):

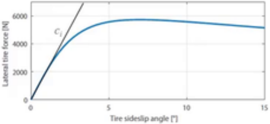
$$ma_x = F_{x1}\cos\delta + F_{x2} - F_{y1}\sin\delta$$

$$ma_y = F_{x1}\sin\delta + F_{y1}\cos\delta + F_{y2}$$

$$J\ddot{\psi} = aF_{x1}\sin\delta + aF_{y1}\cos\delta - bF_{y2}$$

Constitutive relations:

Tire dynamics:



$$\alpha_1 = \delta - \tan^{-1}\left(\frac{v_y + a\dot{\psi}}{v_x}\right)$$

$$\alpha_2 = -\tan^{-1}\left(\frac{v_y - b\dot{\psi}}{v_x}\right)$$

Three phase method

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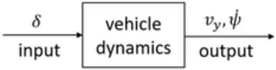

Conservation laws (balance equation):

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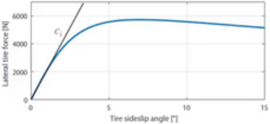
$$J\ddot{\psi} = aF_{x1}\sin\delta + aF_{y1}\cos\delta - bF_{y2}$$

Constitutive relations:

Tire dynamics: $F_{yi} = c_i \alpha_i$



$$\alpha_1 = \delta - \tan^{-1}\left(\frac{v_y + a\dot{\psi}}{v_x}\right)$$
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$$\xrightarrow[\text{input}]{\delta} \boxed{\begin{array}{c}\text{vehicle}\\ \text{dynamics}\end{array}} \xrightarrow[\text{output}]{v_y, \dot{\psi}}$$

Conservation laws (balance equation):

$$ma_x = F_{x1} \cos\delta + F_{x2} - F_{y1} \sin\delta$$

$$ma_y = F_{x1} \sin\delta + F_{y1} \cos\delta + F_{y2}$$

$$J\ddot{\psi} = aF_{x1} \sin\delta + aF_{y1} \cos\delta - bF_{y2}$$

Constitutive relations:

Longitudinal and lateral accelerations (in the inertial reference frame) are:

$$a_x = \dot{v}_x - v_y\dot{\psi}$$

$$a_y = \dot{v}_y + v_x\dot{\psi}$$

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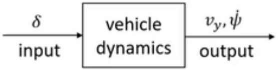

$$ma_x = F_{x1}\cos\delta + F_{x2} - F_{y1}\sin\delta$$

$$ma_y = F_{x1}\sin\delta + F_{y1}\cos\delta + F_{y2}$$

$$J\ddot{\psi} = aF_{x1}\sin\delta + aF_{y1}\cos\delta - bF_{y2}$$

$$\alpha_1 = \delta - \tan^{-1}\left(\frac{v_y + a\dot{\psi}}{v_x}\right)$$

$$F_{yi} = c_i\alpha_i$$

$$a_y = \dot{v}_y + v_x\dot{\psi}$$

$$a_x = \dot{v}_x - v_y\dot{\psi}$$

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Three phase method

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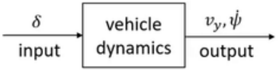
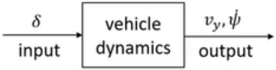


$$m a_x = F_{x1}\cos\delta + F_{x2} - F_{y1}\sin\delta$$
$$m a_y = F_{x1}\sin\delta + F_{y1}\cos\delta + F_{y2}$$
$$J\ddot{\psi} = aF_{x1}\sin\delta + aF_{y1}\cos\delta - bF_{y2}$$
$$\alpha_1 = \delta - \tan^{-1}\left(\frac{v_y + a\dot{\psi}}{v_x}\right)$$
$$F_{yi} = c_i\alpha_i$$
$$a_y = \dot{v}_y + v_x\dot{\psi}$$
$$a_x = \dot{v}_x - v_y\dot{\psi}$$
$$\alpha_2 = -\tan^{-1}\left(\frac{v_y - b\dot{\psi}}{v_x}\right)$$

$$\dot{x} = f(x,u,d)$$
$$y = h(x,u,d)$$

Three phase method



$$ma_x = F_{x1}\cos\delta + F_{x2} - F_{y1}\sin\delta$$
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Choose state variables:

- What is changing?

Three phase method

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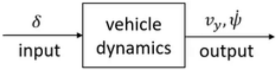

$$ma_x = F_{x1}\cos\delta + F_{x2} - F_{y1}\sin\delta$$
$$ma_y = F_{x1}\sin\delta + F_{y1}\cos\delta + F_{y2}$$
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$$\alpha_2 = -\tan^{-1}\left(\frac{v_y - b\dot{\psi}}{v_x}\right)$$

Choose state variables:

- What is changing?
  - velocity: v_x, v_y
  - yaw rate: $\dot{\psi}$

Three phase method

$$ma_y = F_{x1}\cos\delta + F_{x2} - F_{y1}\sin\delta$$
$$ma_y = F_{x1}\sin\delta + F_{y1}\cos\delta + F_{y2}$$
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$$a_x = \dot{v}_x - v_y\dot{\psi}$$
$$\alpha_2 = -\tan^{-1}\left(\frac{v_y - b\dot{\psi}}{v_x}\right)$$

$$\dot{v}_x = v_y\dot{\psi} + \frac{1}{m}\left(F_{x1}\cos\delta + F_{x2} - c_1\left(\delta - \tan^{-1}\left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right)\sin\delta\right)$$
$$\dot{v}_y = -v_x\dot{\psi} + \frac{1}{m}\left(F_{x1}\cos\delta + c_1\left(\delta - \tan^{-1}\left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right)\cos\delta - c_2\tan^{-1}\left(\frac{v_y - b\dot{\psi}}{v_x}\right)\right)$$
$$\ddot{\psi} = \frac{1}{J}\left(aF_{x1}\cos\delta + ac_1\left(\delta - \tan^{-1}\left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right)\cos\delta + bc_2\tan^{-1}\left(\frac{v_y - b\dot{\psi}}{v_x}\right)\right)$$

Three phase method

$$ma_y = F_{x1}\cos\delta + F_{x2} - F_{y1}\sin\delta$$
$$ma_y = F_{x1}\sin\delta + F_{y1}\cos\delta + F_{y2}$$
$$J\ddot{\psi} = aF_{x1}\sin\delta + aF_{y1}\cos\delta - bF_{y2}$$
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$$a_x = \dot{v}_x - v_y\dot{\psi}$$
$$\alpha_2 = -\tan^{-1}\left(\frac{v_y - b\dot{\psi}}{v_x}\right)$$

$$\dot{v}_x = v_y\dot{\psi} + \frac{1}{m}\left(F_{x1}\cos\delta + F_{x2} - c_1\left(\delta - \tan^{-1}\left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right)\sin\delta\right)$$

$$\dot{v}_y = -v_x\dot{\psi} + \frac{1}{m}\left(F_{x1}\cos\delta + c_1\left(\delta - \tan^{-1}\left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right)\cos\delta - c_2\tan^{-1}\left(\frac{v_y - b\dot{\psi}}{v_x}\right)\right)$$

$$\ddot{\psi} = \frac{1}{J}\left(aF_{x1}\cos\delta + ac_1\left(\delta - \tan^{-1}\left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right)\cos\delta + bc_2\tan^{-1}\left(\frac{v_y - b\dot{\psi}}{v_x}\right)\right)$$

Three phase method

Let's make some assumptions:

$$ma_y = F_{x1}\cos\delta + F_{x2} - F_{y1}\sin\delta$$
$$ma_y = F_{x1}\sin\delta + F_{y1}\cos\delta + F_{y2}$$
$$J\ddot{\psi} = aF_{x1}\sin\delta + aF_{y1}\cos\delta - bF_{y2}$$
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$$\dot{v}_x = v_y\dot{\psi} + \frac{1}{m}\left(F_{x1}\cos\delta + F_{x2} - c_1\left(\delta - \tan^{-1}\left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right)\sin\delta\right)$$

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$$\ddot{\psi} = \frac{1}{J}\left(aF_{x1}\cos\delta + ac_1\left(\delta - \tan^{-1}\left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right)\cos\delta + bc_2\tan^{-1}\left(\frac{v_y - b\dot{\psi}}{v_x}\right)\right)$$

Three phase method

Let's make some assumptions:

- Constant longitudinal velocity, $\dot{v}_x = 0$

$$ma_y = F_{x1}\cos\delta + F_{x2} - F_{y1}\sin\delta$$
$$ma_y = F_{x1}\sin\delta + F_{y1}\cos\delta + F_{y2}$$
$$J\ddot{\psi} = aF_{x1}\sin\delta + aF_{y1}\cos\delta - bF_{y2}$$
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$$\dot{v}_x = v_y\dot{\psi} + \frac{1}{m}\left(F_{x1}\cos\delta + F_{x2} - c_1\left(\delta - \tan^{-1}\left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right)\sin\delta\right)$$

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$$\ddot{\psi} = \frac{1}{J}\left(aF_{x1}\cos\delta + ac_1\left(\delta - \tan^{-1}\left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right)\cos\delta + bc_2\tan^{-1}\left(\frac{v_y - b\dot{\psi}}{v_x}\right)\right)$$

Three phase method

Let's make some assumptions:

- Constant longitudinal velocity, $\dot{v}_x = 0$
- Highway driving, $v_x > 0, \alpha_i$ small

$$ma_y = F_{x1}\sin\delta + F_{y1}\cos\delta + F_{y2}$$

$$J\ddot{\psi} = aF_{x1}\sin\delta + aF_{y1}\cos\delta - bF_{y2}$$

$$\alpha_1 = \delta - \tan^{-1}\left(\frac{v_y + a\dot{\psi}}{v_x}\right)$$

$$F_{yi} = c_i\alpha_i$$

$$a_y = \dot{v}_y + v_x\dot{\psi}$$

$$\alpha_2 = -\tan^{-1}\left(\frac{v_y - b\dot{\psi}}{v_x}\right)$$

$$\dot{v}_y = -v_x\dot{\psi} + \frac{1}{m}\left(F_{x1}\cos\delta + c_1\left(\delta - \tan^{-1}\left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right)\cos\delta - c_2\tan^{-1}\left(\frac{v_y - b\dot{\psi}}{v_x}\right)\right)$$

$$\ddot{\psi} = \frac{1}{J}\left(aF_{x1}\cos\delta + ac_1\left(\delta - \tan^{-1}\left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right)\cos\delta + bc_2\tan^{-1}\left(\frac{v_y - b\dot{\psi}}{v_x}\right)\right)$$

Three phase method

Let's make some assumptions:

- Constant longitudinal velocity, $\dot{v}_x = 0$
- Highway driving, $v_x > 0, \alpha_i$ small
- No external forces, $F_{xi} = 0$

$$ma_y = F_{x1}\sin\delta + F_{y1}\cos\delta + F_{y2}$$

$$J\ddot{\psi} = aF_{x1}\sin\delta + aF_{y1}\cos\delta - bF_{y2}$$

$$\alpha_1 = \delta - \frac{v_y + a\dot{\psi}}{v_x}$$

$$F_{yi} = c_i\alpha_i$$

$$a_y = \dot{v}_y + v_x\dot{\psi}$$

$$\alpha_2 = -\frac{v_y - b\dot{\psi}}{v_x}$$

$$\dot{v}_y = -v_x\dot{\psi} + \frac{1}{m}\left(F_{x1}\cos\delta + c_1\left(\delta - \left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right)\cos\delta - c_2 \quad * \quad \left(\frac{v_y - b\dot{\psi}}{v_x}\right)\right)$$

$$\ddot{\psi} = \frac{1}{J}\left(aF_{x1}\cos\delta + ac_1\left(\delta - \left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right)\cos\delta + bc_2 \quad * \quad \left(\frac{v_y - b\dot{\psi}}{v_x}\right)\right)$$

Three phase method

Let's make some assumptions:

- Constant longitudinal velocity, $\dot{v}_x = 0$
- Highway driving, $v_x > 0, \alpha_i$ small
- No external forces, $F_{xi} = 0$

$$ma_y = F_{y1}\cos\delta + F_{y2}$$

$$J\ddot{\psi} = aF_{y1}\cos\delta - bF_{y2}$$

$$\alpha_1 = \delta - \frac{v_y + a\dot{\psi}}{v_x}$$

$$F_{yi} = c_i\alpha_i$$

$$a_y = \dot{v}_y + v_x\dot{\psi}$$

$$\alpha_2 = -\frac{v_y - b\dot{\psi}}{v_x}$$

$$\dot{v}_y = -v_x\dot{\psi} + \frac{1}{m}\left(c_1\left(\delta - \left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right)\cos\delta - c_2 * \left(\frac{v_y - b\dot{\psi}}{v_x}\right)\right)$$

$$\ddot{\psi} = \frac{1}{J}\left(ac_1\left(\delta - \left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right)\cos\delta + bc_2 * \left(\frac{v_y - b\dot{\psi}}{v_x}\right)\right)$$

Three phase method

Let's make some assumptions:

- Constant longitudinal velocity, $\dot{v}_x = 0$
- Highway driving, $v_x > 0, \alpha_i$ small
- No external forces, $F_{xi} = 0$
- Small steering angles, $\cos\delta \approx 1, \sin\delta \approx \delta$

$$ma_y = F_{y1}\cos\delta + F_{y2}$$

$$J\ddot{\psi} = aF_{y1}\cos\delta - bF_{y2}$$

$$\alpha_1 = \delta - \frac{v_y + a\dot{\psi}}{v_x}$$

$$F_{yi} = c_i\alpha_i$$

$$a_y = \dot{v}_y + v_x\dot{\psi}$$

$$\alpha_2 = -\frac{v_y - b\dot{\psi}}{v_x}$$

$$\dot{v}_y = -v_x\dot{\psi} + \frac{1}{m}\left(c_1\left(\delta - \left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right)\cos\delta - c_2 * \left(\frac{v_y - b\dot{\psi}}{v_x}\right)\right)$$

$$\ddot{\psi} = \frac{1}{J}\left(ac_1\left(\delta - \left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right)\cos\delta + bc_2 * \left(\frac{v_y - b\dot{\psi}}{v_x}\right)\right)$$

Single Track Vehicle Modeling

Three phase method

Let's make some assumptions:

- Constant longitudinal velocity, $\dot{v}_x = 0$
- Highway driving, $v_x > 0, \alpha_i$ small
- No external forces, $F_{xi} = 0$
- Small steering angles, $\cos\delta \approx 1, \sin\delta \approx \delta$

$$ma_y = F_{y1} + F_{y2}$$

$$J\ddot{\psi} = \mathrm{a}F_{y1} + bF_{y2}$$

$$\alpha_1 = \delta - \frac{v_y + a\dot{\psi}}{v_x}$$

$$F_{yi} = c_i\alpha_i$$

$$a_y = \dot{v}_y + v_x\dot{\psi}$$

$$\alpha_2 = -\frac{v_y - b\dot{\psi}}{v_x}$$

$$\dot{v}_y = -v_x\dot{\psi} + \frac{1}{m}\Bigg(\quad c_1\left(\delta - \left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right) - c_2 \; * \; \left(\frac{v_y - b\dot{\psi}}{v_x}\right)\Bigg)$$

$$\ddot{\psi} = \frac{1}{J}\Bigg(\quad ac_1\left(\delta - \left(\frac{v_y + a\dot{\psi}}{v_x}\right)\right) + bc_2 \; * \; \left(\frac{v_y - b\dot{\psi}}{v_x}\right)\Bigg)$$

Three phase method

Let's make some assumptions:

- Constant longitudinal velocity, $\dot{v}_x = 0$
- No external forces, $F_{xi} = 0$
- Small steering angles, $\cos\delta \approx 1, \sin\delta \approx \delta$
- Highway driving, $v_x > 0, \alpha_i$ small

$$ma_y = F_{y1} + F_{y2}$$

$$J\ddot{\psi} = aF_{y1} - bF_{y2}$$

$$\alpha_1 = \delta - \frac{v_y + a\dot{\psi}}{v_x}$$

$$F_{yi} = c_i\alpha_i$$

$$a_y = \dot{v}_y + v_x\dot{\psi}$$

$$\alpha_2 = -\frac{v_y - b\dot{\psi}}{v_x}$$

$$\dot{v}_y = -v_x\dot{\psi} + \frac{1}{m}\left(c_1\delta - c_1\frac{v_y + a\dot{\psi}}{v_x} - c_2\frac{v_y - b\dot{\psi}}{v_x}\right)$$

$$\ddot{\psi} = \frac{1}{J}\left(ac_1\delta - ac_1\frac{v_y + a\dot{\psi}}{v_x} + bc_2\frac{v_y - b\dot{\psi}}{v_x}\right)$$

Three phase method

- Structuring
  - Divide into subsystems
  - Inputs, outputs, internal variables?
- Basic equations
  - Conservation laws
  - Constitutive relations
- Form state-space model
  - Choose state variables
  - Form

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  - Form $\dot{x} = \dots$

Variable change: $x_1 = v_y, x_2 = \dot{\psi}, u = \delta$ *and*
$y_1 = x_1,\ y_2 = x_2$

$$\dot{x}_1 = -v_x x_2 + \frac{1}{m}\left(c_1 u - c_1 \frac{x_1 + a x_2}{v_x} - c_2 \frac{x_1 - b x_2}{v_x}\right)$$

$$\dot{x}_2 = \frac{1}{J}\left(a c_1 u - a c_1 \frac{x_1 + a x_2}{v_x} + b c_2 \frac{x_1 - b x_2}{v_x}\right)$$

Three phase method

- Structuring
  - Divide into subsystems
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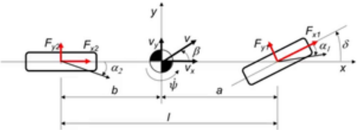
Rewrite in matrix form:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} -\frac{c_1 + c_2}{m v_x} & \frac{-a c_1 + b c_2}{m v_x} - v_x \\ -\frac{a c_1 - b c_2}{J} & -\frac{a^2 c_1 + b^2 c_2}{J}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{c_1}{m}\\ \frac{a c_1}{J}\end{bmatrix} u$$

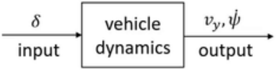
$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 0\end{bmatrix} u$$

Example – Single track vehicle modeling

Derive a mathematical model for studying vehicle handling dynamics:



- Linear single track model:



$$\begin{bmatrix}\dot{v}_y\\ \ddot{\psi}\end{bmatrix} = \begin{bmatrix} -\dfrac{c_1+c_2}{mv_x} & \dfrac{-ac_1+bc_2}{mv_x}-v_x \\ -\dfrac{ac_1-bc_2}{J} & -\dfrac{a^2c_1+b^2c_2}{J}\end{bmatrix}\begin{bmatrix}v_y\\ \dot{\psi}\end{bmatrix} + \begin{bmatrix}\dfrac{c_1}{m}\\ \dfrac{ac_1}{J}\end{bmatrix}\delta$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}v_y\\ \dot{\psi}\end{bmatrix} + \begin{bmatrix}0\\ 0\end{bmatrix}\delta$$

Bibliography

- Karl J. Astrom and Richard M. Murray *Feedback Systems*. Version v3.0i. Princeton University Press. September 2018. Chapter 3.